AN APPROXIMATION RESULT FOR STRONGLY POSITIVE KERNELS

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If μ is finite on R^+, i.e. $\mu \in M(R^+)$, then the sign conditions in (ii) reduce to $R\,\tilde{\mu}(\omega) \geq 0$. A measure μ is obviously of strong positive type if the inequalities in (ii) hold with $\frac{q}{1+\omega^2} I$ on the right-hand sides (instead of 0). For further properties of measures of positive type, see [1, chapter 16].

Strongly positive kernels have their main application in the analysis of Volterra equations. In these applications it is frequently of interest to have access to approximations a_k of a strongly positive kernel a. These approximations should be smoother than a, they should converge to a in a sufficiently strong sense and, in addition, each a_k should be strongly positive with the same constant q as a.

Below we construct such approximations. We formulate three Lemmas; with different smoothness assumptions on the given kernel. Lemma 1 takes $\mu(ds) = a(s)ds$ with a differentiable;; Lemmas 2 and 3 consider kernels with less smoothness. Since the proofs are quite analogous to each other we only give the proof of Lemma 1.

Lemma 1. Assume that

$$a \in AC_{loc}(R^+; \mathbb{C}^{nxn}), \quad a' \in L^1(R^+; \mathbb{C}^{nxn}),$$

and let a be of strong positive type with constant $q > 0$.

Then there exist $\{a_k\}_{k=1}^{\infty}$ satisfying

$$a_k \in C^\infty(R^+; \mathbb{C}^{nxn}),$$

$$\sup_k \| a_k' \|_{L^1(R+)} < \infty, \quad a_k'' \in L^1(R^+; \mathbb{C}^{nxn}),$$

a_k is of strong positive type with constant q, and such that for $k \to \infty$,

$$a_k(t) \longrightarrow a(t) \text{ uniformly on } R^+,$$
$$a_k' \longrightarrow a' \text{ in } L^1(R^+; \mathbb{C}^{n \times n}),$$
$$a_k'(t) \longrightarrow a'(t) \text{ at every Lebesgue point of } a'.$$

Proof. Without loss of generality, take $a(\infty) = 0$ and $q = 1$. Define $a(t) \equiv a'(t) \equiv 0$ for $t < 0$. Note that since $a' \in L^1(R^+)$, then the (distribution) Fourier transform $\tilde{a}$ of a is a function, defined for $\omega \neq 0$. Moreover, the condition $a(\infty) = 0$ implies that the Fourier transform of a has no point mass at the origin. Write $\alpha(\omega) = R\,\tilde{a}(\omega)$.

By the fact that $a \in BUC(R^+) \cap PT(R^+)$ one has, using Bochner's theorem, see [1, chapter 16, Theorem 2.6],

$$\text{(1)} \qquad a(t) = \frac{1}{\pi} \int_R e^{i\omega t} \alpha(\omega)\, d\omega, \quad t \in R^+.$$

In particular, as $0 < |a(o)| < \infty$ and $\alpha \geq 0$, it follows that $\alpha \in L^1(R)$.

Let $\eta(t)$, $t \in R$, be defined by

$$\eta(t) = \frac{1}{\pi t^2}(\cos t - \cos 2t), \quad t \in R \setminus \{0\},$$
$$\eta(o) = \frac{3}{2\pi}.$$

Then $\eta \in C^\infty(R)$, $\eta^{(i)} \in L^1(R)$ for $i = 0, 1, 2, \ldots\ldots$, and $\int_R \eta(t)dt = 1$.

Moreover,

$$\tilde{\eta}(\omega) = \begin{cases} 1, & |\omega| \leq 1 \\ 2 - |\omega| & 1 \leq |\omega| \leq 2, \\ 0 & |\omega| \geq 2. \end{cases}$$

For $k > 0$ and $t \in R$, let $\eta_k(t) = k\eta(kt)$. Clearly

(2) $\| \eta_k \|_{L^1(R)} = \| \eta \|_{L^1(R)}, \quad \eta_k' \in L^1(R)$.

