TCHEBYCHEV NETS ON SPHERES

by

Sandra Samelson
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Research Report No. 89-54

July 1989

Tchebyshev Nets on Spheres

Abstract: We show that the standard sphere on $\mathbb{R}^3$ admits a Tchebychev net which covers an open subset of the sphere which contains a closed hemisphere.

Introduction:

Tchebyshev introduced the notion of what are now called Tchebychev nets in ? [Tch1] in order to model the deformation of cloth. In this model a surface is considered to consist of two families of "fibers" which are inextensible, although the angle between fibers belonging to the two families can vary. Moreover this surface is assumed to be initially plane and to have obtained the current shape after going through a deformation which did not change the lengths of the fibers. The simplest way to describe such a surface is to say that it admits a coordinate chart such that the coordinate vector fields have unit magnitudes with respect to the ambient Euclidean metric.

In ? Bieberback [Bie1] proved that every surface is locally a Tchebychev net i.e. around every point there exists a coordinate chart such that coordinate vector fields have unit magnitudes. This was accomplished by reducing the problem to that of existence of solutions to a pair of quasilinear hyperbolic equations. However, the global issue i.e. existence of a Tchebychev net over a prescribed open set has not adequately been investigated, except for an important equation called Hazzidakias formula, which was first derived in [Ha1] (see [Si1], [Pi1] etc.) which shows that at least on a simply connected, open surface which is complete with respect to its Riemannian metric the integral of the Gaussian curvature must be less than 2π in magnitude in order for a Tchebychev net to cover the whole surface. For example there does not exist a Tchebychev net covering a semi-infinite cylinder with one end is closed.

One important question is, whether we can find a Tchebychev net on the standard sphere. It is easy to show that the open hemisphere and the sphere with north and south poles removed admits global Tchebychev nets e.g. [ ].

The purpose of this paper is to show that there does exist a Tchebychev net which covers a closed hemisphere.

2. Preliminaries and the Main Result:

Let S^2 denote the unit sphere in $\mathbb{R}^3$ endowed with the Riemannian metric g induced from the Euclidean metric in $\mathbb{R}^3$.

Definition 2.1: <u>A Tchebychev net on S^2</u> is a coordinate chart $(U,(x,y))$ in $\mathbb{R}^3$ such that the coordinate vector fields $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ have unit magnitudes with respect to g. These vector fields will be called <u>fiber fields</u> and their integral curves will be called <u>fibers</u>.

Let $(U,(x,y))$ be a Tchebychev net of S^2. Define $\gamma : U \to \mathbb{R}$ such that $\gamma(p)$ is the angle between $\frac{\partial}{\partial x}\big|_p$ and $\frac{\partial}{\partial y}\big|_p$.

Our objective is to obtain a solution to the Sine-Gordon equation and then use this solution to construct a Tchebychev net.

Theorem 2.2 (Main Theorem): There exists a Tchebychev net $(U,(x,y))$ on S^2 such that U contains a closed hemisphere.

3. Characteristic Intitial Value Problem for the Sine Gordon Equation:

Our aim is to construct a Tchebyshev net on S^2 such that orthogonal longitudes through the south pole are fibers. Hence $\gamma = \pi/2$ at the south pole, since $\frac{\partial}{\partial x}$ is parallel along the integral curves of $\frac{\partial}{\partial y}$ and vice-versa (see [Pil]). This forces γ to be $\pi/2$ on each of the two longitudes mentioned above. Therefore we are led to the characteristic initial value problem

$$\frac{\partial^2\gamma}{\partial x\partial y} = -\sin\gamma\ , \quad \gamma(x,0) = \pi/2 = \gamma(0,y) \tag{3.1}$$

for $x,y \in [0,a]$, where a is positive.

