COMPLETE ISOTHERMAL RIEMANNIAN METRICS ON $\mathbb{R}^2$ HAVING COMPACTLY SUPPORTED GAUSSIAN CURVATURE

by

Sandra Samelson
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Research Report No. 89-55 2

July 1989

Complete Isothermal Riemannian Metrics on $\mathbb{R}^2$ having Compactly Supported Gaussian Curvature

Abstract: We consider a smooth Riemannian metric G on $\mathbb{R}^2$ which is assumed to be complete and has compactly supported Gaussian curvature. Using the uniformization theorem one can produce an isometry from $(\mathbb{R}^2, G)$ onto $(\mathcal{U}, g)$ where $\mathcal{U}$ is either the open unit ball of $\mathbb{R}^2$ or $\mathbb{R}^2$ itself and g is an isothermal Riemannian metric, i.e., $g = \lambda^{-2}(dx^2 + dy^2)$ where λ is a positive real valued smooth function. We will prove that $\mathcal{U}$ is necessarily equal to $\mathbb{R}^2$ and the behavior of λ at infinity is determined by the integral of the Gaussian curvature of G on $\mathbb{R}^2$. In particular if this integral is zero then λ is continuous at infinity and bounded away from zero.

1. Introduction:

We will consider a smooth Riemannian metric G on $\mathbb{R}^2$, which is assumed to be complete and has compactly supported Gaussian curvature. By using the Uniformization theorem we obtain an isometry from $(\mathbb{R}^2, G)$ onto $(\mathcal{U}, g = \lambda^{-2}(dx^2 + dy^2))$ where $\mathcal{U}$ is either the unit open ball or all of $\mathbb{R}^2$ and λ is a positive real valued C^∞-function.

We then will show that the hypotheses on $(\mathbb{R}^2, G)$ imply that $\mathcal{U}$ is necessarily equal to $\mathbb{R}^2$ and that the behavior of λ at infinity is determined by the integral of the Gaussian curvature κ of G (equivalently the integral of the Gaussian curvature κ of g). More precisely we will prove that,

$$\lim_{(x,y)\to\infty} \ln(\lambda) + \left(\frac{1}{4\pi}\int_{\mathbb{R}^2} \kappa dA\right)\ln(x^2 + y^2)$$

exists and that

$$\int_{\mathbb{R}^2} \kappa dA$$

is necessarily less than or equal to 2π. Here dA denotes the Riemannian area element of G.

A consequence of the theorem will be that it allows one to produce a homotopy from a given complete Riemannian metric with compactly supported Gaussian curvature to one which is isometric to the Euclidean metric while preserving the completeness and the support of the Gaussian curvature of each Riemannian metric along the path. We use this fact in [Sa]

First let us consider an example.

1.1 Example

$\tilde{\lambda}(x,y) = \dfrac{1}{\sqrt{x^2 + y^2}}$ for $x^2 + y^2 \geq 1$ and a positive C^∞-function on $\mathbb{R}^2$.

It is easy to show that $\int_{\mathbb{R}^2} \kappa dA = 2\pi$.

To show that we consider the map

$$\psi : \mathbb{R}^2 \backslash B_1(0) \to \mathbb{R}^3$$

$$\psi(x,y) := \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, \frac{1}{2} \ln (x^2 + y^2)\right)$$

where $B_1(0)$ denotes the ball of radius 1 on $\mathbb{R}^2$. Clearly this map is an isometry onto the cylinder induced from the Euclidean metric in the ambient space $\mathbb{R}^3$. The curve $x^2 + y^2 = 4$ is mapped to the geodesic $(u_1, u_2, \ln(2))$: $u_1^2 + u_2^2 = 1\}$ on the cylinder and hence the original curve itself must be a geodesic which is closed. Thus by the Gauss Bennett Theorem we obtain $\int_{\mathbb{R}^2} \kappa dA = \int_{B_2(0)} \kappa dA = 2\pi$.

Now we consider the Riemannian metric $\lambda^{-2}(dx^2 + dy^2)$ where $\lambda = (\tilde{\lambda})^\alpha$ for some $\alpha \in \mathbb{R}$. Then for this Riemannian metric,

$$\int_{\mathbb{R}^2} \kappa dA = \int_{\mathbb{R}^2} \Delta \, (\ln \tilde{\lambda}^\alpha) dxdy = \alpha \int_{\mathbb{R}^2} \Delta(\ln \tilde{\lambda}) dxdy = 2\pi\alpha.$$

Thus for this class of metrics we see that

$$\lim_{(x,y)\to\infty} \ln\lambda + \left(\frac{1}{4\pi}\int_{\mathbb{R}^2}\kappa dA\right)\ln(x^2+y^2)$$

exists.

Our objective is to show the existence of this limit for all complete isothermal metrics with compactly supported Gauss curvature.

A consequence of this result is that for a Riemannian metric $\lambda^{-2}(dx^2+dy^2)$ of the type considered here, the associated Riemannian metric $\lambda^{-2t}(dx^2+dy^2)$ for $0 \leq t \leq 1$ is complete also.

2. Main results:

2.1 Theorem: Suppose that G is a complete Riemannian metric on $\mathbb{R}^2$ such that its Gaussian curvature κ is compactly supported. Then there exists an isometry from $(\mathbb{R}^2, G)$ onto $(\mathbb{R}^2, (g = \lambda^{-2}(dx^2+dy^2))$ where λ is a positive C^∞-function on $\mathbb{R}^2$ and the following are satisfied:

(a) $\lim_{(x,y)\to\infty} \ln(\lambda(x,y)) + \frac{1}{4\pi}\left(\int_{\mathbb{R}^2}\kappa dA\right)\ln(x^2+y^2)$ exists, and

(b) $\int_{\mathbb{R}^2}\kappa dA \leq 2\pi$,

where dA denotes the Riemannian area element associated to G.

2.2 Corollary: Suppose that G, g, and λ are as in Theorem 2.1. Then for all $t \in [0,1]$, the Riemannian metric $\lambda^{-2t}(dx^2+dy^2)$ is complete.

In the remainder of the paper we will prove these results. In section 3, we will state some facts which are probably already known; however, we will provide the proofs for completeness. In section 4 we will develop the principal technique for the proofs of the main results. We will generalize the well-known results of Julia on the existence of a Julia ray of a holomorphic function around an essential singularity. Essentially we will replace the term "ray" with "geodesic". The main results will be proved in section 5.

3. Preliminaries:

Let K denote either R^2 or the unit open disk.

In this section we consider a complete Riemannian metric $g = X^{-2}(dx^2 + dy^2)$ on ^ and assume that the Gaussian curvature *K* has compact support and that $\int_{\mathfrak{U}}$ icdA ≤ 0 (we will drop this assumption in section 5). To fix notation, assume that supp(ie) C $B_r(0)$ (Euclidean ball) for some $r > 0$. The goal of this section is to prove some preliminary results which are useful later on. These results are probably valid in a much more general setting, and at least the first one is well-known. For the sake of completeness we present the proofs.

We assume that geodesies are parameterized by arclength. The topological metric induced by the Riemannian metric g is denoted by d and is complete by the Hopf-Renow Theorem.

Note that since g is isothermal, angles between tangent vectors measured with respect to the Euclidean metric agree with angles measured with respect to g. Thus notions such as orthogonality are unambigous. All magnitudes and inner products of tangent vectors are with respect to g unless it is written IhlL or <#.*>o' ln which case the quantities are with respect to the Euclidean metric of R^2.

Lenma 3.1 There is a point p € $dB_r(0)$ such that the geodesic a: R -* % defined by cr(O) = p and $\dot{a}(0)$ = T^-TT has the property that a(t) € $Cl(B_r(O))$ for every $t > 0$.

Proof: Suppose not. For $q \in \partial B_r(0)$ denote by σ^q the geodesic defined by $\sigma^q(0) = q$ and $\dot{\sigma}^q(0) = \frac{q}{\|q\|}$. Let $\delta > 0$ and $\tau > 0$ be such that for all $q \in \partial B_r(0)$ $\sigma^q(t) \notin \partial B_r(q)$ for $t \in (0,\tau)$ and $d(\sigma^q(\tau), \partial B_r(0)) > \delta$. Assume without loss of generality that $\delta < \tau$. By the hypothesis, for every $q \in \partial B_r(0)$ there exists $s > 0$ such that $\sigma^q(s) \in \partial B_r(0)$. It follows that $s > \tau$. Now by the continuity of $\sigma^q(t)$ on t and q, we can find $T > \tau$ such that for every $q \in \partial B_r(0)$ there exists $t \in [\tau,T]$ such that $\sigma^q(t) \in \mathcal{V}$ where $\mathcal{V}: = \{p \in \mathbb{R}^2: d(p, \partial B_r(0)) < \delta\}$. Let $x \in \mathcal{U}$ be given. Since the Riemannian metric is complete find $y \in \partial B_r(0)$ such that $d(x,y) \geq d(x,q)$ for every $q \in \partial B_r(0)$. Let $t_x = d(x,y)$. (Then $x = \sigma^y(tx)$). We claim that $t_x \leq T$. Assume otherwise. Let $t \in [\tau,T]$ be such that $\sigma^y(t) \in \mathcal{V}$. Let $q \in \partial B_r(0)$ be such that $d(q,\sigma^y(t)) < \delta$. Then $d(q,x) \leq d(q, \sigma^y(t)) + d(\sigma^y(t),\sigma^y(t_x)) < \delta + (t_x - t) < \delta + (t_x - \tau) < t_x$ which contradicts the definition of the point y; that is, y is not the closest point to x on $\partial B_r(0)$. But since T is fixed and x is arbitrary, it follows that $\mathcal{U}$ is bounded with respect to d which violates the completeness of d.