In addition, one has $\tilde{\eta}_k(\omega) = \tilde{\eta}(\frac{\omega}{k})$ and so

(3) $$\tilde{\eta}_k(\omega) = \begin{cases} 1, & |\omega| \le k, \\ 2 - |\frac{\omega}{k}|, & k \le |\omega| \le 2k, \\ 0 & 2k \le |\omega|. \end{cases}$$

Define $f_k(t) = (\eta_k * a)(t)$ for $t \in R$. Then

(4) $f_k \in C^\infty(R ; \mathbb{C}^{nxn})$,

and by (2),

$$\| f_k' \|_{L^1(R)} \le [|a(0)| + \| a' \|_{L^1(R^+)}] \, \| \eta \|_{L^1(R)}.$$

Thus

(5) $$\sup_k \| f_k' \|_{L^1(R)} < \infty.$$

Analogously, $f_k'' = a(o)\, \eta_k' + \eta_k' * a'$, which implies

(6) $f_k'' \in L^1(R ; \mathbb{C}^{nxn})$.

From (3) follows

(7) $$\tilde{f}_k(\omega) = \begin{cases} \tilde{a}(\omega), & |\omega| \le k, \quad \omega \ne 0, \\ [2 - |\frac{\omega}{k}|]\, \tilde{a}(\omega), & k \le |\omega| \le 2k, \\ 0 & 2k \le |\omega|. \end{cases}$$

Define $E(t)$, $t \in R$, by

$$E(t) = \begin{cases} e^{-t} I, & t \ge 0, \\ 0, & t < 0, \end{cases}$$

and $g_k = E - \eta_k * E$. Then

(8) $g_k \in C^\infty(R^+) \cap C^\infty(R^-)$,

with

$$(9)\quad \begin{cases} \sup_k \|g_k'\|_{L^1(R^+)} < \infty & \sup_k \|g'_k\|_{L^1(R^-)} < \infty \\ g_k'' \in L^1(R^+), & g_k'' \in L^1(R^-) \end{cases}.$$

Obviously

$$(10)\qquad R\,\tilde{g}_k(\omega) = \begin{cases} 0, & |\omega| \leq k, \\ \left(\frac{1}{1+\omega^2}\right)\left(\left|\frac{\omega}{k}\right|-1\right) I, & k \leq |\omega| \leq 2k, \\ \frac{1}{1+\omega^2}\, I, & 2k \leq |\omega|. \end{cases}$$

For $t \in R$, write $h_k(t) = f_k(t) + g_k(t)$. Then by (7), (10) and since a is strongly positive with constant 1,

$$R\,\tilde{h}_k(\omega) = \begin{cases} \alpha(\omega), & |\omega| \leq k, \quad \omega \neq 0, \\ \left[2 - \left|\frac{\omega}{k}\right|\right]\alpha(\omega) + \frac{1}{1+\omega^2}\left(\left|\frac{\omega}{k}\right| - 1\right) I \geq \frac{1}{1+\omega^2}\, I, & k \leq |\omega| \leq 2k, \\ \frac{1}{1+\omega^2}\, I, & 2k \leq |\omega|. \end{cases}$$

Thus

$$\frac{1}{1+\omega^2}\, I \leq R\,\tilde{h}_k(\omega) \leq \alpha(\omega), \quad \omega \in R\backslash\{0\}.$$

Define the approximations a_k by

$$a_k(t) = \begin{cases} h_k(t) + h_k(-t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

By (4) - (6), (8), (9),

$a_k \in C^\infty(R^+ ; \mathbb{C}^{n\times n})$, $\sup_k \| a_k' \|_{L^1(R^+)} < \infty$, $a_k' \in L^1(R^+ ; \mathbb{C}^{n\times n})$.

