THE DUALS OF HARMONIC
BERGMAN SPACES

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Abstract: In this paper we show that for Ω a starlike Lipschitz domain, the dual of the space of harmonic functions in $L^p(\Omega)$ need not be the harmonic functions in $L^q(\Omega)$ where $1/p+1/q=1$. As a consequence we show that the harmonic Bergman projection for Ω need not extend to a bounded operator on $L^p(\Omega)$ for all $1<p<\infty$.

The duality result is a partial answer to a question of Nakai and Sario [9] posed initially in the Proceedings of the London Mathematical Society in 1978.

We treat the duality question as a biharmonic problem and our result follows from the failure of uniqueness for the biharmonic Dirichlet problem in domains with sharp intruding corners.

THE DUALS OF HARMONIC BERGMAN SPACES

§0. Introduction

In this paper we give a partial answer to a question posed by Nakai and Sario concerning the duals of harmonic Bergman spaces. Their question, initially posed in the Proceedings of the London Mathematical Society ([9],page 345) was to determine whether for $1<p<\infty$, $\Omega \subset R^n$, the dual of the space of harmonic functions in $L^p(\Omega)$ is the space of harmonic functions in $L^q(\Omega)$ wnere $1/p + 1/q = 1$, $1<p,q<\infty$. When the answer is affirmative we will say that the duality property holds (for the given value of p). We note that every domain has the property for $p = 2$ and domains which are bounded and have sufficiently smooth boundary have the property for every p.

In 1981, Nakai and Sario [10] showed that the duality property can fail for $\Omega = R^2 \backslash \{z_1, z_2, \ldots z_n\}$ and one of the authors subsequently discovered that duality also fails for the punctured disk.

In this paper we show that there is a p between one and two and a Lipschitz domain with an intruding corner for which the duality property fails. As a consequence we show that there are Lipschitz domains on which the harmonic Bergman projection does not extend to a bounded operator on a full range of p's between one and infinity.

Our approach is to reduce the duality question to a biharmonic problem. We first show that the duality property is equivalent to the direct sum decomposition of $L^p(\Omega)$ into the sum of the harmonic L^p functions and the L^p closure of the Laplacians of $C_0^\infty(\Omega)$ functions. The duality then fails if the spaces have non—trivial intersection or if the sum fails to exhaust all of L^p.

The non—trivial intersection is seen to occur when uniqueness fails for the biharmonic Dirichlet problem. The direct sum fails to exhaust L^p when the non—homogeneous biharmonic problem with zero Dirichlet data does not have a solution with Laplacian in L^p.

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In either case the counter—examples are constructed from biharmonic functions in wedges with angles greater than 180° and with zero boundary data on the edges of the wedge.

§1. Motivation and Background

For $\Omega \subset \mathbf{R}^2$ and $1<p<\infty$ we define:

$$(1.1) \quad L_h^p(\Omega) = \{u \epsilon L^p(\Omega): \Delta u = 0 \text{ in } \Omega\}.$$

Nakai and Sario showed that $L_h^p(\Omega)$ is a Banach space (see page 344 in [9] for details). They point out that $L_h^2(\Omega)^* = L_h^2(\Omega)$ and ask for which domains does the following property hold:

$$(1.2) \quad L_h^p(\Omega)^* = L_h^q(\Omega) \quad \text{where } 1<p<\infty, \ 1/p + 1/q = 1.$$

Property (1.2) will be called the duality property. Domains Ω for which (1.2) holds will be said to have the duality property.