Lemma 3.2: Let $\sigma : \mathbb{R} \to \mathcal{U}$ be a geodesic such that $\sigma(t) \notin Cl(B_r(0))$ for every $t > 0$. Then there exists $\tau > 0$ such that for every $t \geq \tau$, the geodesic $\nu: \mathbb{R} \to \mathcal{U}$ defined by $\nu(0) = \sigma(t)$ and $\dot{\nu}(0) \perp \dot{\sigma}(t)$ never enters $B_r(0)$.

Proof: For $t > 0$, let $\nu^t: \mathbb{R} \to \mathbb{R}^2$ denote the geodesic defined by $\nu^t(0) = \sigma(t)$ and $\dot{\nu}^t(0) \perp \dot{\sigma}(t)$. (When we write $a \perp b$ we will always assume that (a,b) is positively oriented).

Now we only need to prove that there exists $t_0 > 0$ such that ν^{t_0} does not enter $B_r(0)$, since then $\dot{\nu}^{t_0}$ is an infinite geodesic which lies entirely in $(B_r(0))^c$ on which $\kappa \equiv 0$. Thus ν^{t_0} does not intersect with itself. Then ν^{t_0} separates $\mathcal{U}$ into two connected components of which on one $\kappa \equiv 0$. (Denote this component by $\mathfrak{K}$). Hence $\sigma[t_0,\infty) \cap \nu^{t_0}(\mathbb{R}) = \{\sigma(t_0)\}$, since otherwise we have a geodesic triangle formed by σ and ν^{t_0} in $(B_r(0))^c$ such that the sum of the interior angles is greater than π, thus violating the Gauss Bonnet Theorem. Now $\sigma(t_0,\infty) \subset \mathfrak{K}$ and using the same reasoning as above, $\nu^t(\mathbb{R}) \subset \mathfrak{K}$ for every $t > t_0$ and therefore ν^t does not enter $B_r(0)$ for every $t > t_0$.

Suppose now that ν^t intersects $B_r(0)$ for every $t > 0$. We show that this leads to a contradiction.

Define $T\colon (0,\infty) \to (0,\infty)$ as the smallest value $T(t)$ such that $\nu^t(T(t)) \in \partial B_r(0)$. Note that if $t_1, t_2 > 0$, $\tau_1, \tau_2 > 0$ are such that $\nu^{t_1}(\tau_1) = \nu^{t_2}(\tau_2)$ then the Gauss Bonnet Theorem asserts that $\nu^{t_1}[0,\tau_1] \cap \partial B_r(0) \neq \emptyset$ or that $\nu^{t_2}[0,\tau_2] \cap \partial B_r(0) \neq \emptyset$. Therefore $p_{(\cdot)}\colon [0,\infty) \to \partial B_r(0)$; $p_t = \nu^t(T(t))$ is monotone on $\partial B_r(0)$ and one to one.

Let $\{t_n\}_{n=1}^{\infty}$ be an increasing sequence of positive real numbers such that

(i) $t_n \to \infty$ as $n \to \infty$,

(ii) $\lim_{n\to\infty} p_{t_n}$ exists ($\lim_{n\to\infty} p_{t_n} = p \in \partial B_r(0)$), and

(iii) $\lim_{n\to\infty} \dot{\nu}^{t_n}(T(t_n))$ exists ($\lim_{n\to\infty} \dot{\nu}^{t_n}(T(t_n)) = v$).

Let $\alpha, \beta\colon \mathbb{R} \to \mathcal{U}$ be geodesics defined by

(a) $\alpha(0) = \beta(0) = p$,

(b) $\dot{\alpha}(0) = -v$, and

(c) $v \perp \dot{\beta}(0)$.

Let $\epsilon > 0$ be such that $\beta(-\epsilon,\epsilon) \subset (B_{r/2}(0))^c$. (Note that supp $(\kappa) \subset B_{r/2}(0)$).

By continuity $\beta(-\epsilon,\epsilon) \cap \nu^{t_n}[0,T(t_n)] \neq \emptyset$ for all large n. Without loss of generality assume that this is true for all n. Let $\{s_n\}_{n=1}^{\infty} \subset (-\epsilon,\epsilon)$ and $\{\theta_n \in [0,T(t_n)]\}_{n=1}^{\infty}$ be defined by $\beta(s_n) = \nu^{t_n}(\theta_n)$.

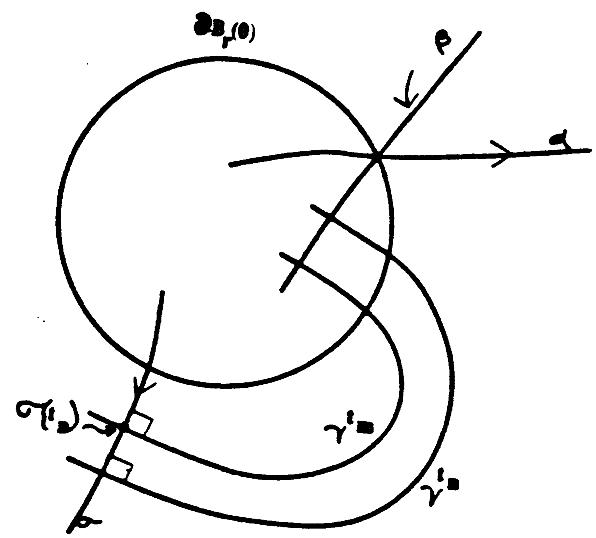

Fig 1

Let $n > m$ and consider the geodesic parallelogram with sides $\sigma[t_m,t_n]$, $\nu^{t_n}[0,\theta_n]$, $\beta[s_n,s_m]$ and $\nu^{t_m}[0,\theta_m]$ as shown in Fig 1. Since this parallelogram bounds a region on which $\kappa \equiv 0$ and since the interior angles at $\sigma(t_n)$ and $\sigma(t_m)$ are both $\pi/2$, it follows that the angle between $\dot{\nu}^{t_n}(\theta_n)$ and $\dot{\beta}(s_n)$ is equal to the angle between $\dot{\nu}^{t_m}(\theta_m)$ and $\dot{\beta}(s_m)$. But since this sequence of angles converges to $\pi/2$, it follows that ν^{t_n} and β meet orthogonally for all n.

Now let $n > m$ be given. Consider the family of geodesics $u^{\theta}: [t_m,t_n] \to \mathcal{M}$, $\theta \in [0,\theta_m]$, $u^{\theta}(t_m) = \nu^{t_m}(\theta)$ and $\dot{u}^{\theta}(t_m) \perp \dot{\nu}^{t_m}(\theta)$. It is easily seen that $u^{\theta}(t_n) = \nu^{t_n}(\theta)$ for every θ and $u^{\theta}(\tau) \cap \nu^{t_n}[0,\theta_n] = \emptyset$ for $\tau \in [t_m, t_n)$. In particular it follows that $u^{\theta_m}(\tau) \cap \nu^{t_n}[0,\theta_n] = \emptyset$ for every $\tau \in [t_m, t_n]$.

Thus $t_n - t_m < 2\varepsilon$. But this is a contradiction because $\{t_k\} \to \infty$ as $k \to \infty$. □

Note that instead of working with a family of geodescis $\{\nu^t\}$ which are orthogonal to σ, we can work with a family which makes a certain fixed angle with σ and obtain the same conclusion. We stated this as a lemma.

Lemma 3.3: Let $\sigma\colon \mathbb{R} \to \mathcal{U}$ be a geodesic such that $\sigma(t) \notin \mathrm{Cl}(B_r(0))$ for every $t > 0$. Let $\alpha \in (-\pi,\pi)$. Then there exists $\tau > 0$ such that for all $t \geq \tau$, the geodesic $\nu\colon \mathbb{R} \to \mathcal{U}$ defined by $\nu(0) = \sigma(t)$ and $\dot{\nu}(0)$ makes an angle α to $\dot{\sigma}(t)$ never enters $B_r(0)$.

Let ν be a geodesic as in the conclusion of lemma 3.2 or 3.3. Then ν separates $\mathcal{U}$ into two connected components and ν is an infinite geodesic. Therefore there exists $\tau > 0$ such that $\nu(t) \notin B_r(0)$ for every $t > \tau$ and thus we may produce a geodesic which makes a desired angle to ν such that this new geodesic never enters $B_r(0)$. Moreover, two such geodesics can be constructed as in Figure 2 below such that $\mathbb{R}^2$ is separated into six connected components and $B_r(0)$ is contained in one of these components.

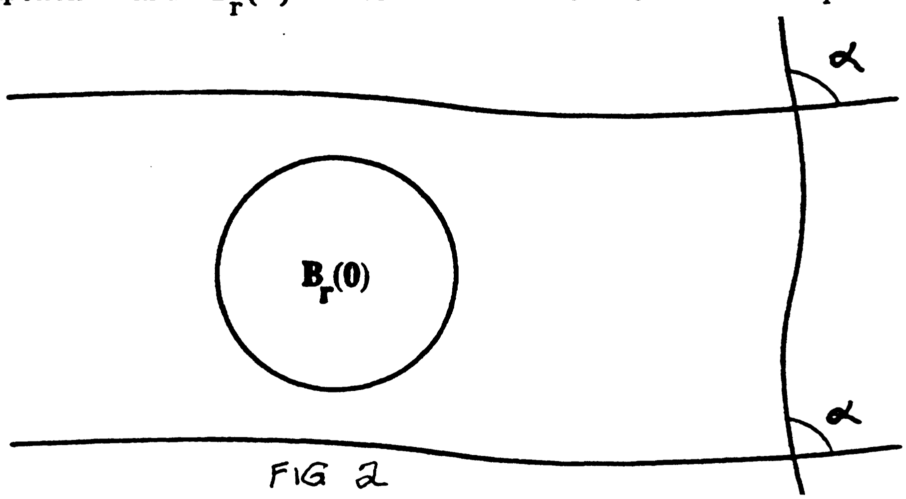

FIG 2

Next we construct a geodesic polygon enclosing $B_r(0)$ such that none of the boundary geodesics enter $B_r(0)$. We will assume that $\int_{\mathbb{R}^2} \kappa dA \leq 0$. If $\mathfrak{R}$ is a geodesic n-gon, then by the Gauss Bonnet theorem ,

$$(n-2)\pi + \int_{\mathfrak{R}} \kappa dA = \text{sum of the interiror angles of } \mathfrak{R}.$$

Since we wish to enclose $B_r(0)$, we pick an n-gon with interior angles $\alpha \in (\pi/4, \pi/2)$ such that

$$(n-2)\pi + \int_{\mathfrak{R}} \kappa dA = n\alpha.$$

Now construct n-geodesics as follows:

Start with any infinite geodesic $\upsilon_1 \colon \mathbb{R} \to \mathfrak{U}$ such that υ_1 separates $\mathfrak{U}$ into two components such that $B_r(0)$ is in one of the components (existence follows from lemma 3.2). Let T_1 be large enough that for all $t > T_1$ a geodesic through $\upsilon_1(t)$ which makes a positively oriented angle α to υ_1 will never enter $B_r(0)$. Let υ_2 be such a geodesic through $\upsilon_1(T_1)$. Now start with υ_2 and repeat the construction to obtain υ_3, and so on, to obtain geodesics $\upsilon_1, \upsilon_2, \ldots, \upsilon_n$.

