RICH MODELS

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Research Report No. 89-45 2

May 1989

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May 2, 1989

Abstract

We define a *rich model* to be one which contains a proper elementary substructure isomorphic to itself. Existence, non-structure, and categoricity theorems for rich models are proved. We show that a countable theory with few rich models is categorical in $\aleph_1$. We also consider a stronger notion of richness, and in discussing it prove: "If T is an unstable theory, then for any saturated model M of T of cardinality $> |D(T)|$ there is an elementary chain of length ω of models isomorphic to M, whose union is not $\aleph_1$-saturated."

We propose here to examine a class of models which we shall call *rich models*.

Definition 1 *A model M is said to be* **rich** *if it contains a proper elementary substructure N such that $N \cong M$.*

Clearly, any rich model is infinite. Also, if T is a theory, then any universal model of T is rich (this is most easily seen by noting that if in the definition we replace "substructure" by "extension" then we have defined the same concept). What if T has no universal models of a particular cardinality λ? The existence of a rich model of cardinality λ is then a little less obvious. One method to prove existence is to consider some extended language containing a lot of set theory, a unary predicate (for a model of T) and a function symbol (for an isomorphism

*Supported by a grant from the NSF

to a proper elementary substructure of the predicate). A suitably large special model of this extended theory, and the downward Löwenheim Skolem theorem then complete the construction. The perceptive reader will wonder why we have left the details of this construction so vague—read on.

Universal models have additional properties:

Definition 2 *A model M is said to be* **weakly α-rich** *(α an ordinal) if it contains a strict chain of length α consisting of proper elementary substructures $N_\beta \prec M$ ($\beta < \alpha$) such that $N_\beta \cong M$ for all $\beta < \alpha$.*

Now it is again easily seen that a universal model M for T is weakly α-rich for all $\alpha < \|M\|^+$. The models produced by the construction outlined above need not have this property. But it is significant that we did not require the chain in the definition above to be continuous, and of course for an arbitrary universal model, there is no reason to suppose that it should be. Instead of banging our heads against a succession of walls, let us now introduce the most appropriate definitions of richness, and see what we can say. Perhaps surprisingly, it turns out that by making the conditions more stringent, we are led to much more basic proofs of the essential results.

Definition 3 *A model M is said to be* **α-rich** *(α an ordinal) if it contains a strict continuous chain of length α consisting of proper elementary substructures $N_\beta \prec M$ ($\beta < \alpha$) such that $N_\beta \cong M$ for all $\beta < \alpha$. We say that M is* **very rich** *if it is α-rich for every $\alpha < \|M\|^+$*

Now that we have the right definition, it is time to begin proving theorems.

Theorem 4 *Suppose that T is a theory which has infinite models. Then for every $\lambda \geq \max(\aleph_0, |T|)$ there is a very rich model M of T such that $\|M\| = \lambda$.*

Proof: We may assume, without loss of generality, that T has built in Skolem functions. The Skolem hull M of a chain X of indiscernibles, order isomorphic to λ is very rich.

To see this, note that for each $\beta < \lambda^+$ we can find a continuous strictly increasing sequence $X_i \, (i < \lambda)$ of subsets of X, each of which is order-isomorphic to X. Applying the Skolem functions yields a chain of submodels M_i within M having the required properties. ∎

We presented the proof above, to outline the very simple idea that answering questions about the existence of very rich models often involves similar questions concerning other structures, only without the requirement that the appropriate substructures are necessarily elementary. This idea is also the foundation of the next result.

Theorem 5 *Suppose that T is not superstable, and $\lambda > |T|$ is regular. Then for all $\mu \geq \lambda$, T has 2^λ very rich models of cardinality μ, none elementarily embeddable in another.*

Proof: For the cognoscenti we remark only that the proof will show that the standard set of 2^λ not mutually embeddable models have this property, and that the argument is essentially the same as the one above.

First we consider the case $\mu = \lambda$. We see from [2] VIII.2 (or less technically in [1]) that 2^λ models, none elementarily embeddable in another are constructed as Ehrenfeucht-Mostowski hulls of sets of indiscernibles (for a Skolemization of T). These indiscernibles have the structure of trees $X = {}^{\omega>}\lambda \cup S$ where S is a stationary subset of λ contained in the set $\{\delta < \lambda : \mathrm{cf}\delta = \omega\}$. The set S is construed as a subset of ${}^{\omega}\lambda$ consisting of some fixed increasing sequences η_δ which converge to δ for each $\delta \in S$. No generality is lost by assuming that these sequences η_δ themselves consist of successors of limit ordinals (which amounts to restricting the original set to those ordinals which are multiples of ω^2). To avoid overcomplicating our notation, we will identify the indiscernibles with the corresponding trees.

