INDISCERNIBLE SEQUENCES IN STABLE MODELS

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Abstract We want to show existence of indiscernible sets in models, without assuming the theory of the model is stable. Among other things we prove the following theorem:

Let M be a model, and let λ be a cardinal satisfying $\lambda^{|L(M)|}=\lambda$. If M does not have the ω-order property then for every $A\subseteq M$, $|A|\leqslant\lambda$, and every $I\subseteq M$ of cardinality λ^+ there exists $J\subseteq I$ of cardinality λ^+ which is an indiscernible set over A.

This is an improvement of a result of S. Shelah.

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Indiscernible sequences and sets are a major tool in model theory. There are many existence theorems (e.g. I 2.8 [Sh1], I 5.2 [Sh2], [Sh3], and [Sh4]). The aim of this paper is to present another existence theorem. We work in first order logic, however since we don't use this fact strongly, and we never use the compactness theorem, our results easily extend to $L_{\omega_1,\omega}$ and other more general contexts (like L_{ω_1,ω_1}).

S. Shelah in (generalizing a result of M. Morley) has shown in Theorem I 2.8 of [Sh1]:

Theorem 1 If T is stable, and $M \models T$ is stable in λ, then for every $A \subseteq M$ of cardinality $\leqslant \lambda$, and every $I \subseteq M$ of cardinality λ^+ there exists $J \subseteq I$ of cardinality λ^+ which is an indiscernible sequence over A.

In Theorems I 5.1, and I 5.2 of [Sh2] he considered a generalization of Theorem 1. Is it possible to waive the assumption that Th(M) is stable? Is it possible to obtain the conclusion of Theorem 1 from a weaker assumption? He had a partial success, proving a theorem which had a weaker conclusion than Theorem 1. Our aim here (see part (1) of the main theorem) is to prove a result which has the same conclusion as Theorem 1. Our argument is quite different from Shelah's. But first we need a definition.

Definition 2 Let k be a positive integer, and let κ be a cardinal number.

(1) M has the <u>(κ,k)- Order property</u> iff there exists a formula $\psi(\mathbf{x};\mathbf{y}) \in L(M)$, and there exists $\{\mathbf{a}_\alpha : \alpha<\kappa\} \subseteq M$, such that for all $\alpha<\kappa$ we have $\ell(\mathbf{x})=\ell(\mathbf{y})=\ell(\mathbf{a}_\alpha)=k$, and

$$\text{for every } \alpha,\beta<\kappa \qquad \alpha<\beta \iff M \models \psi[\mathbf{a}_\alpha;\mathbf{a}_\beta].$$

(3) M has the Order property iff M has the λ-Order property for some infinite λ.

Proposition 3 If $\lambda>\mu\geq\aleph_0$ then "M has the λ-order property" implies "M has the μ-order property".
Proof Trivial.

Notice that in the terminology of [Sh1] T has the order property iff there exists $M\models T$ which has the order property.

Main Theorem Let M be a model, let $\kappa=|L(M)|$, and let λ be a cardinal satisfying $\lambda^{\kappa}=\lambda$.

(1) If M does not have the ω order property then for every $A\subseteq M$, $|A|\leq\lambda$, and every $I\subseteq M$ of cardinality λ^+ there exists $J\subseteq I$ of cardinality λ^+ which is an indiscernible sequence over A.

(2) The conclusion of (1) is valid when χ is a strong limit cardinal, $\lambda^{\chi}=\lambda$, and M fail to have the χ order property.

(3) Let λ,χ be given cardinals such that $\lambda^{\kappa}=\lambda$, and $\lambda>2^{2^{\chi}}\geq\kappa$ If M fail to have the χ^+ order property then or every $A\subseteq M$, $|A|\leq\lambda$, and every $I\subseteq M$ of cardinality λ^+ there exists $J\subseteq I$ of cardinality λ^+ which is an indiscernible sequence over A.

(4) Let χ,λ satisfy $\chi\geq\kappa$, $\lambda^{\chi}=\lambda$, and $\lambda\geq2^{2^{\chi}}$. If M fail to have the χ^+ order property then or every $A\subseteq M$, $|A|\leq\lambda$, and every $I\subseteq M$ of cardinality λ^+ there exists $J\subseteq I$ of cardinality λ^+ which is an indiscernible sequence over A.

Remarks (1) It is possible to get an indiscernible set instead a sequence by proving first the above theorem and then copying the argument from the proof of Lemma II 2.16 in [Sh1].