Lemma 3.4: υ_n intersects υ_1 and υ_k does not intersect υ_1 if $1 < k < n$.

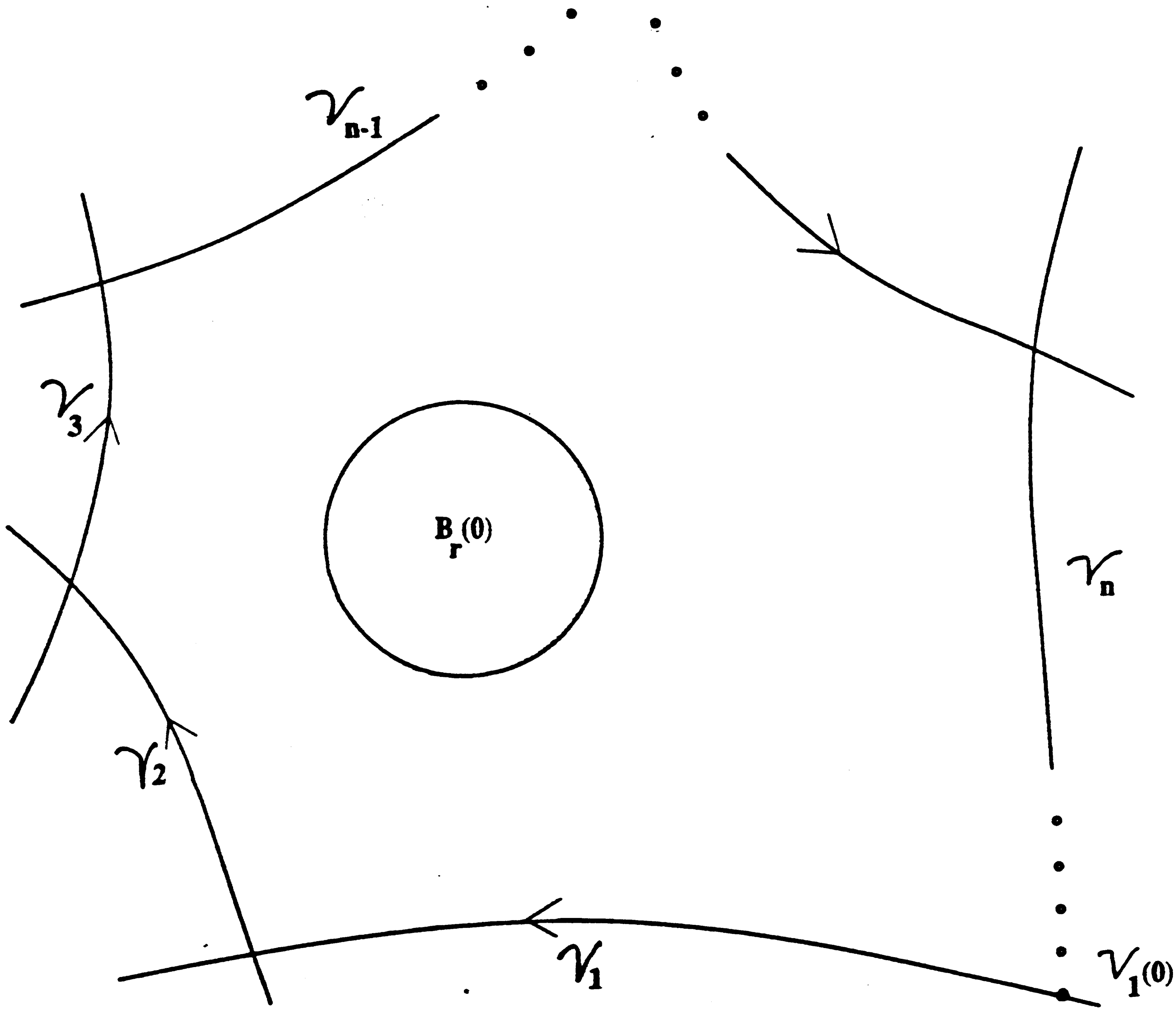

Fig 3

The second assertion follows trivially since otherwise the Gauss Bonnet theorem is violated.

We need to prove that γ_n intersects D_1. Let p_i denote the point of intersection of γ_i and γ_{i+1}. Let $\eta: [0,1] \to M$ be a regular curve such that $\eta(0) = p_{n-1}$, $\eta(1) = p_1$, η does not intersect with itself, η intersects γ_n at p_{n-1} only, η intersects D_1 at p_1 only and η and $\gamma_2, \gamma_3, \ldots, \gamma_{n-1}$ form a polygon containing $B_r(0)$. Such a curve can be constructed easily. If γ_1 intersects γ_n we already have such a curve. Otherwise the piecewise geodesic curve formed by $\gamma_1, \ldots, \gamma_n$ is such that $Cl(B_r(0))$ is contained in one of the connected components of M created by it and since this component is diffeomorphic to M and $K \equiv 0$, we have an isometry from that connected component to an open subset of R^2 with the Euclidean metric.

Now approximate η by a piecewise geodesic curve $f: [0, a] \to M$ for $a > 0$. This approximation can be done, for example, using the existence of geodesically convex neighborhoods of arbitrary points (cf [Mil]).

Our approximation is done such that f does not intersect itself or any of the γ_i that η does not meet and such that $f \cap \gamma_n = p_{n-1}$, $f \cap \gamma_1 = p_1$. Suppose that f has corners at $0 < t_1 < \ldots < t_k = a$. Let $Q_j = f(t_j)$, $1 \le j \le k$.

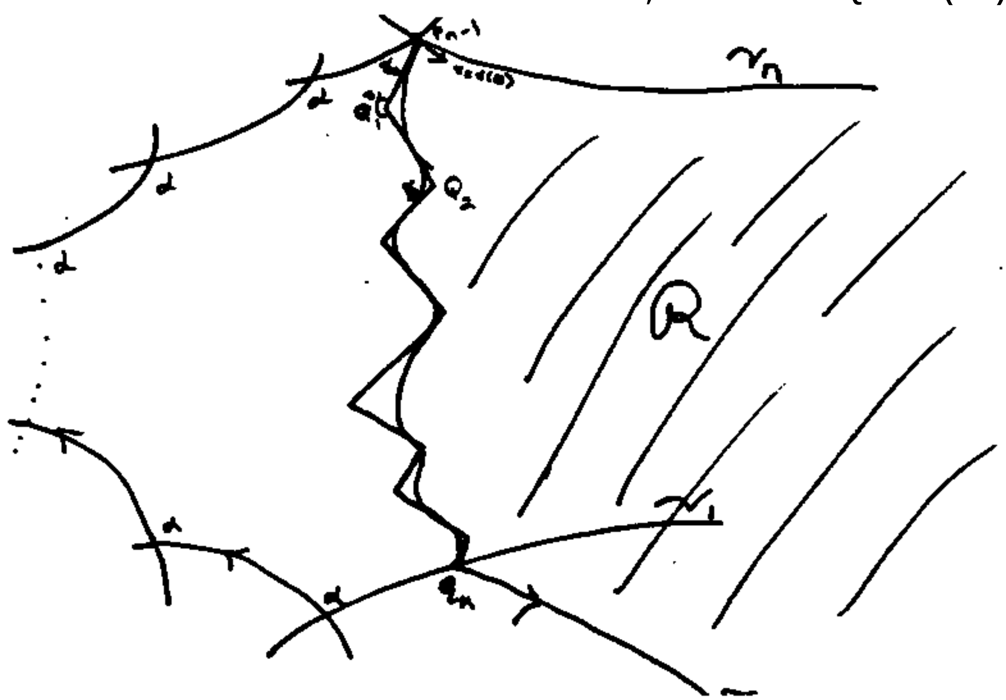

Fig 4

Consider the geodesic polygon G with corners at $p_1, \ldots, p_{n-1}, Q_1, \ldots, Q_k$. Let the interior angles at $p_{n-1}, Q_1, \ldots, Q_k$ be $\gamma_0, \gamma_1, \ldots, \gamma_k$, respectively. Since G encloses $B_r(0)$, by Gauss Bonnet we obtain,

$$(n-2)\alpha + \gamma_0 + \ldots + \gamma_k = \int_{\mathbb{R}^2} \kappa dA + (n+k-3)\pi.$$

since $n\alpha = \int_{\mathbb{R}^2} \kappa dA + (n-2)\pi$, we obtain

$$\gamma_0 + \ldots + \gamma_k - 2\alpha = (k-1)\pi. \tag{1}$$

Consider the unit tangent vector v to υ_n at p_{n-1} and parallel transport it along ξ to obtain a field $v(t)$, $t \in [0,1]$. By equation (1) it follows that $v(1)$ makes an angle $(\pi-\alpha)$ with υ_1 at Q_k. Let $\sigma\colon [0,\epsilon) \to \mathcal{U}$ be the geodesic such that $\sigma(0) = Q_k$ and $\dot{\sigma}(0) = v(1)$. Then since $\dot{\sigma}(0)$ points into the flat side $\mathfrak{R}_1$ of υ_1, it follows that $\sigma[0,\infty) \subset \mathbb{R}_1$. Therefore υ_n, ξ_1 and σ separate $\mathbb{R}^2$ such that on one side $\kappa \equiv 0$. Denote this side by $\mathfrak{R}$. Since $\mathfrak{R}$ is simply connected, absolute parallelism is defined on $\mathfrak{R}$.