For such X and $M = EM(X)$, set

$$X_0 = {}^{\omega>}(\lambda - \{\zeta + 2 : \zeta < \lambda\ \zeta \text{ a limit ordinal}\}) \cup S.$$

It is clear that X_0 is isomorphic to X as a tree, (the isomorphism being induced by a bijection between λ and $\lambda - \{\zeta + 2 : \zeta < \lambda\ \zeta \text{ a limit ordinal}\}$) and that the isomorphism may be chosen so that S and all the branches corresponding to the sequences η_δ are fixed. Since these trees are of height ω the isomorphism preserves the height of each node. Further, beginning with X_0 we may also construct a continuous increasing chain of subtrees of X, each isomorphic to X, of arbitrary length $\alpha < \lambda^+$. This is accomplished by forming a descending chain of subsets A_i ($i < \alpha$) of $\{\zeta + 2 : \zeta < \lambda\ \zeta \text{ a limit ordinal}\}$ and then setting

$$X_i = {}^{\omega>}(\lambda - A_i) \cup S$$

Clearly, the Ehrenfeucht-Mostowski hulls of these subtrees witness the fact that M is very rich.

If $\lambda < \mu$ then our task is easier. This time the models may be constructed as Ehrenfeucht-Mostowski hulls of indiscernibles $X \cup Y$ where X is as above and

$$Y = {}^{\omega>}(\mu - \lambda).$$

The entire construction can then be carried out inside Y, and since X and Y are disjoint we do not even have to worry about branches converging to S. ∎

Corollary 6 *If μ, λ, and M are as above then there is an order preserving embedding from $\mathcal{P}(\mu)$ to the set $\{N : N \prec M \text{ and } N \cong M\}$. This embedding can be chosen to preserve arbitrary intersections.*

Proof: All notation is as above. We give the proof only for the case $\mu = \lambda$, since again $\mu < \lambda$ is similar but easier. Let

$$A = \{\zeta + 2 : \zeta < \lambda \ \zeta \text{ a limit ordinal}\}.$$

Take an infinite subset $B \subseteq A$ such that $|A - B| = \lambda$ and define an order reversing injection from $\mathcal{P}(\lambda)$ to $\mathcal{P}(A)$ by taking any bijection f from λ to $A - B$, and, for $T \subseteq \lambda$ define:

$$A_T = A - \{f(i) : i \in T\}.$$

Let

$$X_T = {}^{\omega>}(\lambda - A_T) \cup S.$$

Finally, set $M_T = EM(X_T)$. Observe that for every pair of finite sequences $\bar{a}$, $\bar{b} \subseteq {}^{\omega>}A$, there is an automorphism φ of X fixing $\bar{a}$, and mapping $\bar{b}$ into $\bar{a} \cup X_\emptyset$ (such an automorphism is induced by any permutation f of λ which fixes the finite set of elements which occur in $\bar{a}$, and maps any others which occur in $\bar{b}$ to B). Thus if s and T are Skolem terms, and

$$s(\bar{a}) = t(\bar{b})$$

then

$$s(\bar{a}) = t(\varphi(\bar{b}))$$

(since X is a tree of indiscernibles with respect to the Skolem functions). This implies that for $\mathcal{T} \subseteq \mathcal{P}(\lambda)$,

$$\cap_{T \in \mathcal{T}} EM(X_T) = EM(\cap_{T \in \mathcal{T}} X_T)$$

(one containment is obvious, and the other follows from the fact that ${}^{\omega>}B \subseteq X_T$ for all $T \subseteq \lambda$). ∎

Note that if T has Skolem functions then the embedding constructed above also preserves suprema (considered as a map into the set of all elementary substructures of M ordered by inclusion). Note also that any acyclic directed graph of cardinality $\leq \mu$, any semilattice of cardinality $\leq \mu$ and any distributive lattice with $\leq \mu$ prime ideals can be embedded in $\mathcal{P}(\mu)$ and hence into $\{N :$

N -< *M* and *N* ≃ *M)* for 2^A pairwise non-elementarily embeddable models *M* of cardinality //.

With a little more care and a different sort of tree then using results in [4] we can prove:

Theorem 7 *If T is superstable but not totally transcendental, then for each uncountable cardinal X > \T\ there are at least* mmp2,2^A) *very rich models of T of cardinality X.*

Proof: See [4] for the appropriate trees. •

Now we can use this to prove:

Theorem *SIfT is a countable theory, and for some uncountable cardinal X there is a unique 1-rich model of T of cardinality X (up to isomorphism of course), then T is categorical in every uncountable cardinal.*

Proof: By Theorems 5 and 7 the hypotheses imply that *T* is totally transcendental In turn this means that the unique 1-rich model of cardinality *X* is the unique universal model in this cardinality and is saturated. Using a result of Shelah ([2] IX.1.14(2)) this implies that *T* is unidimensional. But a unidimensional and totally transcendental theory is categorical in every uncountable cardinal (see [2] IX.1.8). •

If T is incomplete then it is possible to have small numbers of rich models without categoricity. For example take the theory *T* whose models are the disjoint union of an algebraically closed field of characteristic 0, and a set with *n* or fewer elements. Clearly this theory has n -f 1 rich models in every infinite cardinal.