(2) We work inside a given model M, so all notions are relative to it. E.g. $tp(\mathbf{a},A):=tp(\mathbf{a},A,M)$, and if $A\subseteq M$ then $S(A)=\{tp(\mathbf{a},A) : \mathbf{a}\in M\}$.

(3) Notice that part (2) of the main theorem gives an alternative proof to Theorems 2.1 and 2.2 of [Sh3] (take χ=the Hanf number of $L_{\lambda^+,\omega}$).

(4) Part (4) of the main theorem appears implicitly in chapter 1 of [Sh2], we decided to mention it here since it is obtained easily from the other results, and our argument is sufficiently different than Shelah's.

(5) It is natural to ask whether λ^+ in the statement of the main theorem , can be replaced by any regular cardinal? A partial answer is in Theorem 8.

The main tool to obtain indiscernible sequences is the notion of splitting of types.

Definition 4 Let Δ_1, Δ_2 be sets of formulas, $A\subseteq B$ and $\mathbf{a}$ be given. We say that the type $p\subseteq tp(\mathbf{a},B)$ <u>(Δ_1,Δ_2)</u>- <u>splits over</u> A if there are $\mathbf{b}, \mathbf{c} \in B$ such that $tp_{\Delta_1}(\mathbf{b},A)=tp_{\Delta_1}(\mathbf{c},A)$ and there exists a formula $\varphi(\mathbf{x};\mathbf{y})\in\Delta_2$ such that $\varphi(\mathbf{x};\mathbf{b})\in p$ and $\neg\varphi(\mathbf{x};\mathbf{c})\in p$. When $\Delta_1=\Delta_2=L$ we say p splits over A.

Lemma 5 Let $\varphi(\mathbf{x};\mathbf{y})$ be a formula in L(M), $\psi(\mathbf{y};\mathbf{x}):=\varphi(\mathbf{x};\mathbf{y})$, and $\Delta=\{\varphi,\psi\}$, and let $\{A_\alpha\subseteq M : \alpha<\chi\}$ increasing such that for every $B\subseteq A_\alpha$ such that $|B|<\chi$, $p\in S_\Delta(B)\Rightarrow$ p is realized in $A_{\alpha+1}$. <u>If</u> $\exists p\in S(\bigcup_{\alpha<\chi} A_\alpha)$ such that for every $\alpha<\chi$ $p|A_{\alpha+1}$ $(\{\psi\},\{\varphi\})$-splits over every subset of A_α of cardinality less than χ <u>then</u> M has the χ order property.

Proof Let $\mathbf{d}$ be such that $p=tp(\mathbf{d},\bigcup A_\alpha)$. By induction on $\alpha<\chi$ define $\{\mathbf{a}_\alpha,\mathbf{b}_\alpha,\mathbf{c}_\alpha \in A_{2\alpha+2}\}$. At stage α; let $B_\alpha=\bigcup\{\mathbf{a}_\beta,\mathbf{b}_\beta,\mathbf{c}_\beta : \beta<\alpha\}$.

I.e. there are $a_\alpha, b_\alpha \in A_{2\alpha+1}$ such that $tp_{\{\psi\}}(a_\alpha, B_\alpha)=tp_{\{\psi\}}(b_\alpha, B_\alpha)$ and $\models\varphi[d;a_\alpha]\wedge\neg\varphi[d;b_\alpha]$. Let $c_\alpha \in A_{2\alpha+2}$ be an element realizing the type $tp_{\{\varphi\}}(d, B_\alpha \cup a_\alpha \cup b_\alpha)$.

Let $d_\alpha := a_\alpha{}^\frown b_\alpha{}^\frown c_\alpha$, we want to show that

When $\rho(x_1,x_2,x_3;y_1,y_2,y_3) := [\varphi(x_1;y_2)\longleftrightarrow\varphi(x_1;y_3)]$, we have that

$$\beta<\alpha \iff M\models\rho[d_\beta;d_\alpha].$$

Clearly it is enough to show that $\beta<\alpha \iff M\models\varphi[c_\beta;a_\alpha]\longleftrightarrow\varphi[c_\beta;b_\alpha]]$,

(a) Since $\models\varphi[d;a_\alpha]\wedge\neg\varphi[d;b_\alpha]$, by definition of c_α we have that

$\alpha\leqslant\beta \Rightarrow \models\varphi[c_\beta;a_\alpha]\wedge\neg\varphi[c_\beta;b_\alpha]$.