Our approach to the duality question arises from a second consideration. Weyl's lemma (see C.B.Morrey, page 42 [8]) implies that $L_h^q(\Omega) = M^p(\Omega)^\perp$ where $M^p(\Omega)$ is the $L^p(\Omega)$ closure of $\Delta C_o^\infty(\Omega)$. Since $L^p(\Omega)$ is reflexive and $M^p(\Omega)$ is a closed subspace of $L^p(\Omega)$, $M^p(\Omega)^{\perp\perp} = M^p(\Omega)$. Hence $L_h^q(\Omega)^* = L^p(\Omega)/M^p(\Omega)$. For the case p=q=2, the fact that $L_h^2(\Omega) = M^2(\Omega)^\perp$ implies the orthogonal decomposition $L^2(\Omega) = L_h^2(\Omega) \oplus M^2(\Omega)$. It is then natural to ask for which domains the direct sum decomposition

$$(1.3) \quad L^p(\Omega) = L_h^p(\Omega) + M^p(\Omega)$$

holds. The importance of the decomposition (1.3) in this paper is that it is equivalent to the duality property (1.2). To see this note that since L^p is reflexive and M^p is closed we have

$$(1.4) \quad LJJ(ft)^* = L<|Q)^*/Lg(ft)^x = L^QJ/M^Q)^\wedge = L^p(ft)/M^p(ft)$$

It follows immediately that if the decomposition (1.3) holds, then $LJYft) = L^2_{\hat{n}}(ft)^*$. Conversely, if $L\pounds(Q) = L\hat{j}(ft)^*$, then the map $u \to u + M^P(Q)$ is a Banach space isomorphism from $Lj^\wedge(ft)$ to the quotient space $L^p(ft)/M^p(ft)$. This fact together with (1.4) enables us to conclude that the decomposition (1.3) holds.

For $1/p + 1/q = 1$, $L^\wedge(ft) \subset L^2_{\hat{n}}(ft)^*$ in the sense that every element $p \in L^p_{\hat{n}}(ft)$ gives rise to a linear functional A_φ on $L^2_{\hat{n}}(ft)$ given by $A_\varphi(f) = \int_\Omega \text{yrf.}$ We let i_p denote the inclusion map of $LjYft)$ into $L^2_{\hat{n}}(ft)^*$. Then ft has the duality property when i_p is bijective.

For $1 < q < 2$, applying Holder's inequality twice tells us

$$(1.5) \quad \iint \varphi u \leq \|\varphi\|_p \|u\|_q \leq \|\varphi\|_p \|u\|_2 |\Omega|^{1/q - 1/2}$$

where $|ft|$ denotes the measure of ft. Hence for ft of finite measure, $LJj(ft)^* \subset L^2_\lambda(ft)^*$. Finally, as noted earlier $L^{\bar{2}}_{\hat{h}}(ft) = L^2_{\hat{h}}(ft)$ in the sense that every element in the dual of $L^2_{\hat{n}}(ft)$ can be represented uniquely as integration against a function in $L^2_{\hat{n}}(ft)$. We conclude that for $p > 2$ and ft a set of finite measure,

$$(1.6) \quad I|(Q) \subset LJJ(Q)^* \subset ifo)^* = LjJ(Q).$$

This factorization of the inclusion $L^\wedge(ft) \subset L^\wedge Q)$ implies that when ft is bounded or of finite measure, i_p is injective for $p > 2$. Since the adjoint of i_p is i_q where $1/p + 1/q = 1$, we have that the duality property holds for ft if i_p is surjective for $p > 2$.

From now on we assume the domain to be bounded or at least of finite measure. We proceed formally, showing how the solution of a biharmonic Dirichlet problem suffices to

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show that i_p is surjective. For $1/p + 1/q = 1$, $p > 2$ and $\lambda \in L_h^q(\Omega)^*$, there exists $\varphi \in L^p(\Omega)$ satisfying:

$$(1.7) \quad \lambda(u) = \iint_\Omega \varphi(x)u(x)dx \quad \text{for all } u \in L_h^q(\Omega).$$

We want to be able to choose $\varphi \in L^p(\Omega)$ which satisfies (1.6) and is harmonic. First we find a w satisfying $\Delta w = \varphi$. If $\Delta^2 w = 0$ then φ is harmonic and we're done. If not, let v be another function, regular enough to permit integration by parts.