Let X be a vector field on $\mathfrak{R}$ obtained by parallel transporting v on $\mathfrak{R}$ and let Y be the unit vector field orthogonal to X on $\mathfrak{R}$ such that (X,Y) is positively oriented.

Start from $t = 0$ and increase t. If at some $t \neq 0$, $\phi^X_\tau(\xi(t)) \in \xi[0,a]$ (where τ is in the domain of definition of $\phi^X_{(\cdot)}(\xi(t))$ replace the portion of ξ between $\xi(t)$ and $\phi^X_t(\xi(t))$ by $\phi^X_{(\cdot)}(\xi(t))$. The curve obtained after this modification will also be denoted by ξ. Note that ξ is a piecewise geodesic curve.

Now at all $t \in [0,a]$, $\dot{\xi}(t^{\pm})$ are either tangential to X or $(\dot{\xi}(t^{\pm}), X_{\xi(t)})$ is positively oriented. Hence for all $t \in [0,a]$, $\phi_{\tau}^{X}(\xi(t))$ is defined for all $\tau \geq 0$ and $\mathfrak{R} = \{\phi_{\tau}^{X}(\xi(t));\ \tau \geq 0;\ t \in [0,a]\}$.

Since $\Delta(\ln \lambda)) = 0$ on $\mathfrak{R}$, there exists a holomorphic function $f\colon \mathfrak{R} \to \mathbb{C}$ such that $\mathfrak{Re}(f) = \frac{1}{2} \ln \lambda$ (we now identify $\mathbb{R}^2$ with $\mathbb{C}$). Let $\psi\colon \mathfrak{R} \to \mathbb{C}$ be a holomorphic function such that $\psi'(z) = e^{f}(z)$ for all $z \in \mathfrak{R}$. Then $|\psi'(z)|^2 = \lambda^2(z)$ for all $z \in \mathfrak{R}$ and then $g = \psi^{*}(dx^2 + dy^2)$. Thus ψ maps goedesics in $\mathfrak{R}$ into straight lines in $\mathbb{R}^2$. After a suitable rotation of $\mathbb{R}^2$, we assume that ψ maps υ_n to the positive real axis. $\psi \circ \xi$ is a piecewise linear curve such that each piece is either horizontal or slanted downwards. Also $\psi \circ v(t) = \frac{\partial}{\partial x}$ for all $t \in [0,a]$. In particular $\psi \circ \sigma$ is parallel to the positive real axis. Since $\mathfrak{R} = \{\phi_{\tau}^{X}\ (\xi(t))\colon \tau \geq 0,\ t \in [0,a]\}$ and since $\psi \circ \xi$ does not intersect itself (since it always points downward), it follows that ψ is one to one. Hence $\psi\colon \mathfrak{R} \to \psi(\mathfrak{R})$ is an isometry where $\psi(\mathfrak{R})$ is a region in $\mathbb{R}^2$ bounded by two horizontal lines and a piecewise linear curve between them. Now $t \mapsto \psi \circ \upsilon_1(-\tau)$ is a straight line which is positively inclined to the horizontal direction by α and thus it meets $\psi \circ \upsilon_n$ making an angle α. Therefore υ_n and υ_1 meet at an angle α.

4. Julia geodesics for holomorphic functions with essential singularities at infinity.

The purpose of this section is to generalize the following theorem due to Julia [Hil]. This generalization provides the key to the proof of our main result.

Theorem 4.1 [Hil]. Let $f\colon \mathbb{C} \setminus B_r(0) \to \mathbb{C}$ be a holomorphic function which has an essential singularity at infinity. Then there is θ_0 such that for each $\delta > 0$ and each $R > r$, $f^{-1}(\omega) \cap \{z = |z|e^{i\theta}: |z| > R, \theta \in (\theta_0 - \delta, \theta_0 + \delta)$ contains infinitely many points for all $\omega \in \mathbb{C}$ with the possible exception of one point.

Definition 4.2 [Hil]: The ray $\{z = |z|e^{i\theta_0}: |z| \geq R\}$ is called a Julia ray.

Definition 4.3 [Hil]: Let $f: V \to \mathbb{C}$ be a holomorphic function where V is an open subset of $\mathbb{C}$. Then $z \in \mathbb{C}$ is called a Lacunary point for f if $z \notin f(V)$.

In the sequel we shall show that if $\int_{\mathbb{R}^2} \kappa dA \leq 0$ and g is complete, then one can define a Julia geodesic of g, and a corresponding theorem to theorem 4.1 holds once we replace rays by radial geodesics of g.

Definition 4.4: A semi-infinite geodesic means $\sigma|_{[0,\infty)}$ where $\sigma: \mathbb{R} \to \mathcal{U}$ is a geodesic which does not meet $Cl(B_r(0))$.

Definition 4.5: Two semi-infinite geodesics σ_1 and σ_2 are parallel if for all t large enough, the orthogonal geodesic to σ_1 at $\sigma_1(t)$ (which does not enter $Cl(B_r(0))$ by lemma 2.3) meets $\sigma_2|_{[0,\infty)}$ orthogonally.

Definition 4.6: A geodesic sector containing a semi-infinite geodesic σ means a region bounded by two intersecting semi-infinite geodesics σ_1 and σ_2 which are not parallel to each other or to σ, and such that $\sigma[0,\infty)$ is contained in the region.

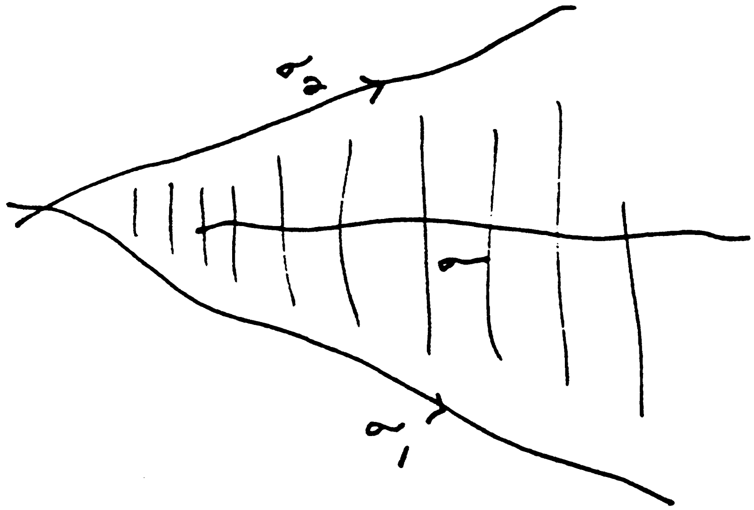

Fig 5

Definition 4.7: Let $h: \mathfrak{A}\setminus B_a(0) \to \mathbb{C}$ be a holomorphic function such that $\lim_{z\to\partial\mathfrak{A}} h(z) \neq \infty$, but there exists a sequence $\{z_n\}$ converging to $z_0 \in \partial\mathfrak{A}$ such that $\lim_{n\to\infty} h(z_n) = \infty$. A semi-infinite geodesic σ is a <u>Julia geodesic</u> for h if it has the following property:

Let $\mathfrak{S}$ be any geodesic sector containing σ. Then $h|_{\mathfrak{S}}$ has at most one Lacunary point.

Consider the following special case which motivates definition 4.6 in particular and this section in general.

Suppose that $F: \mathbb{C}\setminus B_a(0) \to \mathbb{C}$ $(a > 0)$ is a one to one holomorphic map such that F has a simple pole at infinity. Let $\lambda(z) = |\frac{d}{dz}F(z)|$. Extend λ smoothly to $\mathbb{R}^2$ such that $\lambda(z) > 0$ for all z. Consider the Riemannian metric $g = \lambda^{-2}(dx^2 + dy^2)$. It is clear that F maps geodescis of g to straight lines. Thus if $h: \mathbb{C}\setminus B_a(0) \to \mathbb{C}$ is a holomorphic map which has an essential singularity, then $h\circ F^{-1}$ has an essential singularity at infinity and hence it has a Julia ray η. Then $F^{-1}\circ\eta$ is a Julia geodesic of g.

Unfortunately even when $\int_{\mathbb{R}^2} \kappa dA = 0$ and $\mathrm{supp}(\kappa)$ is compact, it does not follow that $\lambda(z) = |\frac{d}{dz} F(z)|$ for some holomorphic function defined on $(B_r(0))^c$ (even though it holds on any simply connected open set).

Theorem 4.6: Let $h\colon \mathfrak{A}\setminus(B_r(0)) \to \mathbb{C}$ be a holomorphic map such that

$\lim_{z\to\partial\mathfrak{A}} h(z) \neq \infty$, but there exists a sequence (z_n) converging to $z \in \partial\mathfrak{A}$ with $\lim_{n\to\infty} h(z_n) = \infty$. Suppose that g is complete and $\int_{\mathbb{R}^2} \kappa dA \leq 0$. Then there exists a Julia geodesic.

Proof: Let $n \in \mathbb{N}$ and $\alpha \in (\frac{\pi}{4}, \frac{\pi}{2}]$ be such that $(n-2)\pi = \int_{\mathbb{R}^2} \kappa dA = n\alpha$.

Let $\gamma_1, \ldots, \gamma_n$ be geodesics which do not enter $B_r(0)$ and which form the sides of a closed n-gon as in the previous section.

