# THE STRONG PERFECT GRAPH CONJECTURE HOLDS FOR DIAMONDED ODD CYCLE-FREE GRAPHS

by

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## ABSTRACT

We define a diamonded odd cycle to be an odd cycle C with exactly two chords and either

a) C has length five and the two chords are non-crossing; or

b) C has length greater than five and has chords (x,y) and (x,z) with (y,z) an edge of C and there exists a node w not on C adjacent to y and C, but not x.

In this paper, we show that given a diamonded odd cycle-free graph G, G is perfect if and only if G does not have an induced subgraph isomorphic to an odd hole with size greater than three.

## § 1 INTRODUCTION

A zero-one matrix A is perfect if $\{x \in \mathbb{R}^n \mid Ax \leq 1, x \geq 0\}$ has all integer extreme points. Matrix A is balanced if all its submatrices are perfect. A graph G is perfect if its clique-node incidence matrix is perfect. In 1961, Claude Berge conjectured that a graph G is perfect if and only if G has no node induced subgraph that is an odd hole of size greater than or equal to five or the complement of an odd hole of size greater than or equal to five. This conjecture, known as the Strong Perfect Graph Conjecture, remains open today. However the conjecture has been proved for several classes of graphs including triangulated graphs [1], comparability graphs [1], circular arc graphs [11], planar graphs [10], torodial graphs [7], $K_{1,3}$–free graphs [8], and $K_4\backslash e$–free graphs [9], [12]. In this paper we show that the conjecture holds for graphs with no induced diamonded odd cycle. This generalizes the results of Parthasarathy and Ravindra [9] and Tucker [12] on $K_4\backslash e$–free graphs.

**Definition 1**: Let C be an odd cycle. C is a **diamonded odd cycle** if C has exactly two chords and either

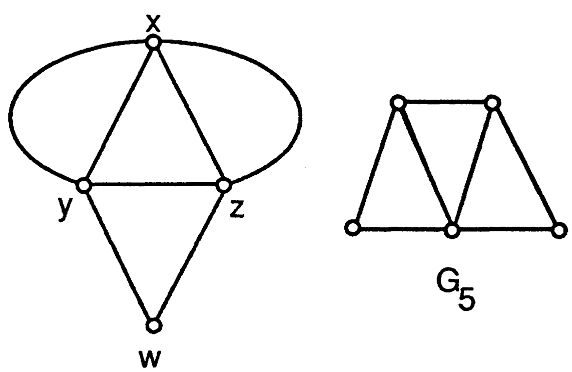



Figure 1

a) C has length 5 and the two chords are non-crossing; or

b) C has length greater than 5 and has chords (x,y) and (x,z) with (y,z) an edge of C and there exists a node w not on C adjacent to y and z but not x (see Figure 1).

We will denote a length 5 diamonded odd cycle by $G_5$.

Let A be a zero-one matrix. In addition to viewing A as the clique-node incidence matrix of a graph G, we can view A as the node-node incidence matrix of a bipartite graph H. H has a node for each row and each column of A with an edge from node i, representing row i, to node j, representing column j, if and only if $a_{ij} = 1$. We will let S be the set of nodes representing the rows of A and T the set of nodes representing the columns of A. Since the rows of A represent cliques in the graph, G, with clique-node incidence matrix A, we will sometimes refer to the nodes in S as **clique nodes**. We will say the bipartite graph H is perfect (balanced) if A is perfect (balanced). Throughout the paper, G will denote a graph with no diamonded odd cycle and A will be G's clique-node incidence matrix. A will have a row for each maximal clique of G only. H will denote a bipartite graph whose node-node incidence matrix is A. We will say H is the **bipartite graph representation** of G.

Since H is bipartite, all cycles of H are even cycles. We will say a cycle C with length 2k is **bi-even** if k is even and **bi-odd** if k is odd. H is balanced if and only if H has no bi-odd holes (Berge [2]). A bi-odd cycle has length congruent to 2 mod 4. In the interest of brevity, the words congruent to will be left out in the future.

For a node u in G, we will let N(u) be u together with the set of nodes adjacent to u. In H, for a node $u \in T$, we will let $N^2(u)$ be the set of nodes at distance less than or equal to two from

u. Note that since u corresponds to a column of A, there will be a node of G, say u′, corresponding to u in H and $N^2(u)$ in H corresponds to N(u′) in G. Throughout this paper we will say G contains a graph G′ when we mean G′ is a node induced subgraph of G. V(G′) will denote the nodes of G′. The complement of a hole (in G) is called an **antihole.**

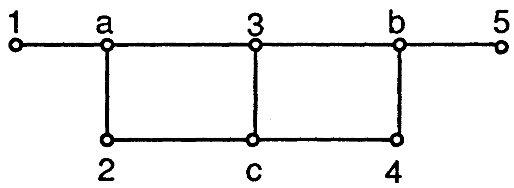
## § 2 THE MAIN RESULTS

**Lemma 1**: G contains no odd antihole of cardinality n with $n \geq 7$.

Proof:

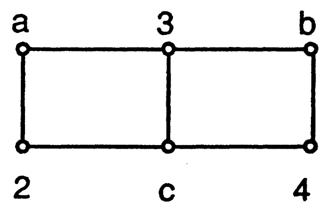
Let G′ be an odd antihole of size n, $n \geq 7$. Label the nodes of G′ such that in the complement (1,2,3,...,n) forms a cycle. The set S = {1,3,5,n,2} induces a $G_5$. ■

The bipartite graph representation of $G_5$ is given in Figure 2 where the nodes labeled by letters are clique nodes and the nodes labeled by numbers are nodes of T.




$G_5^B$

Figure 2




$B_6$

Figure 3

If G has no $G_5$ then the bipartite graph representation of G has no cycles of length 6 with a unique chord. We denote such cycles as $B_6$ (see Figure 3).

It is clear that if H has no $B_6$, then H has no $G_5^B$. It is also true that if H has no $G_5^B$, then H has no $B_6$. This holds because if clique node a is not adjacent to a node of T which is not a neighbor of c, then a does not represent a maximal clique of G. The same is true for b. If a and b have a common neighbor, labeled say 6, which is not a neighbor of c, then the bipartite graph induced by {2,3,4,6,a,b,c} corresponds to a $K_4$ (see Figure 4) and would be represented by a single clique node and four nodes of T. So a and b must each have a neighbor which is not adjacent to any other node of $B_6$.