$$(1.8) \quad \lambda(u) = \iint_\Omega \varphi u$$

$$= \iint_\Omega (\Delta w - \Delta v)u + \iint_\Omega \Delta v \cdot u$$

$$= \iint_{\partial\Omega} (\partial w/\partial n - \partial v/\partial n)u - (w-v)\partial u/\partial n + \iint_\Omega (w-v)\Delta u + \iint_\Omega \Delta v \cdot u$$

If $\Delta v \in L_h^p(\Omega)$ and the other integrals are zero then λ can be represented by an element in $L_h^p(\Omega)$. In other words, λ can be represented by an element in $L_h^p(\Omega)$ if we can find v satisfying:

$$(1.9) \qquad \Delta^2 v = 0 \text{ in } \Omega$$

$$v\big|_{\partial\Omega} = w\big|_{\partial\Omega}$$
$$\partial v/\partial n = \partial w/\partial n$$

and

$$\Delta v \in L^p(\Omega),$$

or, more precisely, we want $v - w \in W_o^{2,p}(\Omega)$.

Since by assumption $p > 2$, the inclusion map $i_p : L_h^p(\Omega) \to L_h^q(\Omega)^*$ is 1–1, and we can conclude that the duality property holds.

It is important to note that the duality property is reflexive in the sense that $L_h^q(\Omega)^* = L_h^p(\Omega)$ if and only if $L_h^p(\Omega)^* = L_h^q(\Omega)$. On the other hand, the existence theory for the biharmonic equation is not reflexive on Lipschitz domains. That is, the theory for data in $L^p(\partial\Omega)$ for $1 < p < 2$ is not like the theory for $2 < p < \infty$ (see Dahlberg, Kenig and Verchota [4], pages 130–133 for details in the range $1 < p < 2$ in $\mathbf{R}^2$ and Pipher and Verchota [11] and [12] for proof of existence in the range $2 - \epsilon < p < \infty$ in $\mathbf{R}^3$). This suggests possible problems on domains with sharp intruding corners.

Some positive results are known for the duality property. For bounded smooth domains, the property follows from the biharmonic equation approach using standard results in elliptic theory.(See Lions and Magenes [7]). One can then easily prove the following:

Proposition (1.10) For Ω a smooth bounded domain the following are equivalent:

(1.11) i_p is surjective

(1.12) For $u \epsilon W^{2,p}$ there is a biharmonic $w \epsilon W^{2,p}$ with $w - u \epsilon W_o^{2,p}$.

For bounded C^1 domains in $\mathbf{R}^2$, Soleve'ev states (Doklady [13])that the duality property holds. For the unit disk there is a substantial literature on the duals of harmonic Bergman spaces including weighted versions and for the range $0 < p < \infty$. (See Coifman and Rochberg [3], E.Ligowcka [5] for a sample of these results).

For negative results, Nakai and Sario [9] showed that $\mathbf{R}^2 \backslash \{z_1, ..., z_n\}$ could be a domain not having the duality property. This follows from the fact that for $\Omega = \mathbf{R}^2 \backslash \{z_1, ..., z_n\}$, $L_h^p(\Omega)$ consists of functions of the form $f(z) = \sum_{j=1}^{n} c_j \ln|z - z_j|$. By choosing the c_j's cleverly one can show that for $1 < p < 2$, $L_h^p(\Omega)$ has dimension zero but $L_h^q(\Omega)$ has dimension at least one for $q > 2$. By a dimension argument $L_h^p(\Omega)^* = L_h^q(\Omega)$ is then impossible.