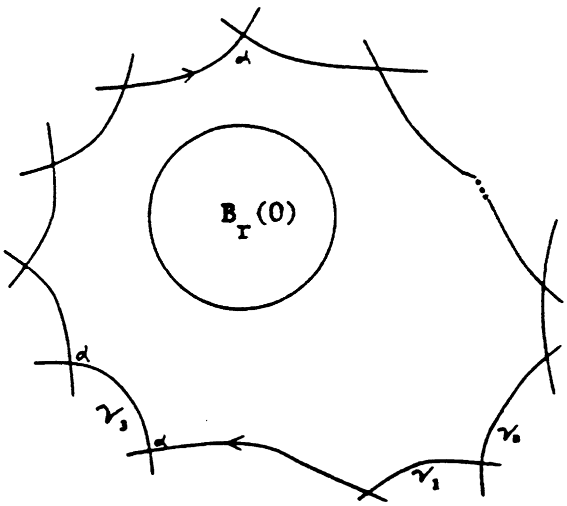

Fig 6

γ_i separates $\mathfrak{A}$ into two connected components. Let $\mathfrak{S}_i$ be the component on which $\kappa \equiv 0$. Consider the map $\psi_i\colon [0,\infty) \times \mathbb{R} \to \mathfrak{A}$ defined by $\psi_i(0,t_2) := \gamma_i(t_2)$ and

$$\psi_i(t_1,t_2) := \exp_{\psi_i(0,t_2)}(t_1\, e^{i\pi/2}\, \dot{\psi}_i(0,t_2)).$$

This map is a diffeomorphism and ψ_i when considered as a complex map is holomorphic.

Now let σ_1^θ: $[0,\infty) \to \mathcal{U}$, $\theta \in [-\pi/2, \pi/2]$ be a family of semi-infinite geodesics defined by,

$$\sigma_1^\theta(t) = \psi_1(t \cos \theta, t \sin \theta).$$

Since $\alpha \in (0, \pi/2]$ it follows that

$$\sigma_1^\theta[0,\infty) \cap \upsilon^2(\mathbb{R}) \neq \phi \quad \text{for all} \quad \theta > \pi/2.$$

Since both ψ_1 and ψ_2 map straight lines to geodesics, $\psi_2^{-1} \circ \psi_1$: $\psi_1^{-1}(\mathcal{R}_1 \cap \mathcal{R}_2) \to \psi_2^{-1}(\mathcal{R}_1 \cap \mathcal{R}_2)$ is a linear map. Therefore $\{\psi_2^{-1}(\sigma_1^\theta)$: $\theta \in (0,\pi/2]\}$ is a family of straight lines eminating from a point $p_2 \in \mathbb{R}^2$. Define the family of geodescis σ_2^θ: $\mathbb{R} \to \mathbb{R}^2$, $\theta \in (-\theta_{21},\theta_{22})$ for suitable $\theta_{21},\theta_{22} \in (-\pi,\pi)$ by

$$\sigma_2^\theta(t) := \psi_2(p_2 + (t \cos \theta, t \sin \theta))$$

for which the right hand side is defined, which occurs when the second coordinate of $p_2 + (t \cos \theta, t \sin \theta)$ is nonnegative.

The following facts are extremely important.

Fact 1: If $\sigma_1^{\theta_1}[0,\infty) \cap \sigma_2^{\theta_2}[0,\infty) \neq \phi$, then $\sigma_1^{\theta_1}$ and $\sigma_2^{\theta_2}$ are the same geodesic (including the parametrization).

Fact 2: The curves μ_1: $(-\pi/2.\pi/2] \to \mathbb{R}^2$ and μ_2: $(-\theta_{21},\theta_{22}) \to \mathbb{R}^2$ defined by

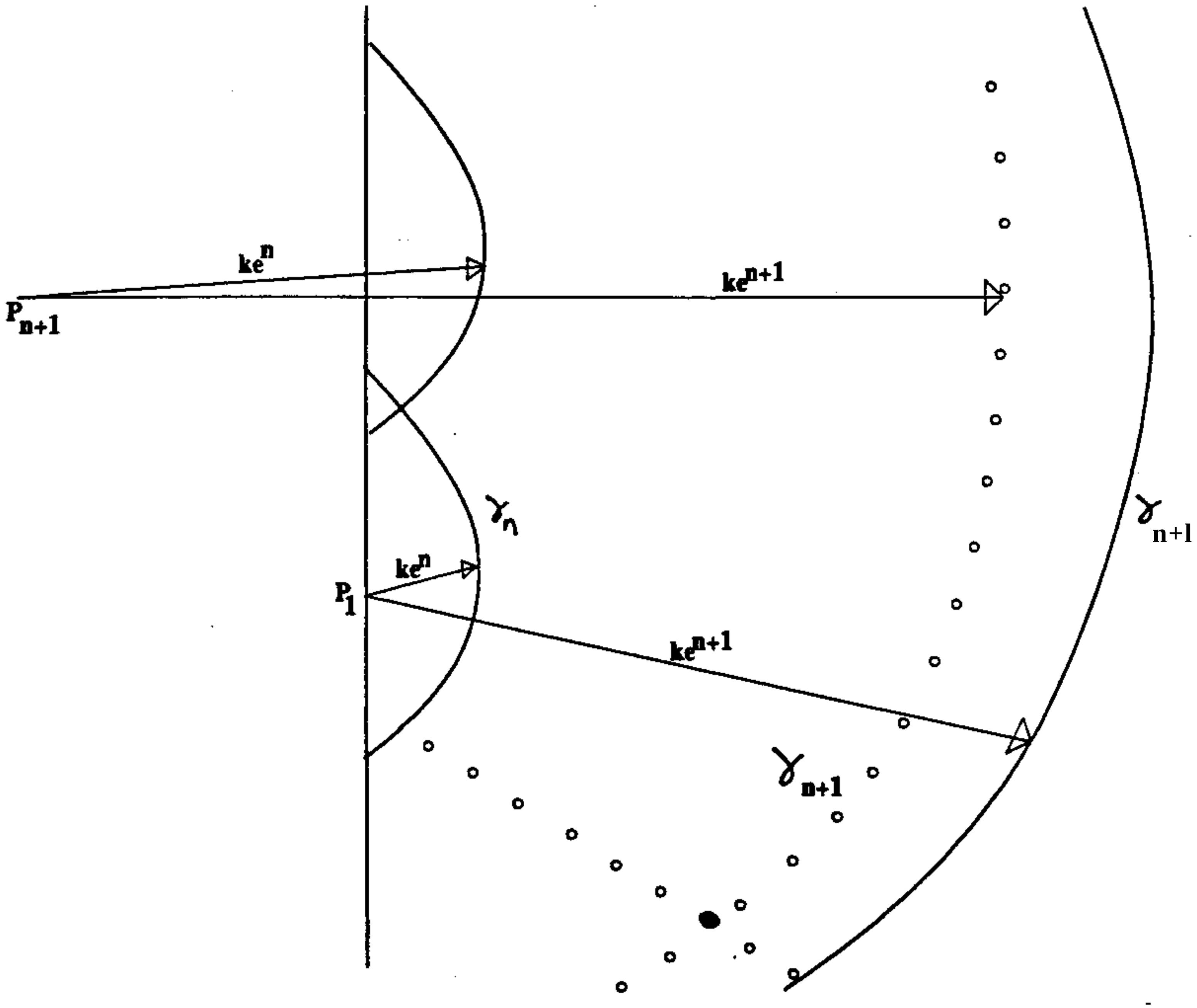

Fig 8

$$\mu_1(\theta) = \sigma_1^\theta(a) \text{ and } \mu_2(\theta) = \sigma_2^\theta(a)$$

are such that their images coincide on $\mathcal{R}_1 \cap \mathcal{R}_2$ whenever a is large enough.

Now using the family $\{\sigma_2^\theta\}$ and ψ_3 define $p_3 \in \mathbb{R}^2$ and a family of geodesics $\{\sigma_3^\theta\}$ such that $\{\psi^{-1}(\sigma_3^\theta)\}$ eminates from p_3 and the family $\{\sigma_2^3\theta\}$ and $\{\sigma_3^\theta\}$ agree on $\mathcal{R}_2 \cap \mathcal{R}_3$. Continue this way to produce families of geodesics $\{\sigma_j^\theta\}_{j=1,2,\ldots,n}$ and points $\{p_j\}_{j=1,2,\ldots,n}$ in $\mathbb{R}^2$. After reindexing the families we may assume that if $\sigma_i^\theta[b,\infty)$ and $\sigma_{i+1}^\theta[b,\infty)$ both lie in $\mathcal{R}_i \cap \mathcal{R}_{i+1}$ for some b, then $\sigma_i^\theta|_{[b,\infty)} = \sigma_{i+1}^\theta|_{[b,\infty)}$. Henceforth we will drop the subscript i from σ_i^θ. Let the range of ρ be $[-\pi/2,\hat{\theta}]$ ($\hat{\theta} \in (-\pi/2,\infty)$). Whenever a is large enough we can define a curve

$$u \quad [-\pi/2,\hat{\theta}] \to \mathcal{U} \text{ by } \mu^a(\theta) = \sigma^\theta(a).$$

The difficulty is that μ need not be a closed curve (even when $\int_{\mathcal{U}} \kappa dA = 0$).

Consider the family of curves $\gamma_n\colon [-\pi/2,\hat{\theta}] \to \mathcal{U}$ defined by $\gamma_n(\theta) = \mu^{ke^n}(\theta)$, $n \in \mathbb{N}$ and k is a positive number to be determined. Note that $\gamma_n(\theta) \in \mathcal{R}$, when θ is small and when θ is large. When θ is large, $\theta \mapsto \psi_1^{-1} \circ \gamma_n(\theta)$ is a family of concentric arcs, each with center at some $p_{n+1} \in \mathcal{U}$.