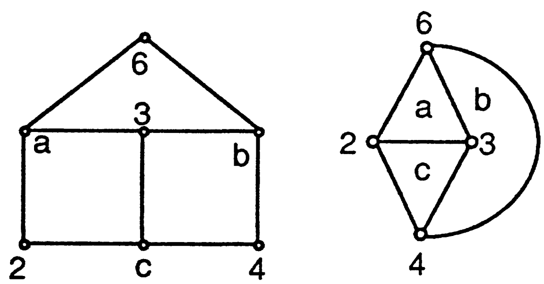


Figure 4

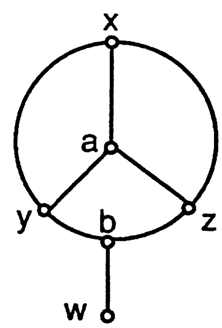


Figure 5a

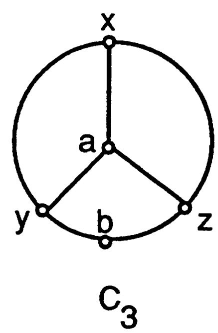


Figure 5b

The bipartite graph representation of a diamonded odd cycle of length greater than five is given in Figure 5a. We will denote the graph given in Figure 5b as $C_3$. If we want to assume G has no diamonded odd cycle of length greater than five then it is sufficient to assume H has no $C_3$ since if b does not have a neighbor different from y and z, then b does not represent a maximal clique.

**Definition 2**: A bi-odd hole C is **minimal** if no subset of its nodes, together with at most three nodes not in C induces a smaller bi-odd hole.

**Definition 3**: A node not belonging to a hole C but having at least two neighbors in C is **strongly adjacent** to C. A node that is strongly adjacent to C and has an odd (even) number of neighbors in C is **odd-strongly** (**even-strongly**) adjacent to C.

**Definition 4**: A bi-odd hole C with length greater than or equal to 10 is an **imperfect bi-odd hole** if there is no clique node strongly adjacent to C with three or more neighbors on C. An imperfect bi-odd hole in H corresponds to an odd hole in G with length at least five.

**Definition 5**: A hole C together with a node v not on C but having at least three neighbors on C form a **wheel** (C,v) with center v. The edges from v to C are the **rays** of the wheel and a subpath from the endnode of one ray of the wheel to the endnode of another ray of the wheel not containing any other neighbor of v is a **sector** of the wheel. The **interior nodes** of a sector S are the nodes of S not adjacent to v (see Figure 6).

In the remainder, we assume that G is a minimally imperfect graph containing no diamonded odd cycles. By Lemma 1, it will suffice to show that G is an odd hole. The technique we will use is to show that if G contains no odd holes, then G has a star cutset. We will then apply Chv£taTs result which says no minimal imperfect subgraph has a star cutset to achieve the desired contradiction. Recall that G is a **minimal imperfect graph** if G is not perfect, but all its induced subgraphs are perfect.

To show that G has a star cutset, we will show that every bipartite graph H containing no B^, no C9, and no imperfect bi-odd holes has a node u e T such that N~(u) contains a cutset of H. To do this, we will use some results of Conforti and Rao to show that there is a node u* € T such that N~(u*) contains all nodes odd-strongly adjacent to a minimal bi-odd hole C. We will then show that N~(u*) contains a cutset, K, which disconnects C. To show K disconnects C, we choose two connected components of C\K and show that if there were a path P connecting

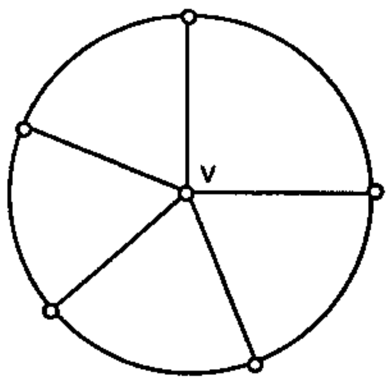



Figure 6

them, then the subgraph induced by $V(C) \cup V(P) \cup \mathcal{S}$ would contain an imperfect bi-odd hole, where $\mathcal{S} = \{s \in S : s$ is odd–strongly adjacent to $C\}$.

Before proving the main results, we will need a few preliminary results.

**Theorem 1** [4]: No minimal imperfect graph has a star cutset.

**Lemma 2** [6]: Let $H$ be a bipartite graph containing no imperfect bi-odd holes. Let $C$ be a minimal bi-odd hole in $H$. All clique nodes odd-strongly adjacent to $C$ have a common neighbor in $C$.

**Lemma 3** [6]: Let $H$ be a bipartite graph containing no imperfect bi-odd holes. Let $C$ be a minimal bi-odd hole in $H$. If $u$ is even-strongly adjacent to $C$, then $u$ has exactly two neighbors in $C$, say $u_1$ and $u_2$, and furthermore there exists a node of $C$ adjacent to both $u_1$ and $u_2$.

Throughout the remainder of the paper, unless otherwise stated, we will assume $H$ contains no $B_6$, no $C_3$, and no imperfect bi–odd holes. If $H$ has no bi-odd holes, then $H$ is balanced and therefore perfect. We will assume $H$ is not balanced. Let $C$ be a minimal bi–odd hole of $H$.

**Lemma 4:** $C$ has length greater than or equal to 10.

Proof:

If C has length 6, label the nodes of C clockwise around C a,1,b,2,c,3 where the nodes labeled with letters are clique nodes and the nodes labeled with numbers are nodes of T. Then nodes 1,2,3 form a triangle in G (see Figure 7) and would all be adjacent to a clique node. So there is a clique node odd–strongly adjacent to C and H contains a $B_6$ (see Figure 8).

Therefore C has length greater than or equal to 10. ■

**Lemma 5:** There exists a node $z \in T \cap C$ such that $N^2(z)$ contains all nodes odd-strongly adjacent to C.

Proof: Postponed to section 3.

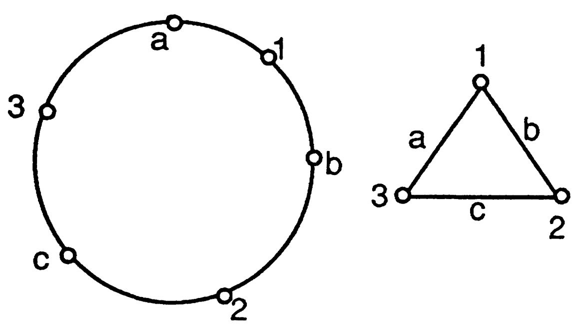



Figure 7

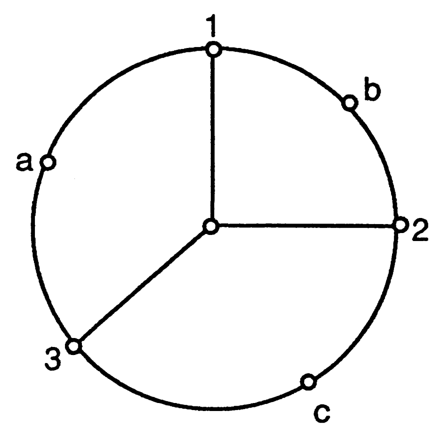



Figure 8

Let Z be the set of nodes in $T \cap C$ such that, for $z \in Z$, $N^2(z)$ contains all nodes odd-strongly adjacent to C. Let $\mathcal{S} = \{s \in S : s$ is odd-strongly adjacent to $C\}$.