The difference between h_k and a_k is an odd function. Therefore, $\hat{a}_k \overset{\text{def}}{=} R\,\tilde{a}_k = R\tilde{h}_k$ and consequently

(11) $$\frac{1}{1+\omega^2} I \le \hat{a}_k(\omega) \le \hat{a}(\omega), \quad \omega \in R\setminus\{0\}.$$

Thus each $\hat{a}_k$ is of strong positive type with constant 1. Moreover, each a_k is bounded and uniformly continuous, hence Bochner's theorem applies and so

(12) $$a_k(t) = \frac{1}{\pi}\int_R e^{i\omega t}\,\hat{a}_k(\omega)\,d\omega, \quad t \in R^+.$$

By (1), (11), (12) and Lebesgue's dominated convergence theorem,

$$\sup_{t\in R^+} |a(t) - a_k(t)| \le \frac{1}{\pi}\int_R |\hat{a}(\omega) - \hat{a}_k(\omega)|\, d\omega \to 0, \quad k \to \infty.$$

To complete the proof it remains to show that a_k' converges to a'. Write $a_k(t) = [g_k(t) + g_k(-t)] + [f_k(t) + f_k(-t)]$ and let $\check{E}(t) = E(-t)$, $\check{a}(t) = a(-t)$ for $t \in R$. Simple calculations, which use the fact that η is even, yield for t nonzero,

$\frac{d}{dt}[g_k(t) + g_k(-t)] = -E(t)\,\eta_k(t) + (\eta_k * E)(t) + (\eta_k * \check{E})(t)$. Since $\eta_k * E \to (\int_R \eta(s)ds)\,E$, $\eta_k * \check{E} \to (\int_R \eta(s)ds)\,\check{E}$ in $L^1(R)$ (and pointwise for $t \ne 0$) and since $\int_R \eta(s)ds = 1$, it follows that

$$\lim_{k\to\infty} \left\| \frac{d}{dt}[g_k(t) + g_k(-t)] \right\|_{L^1(R^+)} = 0.$$

Analogously,

$$\frac{d}{dt}[\ f_k\ (t) + f_k(-t)] = \eta_k * a' - \eta_k * \overline{a'} \longrightarrow a' - \overline{a'} \quad \text{in } L^1(R),$$

and so

$$(13)\quad \lim_{k \to \infty} \| a' - a_k' \|_{L^1\ (R^+)} = 0.$$

Since we have pointwise convergence in (13) at each Lebesgue point of a', the proof is complete.

For the case where $|a(0+)| = \infty$ it is useful to have the following result:

LEMMA 2. Assume that

$$a \in L^1(R^+\ ;\ \mathbb{C}^{nxn})$$

and let

a be of strong positive type with constant $q > 0$.

Then there exist $\{a_k\}_{k=1}^{\infty}$ satisfying

$a_k \in C^\infty(R^+; \mathbb{C}^{nxn})$, $\sup_k \|a_k\|_{L^1(R^+)} < \infty$, $>_k$

$a_k^{(i)} \in L^1(R^+; \mathbb{C}^{nxn})$, $i = 1, 2, \ldots,$

a_k is of strong positive type with constant $f > 0$,

and such that for $k \longrightarrow \infty$

;
$$a_k \longrightarrow a \quad \text{in} \quad L^1(R^+ ; \mathbb{C}^{nxn}),$$

$$a_k(t) \longrightarrow a(t) \quad \text{at every Lebesgue point of} \quad a.$$

If μ does not satisfy any regularity assumptions the method above only yields a rather weak form of convergence:

LEMMA 3. Let $\mu \in M(R^+ ; \mathbb{C}^{nxn})$ be of strong positive type with constant $q > 0$. Then there exist $\mu_k \in C^\infty(R^+ ; \mathbb{C}^{nxn})$ satisfying

$$\sup_k \|\mu_k\|_{L^1(R^+)} < \infty,$$

$$\mu_k^{(i)} \in L^1(R^+ ; \mathbb{C}^{nxn}), \quad i = 1, 2, \ldots,$$

μ_k is of strong positive type with constant $q > 0$,

and such that

$$\mu_k \longrightarrow \mu \quad \text{in} \quad S'.$$

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[1] G. Gripenberg, S-O. Londen and O. Staffans, Volterra Integral and Functional Equations, Cambridge University Press, to appear.