To simplify the equations let

$$\alpha := \pi/2 - \gamma. \tag{3.2}$$

Then

$$\frac{\partial^2\alpha}{\partial x\partial y} = \cos \alpha(x,y),$$
$$\alpha(x,0) = 0 = \alpha(0,y), \tag{3.3}$$

and

$$\alpha(x,y) = \int_0^x \int_0^y \cos(\alpha(x,y))dxdy. \tag{3.4}$$

Let $a > 0$ and $J_a := [-a,a] \times [-a,a]$. A more general type of characteristic initial value problem was considered in [BT]. Therefore the existence of a solution to (3.4) is already known. However we need a sharp estimate of the solution and hence we prove the existence part and the approximation part together in the following lemma, and the proof follows as in [BT]. The fact that the iterations converge rapidly allows us to find a good approximation to the true solution quickly.

Lemma 3.1: For arbitrary $a > 0$, (3.4) admits a unique C^∞-solution.

Proof: Let $F : C(J_a;\mathbb{R}) \to C(J_a;\mathbb{R})$ be defined by

$$F(\alpha)(x,y) = \int_0^x \int_0^y \cos(\alpha(\theta,\varphi))d\theta d\varphi,$$

where $C(J_a;\mathbb{R})$ denotes the space of continuous functions on J_a endowed with the sup norm $\|\cdot\|$. It is easy to see that

$$|(F(\alpha) - F(\beta))(x,y)| \le xy\ \|\alpha-\beta\|$$

and

$$|(F^n(\alpha) - F^n(\beta))(x,y)| \le \frac{x^n y^n}{(n!)^2}\ \|\alpha-\beta\| \ , \tag{3.5}$$

where F^n denotes the n^{th} iterate of F.

Now (3.5) shows that F^n is a contractive mapping whenever n is large enough. Hence there exists a unique $\alpha \in C(J_a;\mathbb{R})$ satisfying (3.4). This α is necessarily C^∞, also by (3.4). □

As a consequence of the fact that F^n in the proof of lemma 3.1 is a contraction mapping whenever n is large enough, it follows that the unique solution of (3.4) on J_a is obtained as the uniform limit of the sequence (α_n) in $C[J_a;\mathbb{R}]$ defined by

$$\alpha_0 = 0$$

$$\alpha_{n+1}(x,y) = \int_0^x \int_0^y \cos \alpha_n(\theta,\varphi)d\theta d\varphi. \tag{3.6}$$

Since

$$|(\alpha_{n+1} - \alpha_n)(x,y)| \le \left|\frac{(xy)^{n+1}}{((n+1)!)^2}\right|,$$

if $a < 4$, we have

$$|(\alpha - \alpha_2)(x,y)| \le \sum_{n=2}^{\infty} \frac{|xy|^{n+1}}{((n+1)!)^2} \le \frac{|xy|^3}{(3!)^2} \left(1 + \frac{xy}{4^2} + \frac{x^2y^2}{(4\cdot5)^2} + \ldots\right)$$

$$\le \frac{|xy|^3}{(3!)^2} \left(\frac{1}{1 - \frac{|xy|}{4^2}} \right). \tag{3.7}$$

Of course

$$\alpha_2(x,y) = \int_0^x \int_0^y \cos(\theta\varphi)\, d\theta\, d\varphi. \tag{3.8}$$

Let

$$\mathcal{D}_a = \{(x,y) \in J_a : |xy| < \pi/2\} \text{ for } 0 < a < 4.$$

We may bound the 4^{th} and higher order terms of the Taylor series of $\cos(\theta\varphi)$ on $\mathcal{D}_a$ as follows:

$$\left|\frac{\theta^4\varphi^4}{4!} - \frac{\theta^6\varphi^6}{6!} + \ldots \right| \le \frac{\theta^4\varphi^4}{4!} \left(1 + \frac{\theta^2\varphi^2}{5\cdot6} + \frac{(\theta^2\varphi^2)^2}{5\cdot6\cdot7\cdot8} + \ldots\right)$$

$$\le \frac{\theta^4\varphi^4}{4!} \left(\frac{1}{1 - \frac{\theta^2\varphi^2}{5\cdot6}}\right) \le \frac{\theta^4\varphi^4}{4!} \left(\frac{1}{1 - \frac{\pi^2}{120}}\right)$$

$$\le \frac{\theta^4\varphi^4}{12}.$$

Therefore

$$\left|\cos(\theta\varphi) - \left(1 - \frac{\theta^2\varphi^2}{2!}\right)\right| \le \frac{\theta^4\varphi^4}{12}. \qquad (3.9)$$

So

3 3 5 5

Now by (3.7)

i # \ x y i , i2(x y) x y i ro i1A

|a(x.y) - xy + -j^l < |-l9-7-i. + ^ _12|. (3.11)

The approximation to a on $\mathcal{D}_a$ given in (3.11) will be important throughout the rest of the paper.