Fix $z \in Z$. C is not imperfect, so there is a clique node odd-strongly adjacent to C. Let v be a clique node odd-strongly adjacent to C with the property that when traversing C counterclockwise from z a node adjacent to v is encountered before a node adjacent to any other clique node odd-strongly adjacent to C. Let a and b be the neighbors of z on C and let c (d) be the neighbor of a (b) on C different from z. Let $S_1$ and $S_2$ be the sectors of (C,v) containing z (see Figure 9). Since H does not contain a $B_6$, at least one of $S_1$ or $S_2$ has length greater than two. If both $S_1$ and $S_2$ have length greater than two, let $K = N^2(z) \backslash \{c,d\}$. If one of $S_1$ or $S_2$ has length two, assume without loss of generality that the sector containing a has length two and let $K = N^2(z) \backslash \{d\}$.

**Lemma 6**: Either, (i) For some $z \in Z$, K is a cutset of H with the property that at

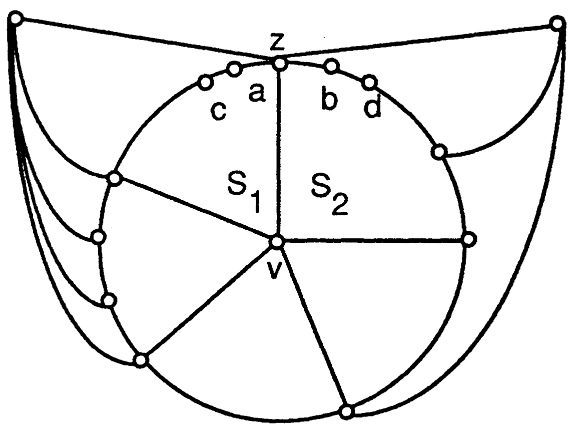



Figure 9

least two connected components of H\K contain a node of T; or

(ii) There exists $z \in Z$ and two connected components of C\K such that if P is a shortest path in H\K connecting these two components, the subgraph of H induced by $V(P) \cup V(C) \cup \mathcal{S}$ contains a minimal bi–odd hole C′ with the property that no $s \in \mathcal{S}$ is odd-strongly adjacent to C′.

Proof: Postponed to section 4.2.

Lemma 6 says that if K is not a cutset of H, then H contains a bi-odd hole. However, the fact that H contains a bi-odd hole does not contradict perfection; we need an imperfect bi-odd hole. The following theorem shows that if K is not a cutset of H, then H contains an imperfect bi-odd hole. But H does not contain an imperfect bi-odd hole, so K is a cutset of H.

**Theorem 2**: There exists $z \in Z$ such that $N^2(z)$ contains a cutset, K, of H with the property that at least two of the connected components of H\K contain a node of T.

Proof:

Suppose the theorem is not true. By Lemma 6, H contains a minimal bi-odd hole C′ with the property that no $s \in \mathcal{S}$ is odd-strongly adjacent to C′. Since H has no imperfect bi–odd holes, there is a clique node x in H which is odd–strongly adjacent to C′. x is adjacent to three or more nodes of $T \cap C'$. All nodes of $T$ on $C'$ are either on $P$ or $C \cap C'$. So for any clique node odd–strongly adjacent to $C'$ either $N(x) \cap C' \subset P$ or $N(x) \cap C' \cap C \neq \emptyset$. Let $p_1$ and $p_2$ be the nodes of $P \cap C$ and let $c_i$ be the component of $C$ containing $p_i$, $i = 1,2$. Figures 10a-10d illustrate the possible configurations for $x$. Note that $x$ has at most two neighbors on $C$ since $x$ is not in $\mathcal{S}$ and so is not odd-strongly adjacent to $C$ and if $x$ is even-strongly adjacent to $C$, $x$ has two neighbors on $C$ by Lemma 3. Also, $x$ is not adjacent to $z$ since $x$ has a neighbor on $P$.

In figures 10a and 10b, there is a $(c_1,c_2)$-path containing $x$ that is shorter than $P$; contradicting the choice of $P$. In figure 10c, $(C',x)$ is a $C_3$. In figure 10d, the $(x_3,p_1)$-subpath of $P$ must have length less than or equal to two, since otherwise there would be a shorter $(c_1,c_2)$-path than $P$. If the $(x_3,p_1)$-subpath of $P$ has length two, replace the path $(x_1,y,x_2)$ on $C$ with the path $(x_1,x,x_2)$, shortening $P$. If the $(x_3,p_1)$-subpath of $P$ has length one, again replace the path

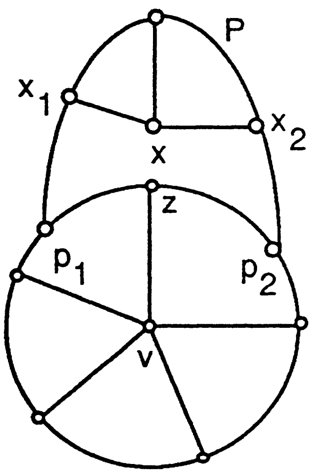



Figure 10a

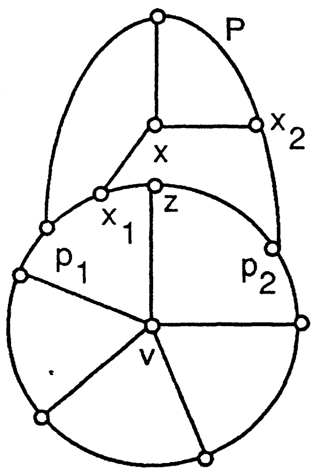



Figure 10b

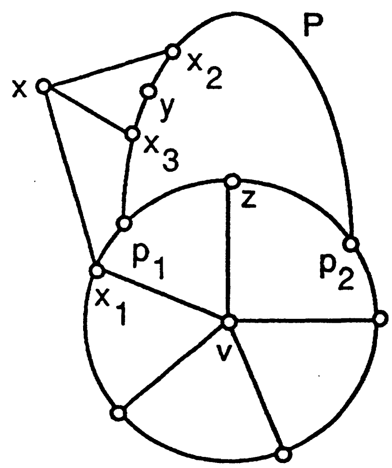



Figure 10c

$(x_1,y,x_2)$ on C with the path $(x_1,x,x_2)$ forming C* (see figure 10e). $x_3$ is even-strongly adjacent to C* so by Lemma 3, $p_1$ is adjacent to $x_1$ and $\{x_1,y,x_2,x,x_3,p_1\}$ induces a $B_6$. ■

**Theorem 3**: The strong perfect graph conjecture holds for the class of graphs not containing a diamonded odd cycle.

Proof:

Let G be a minimal imperfect graph not containing a diamonded odd cycle. By Lemma 1, it suffices to show G is an odd hole of size greater than or equal to 5. Assume not. Let H be the bipartite graph representation of G. H is not balanced, so H has a bi-odd hole. G has no diamonded odd cycle and no odd hole of size greater than or equal to 5, so H has no $B_6$, no $C_3$, and no imperfect bi-odd hole. By Theorem 2 H has a node $u \in T$ such that $N^2(u)$ contains a

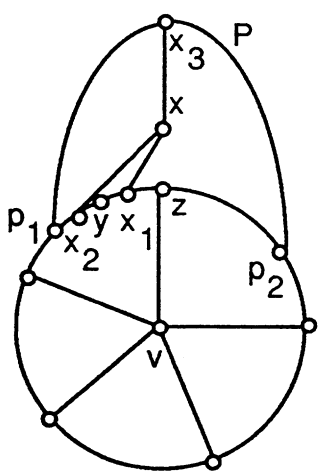


Figure 10d

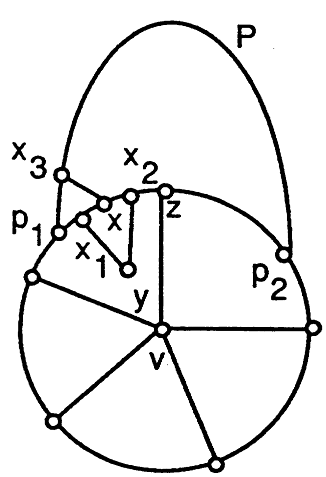


Figure 10e

cutset K. Also, there exist nodes tj and t2 in T with tj in one component of H\K and t^ in another. Then the node u' of G corresponding to u is such that N(u') contains a star cutset of G. By Theorem 1 this contradicts the choice of G. _

## § 3 THE PROOFS OF LEMMAS 5 AND 6

Before proving Lemma 5 we will state some results of Conforti and Rao that we will need in the proof.

Let K be a bipartite graph containing no imperfect bi-odd holes, let C be a minimal bi-odd hole and let w be a clique node odd-strongly adjacent to C Conforti and Rao have shown the following:

**Lemma 7** [6]: If u is a clique node odd-strongly adjacent to C with a neighbor in the interior of a sector of (C,w), then u has at least one other neighbor in the same sector.

**Lemma 8** [6]: All nodes in T odd-strongly adjacent to C have a common neighbor in C.

**Lemma 9** [6]: If |C|≥ 10 then for every node u *e* T odd-strongly adjacent to C but not adjacent to w, u has exactly one neighbor u* in some sector Sj of C and an even number of neighbors in an adjacent sector $S_{i+1}$. Moreover, u* is adjacent to the common node in the two sectors.

**Lemma 10** [6]: If $|\mathcal{C}| \geq 10$ then for every node $u \in T$ odd-strongly adjacent to $\mathcal{C}$ but not adjacent to w, the nodes of $N(u) \cap \mathcal{C}$ are contained in the same two sectors, say $S_{i-1}$ and $S_i$, of the wheel $(\mathcal{C}, w)$.

**Lemma 11** [6]: If $u \in T$ is odd-strongly adjacent to $\mathcal{C}$, then one of the following holds:

(i) u is adjacent to all clique nodes odd-strongly adjacent to $\mathcal{C}$; or

(ii) u has a neighbor, say u*, in $\mathcal{C}$ such that all clique nodes odd-strongly adjacent to $\mathcal{C}$ are adjacent to one of the two neighbors of u* in $\mathcal{C}$.

Now, the proof of Lemma 5. Recall:

**Lemma 5:** There exists a node $z \in T$ such that $N^2(z)$ contains all nodes odd-strongly adjacent to C.

Proof:

We will consider two cases:

i) There is no node of T odd-strongly adjacent to C satisfying (ii) of Lemma 11;

ii) There is a node of T odd-strongly adjacent to C satisfying (ii) of Lemma 11.

Case (i): Let z be the node of C adjacent to all clique nodes odd-strongly adjacent to C. Every node, $x \in T$, odd-strongly adjacent to C is adjacent to v. z is adjacent to v, so $x \in N^2(z)$.

Case (ii): Let z be the neighbor of u* described in Lemma 11 (ii). z is adjacent to all clique nodes odd-strongly adjacent to C. In particular $v \in N(z)$. Let $S_k$, $S_{k+1}$ be the sectors of (C,v) containing z. By Lemma 10, if $x \in T$ is odd-strongly adjacent to C, then either x is adjacent to v or $N(x) \cap C$ is contained in $S_k \cup S_{k+1}$, so by Lemma 9 a neighbor of x is adjacent to z and $x \in N^2(z)$. ■

**Definition 6**: Let r and s be clique nodes odd-strongly adjacent to C. Let p (q) be a neighbor of r (s) on C. We will say p is **next to** q if there exists a (p,q)–path on C containing no neighbors of any clique node odd-strongly adjacent to C.

**Definition 7**: A cycle $\mathcal{C}$ is **starred** if its set of chords satisfies the following properties:

(a) there exist two nodes x and y in $\mathcal{C}$, called the **stars of** $\mathcal{C}$, such that every chord of $\mathcal{C}$ has either node x or node y but not both as its endpoint;

(b) no other node of $\mathcal{C}$ is the endpoint of two distinct chords;

(c) no two endpoints of chords are adjacent.

**Theorem 4** [5]: Let $\mathcal{C}$ be a starred cycle. If the graph induced by the nodes of $\mathcal{C}$ has no bi-odd holes, then $\mathcal{C}$ has length 2 mod 4 if and only if $\mathcal{C}$ has an odd number of chords.

On to Lemma 6. Recall:

**Lemma 6**: Either, (i) For some $z \in Z$, K (as defined on page 9) is a cutset of H with the property that at least two connected components of $H \backslash K$ contain a node of T; or

(ii) There exists $z \in Z$ and two connected components of $C \backslash K$ such that if P is a shortest path in $H \backslash K$ connecting these two components, the subgraph of H induced by $V(P) \cup V(C) \cup \mathcal{S}$ contains a minimal bi–odd hole $C'$ with the property that no $s \in \mathcal{S}$ is odd-strongly adjacent to $C'$.

The proof of Lemma 6 involves several cases, but the basic argument is the same in each case. We will present one case in detail here and sketch the rest. Complete details can be found in [3].

