ON CHAINS OF RELATIVELY SATURATED SUBMODELS OF A STABLE MODEL

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Abstract Let M be a given model, we call $N \prec M$ relatively saturated iff for every $B \subseteq N$ of cardinality less than $\|N\|$ every type over B which is realized in M is realized in N. We discuss the existence of such submodels.

The following are corollaries of the existence theorems.

(1) If M is of cardinality at least $\beth_{\omega_1}$, and fails to have the ω order property then there exists $N \prec M$ which is relatively saturated in M of cardinality $\beth_{\omega_1}$.

(2) Let T be a countable $L_{\omega_1,\omega}$ theory. If there exists an uncountable cardinal χ such that $I(\chi,T)<2^{\chi}$ then every model $M \models T$ of cardinality greater or equal to $\beth_{\omega_1}$ has a relatively saturated submodel of cardinality $\beth_{\omega_1}$.

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Here we continue the study of stability inside a model (as started in [Sh2] and [Gr]) rather than of stability inside an arbitrarily large saturated model $\mathfrak{C}$ as in [Sh1]. In general a union of a chain of saturated models need not be saturated (see [AlGr]). However from Theorem III 3.11 of [Sh1] it follows that when a first order theory T is stable then a union of a chain of cofinality greater or equal than $|T|^+$ of saturated models is saturated. Our goal here is to generalize this result, this is done in Theorem 2(2). This seems to be of greater interest than just a simple generalization of [Sh1] to this context. We hope that this may serve as a beginning of a classification theory for some non elementary classes.

A word on notation: Since our work will be carried out inside a given model M, when $A \subseteq M$ we use $S(A)$ to denote the set of types with parameters from A realized in M. We follow the notation of [Gr]. M has the μ- order property iff there exists a formula $\varphi(x;y) \in L(M)$ and a subset of finite sequences of M $\{a_\alpha : \alpha < \mu\}$ satisfying:

$$\text{for every } \alpha,\beta<\mu \qquad \alpha<\beta \iff M \models \psi[a_\alpha;a_\beta].$$

Definition 1 We call $N \prec M$ $\_\kappa$ <u>relatively saturated</u> iff for every $B \subseteq N$ of cardinality less than κ every type over B which is realized in M is realized in N. We denote this by $N \prec_{<\kappa} M$. When N is $\|N\|$-relatively saturated we say that N is a <u>relatively saturated</u> substructure of M. When N is relatively κ^+ saturated we denote this by $N \prec_\kappa M$.

Our aim here is to find conditions on M, $\|N\|$, and κ which will imply the existence of a relatively saturated submodel N of the structure M.

Main Theorem 2 Let M be a structure whose similarity type is of cardinality κ.

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(1) Suppose M does not have the oa order property.

(I.a) If X, and ji satisfy X^=X (and JJ>X) then M has a submodel N of cardinality X which is p^+ relatively saturated.

(I.b) If x<cfX, X=2 X: , and for every i<cfX
i<cfX '
Xj^X=Xj, then for every {M'j<r1 = i<cfX} increasing chain of relatively saturated submodels such that IT I=X: the model M'-U IT: is
i ' i<cfX '
relatively saturated in M.

(2) Suppose M fail to have the a>- order property. Let X be a cardinal such that X^K=X. if-{Mj<r1 = i<cx} is an increasing chain of X-relatively saturated submodels, and cf<x>x ~~then~~ U fT is a X-
i<cx '
relatively saturated submodel of M.

(3) Let T=„rn' 0L) 1, c&(^' and x=Xo. -Suppose there exists an uncountable cardinal X such that I(X,T)<2* . If X satisfy $X=X^{<\beth_{\omega_1}}$ then every model NI=T of cardinality at least X has a relatively saturated submodel of cardinality X.

It is not hard to derive the following corollary from parts (I.a).(I.b) and (3) of Theorem *2-*

Corollary 3 Suppose that M has a countable similarity type, and is of cardinality at least $^_1$.

(1) If M does not have the G) order property then M always has a relatively saturated substructure of cardinality $^_1$.

(2) If $3X^{>x}$o such that KXJh^ $_{1,a)}$(M))<2^ then M always has a relatively saturated substructure of cardinality 3_{ω_1}. o3

Remarks (1) There is a natural example showing that $\beth_{\omega_1}$ in Corollary 3 can not be replaced by $\beth_\omega$, namely the assumption in the Main Theorem that $cf\lambda>|L(M)|$ $(=\kappa)$ is essential. For this see Theorem 2 of [AlGr] when Th(M) is stable but not superstable.