Lemma 3.2: $\mathrm{sgn}(xy)a(x,y) \ge 0$ on $\mathcal{D}_a$.

Proof: Because of (3.11) it suffices to show that the function

$f : [0,\pi/2] \to \mathbb{R}$ defined by

f(X) := (X - Ffg^j X^3 - X^5 / (25x12)) .

is a nonnegative function. Clearly $f(0) = 0$ and $f'(X) \ge 1 -$ 1g 14(TT/2)° - l/6O(Tr/2y ≥ 0; therefore $f(X) \ge 0$ for every $X \in [0,\pi/2]$.

Lemma 3.3: $\operatorname{sgn}(xy)\ \alpha(x,y) < \pi/2$ on $\mathfrak{D}_a$.

Proof: By (3.11) it suffices to prove that

$$g : [0,\pi/2] \to \mathbb{R} \text{ defined by}$$
$$g(\lambda) := \lambda - \frac{6}{14\times18}\lambda^3 + \frac{\lambda^5}{25\times12}$$

is bounded above by $\pi/2$. Since

$$g'(\lambda) = 1 - \frac{\lambda^2}{14} + \frac{\lambda^4}{60} > 0 \quad \forall\ \lambda \in [0,\pi]$$

and

$$g(\pi/2) = \pi/2 - \frac{6}{14\times18}(\pi/2)^3\left(1 - (\pi/2)^2\frac{14\times18}{6\times25\times12}\right) < \pi/2$$

the lemma is proved.

Lemma 3.4: α depends on (xy) only and is an odd function of (xy).

Proof: Consider the iteration of $C(\mathfrak{D}_a;\mathbb{R})$,

$$\alpha_0 = 0\ ,$$
$$\alpha_{n+1}(x,y) = \int_0^x\int_0^y \cos(\alpha_n(\theta,\theta))\ \ d\theta d\varphi, \qquad n \geq 0.$$

Clearly α_0 is expressible as a function of xy and is odd. Now suppose that the α_k also has these properties: we will prove that α_{k+1} satisfies this property as well.

Let (x_1,y_1) and $(x_2,y_2) \in \mathcal{D}_a$ be such that $x_1y_1 = x_2y_2$. We may assume that $x_1 \neq 0 \neq y_1$ since otherwise $\alpha_{n+1}(x_1,y_1) = \alpha_{n+1}(x_2,y_2) = 0$.

Now

$$\alpha_{k+1}(x_1,y_1) = \int_0^{x_1} \int_0^{y_1} \cos(\alpha_n(\theta,\varphi))d\varphi d\theta$$

$$= \int_0^{x_2} \int_0^{y_2} \cos(\alpha_k(\frac{x_1}{x_2} s, \frac{y_1}{y_2} \sigma))(\frac{x_1}{x_2})(\frac{y_1}{y_2})d\sigma ds$$

$$= \int_0^{x_2} \int_0^{y_2} \cos(\alpha_k(s,\sigma))d\sigma ds$$

$$= \alpha_{k+1}(x_2,y_2).$$

By induction we have proved that $\alpha_n(x,y)$ depends only on (xy) for every n. A similar induction argument proves that α_n is an odd function of (xy) for every n.

Since $\{\alpha_n\} \to \alpha$ uniformly, the lemma is proved. □

Now let $\gamma = \pi/2 - \alpha$ on $\mathcal{D}_a$. Then γ satisfies

$$\frac{\partial^2\gamma}{\partial x \partial y} = -\sin \gamma \ , \quad \gamma(x,0) = \pi/2 = \gamma(0,y) \ ,$$

and $\gamma(x,y) \in (0,\pi)$ for every $(x,y) \in \mathcal{D}_a$. Thus γ is a candidate for the angle between the fibers of a Tchebychev net on S^2, where x and y are now coordinates with respect to a Tchebychev chart. We must produce an open subset $\mathcal{U}$ of S^2 which is the domain of this Tchebychev chart.