Another interesting example is that of the punctured disk

$D_o = D \backslash \{0\} = \{z \epsilon \mathbf{R}^2 : 0 < |z| < 1\}$ which also does not satisfy the duality property. This example is more complicated than the plane minus a finite set of points since the space

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$L_h^p(D_O)$ is infinite dimensional. However, in D_O, $(2r-1/r)\cos\theta$ and $(2r-1/r)\sin\theta$ are in $L_h^p(D_O)$ for $1<p<2$ and additionally satisfy

$$(1.13) \qquad \int_0^{2\pi}\int_0^1 (2r-1/r)\cos\theta\cdot u(r,\theta)rdrd\theta = 0$$

$$\int_0^{2\pi}\int_0^1 (2r-1/r)\sin\theta\cdot u(r,\theta)rdrd\theta = 0$$

for all $u\epsilon L_h^q(D_O)$ where $q>2$. In other words, for $1/p + 1/q = 1$, $1<p<2$, $2<q<\infty$ every element in $L_h^q(D_O)$ annihilates $(2r-1/r)\cos\theta$ and $(2r-1/r)\sin\theta$. But by the Hahn Banach theorem there exists a function $g\epsilon L^p(D_O)^* = L^q(D_O)$ which doesn't vanish on $(2r-1/r)\sin\theta$. As an element of $L_h^p(D_O)^*$ g cannot be represented by a harmonic function because this would violate (1.13).

It is also true that $\Delta(r\ln r - r^3/2 + r/2)\sin\theta = (3/r-4r)\sin\theta \epsilon L_h^p(D_O) \cap M^p(D_O)$ and so property (1.3) is seen to fail as well.

Another interesting property of D_O is that the projection operator which maps $L^2(D_O) \rightarrow L_h^2(D_O)$ does extend to a bounded operator on $L^p(D_O)$ since an orthonormal basis for $L_h^2(D_O)$ can be obtained from the usual basis for $L_h^2(D)$ by adjoining the element $\ln r$.

In this paper we show that there is a simply connected Lipschitz domain for which the duality property fails for some p between one and two. Furthermore we show that the Bergman projection must also fail to be bounded on L^p for this particular p. The last result is the opposite situation from the punctured disk where the boundary point at the origin allows for the duality property to fail without affecting the boundedness of the Bergman projection.

§2 The failure of the duality property.

In this section we show how the duality property can fail on a Lipschitz domain with a

sharp intruding corner. We construct our counter–example through the solution of a biharmonic equation. However, rather than showing that i_p is not surjective for some p>2 we show that i_p can fail to be injective for 1<p<2. From the point of view of the biharmonic Dirichlet problem this amounts to showing a failure of uniqueness.

Dahlberg, Kenig and Verchota [4] have shown that there exists a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a biharmonic function $u \in \Omega$ whose Dirichlet data is zero. ($\Delta^2 u=0$ in Ω, u=0 and $\partial u/\partial n=0$ on $\partial\Omega$). One would like to conclude then that Δu is a non–trivial element in $M^P(\Omega) \cap L_h^P(\Omega)$. Unfortunately their construction yields a function $u \in W^{3/2,2}(\Omega)$. Hence one cannot conclude that $\Delta u \in L^P(\Omega)$. In fact one has $\Delta u \in W^{-1/2,2}(\Omega)$. Instead we use the existence theorem (page 302 of Grisvard [6]) to construct a biharmonic function w with w and its normal derivative vanishing on the boundary of Ω and for which $\Delta w \in L^P(\Omega)$ where 1<p<2. It then follows easily that $\Delta w \in L_h^q(\Omega)^\perp$ and so $L_h^q(\Omega)^* \neq L_h^P(\Omega)$.