Corollary 9 *Under the assumptions of the theorem above, T has a unique countable rich model.*

Proof: In [2] IX.2.2 it is shown that the isomorphism types of countable models of a theory which is categorical in *u* form **a** chain M_n $(n \leq$ a;) so that M_n+\ is a prime extension of M_n. Under these circumstances, only M_w can be rich. •

In the light of all these positive results concerning richness it is perhaps natural to define a stronger notion (we were only kidding when we said that the previous definition would be the last one).

Definition 10 *A model M is said to be* **strongly α-rich** *if, it is α-rich, and for all $\beta < \alpha$, and every strict continuous chain $M_i\,(i < \beta)$ of proper elementary submodels of M, each isomorphic to M, $\bigcup_{i<\beta} M_i \cong M$*

The notion strongly α-rich is essentially the conjunction of α-rich, and $< \alpha$-strongly limit in the class of models isomorphic to M (see [3]). To prove the existence of strongly α-rich models one may use the conjunction of a preservation theorem, and a uniqueness theorem. Thus for example:

Proposition 11 *Let T be a theory, and let λ be a cardinal, $cf(\lambda) = \omega$. If there is a special model M of T of cardinality λ then M is strongly ω-rich.*

Proof: Special models of a particular cardinality, when they exist, are unique. Further, the union of a chain of $\mathrm{cf}\,\lambda$ special models of cardinality λ is special. ■

Similarly we have:

Proposition 12 *Let T be a countable, superstable theory, and let $\lambda \geq \aleph_1$. If there is a saturated model M of T with $\|M\| = \lambda$ them M is strongly α-rich for every $\alpha < \lambda^+$.*

Proof: See [2] III.3.11 for the appropriate preservation theorem. ■

However, on the negative side we get:

Proposition 13 *Let T be an unstable theory. If M is a saturated model of T and $\|M\| > 2^{|T|}$ then M is not strongly ω-rich. In fact, there is an elementary chain of models $M_n\,(n \in \omega)$, each isomorphic to M, but $\cup_{n\in\omega} M_n$ is not $\aleph_1$-saturated.*

Proof: Let $\lambda = \|M\|$. Since the theory T is unstable, there exists $n < \omega$, a formula φ with $2n$ free variables, and an infinite subset of $|M|^n$ which is linearly ordered by φ. It will be helpful, and not misleading, to assume that $n = 1$, and to denote $\varphi(x, y)$ by $x < y$. Add to $L(T)$ a new unary predicate U, and form the theory

$$T' = T \cup \text{“}U \text{ is infinite”} \cup \text{“}< \text{ linearly orders } U\text{”} \cup \text{“No proper superset of } U \text{ is linearly ordered by } <\text{”}$$

Since $\lambda > \|T\| \geq |D(T)|$ by [2] VIII.4.7 the existence of a saturated model for T of cardinality λ implies that $\lambda = \lambda^{<\lambda}$ and hence T' will also have a saturated model M_0' such that $||M_0'|| = \lambda$, and in which $|U^{M_0'}| = \lambda$. We define the subset $B \subseteq U^{M_0'}$ to be the smallest final segment of $U^{M_0'}$ such that $U^{M_0'} - B$ has no maximal element (B may be empty).

By compactness, and the universality of saturated models, there is an elementary extension M_1' of M_0' which is saturated, has cardinality λ, and contains an element a_1 such that $a < a_1$ for all $a \in U^{M_0'} - B$ and $a_1 < b$ for all $b \in B$. (Note that by the saturation of M_0', no elements can be added to U which lie above any element of B in any case). This process can be continued through ω stages. If we let $M_\omega' = \bigcup_{n\in\omega} M_n'$, and if we take the type p to be

$$\{x > a_i : 1 \leq i < \omega\} \cup \{U(x)\} \cup \{x < b : b \in B\},$$

then p is not realized in M_ω' and hence M_ω' is not even $\aleph_1$-saturated. But, by the maximality of U, $p \restriction L(T)$ is not realized in $M_\omega := M_\omega' \restriction L(T)$. On the other hand, for each $n < \omega$, $M_n := M_n' \restriction L(T)$ is saturated. Since we can embed M_ω into M, this completes the proof. ∎

References

[1] Saharon Shelah. The lazy model theoretician's guide to stability theory. *Logique et Analayse, Nouvelle Série*, 18:241–308, 1975.

[2] Saharon Shelah. *Classification Theory*. North Holland, Amsterdam, 1978.

[3] Saharon Shelah. Classification of non elementary classes II. In J.T. Baldwin, editor, *Classification Theory*, pages 419–497. Springer-Verlag, 1985.

[4] Saharon Shelah. Number of pairwise non-elementarily embeddable models. *Journal of Symbolic Logic*, to appear.