It is well known that the Riemannian metric can be written on a Tchebychev chart (see [Pi1], [St1]) as

$$g_{(x,y)} = dx^2 + 2\cos\gamma(x,y)dxdy + dy^2. \tag{3.12}$$

Since $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ have unit magnitude with respect to g on $\mathcal{D}_a$ it follows that $\mathcal{D}_a$ endowed with the Riemannian metric $g = dx^2 + 2\cos\gamma dxdy + dy^2$ admits a Tchebychev net $(\mathcal{D}_a, (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}))$. If we can find an open subset $\mathcal{U}$ of S^2 and an isometry φ from $\mathcal{U}$ into or onto $\mathcal{D}_a$ endowed with g, then $(\mathcal{U}, \varphi)$ is a Tchebychev net on S^2. We will produce such an open subset by using the exponential mapping.

First let us recall some facts from Riemannian geometry.

Let (M, ρ) be a Riemannian manifold and $p \in M$. Then $\exp_p$ is a mapping having an open subset of T_pM as its domain and M as its codomain, and such that

$$\exp_p(v) = \sigma(1) \ ,$$

where σ is the geodesic of (M, ρ) such that $\sigma(0) = p$ and $\dot{\sigma}(0) = v$.

The Riemannian metric ρ induces a unique Riemannian connection ∇ on M. Many of the computations needed can be done conveniently by using the Christoffel symbols with respect to some coordinate chart.

Let $(U, (x^1, x^2))$ be a local coordinate chart on M. Then the Christoffel symbols $\{\Gamma^k_{ij}\}_{1 \leq j, k \leq 2}$ are C^∞-functions on U defined by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}, \tag{3.13}$$

where the summation convention is used.

Let $\rho = \rho_{ij} dx^i dx^j$ with respect to the local coordinates on U. Define $\{\rho^{ij}\}_{1 \le i,j \le n} \subset C^\infty(U)$ by

$$\rho^{ij} \rho_{j\ell} = \begin{cases} 1 & \text{if } i = \ell \\ 0 & \text{if } i \ne \ell \end{cases} \tag{3.14}$$

The Christoffel symbols are easily computed using the formula

$$\Gamma^k_{ij} = \frac{1}{2} \rho^{k\ell} \left[\frac{\partial}{\partial x^i} \rho_{\ell j} + \frac{\partial}{\partial x^j} \rho_{i\ell} - \frac{\partial}{\partial x^\ell} \rho_{ij}\right]. \tag{3.15}$$

Let $\sigma : [a,b] \to M$ be a geodesic such that $\sigma([a,b]) \subset U$. Then, writing the components of σ with respect to the local coordinates as $\sigma = (\sigma^1, \sigma^2)$, we have

$$(\ddot{\sigma}^i + \dot{\sigma}^j \dot{\sigma}^k \Gamma^i_{jk}) = 0, \quad i = 1, \ldots, n. \tag{3.16}$$

For the remainder of this paper we fix $a = \pi$ and let

$$\mathfrak{D} = \mathfrak{D}_\pi = \{(x,y) \in \mathbb{R}^2 : |x| \le \pi, \ |y| \le \pi, \ |xy| \le \pi/2\}.$$

Let's consider the Riemannian manifold $(\mathfrak{D}, g)$. Using (3.15) we compute the Christoffel symbols:

$$\Gamma^1_{11} = \Gamma^2_{22} = \cot(\gamma)\gamma y$$

$$\Gamma^2_{11} = -\csc(\gamma)\gamma_x$$

$$\Gamma^1_{22} = -\csc(\gamma)\gamma_y$$

$$\Gamma^1_{12} = \Gamma^2_{12} = \Gamma^1_{21} = \Gamma^2_{21} \equiv 0. \tag{3.17}$$

Remark 3.5: The fact that $\Gamma^k_{ij} = 0$ for $i = k$ is equivalent to the statement that on a Tchebychev net each fiber field is parallel along the integral curves of the remaining vector fields.