The main ideas of the proof are as follows. Assume K is not a cutset of H. First, we will carefully choose the two components of $C \backslash K$ that P will connect. We will choose two connected components so each component will be a path, say $P_1$ and $P_2$, containing no neighbors of clique nodes odd–strongly adjacent to C. We will chose a shortest path P from $P_1$ to $P_2$ in H\K. P may contain nodes of $C \setminus (P_1 \cup P_2)$. We will then show that P does not contain any nodes strongly adjacent to $P_1 \cup P_2$. In particular, the endpoints of P, $s_1$ and $t_1$, are not strongly adjacent to $P_1 \cup P_2$. We will let s (t) be the node of $P_1$ ($P_2$) adjacent to $s_1$ ($t_1$). We will then consider four (s,t)-paths in the graph induced by $V(P_1) \cup V(P_2) \cup \mathcal{S}$ and the cycles closed by these (s,t)–paths with P. All four cycles are starred cycles, so Theorem 4 applies. Three of the four cycles will have the same length mod 4 and the fourth will have a different length mod 4. By Theorem 4 the three cycles with the same length mod 4 must have the same number of chords mod 2 and the fourth must have a different number of chords mod 2. Which three cycles have the same length mod 4 varies depending on which sides of the bipartition s and t are on, but in every case, the number of chords must satisfy an equation which is impossible to satisfy. So by Theorem 4 one of the cycles must contain a bi–odd hole. This bi–odd hole has the property that no $s \in \mathcal{S}$ is odd–strongly adjacent to it.

Proof of Lemma 6:

Assume without loss of generality that $S_1$ contains no neighbors of clique nodes odd-strongly adjacent to C. Let $z_1$ ($z_2$) be the endpoint of $S_1$ ($S_2$) different from z. For the purposes of the argument, two clique nodes u and w with $N(u) \cap V(C) = N(w) \cap V(C)$ are redundant, so we will assume that $N(u) \cap V(C) \neq N(w) \cap V(C)$ for all clique nodes $u$ and $w$ odd-strongly adjacent to $C$.

Case I): $S_1$ and $S_2$ both have length greater than two and no clique node odd-strongly adjacent to $C$ has four or more neighbors in $S_2$.

Let $\mathcal{P}$ be the collection of chordless paths from $S_1$ to $S_2$ in $H \setminus K$. If $\mathcal{P} = \emptyset$, then $K$ is a cutset of $H$ disconnecting $S_1$ from $S_2$. $S_1$ and $S_2$ both have length greater than two so the components of $H \setminus K$ containing $S_1$ and $S_2$ each contain a node of $T$. If $\mathcal{P} \neq \emptyset$ let $P$ be the shortest $(S_1,S_2)$-path in $\mathcal{P}$. Let $s_1$ $(t_1)$ be the endpoint of $P$ adjacent to $S_1$ $(S_2)$. If $s_1$ is even strongly adjacent to $C$ with both nodes of $N(s_1) \cap C$ in $S_1$, replace $S_1$ so that $s_1$ is in $S_1$ and shorten $P$. If $s_1$ is even-strongly adjacent to $C$ with one neighbor in $S_1$ and the other neighbor in an adjacent sector, then if $s_1$ is not adjacent to $v$ there is a smaller bi-odd hole including $s_1$ contradicting the minimality of $C$ and if $s_1$ is adjacent to $v$, $s_1 \in N^2(z)$. A similar argument holds for $t_1$, so we can assume without loss of generality that $s_1$ and $t_1$ are not strongly adjacent to $C$. Let $s$ $(t)$ be the node of $S_1$ $(S_2)$ adjacent to $s_1$ $(t_1)$ (see Figure 11).

Consider the (s,t)-paths in C: $P_{ab} = (s,\ldots,a,z,b,\ldots,t)$, $P_{az_2} = (s,\ldots,a,z,v,z_2,\ldots,t)$,

$P_{z_1b} = (s,\ldots,z_1,v,z,b,\ldots,t)$, $P_{z_1z_2} = (s,\ldots,z_1,v,z_2,\ldots,t)$. Let $C_{ab}$, $C_{az_2}$, $C_{z_1b}$, $C_{z_1z_2}$ be the cycles closed by P with $P_{ab}$, $P_{az_2}$, $P_{z_1b}$, $P_{z_1z_2}$, respectively. Note that no $s \in \mathcal{S}$ is odd-strongly adjacent to $C_{ij}$, $i \in \{a,z_1\}$, $j \in \{b,z_2\}$. There may be chords from P to $\{z_1,z_2\}$. Since H does not contain a $C_3$, both $(z_1,z_2)$–paths on C have length greater than 2. No node y of P is adjacent to both $z_1$ and $z_2$ since y cannot be even-strongly adjacent to C by Lemma 3 and if y were odd-strongly adjacent to C, $y \notin H\backslash K$. Let $T_{z_1}$, $T_{z_2}$ be the set of edges having one endpoint in P and the other endpoint as $z_1$, $z_2$, respectively. The cycles $C_{ij}$, $i \in \{a,z_1\}$, $j \in \{b,z_2\}$, are starred cycles with the subscripts indicating the star nodes. The set of chords in

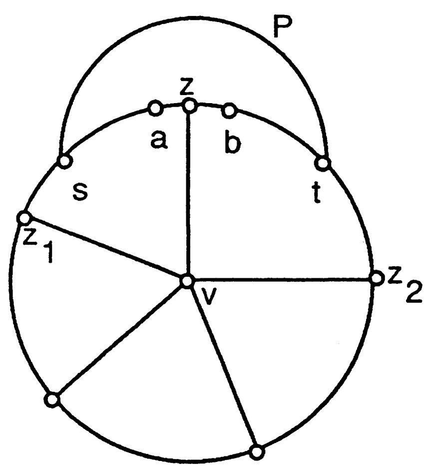



Figure 11

$C_{\bar{1j}}$ is given by Tj u Tj where $T_a$ and $T_b$ are defined to be empty.

If s and t are on the same side of the bipartition, then either the lengths of C ^, $C_{Zib}$, $C_{z_1 z_2}$ are the same mod 4, say p mod 4, and the length of C^ is (p + 2) mod 4 (s and t in S) or the lengths of $C_{ab}$, C ^, $C_{ZJb}$ are the same mod 4, say p mod 4, and the length of $C_{ZjZ2}$ is (p + 2) mod 4. By Theorem 4, one of the following relations holds or else there is a bi-odd hole.

$$|T_{z_1}|+|T_{z_2}|=|T_{z_1}|+|T_b|=|T_a|+|T_{z_2}|\neq|T_a|+|T_b| \pmod 2 \quad \text{or}$$

$$|T_a|+|T_b|=|T_{z_1}|+|T_b|=|T_a|+|T_{z_2}|\neq|T_{z_1}|+|T_{z_2}| \pmod 2$$

Neither relation holds, so the graph induced by V(C) u V(P) u /8 contains a bi-odd hole C with no clique node odd-strongly adjacent to C also odd-strongly adjacent to C.