(b) By the choice of a_α and b_α we have in particular $tp_{\{\psi\}}(a_\alpha, c_\beta)=tp_{\{\psi\}}(b_\alpha, c_\beta)$ (when $\beta<\alpha$). So we have $\beta<\alpha \Rightarrow \models\varphi[c_\beta;a_\alpha]\longleftrightarrow\varphi[c_\beta;b_\alpha]$. $\square_5$

Lemma 6 (1) If M does not have the ω- order property then for every λ satisfying $\lambda^\kappa=\lambda$, M is stable in λ. Moreover for every finite set of formulas Δ, M is Δ- μ -stable for every $\mu\geqslant\aleph_0$.

(2) When χ is a strong limit cardinal, and $\lambda^\chi=\lambda$ the conclusion of (1) follows if M fail to have the χ order property.

(3) If M does not have the χ^+ order property then for every λ satisfying $\lambda^\kappa=\lambda$, and $2^{2^\chi}\leqslant\lambda^\chi=\lambda$ M is stable in λ. Moreover for every finite Δ M is Δ-μ stable for μ satisfying $2^{2^\chi}\leqslant\mu^\chi=\mu$.

Proof (1) Suppose there exists $A\subseteq M$ such that $|S(A)|>\lambda=|A|$.

Let λ be a cardinal satisfying $\lambda^\kappa=\lambda$, let A be of cardinality λ such that $|S(A)|\geqslant\lambda^+$. Fix $\{\varphi_i : i<\kappa\}=L(M)$.

Consider the natural function $f:S(A)\rightarrow \times\{S_{\{\varphi_i\}}(A) : i<\kappa\}$.

It is easy to check that f is one to one. Hence $|\times\{ S_{\{\varphi_i\}}(A) : i<\kappa \}|\geqslant\lambda^+$.

Since $\lambda^\kappa=\lambda$ there exists $i_0<\kappa$ such that $|S_{\{\varphi_{i_0}\}}(A)| \geqslant \lambda^+$ Let $\varphi:=\varphi_0$.

Clearly it is enough to show:

Claim For every $\mu\geqslant\aleph_0$, if there exists $A\subseteq M$ such that $|A|=\mu$ and $|S_\varphi(A)|\geqslant\mu^+$ then M has the ω- order property.

Proof Suppose that A refute the claim. Since $|S_{\{\varphi\}}(A)| \geqslant \mu^+$ fix $\{\mathbf{a}_i: i<\mu^+\}$ be such that $i\neq j\Rightarrow tp_{\{\varphi\}}(\mathbf{a}_i,A)\neq tp_{\{\varphi\}}(\mathbf{a}_j,A)$.

Define $\{A_n : n<\omega\}$ by induction on $n<\omega$ with the following properties:

(i) $A\subseteq A_n\subseteq A_{n+1}$,

(ii) $|A_n|=\mu$, and the main requirement:

(iii) for every finite $B\subseteq A_n$, and for every $p\in S_{\{\varphi(\mathbf{x};\mathbf{y})\}}(B)\cup S^{\ell(\mathbf{y})}{}_{\{\varphi(\mathbf{y};\mathbf{x})\}}(B)$ the type p is realized by a sequence from A_{n+1}.

Since $|S_{\{\varphi(\mathbf{x};\mathbf{y})\}}(B)\cup S^{\ell(\mathbf{y})}{}_{\{\varphi(\mathbf{y};\mathbf{x})\}}(B)|$ is finite the inductive definition of sets as above can be carried out.

Sub Claim There exists $i<\mu^+$ such that for every $n<\omega$ and every finite $B\subseteq A_n$ the type $tp_{\{\varphi\}}(\mathbf{a}_i,A_{n+1})$ $(\{\psi(\mathbf{y};\mathbf{x})\},\{\varphi(\mathbf{x};\mathbf{y})\})$-splits over B

Proof For the sake of contradiction suppose that for every $i<\mu^+$ there are $n(i)<\omega$ and a $B_i\subseteq A_{n(i)}$ of cardinality less than ω such that the type $tp_{\{\varphi\}}(\mathbf{a}_i,A_{n(i)+1})$ does not $(\{\psi(\mathbf{y};\mathbf{x})\},\{\varphi(\mathbf{x};\mathbf{y})\})$-split over B_i.