Lemma 5.1 of Dahlberg, Kenig and Verchota [4] proves that there exist constants γ, λ, α with $0<\lambda<1/2$, $\pi/2<\alpha<\pi$ and a function $\psi(r,\theta) = r^\lambda(\sin\lambda\theta - \gamma\sin(\lambda-2)\theta)$ satisfying

$$(2.1) \qquad \Delta^2 \psi(r,\theta) = 0 \quad r>0, \; -\alpha<\theta<\alpha$$

$$\psi(r,\pm\alpha) = 0$$

$$\psi_\theta(r,\pm\alpha) = 0$$

For A a bounded neighborhood of the corner, $\Delta\psi \in L^P(A)$ provided $p < 2/(2-\lambda)$ and $\Delta\psi \notin L^P(A)$ when $p \geq 2/(2-\lambda)$. In particular, for $\lambda<1/2$, $\Delta u \notin L^2(A)$. (In fact $\Delta u \notin L^2$ for $2>2/(2-\lambda)$ if and only if $\lambda<1$). On the other hand, for any $0<\lambda<1/2$ there is a p>1 for which $p < 2/(2-\lambda)$. We point out that Grisvard shows (lemma 7.3.2.4, page 338 of [6]) that it is necessary that $\alpha > \pi/2$.

Consider the domain Ω whose boundary $\partial\Omega = \{(r,\pm\alpha): 0 \leq r \leq 1\} \cup S$ where S is C^∞ and completes the domain so that Ω is starlike. (e.g. S can be a circular arc smoothed out when it meets the edge of the sector so Ω looks like PACMAN).

Let $\varphi \in C^\infty(\Omega)$ with $\varphi = 0$ in a small neighborhood of 0 and $\varphi \equiv 1$ outside a small

neighborhood of the origin which does not intersect S. By Grisvard (page 302, lemma 7.1 [6]) there exists w satisfying $\Delta^2 w = \Delta^2 \varphi \psi$, with w and its normal derivative vanishing on the boundary and with the regularity $\Delta w \epsilon L^2(\Omega)$; or more precisely, $w \epsilon W_0^{2,2}(\Omega)$. Set $v = \psi + (w - \varphi \psi)$. Then $\Delta^2 v = \Delta^2 \psi + \Delta^2 w - \Delta^2 \varphi \psi = 0$, $v|_{\partial\Omega} = 0$, $\partial v / \partial n|_{\partial\Omega} = 0$. Again more precisely, since obviously $\psi - \varphi \psi \epsilon W_0^{2,P}(\Omega)$, we have $v \epsilon W_0^{2,P}(\Omega)$. On the one hand $\Delta \psi \epsilon L^P(\Omega)$ for some $1 < p < 2$. On the other hand $\Delta \psi \not\epsilon L^2(\Omega)$. Hence we conclude that $v \neq 0$ since $\Delta(w - \varphi \psi) \epsilon L^2(\Omega)$ and $\Delta \psi \not\epsilon L^2(\Omega)$.

As indicated earlier, we now have a function v with zero biharmonic dirchlet data and with $\Delta v \epsilon L_h^P(\Omega)$. For any $u \epsilon L_h^q(\Omega)$, $1/p + 1/q = 1$,

$$(2.2) \quad \int\int_\Omega \Delta v \cdot u = \int_{\partial\Omega} \partial v / \partial n \cdot u - \int_{\partial\Omega} v \cdot \partial u / \partial n + \int\int_\Omega v \cdot \Delta u = 0$$

Alternatively, the same follows from the fact that $v \epsilon W_0^{2,P}(\Omega)$. But $\Delta v \neq 0$ and so there exists a function $g \epsilon L^q(\Omega)$ with $\int\int g \cdot \Delta v \neq 0$. By (2.2), g cannot be harmonic. In other words we have proved the existence of a linear functional on $L_h^P(\Omega)$ which can't be represented by a harmonic function. This means that the duality property fails.

§3 The failure of the harmonic projection in L^P.

Let Ω be a bounded domain in $\mathbf{R}^2$. The harmonic kernel function is the reproducing kernel function for the space $L_h^2(\Omega)$. A detailed description of reproducing kernel spaces can be found in [1] but for our purposes the following short explanation will suffice.