If s and t are on opposite sides of the bipartition, either the lengths of $C_{a0}$u, $C_{z_1 b}$, $C_{z_1}$^ are the same mod 4, say p mod 4, and the length of C^ is (p+2) mod 4 (s e T and t e S) or the lengths of $C_{ab}$, C ^, $C_{ZjZ2}$ are the same mod 4, say p mod 4, and the length of $C_{z_1 b}$ is (p + 2) mod 4. By Theorem 4 one of the following relations holds or there is a bi-odd hole.

$$|T_a|+|T_b|=|T_{z_1}|+|T_{z_2}|=|T_{z_1}|+|T_b|\neq|T_a|+|T_{z_2}| \pmod 2 \quad \text{or}$$

$$|T_a|+|T_b|=|T_{z_1}|+|T_{z_2}|=|T_a|+|T_{z_2}|\neq|T_{z_1}|+|T_b| \pmod 2$$

Neither relation can hold, so the graph induced by V(C) u V(P) u /8 contains a bi-odd hole C with no clique nodes odd-strongly adjacent to C also odd-strongly adjacent to C′.

Case II): Both $S_1$ and $S_2$ have length greater than two and there is a clique node odd-strongly adjacent to C with four or more neighbors in $S_2$.

Let x be such a clique node with the property that when traversing C counterclockwise from $z_2$ a neighbor of x, say $x_1$, is encountered before a neighbor of any other clique node odd-strongly adjacent to C. Let x* be the neighbor of x on $S_2$ closest to z. If x is adjacent to $z_1$, the length of the (z,x*)-path in $S_2$ must be greater than two since if y is the common neighbor of z and x* on C, the set $\{x,x^*,y,z,v,z_1\}$ induces a $B_6$ (see Figure 12). So if x is adjacent to $z_1$, let x be the center of the wheel and the new wheel is either in Case I or Case II. No node will appear as the center of the wheel twice, so the procedure will terminate. Let 𝒫 be

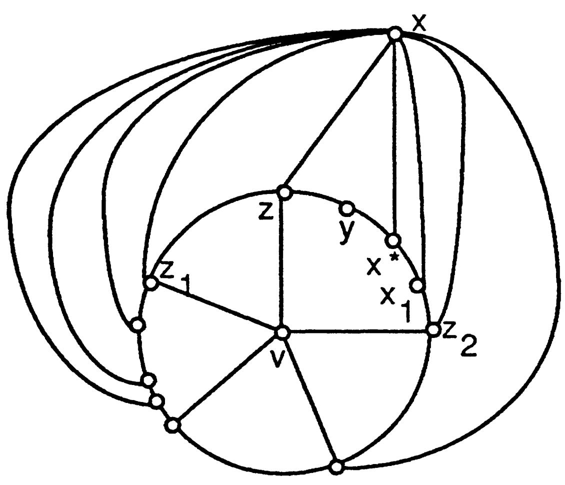



Figure 12

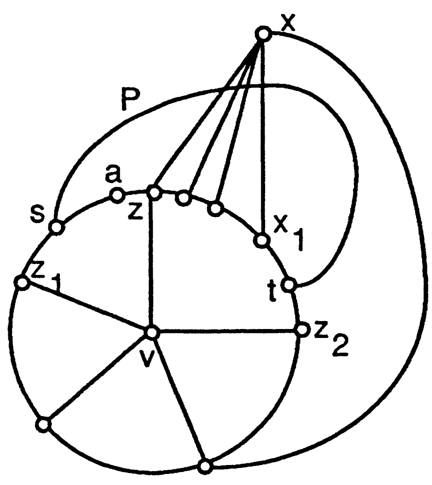



Figure 13

the collection of chordless paths from $S_1$ to the $(x_1,z_2)$–subpath of $S_2$ in H\K. If $\mathcal{P} = \emptyset$, K is a cutset of H disconnecting $S_1$ from the $(x_1,z_2)$-subpath of $S_2$. The component of H\K containing $S_1$ and the component of H\K containing the $(x_1,z_2)$-subpath of $S_2$ each contain a node of T. If $\mathcal{P} \neq \emptyset$, let P be the shortest path in $\mathcal{P}$. Let $s_1$ $(t_1)$ be the endpoint of P adjacent to $S_1$ $(S_2)$. As in Case I, we can assume without loss of generality that $s_1$ and $t_1$ are not strongly adjacent to C. Let s (t) be the node of $S_1$ $(S_2)$ adjacent to $s_1$ $(t_1)$ (see Figure 13).

Apply the argument from Case I to the paths $P_{ax_1} = (s,...,a,z,x,x_1,...,t)$, $P_{az_2} = (s,...,a,z,v,z_2,...,t)$, $P_{z_1x_1} = (s,...,z_1,v,z,x,x_1,...,t)$, $P_{z_1z_2} = (s,...,z_1,v,z_2,...,t)$ and the cycles $C_{ax_1}, C_{az_2}, C_{z_1x_1}, C_{z_1z_2}$ formed by P with $P_{ax_1}, P_{az_2}, P_{z_1x_1}, P_{z_1z_2}$, respectively.

Case III) $S_1$ has length two and there exists a node $w \in T$ odd-strongly adjacent to C, not adjacent to v.

Let the length two sector of C be $(z,y,z_1)$. By Lemmas 9 and 11 w is adjacent to y and has an even number of neighbors in $S_2$. Let $w_1$ be the neighbor of w closest to $z_2$. There is no clique node u odd-strongly adjacent to C with a neighbor of u on the $(b,w_1)$ path of C since taking u as the center of the wheel would violate Lemma 9. w is not adjacent to b since if it were the set $\{z,b,w,y,z_1,v\}$ would induce a $B_6$. Let u be the clique node odd-strongly adjacent to C with the property that a neighbor of u (different from $z_1$) is closer to $z_1$ than any neighbor of any other clique node odd-strongly adjacent to C (u may be v). Let $u_1$ be the neighbor of u closest to $z_1$. If there is a clique node x odd-strongly adjacent to C with neighbors of x on the interior of the $(w_1,z_2)$-path of C, take x to be the center of the wheel. x is adjacent to $z_1$ since otherwise both sectors of (C,x) containing z would have length greater than 2 and the sector containing $w_1$ would have no neighbors of a clique node odd-strongly adjacent to C. So we can assume without loss of generality that the $(w_1,z_2)$-subpath of C contains no neighbors of clique nodes odd-strongly adjacent to C. Let $\mathcal{P}$ be the collection of chordless paths from $S_2$ to the $(z_1,u_1)$-path of $C\backslash K$ in $H\backslash K$. If $\mathcal{P} = \emptyset$, K is a cutset of H disconnecting $S_2$ from the $(z_1,u_1)$-path of $C\backslash K$. The component of $H\backslash K$ containing $S_2$ and the component of $H\backslash K$ containing the $(z_1,u_1)$-path of $C\backslash K$ each contain a node of T. If $\mathcal{P} \neq \emptyset$, let P be the shortest path in $\mathcal{P}$. Let $s_1$ $(t_1)$ be the endpoint of P adjacent to $S_2$ (the $(z_1,u_1)$-path of $C\backslash K$). As in Case I, we can assume without loss of generality that $s_1$ and $t_1$ are not strongly adjacent to C. Let s (t) be the node of $S_2$ (the $(z_1,u_1)$-path of $C\backslash K$) adjacent to $s_1$ $(t_1)$.