Let $t \mapsto (x(t), y(t))$ be a geodesic of $(\mathcal{D}, g)$. Then by (3.16) and (3.17) we get

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= -\cot(\gamma)\gamma_x u^2 + \csc(\gamma)\gamma_y v^2 \\ \dot{v} &= \csc(\gamma)\gamma_x u^2 - \cot(\gamma)\gamma_y v^2. \end{aligned} \tag{3.18}$$

The following facts are obtained from lemmas (3.3) and (3.4) and equations (3.18).

Fact 3.6: The curves $t \mapsto (t,0)$, $|t| \leq \pi$,

$t \mapsto (0,t)$, $|t| \leq \pi$.

are both geodesics.

Fact 3.7: If $t \mapsto (x(t),y(t))$, $|t| < b$, is a geodesic, then the curves

$$t \mapsto (\pm\, x(t),\ \pm\, y(t))$$

are all geodesics.

Because of this we consider geodesics in the first quadrant unless such a geodesic crosses the x or the y axis.

Fact 3.8: If $t \mapsto (x(t),y(t))$ is a geodesic, then $t \mapsto (y(t),x(t))$ is a geodesic.

In particular the geodesic $\sigma : [-b,b] \to \mathfrak{D}$ satisfying $\sigma(0) = 0$ and $\dot{\sigma}(0) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ lies on the line $x = y$.

Fact 3.9: $\gamma_x(x,y) = -\int_0^y \cos(\alpha(x,\varphi))d\varphi$. Thus

$$0 \geq \operatorname{sgn}(y)\gamma_x(x,y) \geq -|y| \quad \text{and} \quad \gamma_x(x,y) = 0 \quad \text{iff} \quad y = 0,$$

and

$$0 \geq \operatorname{sgn}(x)\gamma_y(x,y) \geq -|x| \quad \text{and} \quad \gamma_y(x,y) = 0 \quad \text{iff} \quad x = 0.$$

Lemma 3.10: The domain of $\exp_0$ contains $(\delta\pi/2)(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ for some $\delta > 1$.

Proof: Let $\sigma : [-b,b] \to \mathbb{R}^2$, $\sigma(t) = (x(t),x(t))$ be the geodesic satisfying $\sigma(0) = 0$ and $\dot{\sigma}(0) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Let $u(t) = \dot{x}(t)$ for all t.

Since $\|\sigma(t)\|_\infty = 1$ for all t,

$$u(t) = \frac{1}{\sqrt{2(1 + \cos \gamma(x(t),x(t)))}}$$

$$= \frac{1}{2 \cos \frac{\gamma}{2} (x(t),x(t))} \, ,$$

and so,

$$\dot{x}(t) = u(t) = \frac{1}{2 \cos(\pi/4 - \frac{1}{2}\alpha(x(t),x(t)))}$$

$$= \frac{1}{\sqrt{2}\,(\cos(\frac{1}{2}\alpha(x(t),x(t))) + \sin(\frac{1}{2}\alpha(x(t),x(t))))} \qquad (3.19)$$

Let $\theta \in [0,1]$. Then,

$$\frac{\theta^6}{6!} - \frac{\theta^8}{8!} + \dots + (-1)^{n+1} \frac{\theta^{2n}}{(2n)!} + \dots \le \frac{\theta^6}{6!} \left(\frac{1}{1 - \frac{\theta^2}{7 \cdot 8}}\right) \le \frac{\theta^6}{6!} \left(\frac{56}{55}\right). \qquad (3.20)$$

$$\frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots + (-1)^n \frac{\theta^{2n}}{(2n)!} + \dots \ge \frac{\theta^4}{4!} \left(1 - \frac{\theta^2}{5 \cdot 6}\right) \ge 0. \qquad (3.21)$$

and hence,

$$1 - \frac{\theta^2}{2} \le \cos \theta. \qquad (3.22)$$

Similarly

$$\theta - \frac{\theta^3}{3!} \le \sin\theta. \tag{3.23}$$

Now suppose that $t > 0$ is small enough that $0 < x(t) \le 1$. Then by (3.11),

$$\frac{1}{2}\alpha(x(t),x(t)) \le \frac{1}{2}\left(1 + \frac{1}{18} + \frac{2}{63} + \frac{1}{25\times 12}\right) < 1.$$