For $x \epsilon \Omega$, the solid mean value theorem and Schwartz' inequality imply that point—wise evaluation of an $L_h^2(\Omega)$ function at x is a bounded liniear functional on $L_h^2(\Omega)$. Hence by the Riesz representation theorem, for each $x \epsilon \Omega$, there is a unique, symmetric $K(x,z)$ satisfying $\Delta_z K(x,z) = 0$ for all $z \epsilon \Omega$, $\int\int_\Omega |K(x,z)|^2 dz = 0$, and $u(x) = \int\int_\Omega K(x,z) u(z) dz$ for all $u \epsilon L_h^2(\Omega)$. (Note that dz represents area measure in Ω).

As remarked earlier it follows from Weyl's lemma that $L^2(\Omega) = L^2_h(\Omega) \oplus M^2(\Omega)$. This guarantees that the operator $Tf(x) = \int\int_\Omega K(x,z)f(z)dz$ is a bounded operator on $L^2(\Omega)$. In fact T is the projection from $L^2(\Omega)$ onto $L^2_h(\Omega)$.

In this section we consider the question of determining domains on which the operator T extends to a bounded operator on $L^p(\Omega)$ for $1<p<\infty$.

We recall that D_0 denotes the punctured unit disk. Now we let $K_D(x,z)$ denote the kernel function for the full unit disk and $K_0(x,z)$ the kernel function for the punctured unit disk. Clearly,

$$(3.1) \qquad K_0(x,z) = K_D(x,z) + C_1\ln(C_2|x|)\ln(C_2|z|).$$

Letting $T_0f(x) = \int\int_\Omega K_0(x,z)f(z)dz$ we see that T_0 extends to a bounded operator on $L^p(\Omega)$ for $1<p<\infty$. This means that it is possible for the harmonic Bergman projection to extend to a bounded operator on all of L^p but for the duality property to fail.

We next consider the starlike domain Ω_P described in §2 and make a small translation so that the domain is starlike with respect to the origin. We let $K_P(x,z)$ denote the kernel function for the domain Ω_P. If the operator $T_Pf(x) = \int\int_\Omega K_P(x,z)f(z)dz$ extends to a bounded operator on all of $L^p(\Omega_P)$ then by duality $\int\int_{\Omega_P}|K_P(x,z)|^q dz<\infty$ for $1/p + 1/q = 1$ and for all $x\epsilon\Omega$. Hence as a function of z, $K_P(x,z)\epsilon L^q_h(\Omega_P)$ for all $x\epsilon\Omega_P$.

We recall from §2, the existence of a biharmonic function v with zero Dirichlet data and for which $\Delta v\epsilon L^p(\Omega_P)\backslash L^2(\Omega_P)$ We let $w(x) = \Delta v(x)$ and define $w_t(x) = w(tx)$. Then w_t is harmonic and bounded in Ω_P for $0<t<1$. In particular $w_t\epsilon L^2_h(\Omega_P)$. Hence

$$(3.2) \qquad w_t(x) = \int\int_{\Omega_P} K_P(x,z)w_t(z)dz$$

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$$= \int_{\Omega_p} \int K_p(x,z)(w_t(z)-w(z))dz$$

since as we observed in §2 $w \in L^2(Q_p)^x$. If the Bergman operator Tp is bounded on $L^p(\Omega_p)$ we get

(3.3) $||w_t||_p < c||w_r w||_p = o(1)$ as t-1.

This would imply that $||w||_p = 0$ which we know is false. So for this example, the starlike nature of the domain allows us to infer the failure of the boundedness of the Bergman operator in $L^\wedge(Q_p)$ from the failure of the duality property in Op. We point out that this is in sharp contrast to the punctured disk where duality fails but the Bergman operator is bounded on $L^\wedge(D_o)$.

Concluding Remark:

The above argument shows that a starlike domain Q has the duality property for a given p if and only if the (harmonic) Bergman operator is bounded on $1^p(0)$. A slightly more involved argument shows that this remains true for domains with continuous boundary in the sense of Grisvard [6], p. 5, i.e. equivalently domains with the uniform segment property.

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