We will consider two subcases.

IIIa) s is on the $(z,w_1)$-subpath of $S_2$;

IIIb) s is on the $(w_1,z_2)$subpath of $S_2$.

Case IIIa): s is on the $(z,w_1)$-subpath of $S_2$ (see Figure 14a).

Let $w_k$ be the first neighbor of w encountered when traversing C from s toward $z_2$.

Apply the argument from Case I to the paths $P_{bz_1} = (s,...b,z,v,z_1,...,t)$, $P_{bu_1} = (s,...b,z,u,u_1,...,t)$, $P_{wz_1} = (s,...,w_k,w,y,z_1,...,t)$, $P_{wu_1} = (s,..., w_k,w,y,z,u,u_1,...,t)$ and the cycles $C_{bz_1}, C_{bu_1}, C_{wz_1}, C_{wu_1}$, formed by P with $P_{bz_1}, P_{bu_1}, P_{wz_1}, P_{wu_1}$, respectively.

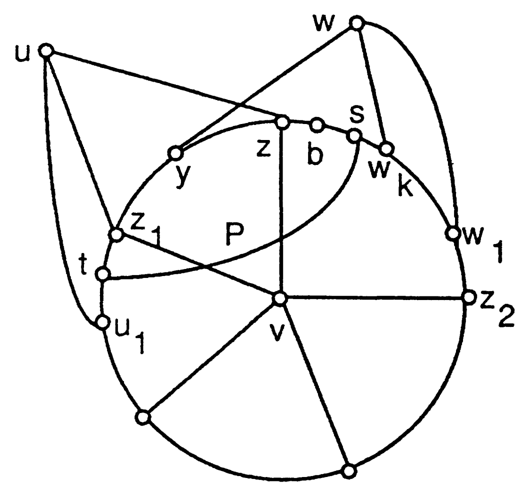



Figure 14a

Case IIIb): s is on the $(w_1,z_2)$-subpath of $S_2$.

Case IIIbi): $u \neq v$ (see Figure 14b).

Apply the argument from Case I to the paths $P_{bz_1} = (s,...b,z,y,z_1,...,t)$, $P_{bu_1} = (s,...b,z,u,u_1,...,t)$, $P_{z_2z_1} = (s,...,z_2,v,z_1,...,t)$, $P_{z_2u_1} = (s,..., z_2,v,z,u,u_1,...,t)$ and the cycles $C_{bz_1}, C_{bu_1}, C_{z_2z_1}, C_{z_2u_1}$ formed by P with $P_{bz_1}, P_{bu_1}, P_{z_2z_1}, P_{z_2u_1}$ respectively.

Case IIIbii): $u = v$ (see Figure 14c).

Apply the argument from Case I to the paths $P_{wz_1} = (s,...w_1,w,y,z_1,...,t)$, $P_{wu_1} = (s,...,w_1,w,y,z,u,u_1,...,t)$, $P_{z_2z_1} = (s,...,z_2,v,z_1,...,t)$, $P_{z_2u_1} = (s,..., z_2,v,u_1,...,t)$ and the

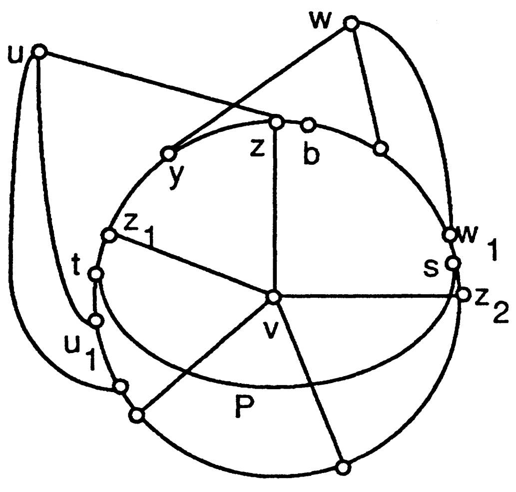



Figure 14b

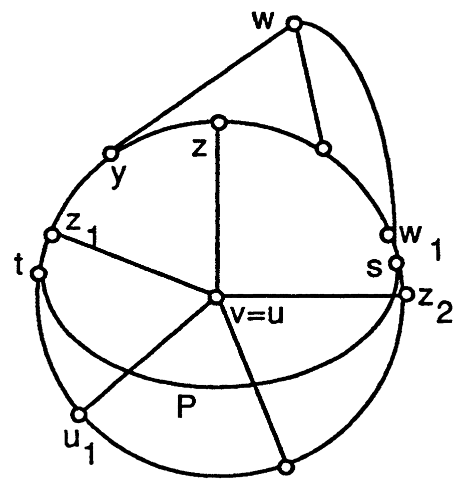



Figure 14c

cycles $C_{wz_1}, C_{wu_1}, C_{z_2z_1}, C_{z_2u_1}$ closed by P with $P_{wz_1}, P_{wu_1}, P_{z_2z_1}, P_{z_2u_1}$ respectively.

Case IV: $S_1$ has length two and every node in T odd-strongly adjacent to C is adjacent to v.

There must be a clique node w odd-strongly adjacent to C with $N(v) \cap C \not\subset N(w) \cap C$ since if all clique nodes were adjacent to all neighbors of v and every node of T odd-strongly adjacent to C is adjacent to v, then any neighbor of v could be chosen as z. In particular, since H does not contain a $B_6$, z could be chosen so that $S_1$ and $S_2$ both have length greater than 2.

Let w be a clique node odd-strongly adjacent to C with $N(v) \cap C \not\subset N(w) \cap C$. Let x be a clique node odd-strongly adjacent to C with neighbors in $S_2$ such that x has a neighbor $x_1$ in $S_2$ next to z (x may be v).

Claim: There is a node, $v_k$, on C different from z adjacent to both v and x.

Proof:

Trivial if $x = v$. Suppose $x \neq v$. x has an even number of neighbors in $S_2$ and no neighbors in the interior of $S_1$ so x either has a unique neighbor $x'$ in some sector with $x'$ adjacent to v (by Lemma 7) or x has an even number of neighbors in two adjacent sectors and one of x's neighbors is the common endpoint of the sector.

The length of the (z,xj)-path on C cannot be two since if it were and x * v, the set

**{z,v,$v_k$,x,x**pb} would induce a B^ and if x = v, then the (z,Xj)-path on C would be S2

which has length greater than two.

Case IVa: x = w.

Clearly, some neighbor of w, w^, not in S2 is next to some neighbor Uj of a clique node u

odd-strongly adjacent to C with $u^*_1$ not adjacent to w. (u maybe v.)