(2) Corollary 3 can easily be generalized to uncountable similarity types.

(3) Let $\psi\in L_{\omega_1,\omega}$ have countable similarity type. If ψ has a model of cardinality $\beth_{\lambda^+}$ and there exists an uncountable χ such that $I(\chi,Th_{\omega_1,\omega}(M))<2^\chi$ then by Theorem 1.6(1) of [GrSh] there is no $\varphi(x;y)\in L_{\lambda^+,\omega}$ which has the $\beth_\delta$ order property for some $\delta<\lambda^+$. This can be used to repeat the relevant parts of [Gr] which are needed here, and by proving a version of the main Theorem we can get submodels which are <u>strongly</u> - relatively saturated submodels I.e. the types can include also formulas from some countable fragment of $L_{\lambda^+,\omega}$.

(4) The last remark can be generalized to uncountable similarity types. Suppose $\psi\in L_{\kappa^+,\omega}$ when L is of cardinality $\leqslant\kappa$. In this case in order to apply Theorem 4.2 of [GrSh] we should start with a model of cardinality $\beth_{\delta(\lambda,\kappa)}$ (for definition of $\delta(\lambda,\kappa)$ see Definition 4.1(1) of [GrSh]), and the fragment of $L_{\lambda^+,\omega}$ can be of cardinality κ.

We will make use of the following

Lemma 4 Suppose B is a set of cardinality κ. Let $N\supseteq B$ satisfy $N\prec_\kappa M$, and let $C\subseteq M$ contain N. If $p_1,p_2\in S(C)$ both do not split over B, and $p_1|N=p_2|N$ then $p_1=p_2$.

Proof of 4 Follows easily from Exercise I.2.3 of [Sh1]. $\square_4$

Proof of the main Theorem (1.a) Since $\lambda^{\mu}=\lambda$ by Lemma 6(1) of [Gr] M is stable in λ. Define by induction on $i<\mu^+$ M_i increasing and continuous chain of submodels of M of cardinality λ, such that M_{i+1} contains realizations to every type from $\mathbf{S}(M_i)$. Clearly (by regularity of μ^+) $N:=\bigcup_{i<\kappa^+} M_i$ is as required.

(1.b) Let λ_i, M'_i, and M' be as in the assumption. We will prove that M' is a relatively saturated submodel of M. Let $A\subseteq M'$ be a given set such that $|A|<\|M'\|$, and suppose $p\in\mathbf{S}(A)$, we will find below a finite sequence (in M') realizing this type (in fact λ^+_i many elements). Let $\mathbf{a}\in M$ be an element such that $p=tp(\mathbf{a},A)$. Define $p':=tp(\mathbf{a},M')$, since $cf\lambda>\kappa$ and M' is a union of relatively saturated models we have that $M'\prec_\kappa M$. By Lemma 7(1) of [Gr] there exists $B\subseteq M'$ of cardinality at most κ such that p' does not split over B.

Since $cf\lambda>\kappa$ there exists $i<cf\lambda$ such that $B\subseteq M'_i$. Let $\lambda:=\lambda_i$, $N^*:=M'_{i+1}$. Define $\{N_i\prec N^* : i<\lambda^+\}$ increasing and continuous, and $\{\mathbf{a}_i\in N_{i+1} : i<\lambda^+\}$ such that

(i) $N_0\supseteq B$,

(ii) $\|N_i\|=\lambda$,

(iii) $N_{i+1}\prec_\kappa N^*$,

(iv) $tp(\mathbf{a}_i,|N_i|)=tp(\mathbf{a},|N_i|)$, and

(v) $N_{i+1}\supseteq N_i\cup\{\mathbf{a}_i\}$.

The construction, can be carried out since $\lambda^\kappa=\lambda$, the relative saturation of N^*, and Lemma 6(1) in [Gr]. Let $N:=\bigcup_{i<\lambda^+} N_i$. (It is easy to check that $\mathbf{I}:=\{\mathbf{a}_i : i<\lambda^+\}$ is an indiscernible sequence over N_1 .)

Notation 5 Let CQM, and I<=M be a set of finite sequences (all of the same length).