Since μ^+ is regular, and there are at most μ many finite subsets of $A_{n(i)}$ there exist $S\subseteq\mu^+$ of cardinality μ^+, $n_0<\omega$ and $B\subseteq A_{n_0}$ such that $i\in S \Rightarrow n(i)=n_0$ and $B_i=B$. Since $B\subseteq A_{n_0}$ is finite, by requirement (iii) there exists a finite set C, $B\subseteq C\subseteq A_{n_0+1}$ such that every peS ℓ(y) {VI/(M-X)}^ ;s rea,,zecl bU an element of C. For i<jj+ let Pj;=tp(aj,A$_{n_0+1}$). Another use of the pigeon hole principle gives an S'QS of cardinality \i+ such that i*jeS'=>pj|C=pj|C; denote the latter type by p. Let <x*£eS\ since P^IA^PRIA there exists aeA such that f=tp[a;a] and l=--<p[a$_R$;a] . By the choice of C there exists a'eC such that tpr^(u·x yi(a,B)=tpr^(u·x\j(a\B) , since p does not split over B we have that <Kx;a)ep<=>-<p(x;a')ep ; since l^ffa^a] and t=-*<p[a«;a] also f=<p[a ;a'] and t=--f[a$_R$;a[1]]. But the last conjunction contradicts

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p lC=p$_R$|C . Let i<ji+ be from the subclaim, apply Lemma 5 to obtain the order property.

(2) When X 1S strong limit repeat the argument of (1) when u> is replaced by X and "finite" is replaced by "of cardinality less than X". The assumption that X*=X is used in (iii), the assumption that X is strong limit is used in the choice of C.

(3) As in (1) we want to use Lemma 5 to obtain the order property from instability in ji. Repeat the proof of the Claim with the following changes: define {A^ = cx<X+) increasing and continuous such that (i) ACA . (ii) |A |=ji, and (iii) for every BCA of cardinality <X p∈S$_{\{\varphi(x;y)\}}^{cx}$(B)US ℓ(y) {P(U'x)}^ the ^Pe P ,s real*zed bU a sequence from A^ , . Since ISf.pCx^MCBWsWy^^^j^B)! < Z^KBSA^ = |B|≤χ}|=2$^{\chi}\cdot\mu^{\chi}=\mu^{\chi}$ the construction is possible. After fixing BGA^$_0$ as in the subclaim, choose CCA$_{(XQ}$ of cardinality <2^. Since the number of {f}- types over C is at most $2^{2^{X}}$ which is less or equal to \i, the choice of S'Cji+ is possible. The rest is as before. n$_6$

Notation We call $N \prec M$ $\_\kappa^+$ relatively saturated iff for every $B \subseteq N$ of cardinality κ every type over B which is realized in M is realized in N. We denote this by $N \prec_\kappa M$. When N is relatively κ saturated we denote this by $N \prec_{<\kappa} M$.

Lemma 7 (1) Suppose M does not have the ω order property. Let $N \prec_\kappa M$ of cardinality λ (when λ satisfying $\lambda^\kappa = \lambda$). If $p \in S(N)$ then there exists $B \subseteq N$ of cardinality at most κ such that p does not split over B.

(2) Let χ be a strong limit cardinal such that $\lambda^\chi = \lambda$. If M does not have the χ order property then for every $N \prec_\chi M$, and every $p \in \mathbf{S}(N)$ there exists $B \subseteq N$ of cardinality at most χ such that p does not split over B.

(3) Let μ, κ be cardinals such that $\mu^\chi = \mu \geq \kappa + 2^{2^\chi}$. Suppose M does not have the χ^+ order property, and let $N \prec_\mu M$. If $p \in \mathbf{S}(N)$ there exists $B \subseteq N$ of cardinality at most μ such that p does not split over B.

(4) Let $\chi \geq \kappa$ $(= |L(M)|)$, and $N \prec_\chi M$. Suppose that M fail to have the χ^+ order property. If $p \in \mathbf{S}(N)$ then there exists $B \subseteq N$ of cardinality at most χ such that p does not split over B.