It is clear that p_{n+1} is independent of k, and if k is large enough, $\psi_1^{-1}(\gamma_n)$ and $\psi_1^{-1}(\gamma_{n+1})$ do not intersect. Fix k at such a value. Let $\mathcal{A}_n := \{\sigma^\theta(t)\colon \theta \in [-\pi/2,\hat{\theta}],\ ke^n < t < ke^{n+1}\}$. By the above there exists a closed curve $s_n \in \mathcal{A}_n$ which does not meet γ_n or γ_{n+1}.

Now let $\hat{\mathcal{R}} = \{z \in \mathbb{C}\colon \mathrm{Re}(z) \geq \ln k,\ \mathrm{Im}(z) \in [-\pi/2,\hat{\theta}]\}$ and let $\hat{\mathcal{R}}_n =$

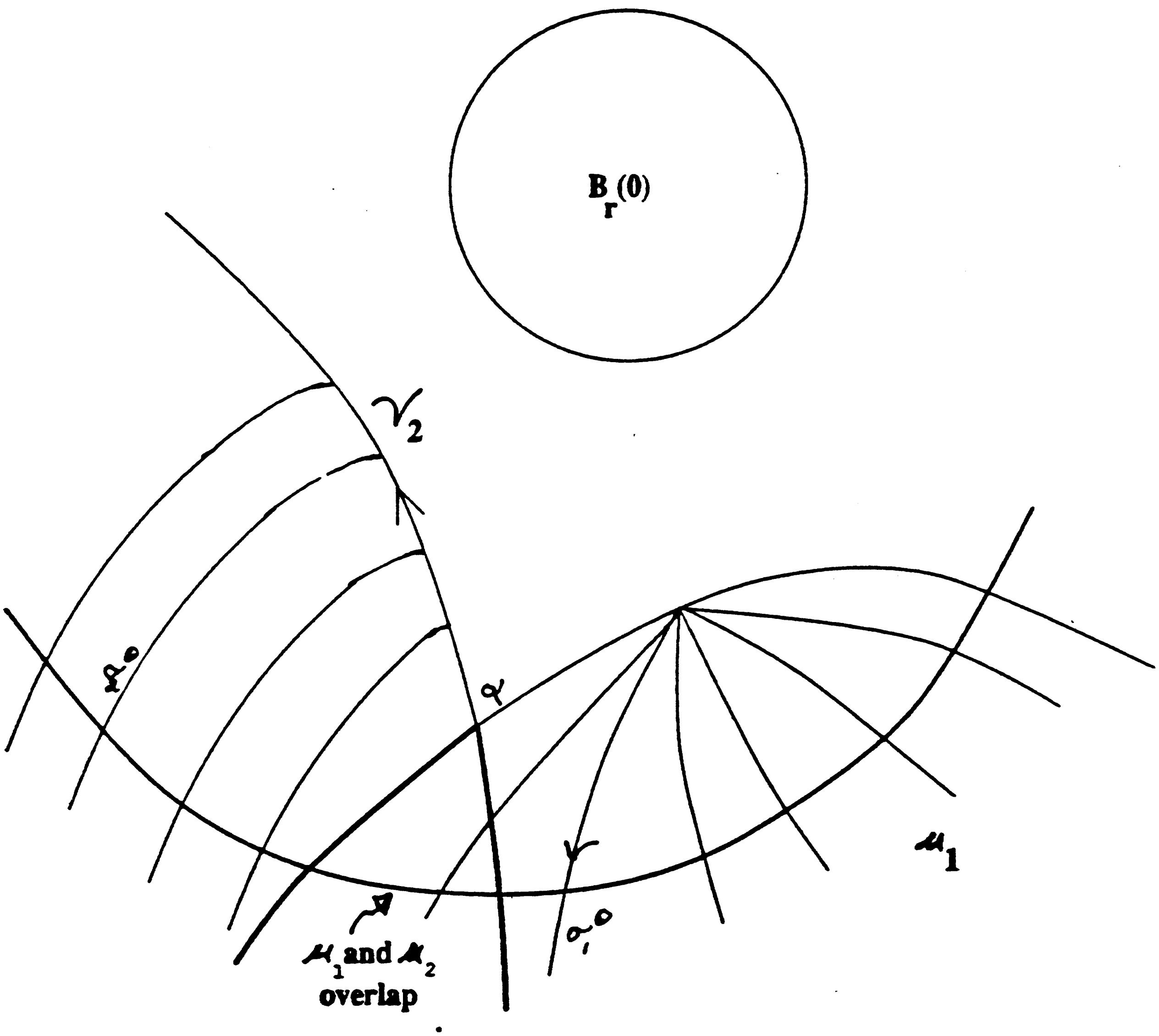

Fig 7

$\{z \in \hat{\mathfrak{R}}: n + \ln(k) \le \mathfrak{Re}(z) \le (n+1) + \ln k\}$. Let $H: \hat{\mathfrak{R}} \to \mathbb{C}$ be $H(x + iy) = h(\sigma^y(\exp x))$. Clearly $(x + iy) \mapsto \sigma^y(\exp x)$ is holomorphic and hence H is holomorphic. Let $H_n: \hat{\mathfrak{R}}_0 \to \mathbb{C}$ be $H_n(x + iy) = H(n + x + iy)$.

Consider the family of holomorphic functions $\{H_n: \text{int}(\hat{\mathfrak{R}}_0) \to \mathbb{C}\}_{n\in\mathbb{N}}$. We claim that this is not a normal family of meromorphic functions. Suppose that it is.

Case 1: There exists a subsequence of $\{H_n\}$ (say $\{H_{n_k}\}_{k=1}^{\infty}$) converging to some $f: \hat{\mathfrak{R}}_0 \to \mathbb{C}$ which is holomorphic on $\text{int}(\hat{\mathfrak{R}}_0)$.

Let $\epsilon > 0$ be small enough such that (after possibly redefining it), s_n is in $\{\sigma^{\theta}(t): -\pi/2 + \epsilon \le y \le \hat{\theta} - \epsilon,\ ke^{\epsilon+n} \le t \le (k+1)e^{n+1-\epsilon}\}$. Let $\tilde{\mathfrak{R}}_0 = \{x + iy \in \hat{\mathfrak{R}}_0: \epsilon + n + \ln k \le x \le (n+1) + \ln k-\epsilon;\ -\pi/2 + \epsilon \le y \le \hat{\theta} - \epsilon\}$. then $f|_{\tilde{\mathfrak{R}}_0}$ is bounded; say $|f|_{\tilde{\mathfrak{R}}_0}| < M$. Then whenever k is large enough $|H_{n_k}|_{\tilde{\mathfrak{R}}_0}| < M$.

By the definition of $\{H_n\}$ it follows that $|h|_{s_{n_k}}| < M$ whenever k is large enough. But now by the Maximum modulus theorem, h is bounded in a neighborhood of infinity, contradicting the fact that h has an essential singularity at infinity.

Case 2: There exists a subsequence $\{H_{n_k}\}_{k=1}^{\infty}$ of $\{H_n\}$ converging to infinity on $\text{int}(\hat{\mathfrak{R}}_1\}$. Then $\{\frac{1}{H_{n_k}}\}_{k=1}^{\infty}$ converges to zero on $\hat{\mathfrak{R}}_1$; as in case 1, this implies that $(\frac{1}{h})$ is bounded in a neighborhood of infinity, yielding a contradiction.

We have proved that $\{H_n\}_{n=1}^{\infty}$ is not a normal family; and therefore, there exists $z \in \operatorname{int}(\hat{\mathcal{R}}_1)$ such that for an arbitrary neighborhood D of z_0 in $\operatorname{int}(\hat{\mathcal{R}}_1)$ and arbitrary $k \in \mathbb{N}$, $\bigcup_{n=k}^{\infty} H_n(D)$ is either $\mathbb{C}$ or the complement of a singleton [Hil]. Now let $z_0 = x_0 + iy_0$ and consider the semi-infinite geodesic $\sigma^{y_0}\big|_{[0,\infty)}$. We claim that it is a Julia geodesic. Let $1 \leq j \leq n$ be such that $\sigma^{y_0} \subseteq \mathcal{R}_j$. Let $\mathcal{A}$ be a geodesic sector containing $\sigma^{y_0}\big|_{[a,\infty]}$ for some a..

By making the sector smaller if necessary we may assume that $\mathcal{A} \subset \mathcal{R}_j$. Since $\mathcal{A}$ is bounded by semi-infinite geodesics it follows that $\psi_j^{-1}(\mathcal{A})$ is a sector in $\mathbb{R}^2$ bounded by two straight lines and it contains $\psi_j^{-1} \circ \sigma^{y_0}\big|_{[a,\infty)}$. Moreover, by the definition of a geodesic sector (definition 4.5) neither of the straight lines bounding $\psi_j^{-1}(\mathcal{A})$ are parallel to $\psi_j^{-1}\,\sigma^{y_0}$. It is clear when ϵ is small and b is large, the set $\Omega = \{\sigma^{\theta}(t)\colon \theta \leq (y_0-\epsilon, y_0+\epsilon),\ t \geq b\}$ is contained in $\mathcal{A}$, since ψ_j^{-1} maps such a set into $\psi_j^{-1}(\mathcal{A})$ when ϵ and $\frac{1}{b}$ are small enough. Consider the set $B_1 = \{x + iy \in \mathcal{R}_1\colon y \in (y_0 - \epsilon, y_0 + \epsilon)\}$. Let D be a neighborhood of z_0 in B_1, of the form $D = (x_0 - \delta, x_0 + \delta) \times (y_0 - \epsilon, y_0 + \epsilon)$. Let $\ell \in \mathbb{N}$ be such that $\ell + \ln(k) \geq \ln(b)$. Then

$$\bigcup_{n=\ell}^{\infty} H_n(D) = \bigcup_{n=\ell}^{\infty} H(D+n) = \bigcup_{n=\ell}^{\infty} \{h(\sigma^{y}(\exp x))\colon n + x_0 - \delta \leq x$$

$$\leq x_0 + \delta + n;\ y_0 - \epsilon \leq y_0 \leq y_0 + \epsilon\} \subset h(\Omega).$$

Hence $h(\Omega)$ contains the complement of a singleton. Since $\Omega \subset \mathcal{A}$ it follows σ^{y_0} is a Julia geodesic.