Av(I.C):= (<P(x;c) : ceC, there exists JCI of cardinality less than the cardinality of I satisfying: jl there exists a sequence ael-J such that ř1l=<p[a;c] then for every ael-J =* Mt=<p[a;c]}.

Claim 6 For I as above, and every C of cardinality <X, Av(I,C) is a complete type, realized by an element of I.

Proof Let ceC be given, consider q;=tp(c,N). By Lemma 7(1) of [Gr] there exists B'QN of cardinality at most x, such that q does not split over B'. There exists $i<X^+$ such that B'QNj . Let J_c:=inNj , we will show that if there exists a sequence $a^£I"J_c$ such that Mt=<p[a;c] then for every ael-J => Mt=<p[a;c] for every <p . Let *<P(x;c)* be formula over C. such that Mt=<p[a;c] for some $ael-J_c$. Let $bel-J_c$ be an arbitrary sequence. We have that

Mt=<pfb;c] <=><p(b;y)<Eq (1)

By the choice of I , and Lemma 4 we have

$tp(a,N_j)=tp(b,N_j)$. (2)

However since q does not split over Nj certainly also tp(c.I) does not split over Nj (remember IQN). This together with (2) implies

<P(b;y)eq<=><p(a;y)eq (3)

Now (1) and (3) together imply what we wanted , namely:

Mt=<p[b;c] «<p(b;y)eq <=><p(a;y)eqc*Mt=<p[a;c] (4)

Since $|I|=X^+$ is a regular cardinal and greater than |C|, $J=\bigcup_{C€C} J_c$ is as required, and Av(I.C) is realized by any element of I-J. ^{D}6

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Apply Lemma 6 to $C:=A$. Let $\delta<\lambda^+$ be such that J from Lemma 6 is included in N_δ. We may assume that N_δ also contains B (the set from the beginning of the proof). Apply Lemma 6 again to $C:=A\cup N_\delta$. Let $\xi<\lambda^+$ be such that the corresponding J is included in $N_{\xi+1}$ and $N_\delta\subseteq N_{\xi+1}$.

Claim 7 $Av(I,N_{\xi+1}\cup A)=p'|(N_{\xi+1}\cup A)$.

Proof By Lemma 6 $q:=Av(I,N_{\xi+1}\cup A)$ is a complete consistent (=realized in M) type over $C:=N_{\xi+1}\cup A$. Since $p'|C$ does not split over B, and the choice of I (remember they all realize $p'|N_{\xi+1}$), by Lemma 4 it is enough to show that q does not split over B.

Suppose $\mathbf{c}_\ell:=\mathbf{n}_\ell{}^\frown\mathbf{a}_\ell$ ($\ell=1,2$) are such that $tp(\mathbf{c}_1,B)=tp(\mathbf{c}_2,B)$, and $\varphi(\mathbf{x};\mathbf{c}_1)\wedge\neg\varphi(\mathbf{x};\mathbf{c}_2)\in q$; when $\mathbf{n}_\ell\in N_{\xi+1}$, and $\mathbf{a}_\ell\in A$. By the κ^+-relative saturation of $N_{\xi+1}$ there are $\mathbf{a}'_\ell\in N_{\xi+1}$ such that $tp(\mathbf{a}'_\ell,B\cup\mathbf{n}_1\cup\mathbf{n}_2)=tp(\mathbf{a}_\ell,B\cup\mathbf{n}_1\cup\mathbf{n}_2)$. Hence there are $\mathbf{c}'_\ell:=\mathbf{n}_\ell{}^\frown\mathbf{a}'_\ell\in N_{\xi+1}$ realizing the same type over B such that $\varphi(\mathbf{x};\mathbf{c}'_1)\wedge\neg\varphi(\mathbf{x};\mathbf{c}'_2)\in q|N_{\xi+1}$, a contradiction to the fact that $q|N_{\xi+1}$ does not split over B. $\square_7$

This completes the proof of (1.b).

(2) Similar to (1.b).

(3) By [Sh3] (see also [GrSh]) If $\exists\chi>\aleph_0$ such that $I(\chi,T)<2^\chi$ then there exists a limit ordinal $\delta<\omega_1$ such that if $M\models T$ then M fail to have the $\beth_\delta$ - order property. By our assumption on λ we have that $\lambda=\lambda^{\beth_\delta}$, we can now use Lemma 7(2) of [Gr] and repeat the argument of (1.b). $\square_2$

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