Let P be the collection of chordless paths in H\K from the (z,Xj)-subpath of S2 to the

**($u_1$,$w_k$)-path** in C\K. If P = 0, then K is a cutset of H disconnecting the (zpXj)-subpath of

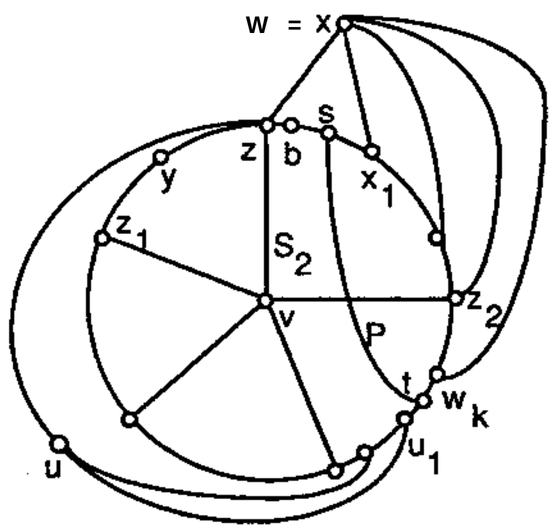



Figure 15

$S_2$ from the $(u_1,w_k)$-path in $C\backslash K$. The components of $H\backslash K$ containing the $(z_1,x_1)$-subpath of $S_2$ and the component of $H\backslash K$ containing the $(u_1,w_k)$-path in $C\backslash K$ contain a node of T. If $\mathcal{P} \neq \emptyset$, let P be the shortest path in $\mathcal{P}$. Let $s_1$ $(t_1)$ be the endpoint of P adjacent to the $(z,x_1)$-path $((u_1,w_k)$-path) in C. As in Case I, we can assume without loss of generality that $s_1$ and $t_1$ are not strongly adjacent to C. Let s (t) be the node of the $(z,x_1)$-subpath $((u_1,w_k)$-subpath) adjacent to $s_1$ $(t_1)$ (see Figure 15).

Apply the argument from Case I to the paths $P_{bu_1} = (s,...,b,z,u,u_1,...,t)$, $P_{bw_k} = (s,...,b,z,w,w_k,...,t)$, $P_{x_1u_1} = (s,...,x_1,x,z,u,u_1,...,t)$, $P_{x_1w_k} = (s,..., x_1,w,w_k,...,t)$ and the cycles $C_{bu_1}, C_{bw_k}, C_{x_1u_1}, C_{x_1w_k}$ formed by P with $P_{bu_1}, P_{bw_k}, P_{x_1u_1}, P_{x_1w_k}$, respectively.

Case IVb: $x = v$.

$N(w) \cap C \not\subset N(v) \cap C$ by choice of v, so w has neighbors in the interior of some sector $S_j$ of C. Let $z_j$ and $z_{j-1}$ be the endpoints of $S_j$. Since H contains no $C_3$ one of $z_j$, $z_{j-1} \neq z_1, z_2$. Assume without loss of generality that $z_j \neq z_1, z_2$. If possible, choose $z_j$ so that $z_2$ and $z_j$ are not endpoints of the same sector. Choose w so some neighbor $w_k$ of w is as close to $z_j$ as possible. If there exists a clique node q odd-strongly adjacent to C with a neighbor of q between $z_j$ and $w_k$, either $N(v) \cap C \not\subset N(q) \cap C$ in which case replace w with q; or $N(v) \cap C \subset N(q) \cap C$ in which case, let q be the center of the wheel, and $x = q$. If q becomes the center of the wheel, q is not adjacent to $w_k$ since if q is adjacent to $w_k$, the set $\{z,w,w_k,v,q,v^*\}$ induces a $B_6$ where $v^*$ is a neighbor of v on C not adjacent to w.

Assume without loss of generality there are no clique nodes odd-strongly adjacent to C with neighbors on the $(z_j,w_k)$-subpath of C. Let $\mathcal{P}$ be the collection of chordless paths in H\K from $S_2$ to the $(z_j,w_k)$-path of C\K. If $\mathcal{P} = \emptyset$, K is a cutset of H disconnecting $S_2$ from the $(z_j,w_k)$-path in C\K. The component of H\K containing $S_2$ and the component of H\K containing the $(z_j,w_k)$-path in C\K each contain a node of T. If $\mathcal{P} \neq \emptyset$ let P be the shortest path in $\mathcal{P}$. Let $s_1$ $(t_1)$ be the end-node of P adjacent to $S_2$ (the $(z_j,w_k)$-path of C). As in Case I, we can assume without loss of generality that $s_1$ and $t_1$ are not strongly adjacent to C. Let s (t) be the node of $S_2$ (the $(z_j,w_k)$-subpath of C) adjacent to $s_1$ $(t_1)$ (see Figure 16).

Apply the argument form Case I to the paths $P_{bz_j} = (s,...,b,z,v,z_j,...,t)$, $P_{bw_k} = (s,...,b,z,w,w_k,...,t)$, $P_{x_1z_j} = (s,...,x_1,v,z_j,...,t)$, $P_{x_1w_k} = (s,..., x_1,v,z,w,w_k,...,t)$ and the cycles $C_{bz_j}$, $C_{bw_k}$, $C_{x_1z_j}$, $C_{x_1w_k}$ closed by P with $P_{bz_j}$, $P_{bw_k}$, $P_{x_1z_j}$, $P_{x_1w_k}$, respectively.

Case IVc): $x \neq v,w$.

Claim: $N(w) \cap N(x) \cap C \subseteq N(v) \cap C$.

Let $w_k \in (N(w) \cap N(x) \cap C) \backslash (N(v) \cap C)$. Since $x \neq v,w$, $N(v) \cap C \subset N(x) \cap C$. Then the set $\{w,w_k,x,z,v,v^*\}$ induces a $B_6$ where $v^*$ is a neighbor of $v$ on $C$ not adjacent to $w$.

Let $x$ be the center of the wheel. $N(w) \cap C \not\subset N(x) \cap C$, so $w$ has neighbors in the interior of some sector of $(C,x)$. Now we are in case IVb where $x$ is the center of the wheel. ■

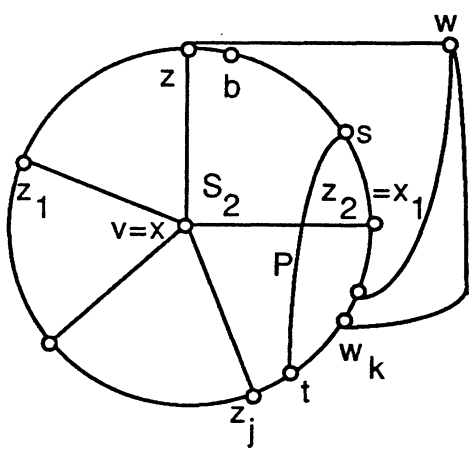



Figure 16

## Acknowledgement

I would like to thank Gérard Cornuéjols for many very helpful conversations regarding these results and also for all his suggestions for improving the presentation of this work. His input has greatly improved this paper.

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