5. Proof of the Main Theorem:

First let us assume that $\mathcal{U} = \mathbb{R}^2$.

Let us first consider the special case when $\int_{\mathbb{R}^2} \kappa dA = 0$.

On $(B_r(0))^c$, $\ln \lambda$ is a harmonic function, since $\kappa = -\frac{\Delta(\ln \lambda)}{2\lambda}$. Since $\int_{B_r(0)} \Delta \ln \lambda \, dxdy = \int_{\mathbb{R}^2} \kappa dA = 0$, there is a holomorphic function $f\colon (B_r(0))^c \to \mathbb{C}$ such that $\ln \lambda = \mathrm{Re}(f)$. We claim that f has a removable singularity at ∞. So suppose not. Then e^f has an essential singularity at ∞. Expand e^f in terms of the Laurent's series at ∞: $e^{f(z)} = \sum_{n=-\infty}^{\infty} a_n z^n$, where $a_n \neq 0$ for infinitely many $n > 0$. Define

$h(z) := \sum_{\substack{n=-\infty \\ n\neq -1}}^{\infty} \frac{a_n}{n+1} z^{n+1}$. Note that $h\colon (B_r(0))^c \to \mathbb{C}$ is holomorphic and has an essential singularity at ∞.

Let $\sigma_0 = \sigma^{y_0}$ be a Julia geodesic for h and suppose that $\sigma_0[0,\infty)$ lies on $\mathfrak{R}_i$ corresponding to the geodesic υ_i, as in section 4. Let us drop the subscript i in $\mathfrak{R}_i$, υ_i, and ψ_i. Since $\mathfrak{R}$ is simply connected, the map

$$F := \int e^f \, dz\colon \ \mathfrak{R} \to \mathbb{C}$$

is well-defined.

Furthermore, since $\left|\frac{d}{dz}\left(\int e^{f(z)} dz\right)\right| = \left|e^{f(z)}\right| = \lambda(z)$, it follows that F maps geodesics to straight lines and is one-to-one onto some half plane. Since $\mathfrak{R}$ is simply connected, define $\ln z$ on $\mathfrak{R}$ by fixing $\mathrm{Im}(\ln(\sigma_0(0))) \in [0,2\pi)$. Now

$$F(z) = a_{-1}\ln(z) + h(z) \quad \text{on } \mathfrak{R} \qquad (*)$$

Case 1: Suppose that $a_{-1} = 0$.

In this case $F(z) = h(z)$. But F maps $\mathfrak{R}$ into a half plane thus contradicting the fact that h maps arbitrary geodesic sectors of $\sigma_0\big|_{[0,\infty)}$ in $\mathfrak{R}$ onto $\mathbb{C}$ or $\mathbb{C}\backslash\{\text{singleton}\}$.

Case 2: Suppose that $a_{-1} \neq 0$. Denote $\frac{1}{a_{-1}} F$ by G. Then

$$G - \ln(z) = \frac{1}{a_{-1}} h(z) \quad \text{on} \quad \mathfrak{R}. \tag{**}$$

Note that $G \circ \nu$ is not necessarily a horizontal or vertical line; however, $G(\mathfrak{R})$ is a half plane and $G \circ \sigma_0$ is a semi-infinite straight line in $G(\mathfrak{R})$. Let μ be either a horizontal or a vertical line which makes an angle with $\eta{:} = G \circ \sigma_0$ less than or equal to $\pi/2$. Now $G \circ \mu$ is a semi-infinite geodesic in $\mathfrak{R}$. By parallel translating μ (in the Euclidean sense) if necessary, we may assume that $G^{-1} \circ \mu$ extends to an infinite geodesic which does not meet $B_r(0)$. Denote the flat side of μ by $\mathcal{S}$. By parallel translating μ into $\mathcal{S}$, if necessary, we may assume that $|z| > 1$ on $\mathcal{S}$, so that $\mathfrak{R} \in (\ln z) > 0$ on $\mathcal{S}$.

Now if $G(\mathcal{S})$ is contained in a left half plane, then so is $G(\mathcal{S}) - \ln(\mathcal{S})$ and hence σ_0 is not a Julia geodesic.

Suppose that $G(\mathcal{S})$ is contained in the right half plane $\mathbb{C}_+ = \{z \in \mathbb{C}: \mathfrak{R}(z) \geq 0\}$. By adding a real constant to G (which will be added to $\frac{1}{a_{-1}} h(z)$ also) we assume that $G(\mathcal{S}) = \mathbb{C}_+$. Then by (**) we obtain

$$\text{id} - \ln(G^{-1}) = \frac{1}{a_{-1}} h \circ G^{-1} =: \theta \quad \text{on} \quad \mathbb{C}_+ \, . \tag{+}$$

where 0 has the property that any sector containing 17 is mapped to <C or <C\{singleton}.

Since ln(G~ ^{1}) maps <C+ into <C+, it follows [Dono], that there exists a positive Borel measure T on R such that

$$9te(\ln(G^{-*})) = ax + xf \frac{1}{\backslash (q-Y) + x} \wedge dr(\pounds)$$

where $a \geq 0$ and $\int_{\mathbb{R}} \frac{1}{5} dT(f) < \infty$. Let si := {(x,y): |y| ≤ TX} where T € R is greater than 1/2.

We claim that

$$\lim_{\mathcal{A} \ni (x,y) \to \infty} \int_{\mathbb{R}} \frac{1}{(\xi-y)^2 + x^2} dr(f) = 0.$$

Note that if x > 2, then $\frac{1}{(f-y)^2 + x^2} \leq \frac{8 \cdot Y^2}{r + 1}$ on *si,* since

if |f I ≤ 2|y| then

$$(f-y)^2 + x^2 \; 1 \; x^2 = | - + \backslash > 2 + \frac{1}{2T^2} \geq 2 + \frac{1}{8T^2} \geq \frac{1}{8T^2}$$

and

if |f I > 2|y| then

$$(\xi-y)^2 \bullet \; ? \bullet (\xi/2 + \frac{\xi}{2} - y)^2 + x^2 \geq \frac{\xi^2}{2} + x^2 \geq \frac{\xi^2 + 1}{2} \geq \frac{1}{8r^2}.$$

Now the dominated convergence theorem applies; and therefore,

$$\lim_{\mathscr{A}\ni(x,y)\to\infty}\int_{\mathbb{R}}\frac{1}{(\xi-y)^2+x^2}\,d\tau(\xi)=\int_{\mathbb{R}}\lim_{(x,y)\to\infty}\frac{1}{(\xi-y)^2+x^2}\,d\tau(\xi)=0 \qquad (++)$$

and

$$\mathscr{Re}(\mathrm{id}-\ln(G^{-1}))(x+iy)=x\Big(1-\alpha-\int_{\mathbb{R}}\frac{1}{(\xi-y)^2+x^2}\,d\tau(\xi)\Big). \qquad (+++)$$

Now let $\mathscr{A}$ as above be such that $\eta \subseteq \mathscr{A}$. Then $\theta(\mathscr{A}) \supseteq \mathbb{C}\backslash\{\text{singleton}\}$. However, by (++) and (+++), $\mathscr{Re}(\mathrm{id}-\ln(G^{-1}))(x+iy)$ does not change sign wherever x is large enough and $(x,y)\in\mathscr{A}$. Therefore $\theta(\mathscr{A}) \subseteq$ (a half plane) $\cup$ (a compact subset) giving the desired contradiction.

The only remaining case is that for which $G(\mathscr{S})$ is contained in the upper half plane $\mathscr{U}=\{x+iy \; y\geq 0\}$ and η is the imaginary axis. Again by adding constants to h and G we assume that G maps $\mathscr{S}$ onto $\mathscr{U}$. Since $\ln(G^{-1})(\mathscr{U}) \subseteq \mathbb{C}^-$, it follows that [Dono],

$$\ln(G^{-1})(z)=i\Big(\alpha z+\beta+\int_{\mathbb{R}}\Big(\frac{1}{(\xi-z)}-\frac{\xi}{1+\xi^2}\Big)d\tau(\xi)\Big),$$

where τ is a positive Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}\frac{1}{1+\xi^2}\,d\tau(\xi)<\infty$, $\alpha\geq 0$ and β is a constant. As in the previous case, if $\mathscr{A}=\{(x+iy)\in\mathbb{C}:\; x\geq 0,\ |y|\leq \gamma x\}$ for some γ then $\lim_{\mathscr{A}\ni(x,y)\to\infty}\frac{1}{y}\int_{\mathbb{R}}\Big(\frac{1}{\xi-z}-\frac{1}{1+\xi^2}\Big)d\tau(\xi)=0.$

There $\mathrm{im}(\mathrm{id}-\ln(G^{-1}))(x+iy)$ does not change sign in $\mathscr{A}$ when y is large enough and we have a contradiction of (+) as in the previous case since $\eta\subseteq\mathscr{A}$.

Hence the hypothesis that f has a pole at infinity leads to a contradiction. We conclude that f has a removable singularity at infinity.

Now since $\lim_{x^2+y^2\to\infty} \ln \lambda = \lim_{x^2+y^2\to\infty} \mathrm{Re}(f)$ exists, we have proved theorem 2.1 for the special case $\int_{\mathbb{R}^2} \kappa dA = 0$.

Next let us consider the case $\int_{\mathbb{R}^2} \kappa dA > 0$. Let $\alpha = \frac{1}{2\pi}\int_{\mathbb{R}^2} \kappa dA$. We can construct $\hat{\lambda} \in C^\infty(\mathbb{R}^2)$ such that $\hat{\lambda} > 0$ and $\hat{\lambda}(x,y) = \frac{1}{(x^2+y^2)^\alpha}$ on $(B_r(0))^c$. Consider the Riemannian metric $\tilde{g} = (\tilde{\lambda})^2(dx^2 + dy^2)$, where $\tilde{\lambda} = (\frac{\lambda}{\hat{\lambda}})$. Let K, $\tilde{K}$ and $\hat{K}$ denote the Gaussian curvatures of $g, \tilde{g}$ and $\hat{g} := (\hat{\lambda})^2(dx^2 + dy^2)$ respectively. Similarly, let dA, $d\tilde{A}$ and $d\hat{A}$ denote the corresponding area elements.