Thus

$$\cos(\tfrac{1}{2}\alpha(x(t),x(t)) + \sin(\tfrac{1}{2}\alpha(x(t),x(t))$$

$$\ge 1 - \tfrac{1}{2}(\tfrac{1}{2}\alpha(x(t),x(t))^2 + \tfrac{1}{2}\alpha(x(t),x(t)) - \tfrac{1}{6}(\alpha(x(t),x(t))$$

$$\ge 1 + \frac{17}{48}\alpha(x(t),x(t))$$

$$\ge 1 + \frac{17}{48}\left((x(t))^2 - \frac{22}{18\times 14}(x(t))^6 - \frac{(x(t))^{10}}{25\times 12}\right). \tag{3.24}$$

Hence in (3.19),

$$\dot{x}(t) \le \frac{1}{\sqrt{2}\,(1 + \frac{17}{48}(x^2 - \frac{2}{21}x^6)},$$

and

$$\sqrt{2}\,(x(t) + \frac{17}{48}\frac{(x(t))^3}{3} - \frac{34}{7\times 21\times 48}x(t))^7) \le t.$$

By taking derivatives we see that the left hand side is monotone increasing when x increases from 0 to 1. When $x = 1$, the left hand side is greater than $\pi/2$. Since $(1,1) \in \mathfrak{D}$ the integral curve $t \mapsto \sigma(t)$ exists for all $t \in [-\delta\pi/2, \delta\pi/2]$ where $\delta > 1$.

Remark 3.11: In the previous lemma there exists $\delta > 1$ such that $|x(t)| \leq 1$ for all $|t| \leq \delta\pi/2$ along the integral curve σ.

We would like to find an isometry from some subset of S^2 into $\mathfrak{D}$ such that the south pole is mapped to the origin. Hence some longitude would be mapped to the geodesic σ which we had just considered and in particular $\sigma(\pi/2)$ would correspond to a point on the equator on S^2. We now produce as a candidate for the equator a geodesic passing through $\sigma(\pi/2)$ and orthogonal to σ.

Lemma 3.12: Consider the geodesic μ of $(\mathfrak{D},g)$ such that $\mu(0) = \sigma(\pi/2)$ and $\dot{u}(0)$ is a unit vector such that $(\dot{\mu}(0),\dot{\sigma}(\pi/2))$ forms a positively oriented orthonormal basis. Then μ can leave $\mathfrak{D} \cap \{(x,y) : x \geq 0, y \geq 0\}$ by crossing the x-axis and no other part of the boundary.

Proof: Let

$$\mathfrak{D}^+ = \mathfrak{D} \cap \{(x,y) : x \geq 0, y \geq 0\},$$

$(\bar{x},\bar{x}) = \exp_0(\frac{\pi}{2}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}))$, and $\mu(t) = (x(t),y(t))$, $t \in [0,T]$, where T is chosen such that $\mu(t) \in \text{int}(\mathcal{D}^+)$ for all $t \in [0,T]$. Then $\sin(\gamma(\mu(t))$ is bounded away from 0 and $x(t)$ is bounded above for all $t \in [0,T]$. In particular,

$$x(t) \csc(\gamma(\mu(t))) \text{ is bounded on } [0,T].$$

Now consider u and v as in equation (3.18). By fact 3.9 we obtain,

$$\dot{v}(t) < x(t) \cot(\gamma(\mu(t)))v^2.$$

Hence $-\frac{1}{v(t)} + \frac{1}{v(0)} \le \int_0^t x(\tau) \cot \gamma(\mu(\tau))d\tau < \infty$ for any $t \in [0,T]$.