Proof (1) If for every $B \subseteq N$ such that $|B| \leq \kappa$ p splits over B, we can define by induction on $i < \kappa^+$ $A_i \subseteq N$ such that $|A_i| \leq \kappa$, for every finite Δ, every $p \in S_\Delta(A_i)$ is realized in A_{i+1}, and $p|A_{i+1}$ splits over A_i. By Lemma 6(1) $|\mathbf{S}_\Delta(A_i)| \leq \kappa$; in general it is possible that there exists a Δ type p over A_i which is not realized in N, so we use here the assumption $N \prec_{\kappa} M$. Suppose φ_i is the formula which exemplifies the fact that $p|A_{i+1}$ splits over A_i. There exists $S \subseteq \kappa^+$ of cardinality κ^+ and a formula φ such that $i \in S \Rightarrow \varphi_i = \varphi$. By renumbering, we may assume without loss of generality that $S = \kappa^+$. Since $\{A_\alpha : \alpha < \kappa^+\}$ and φ satisfy the hypothesis of Lemma 5, M has the κ^+- order property. Hence by Proposition 3 M has the ω order property.

(2) Similar to (1).

(3) Define $\{A_i \subseteq N : i < \mu\}$ $|A_i| = \mu$, such that for every finite Δ $p \in S(A_i)$ p is realized in A_{i+1}. This is possible by Lemma 6(3), as before now we can get the μ^+ order property.

(4) Carry out the argument of Lemma 5 inside the model N. For the sake of contradiction suppose that $p \in S(N)$ is such that for every $B \subseteq N$ of cardinality χ p splits over B. Define $\{\mathbf{a}_\alpha, \mathbf{b}_\alpha, \mathbf{c}_\alpha \in N : \alpha < \chi^+\}$, $B_\alpha = \cup\{\mathbf{a}_\beta, \mathbf{b}_\beta, \mathbf{c}_\beta : \beta < \alpha\}$, and $\varphi_\alpha \in L(M)$ such that the fact that p splits over B_α is demonstrated by φ_α, and $\mathbf{a}_\alpha$, $\mathbf{b}_\alpha$. $\mathbf{c}_\alpha$ is chosen as an element realizing $p|A_\alpha$ (possible since $N \prec_\chi M$). Since $\chi^+ > \kappa$ there exists $S \subseteq \chi^+$ of cardinality χ^+ such that $\alpha \neq \beta \in S \Rightarrow \varphi_\alpha = \varphi_\beta$. The rest is like Lemma 5. $\square_7$

Proof of the Main Theorem (1) Let A, and **I** be given. Suppose $\mathbf{I} = \{\mathbf{a}_i : i < \lambda^+\}$. Define $\{M_i \prec M : i < \lambda^+\}$ increasing and continuous such that

(i) $M_0 \supseteq A$,

(ii) $\|M_i\| = \lambda$,

(iii) $M_{i+1} \prec_\kappa M$, and

(iv) $M_{i+1} \supseteq M_i \cup \{\mathbf{a}_i\}$.

Since $\lambda^{\kappa}=\lambda$, and Lemma 6(1) the construction, can be carried out. By Zermelo - Konig's Theorem since $\lambda^{\kappa}=\lambda$ we have that $\lambda>\kappa$, hence the set $\{\delta<\lambda^{+} : cf\delta=\kappa^{+}\}$ is stationary. Observe that for $\delta<\lambda^{+}$ of cofinality κ^{+} we have that $M_{\delta} \prec_{\kappa} M$.

Consider $f(\delta) = Min\{i<\lambda^{+} : tp(\mathbf{a}_{\delta},M_{\delta})$ does not split over $M_i\}$. By Lemma 7(1) $f(\delta)<\delta$ for every $\delta<\lambda^{+}$ of cofinality κ^{+} . By Fodor's theorem there is a stationary set S and i_0 such that $\delta\in S\Rightarrow f(\delta)=i_0$. Denote $p_{\delta}=tp(\mathbf{a}_{\delta},M_{\delta})$. Since by Lemma 6 M is stable in λ, there is $S'\subseteq S$ of cardinality λ^{+} such that $\alpha,\beta\in S' \Rightarrow p_{\alpha}|M_{i_0+1}=p_{\beta}|M_{i_0+1}$.
For $\delta\in S'$ let $B_{\delta}\subseteq M_{i_0}$ be the set provided by Lemma 7 i.e. p_{δ} does not split over B_{δ}. Since the number of subsets of M_{i_0} of cardinality κ is λ^{κ} which is λ by the assumption on λ; there exists $S''\subseteq S'$ of cardinality λ^{+} , and $B\subseteq M_{i_0}$ such that $\delta\in S''\Rightarrow B_{\delta}=B$. Recall:

Fact (see [Sh1]) Let $n<\omega$, Let $\Delta\subseteq L(M)$, let A be a set, and let $\mathbf{I}=\{\mathbf{a}_i : i<\alpha\}$ be a set of finite sequences all of the same length. Let $A_i:=A\cup\{\mathbf{a}_j :j<i\}$, and let $p_i:=tp(\mathbf{a}_i,A_i)$. **If** for every $i<\alpha$ p_i does not split over A and $i<j \Rightarrow p_i\subseteq p_j$ **then** $\mathbf{I}$ is an indiscernible sequence over A.
Let $\Delta:=L(M)$. Thus, it is enough to show that $\alpha<\beta\in S''\Rightarrow p_{\alpha}\subseteq p_{\beta}$. Suppose this is not the case, i.e. there exists $\psi(\mathbf{x};\mathbf{a})$ such that $\psi(\mathbf{x};\mathbf{a})\in p_{\alpha}$ and $\neg\psi(\mathbf{x};\mathbf{a})\in p_{\beta}$. Since $M_{i_0+1} \prec_{\kappa} M$ there exists $\mathbf{a}'\in M_{i_0+1}$ such that $tp(\mathbf{a}',B)=tp(\mathbf{a},B)$. Since both p_{α} and p_{β} do not split over B we have that $\psi(\mathbf{x};\mathbf{a}')\in p_{\alpha}$ and $\neg\psi(\mathbf{x};\mathbf{a}')\in p_{\beta}$. Hence $p_{\alpha}|M_{i_0+1}\neq p_{\beta}|M_{i_0+1}$ which is a contradiction to the fact that both α and β are members of S'.

(2) Exercise.

(3) Repeat the argument in (1) using Lemma 7(3), when $\mu := 2^{2^{\chi}}$.

(4) Define $\{M_i : i<\lambda^+\}$ increasing and continuous, $M_{i+1} \prec_{\chi} M$, and $\|M_i\|=\lambda$. Use Lemma 7(4) to show that f is regressive, apply Lemma 6(3) that there few types over M_{i_0}. This completes the proof of the main theorem. □

Theorem 8 Let M a given model, let Δ be a finite set of formulas in L(M). **If** λ is regular greater than $|L(M)|$, $\lambda<\|M\|$, and M fail to have the ω order property (or alternatively χ is a strong limit cardinal, such that $\lambda^{\chi}=\lambda$, and M fail to have the χ order property) **then** for every $A \subseteq M$ of cardinality less than λ, and every $I \subseteq M$ of cardinality λ there exists $J \subseteq I$ of cardinality λ which is an indiscernible sequence over the set A.

Proof By Lemma 6(1), M is Δ-μ stable for every $\lambda \geq \mu \geq \aleph_0$. A similar argument to that in the proof of Lemma 7 shows:

(*) for every $N \prec M$, and every $p \in S_{\Delta}(N)$ there exists a finite $B \subseteq N$ such that p does not split over B.

Now look at the proof of the main theorem, and make the following changes: $I=\{a_i : i<\lambda\}$, define $\{M_i \prec M : i<\lambda\}$ increasing and continuous such that (1) $M_0 \supseteq A$, (2) $\|M_i\| \leq |i|+|A|\cdot|L(M)|$, (3) $M_{i+1} \prec M$, and (4) $M_{i+1} \supseteq M_i \cup \{a_i\}$.

I.e. λ^+ is replaced by λ, κ^+ is replaced by $\aleph_0$, since $\{\delta<\lambda : cf\delta=\aleph_0\}$ is stationary the Fodor lemma argument can be carried out when (*) replaces Lemma 7. When χ is strong limit use the failure of the χ order property to show for every $N \prec_{<\chi} M$, and every $p \in \mathbf{S}_{\Delta}(N)$ there exists a $B \subseteq N$ of nality less than χ such that p does not split over B.

est of the proof is easy. $\square_8$

It is easy to verify that the following variant of the main theorem is

rem 9 Let M be a model, let λ be an inaccessible cardinal er than $|L(M)|$.

(1) If M does not have the ω order property then for every $|A|<\lambda$, and every $I \subseteq M$ of cardinality λ there exists $J \subseteq I$ of nality λ which is an indiscernible sequence over A.

(2) The conclusion of (1) is valid when χ is a strong limit nal, $\lambda^{\chi}=\lambda$, and M fail to have the χ order property.

(3) The conclusion of (1) is valid when M fail to have the χ^{+} property, and $\lambda^{\chi}=\lambda$. $\square_9$

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(3) The conclusion of (1) is valid when M fail to have the X+
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