Thus $v(t)$ never changes sign in $[0,T]$ and since $v(0) < 0$, $v(t) < 0$ for all $t \in [0,T]$. Therefore,

$$y(t) < y(0) = \bar{x} \quad \text{for all} \quad t \in (0,T].$$

Also by fact 3.9,

$$\frac{d}{dt}(u(t) + v(t)) = \gamma_x \frac{(1 - \cos \gamma)}{\sin \gamma} u^2 + \gamma_y \frac{(1 - \cos \gamma)}{\sin \gamma} u^2 < 0 \text{ for } t \in [0,T].$$

Since $u(0) + v(0) = 0$, it follows that

$$u(t) + v(t) < 0 \quad \text{for all} \quad t \in [0,T].$$

Now the conditions $v(t) < 0$ and $u(t) + v(t) < 0$ for all $t \in (0,T]$ imply that the curve $\mu(t)$, $(t \in (0,T])$ is in the triangle Λ, bounded by the lines

$$\{(x,0) : 0 \leq x \leq \sqrt{2}\,\bar{x}\},\ \{(x,x) \in 0 \leq x \leq \bar{x}\} \text{ and } \{(x,2\bar{x} - x) : \bar{x} \leq x \leq 2\bar{x}\}.$$

Since the line $\{(x,2\bar{x} - x) : 0 \leq x \leq 2\bar{x}\}$ is in the interior of $\mathfrak{D}^+$, μ does not leave $\mathfrak{D}^+$ by crossing the curve $xy = \pi/2$ or $y = \pi$.

Moreover, the curve μ does not cross the line $\{(x,x) : 0 \leq x \leq \bar{x}\}$ for $t \in [0,T]$, since if it does, there is a geodesic triangle contained in Λ the sum of whose interior angles is greater than $\pi/2$. However, since the Gaussian curvature of g is equal to 1 and since the measure of Λ (with respect to the measure induced by g) is less than 1, this contradicts the Gauss Bonnet Theorem.

Since $v(t) < 0$, $t \in [0,T]$, it follows that the geodesic μ ultimately leaves $\mathfrak{D}^+$ by crossing the x-axis.

Let $\tilde{\mathfrak{D}}$ be the closed region bounded by the line $x = y$, the geodesic μ, and the positive x-axis.

We now show that we can construct a diffeomorphism of a segment of the hemisphere into $\tilde{\mathfrak{D}}$. We introduce Gauss coordinates to do so.

Lemma 3.13: The map

$$\psi : [0,\pi/4] \times [0,\pi/2] \to \mathfrak{D} \quad \text{defined by}$$
$$\psi(\theta,t) = \exp_0(t(\cos\theta,\ \sin\theta))$$

is well defined, and it is the inverse of a Gauss coordinate system on the sector $\tilde{\mathfrak{D}}$.

Proof: Let μ be the geodesic constructed in lemma 3.11.

Thus we wish to show that all geodesics in $\tilde{\mathfrak{D}}$ which start radially from the origin meet μ orthogonally; i.e., that,
$U := \{\theta \in [0,\pi/4] : \psi(\theta,[0,\frac{\pi}{2}] \in \mathfrak{D},\ \psi(\theta,\pi/4) \in \mu,$ and $\frac{\partial}{\partial t}\psi(\theta,\pi/2)$ is orthogonal to $\mu\}$ is all of $[0,\pi/4]$.

Clearly $\pi/4 \in U$. Let $\theta_0 \in U$. Then the map $t \mapsto \psi(\theta_0,t)$, $t \in [0,\pi/2]$ is a geodesic by definition. Also by our construction the curvature on $\tilde{\mathfrak{D}}$ is 1 and hence for each $t_0 \in [0,\pi/2]$ there exists an isometry from a neighborhood of $\psi(\theta_0,t_0)$ in $\tilde{\mathfrak{D}}$ onto an open subset of the standard sphere. It follows from the compactness of $[0,\pi/2]$ that there exists an open subset V of θ_0 in $[0,\pi/4]$ such that $\psi(V \times [0,\pi/2])$ is isometric to a triangular sector around $\psi(\theta_0 \times [0,\pi/2])$ of the standard sphere with a vertex at the south pole and a base on the equator. Now $\psi(V \times \pi/2)$ is mapped onto the equator, thus $V \subset U$. Therefore U is an open subset of $[0,\pi/4]$. Since U is closed by the continuity of ψ, $U = [0,\pi/4]$. In particular, ψ is defined and $\mathrm{Rng}(\psi) \subset \Delta$. Since $\psi(0 \times [0,\pi/2])$ is a subset of the horizontal axis and $\psi(\pi/4 \times [0,\pi/2])$ is along the line $x = y$ and $\psi([0,\pi/4] \times \pi/2)$ contains μ, it follows that $\psi([0,\pi/4] \times [0,\pi/2])$ contains $\tilde{\mathfrak{D}}$. We claim that ψ is one to one. In order to show this we use the following fact from [OKU1]. Let $p \in M$ where M is a Riemannian manifold, and let $r(p) = \sup_{r\in\mathbb{R}}\{r \geq 0$ such that $\exp : B_r(0) \subset T_pM \to M$ is injective$\}$. $r(p)$ is called the injectivity radius at p. Then $\exp_p : B_{r(p)}(0) \subset T_pM \to M$ is a diffeomorphism onto some open set (and this is a Gauss coordinate system).

Now consider $\varphi : \overline{B_{\pi/2}(0)} \to \mathfrak{D}$ given by

$$\varphi(\theta,t) = \exp_0(t(\cos\theta,\ \sin\theta))$$

where (t,θ) are polar coordinates on the disc. If φ is not one to one, then the injectivity radius is less than or equal to $\pi/2$. It follows [OKU1] that there are 2 geodesics η_1, η_2 in $\mathfrak{D}$ eminating from the origin of the same length $r(0) \leq \pi/2$ which meet. Let the parameter t denote arclength parameterization on each geodesic. Since $\exp_0 : B_{r(0)}(0) \rightarrow \mathfrak{D}$ is the inverse of a Gauss coordinate system, we have an isometry from $\exp_0(B_{r(0)}(0))$ to a southern polar cap of radius $r(0)$ of the sphere. By considering the isometry, we conclude that the distance $d(\eta_1(t),\eta_2(t))$ is increasing with t.

But this is a contradiction since $\eta_1(r(0)) = \eta_2(r(0))$. Since φ is one to one it follows that ψ is one to one. We have now seen that the injectivity radius at the origin is greater than $\pi/2$ and hence ψ is a diffeomorphism providing a sectorial Gauss coordinate system.

We have now established that $\psi : [0,\pi/4] \times [0,\pi/2] \rightarrow \tilde{\mathfrak{D}}$ is a bijection. Also, no point in $\tilde{\mathfrak{D}}$ is antipodal to the origin along any geodesic. Hence ψ^{-1} is a Gauss coordinate system on $\tilde{\mathfrak{D}}$ and therefore $\tilde{\mathfrak{D}}$ can be mapped isometrically onto the subset of S^2 in the closed southern hemisphere bounded between longitudes 0 and 90 degrees.

By symmetry there exists an isometry from the closed southern hemisphere into $\mathfrak{D}$ such that the image is contained in the interior of $\mathfrak{D}$. Since Gauss coordinates have open domain, we can increase the domain slightly and obtain a surjective isometry $\varphi : W \longrightarrow \hat{W}$ such that W is an open subset of S^2 and contains the closed southern hemisphere and $\hat{W}$ is an open subset of $\mathfrak{D}$. Moreover, we can use the inverse of $\varphi : W \rightarrow \mathbb{R}^2$ to construct a Tchebychev net on S^2. This concludes the proof of the Main Theorem.

References

[Teh] P.L. Tchebychev, Sur la coupes des ve tements, Assoc. *fran$ -pour I'advancement des* set., Congres de Paris, 154-155 (1878).

[Bie] L. Bieberbach, Uber Tchebychefsche netz auf flächen negative **Kriiiranung, Preuss. Afcod. Wiss., Phys. Math. Kl. 23, 294-321 (1962).**

[Haz] J.N. Hazzidakis, Ueber einige eigenchaften des f lachen mit **constantem krlimmungs maass, J. flir *die reine undangen. Math.* (Crelle's J.) 88, 68-73 (188).**

[Ser] M. Servant, sur l'habillage des serfaces, *C.R. Akad.* Sci., 137, 112-115 (1903).

[St] J.J. Stoker, Differential Geometry, John Wiley and Sons, 1969.