Note the following:

(i) $\tilde{K} = \frac{\Delta \ln \tilde{\lambda}}{2\tilde{\lambda}} = \frac{\Delta \ln \lambda - \Delta \ln \hat{\lambda}}{2\tilde{\lambda}} = 0$ on $(B_r(0))^c$.

(ii) $$\int_{\mathbb{R}^2} \tilde{K} d\tilde{A} = \int_{\mathbb{R}^2} \Delta \ln \tilde{\lambda}\, dxdy = \int_{\mathbb{R}^2} \Delta \ln \lambda\, dxdy = - \int_{\mathbb{R}^2} \Delta \ln \hat{\lambda}\, dxdy$$

$$= 2\pi\alpha - 2\pi\alpha = 0.$$

Therefore there exists a holomorphic function $f: (B_r(0))^c \to \mathbb{C}$ such that $\ln \tilde{\lambda} = \mathrm{Re}(f)$.

(iii) $\tilde{g}$ is complete.

Perhaps the easiest way to prove this is to prove that the assoicated topological metric $\tilde{d}$ is complete. Since d is complete, it sufficies to show that there is a constant $a > 0$ such that $d \le a\tilde{d}$. Since

$$\tilde{d}(x,y) := \inf_{\substack{\sigma:[0,1]\to\mathbb{R}^2 \\ \sigma(0)=x \\ \sigma(1)=y \\ \sigma \text{ is piecewise regular}}} \int \|\dot{\sigma}(t)\|_{\tilde{g}}\, dt$$ (similar definition holds for d), it suffices to prove that there exists $a > 0$ such that for an arbitrary tangent vector $v \in \mathbb{R}^2$, $\|v\|_g \le a\|v\|_{\tilde{g}}$.

Let $v = (v_1, v_2) \in T_p(\mathbb{R}^2)$ i.e. written in natural coordinates of $\mathbb{R}^2$. Then $\|v\|_g^2 = \lambda^2(p)(v_1^2 + v_2^2) = (\hat{\lambda}(p))^2(\tilde{\lambda}(p)^2(v_1^2 + v_2^2) = (\hat{\lambda}(p))^2 \|v\|_{\tilde{g}}^2$.

But $\hat{\lambda}$ is bounded on the compact set $B_r(\gamma)$ and $\hat{\lambda}(x,y) = \dfrac{1}{(x^2 + y^2)^{\alpha/2}}$ (where $\alpha > 0$) on $(B_r(0))^c$.

Therefore there is an $a > 0$ such that $\hat{\lambda}(x,y) \le a$ for all $(x,y) \in \mathbb{R}^2$ hence, $\|v\|_g^2 \le a^2 \|v\|_{\tilde{g}}^2$ for an arbitrary tangent vector v of $\mathbb{R}^2$, thus proving the completeness of $\tilde{g}$.

Now, by theorem 2.1, in the special case $\int \kappa dA = 0$ (already proved), f has a removable singularity at ∞.

Therefore $\lim\limits_{(x^2+y^2)\to\infty} \ln(\tilde{\lambda}(x,y))$ exists.

$$\begin{aligned} \text{But } \ln(\tilde{\lambda}(x,y)) &= \ln(\lambda(x,y)) + \ln((x^2 + y^2)^{\alpha/2}) \\ &= \ln(\lambda(x,y)) + \alpha/2 \ln(x^2 + y^2) \\ &= \ln(\lambda(x,y)) + \frac{1}{4\pi}\int_{\mathbb{R}^2} \kappa dA) \ln(x^2 + y^2) \quad \text{on} \quad (B_r(0))^c. \end{aligned}$$

Therefore $\lim\limits_{(x^2+y^2)\to\infty} (\ln(\lambda(x,y)) + (\frac{1}{4\pi}\int_{\mathbb{R}^2} \kappa dA) \ln(x^2 + y^2))$ exists, which is what we wanted to show.

The case $\int_{\mathbb{R}^2} \kappa dA < 0$ is the only remaining case.

Let $\alpha = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \kappa dA$ and let $\tilde{\lambda} = \lambda\hat{\lambda} \in C^\infty(\mathbb{R}^2)$ where $\hat{\lambda}$ is as in the previous case. As before $\int_{\mathbb{R}^2} \Delta(\ln \tilde{\lambda})dxdy = 0$; hence, $\ln(\tilde{\lambda}) = \mathrm{Re}(f)$ on $(B_r(0))^c$ for some holomorphic map $f\colon (B_r(0))^c \to \mathbb{R}^2$. We will prove that f has a removable singularity at ∞ so that the proof follows as in the previous case. However we cannot prove the completeness of $(\tilde{\lambda})^{-2}(dx^2 + dy^2)$ directly in this case.

Consider the case that d is an integer. Then $|\lambda| = |\frac{\tilde{\lambda}}{\hat{\lambda}}| = r^{\alpha}|e^{f(z)}|$ on $(B_r(0))^c$, where $r = \sqrt{x^2+y^2} = |z|^{\alpha}\ |e^{f(z)}|$.

The proof proceeds as in the case $\int \kappa dA = 0$ and we contradict the supposition that $f(z)$ has a nonremovable singularity at ∞.

If α is not an integer, then $|\lambda| = |z|^{\alpha}\ |e^{f(z)}|$ on $(B_r(0))^c$.

Suppose that $f(z)$ has a nonremovable singularity at infinity. Then $e^{f(z)}$ has an essential singularity at infinity. Let $e^{f(z)} = \sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent expansion of $e^{f(z)}$ at infinity. Let $h(z) = \sum_{n=-\infty}^{\infty} \frac{a_n}{n+1+\alpha} z^{n+1+k}$, where $k \in \mathbb{N}$ and $k > \alpha$. Clearly $h(z)$ converges on compact subsets of $(B_r(0))^c$ and hence defines a holomorphic function on $(B_r(0))^c$. Also, since $a_n \neq 0$ for infinitely many positive n, $h(z)$ has an essential singularity at infinity. Let σ_0 be a Julia geodesic of h with respect to the Riemannian metric g. Let υ be a geodesic which separates $\mathbb{R}^2$ into two components such that on one component, say $\mathfrak{R}$, $\sigma_0 \subset \mathfrak{R}$. Since $\mathfrak{R}$ is simply connected and $0 \notin \mathfrak{R}$, $z \mapsto z^{\alpha}\, e^{f(z)}$ is a well defined holomorphic function on $\mathfrak{R}$, and so is

$$F(z) = \int z^{\alpha} e^{f(z)} dz = \sum_{n=-\infty}^{\infty} \frac{a_n z^{n+1+\alpha}}{n+1+\alpha} \quad \text{on } \mathfrak{R}.$$

Also $|F'(z)| = |z^{\alpha}f(z)| = |\lambda(z)|$; hence F maps geodesics on $\mathcal{R}$ to straight lines. Therefore F maps $\mathcal{R}$ injectively onto a half plane.

Let $\mathcal{A}$ be a small geodesic sector containing σ_0. By removing a compact subset of $\mathcal{A}$ we can ensure that

$$F(\mathcal{A}) \subset \{z \in \mathbb{C}: |z| \geq a\} \text{ for some } a > 0.$$

Then

$$|h(z)| = z^{k-\alpha}F(z) \geq r^{k-\alpha} a \geq a \quad \text{for all } z \in \mathcal{A},$$

contradicting the fact that $h(\mathcal{A}) \supset \mathbb{C}\backslash\{\text{singleton}\}$.

This proves Theorem 2.1 for $\mathcal{U} = \mathbb{R}^2$. □

Now we will prove that $\mathcal{U}$ is necessarily equal to $\mathbb{R}^2$. Assume the contrary; i.e., $\mathcal{U} = B_1(0)$. In the following, we continue to use $B_a(p)$ to denote the ball of radius a and center p determined by the Euclidean metric, while $\mathcal{B}_a(p)$ will be the ball of radius a and center p determined by the metric g. We denote the boundary of $\mathcal{B}_a(p)$ by $\mathcal{S}_a(p)$ and the boundary of $B_a(p)$ by $S_a(p)$.

Let $\hat{\lambda}: \mathcal{U} \to (0,\infty)$ be a smooth function such that $\hat{\lambda}(x,y) = \frac{1}{x^2+y^2}$ on $\mathcal{U}\backslash B_r(0)$. Let $\alpha = \int \kappa dA$ and $\hat{\lambda}: = \lambda(\hat{\lambda})^{-\alpha/2\pi}$. Since $\hat{\lambda}$ is positive and bounded above and below on $\mathcal{U}$, it follows that $\tilde{g}: = (\tilde{\lambda})^{-2}(dx^2 + dy^2)$ is also a complete metric and $\tilde{g}$ has Gaussian curvature $\tilde{\kappa}$ with support in $B_r(0)$. It is easily seen that $\tilde{g}$ has the property that the integral of the Gaussian curvature of $\tilde{g}$ is zero. Now replace λ by $\tilde{\lambda}$ and define $g: = (\lambda)^{-2}(dx^2 + dy^2)$. Let $f: \mathcal{U}\backslash B_r(0) \to \mathbb{C}$ be a holomorphic function such that

$\ln \lambda = \mathrm{Re}(f)$.

Case 1: $\int_{S_r} e^{f(z)}\,dz = 0$.

In this case let $F: \mathcal{U}\setminus \bar{B}_r(0) \to \mathbb{C}$ be a holomorphic function such that $F'(z) = e^{f(z)}$. Since $|\lambda| = |F'(z)|$, it follows that $F^*(dx^2 + dy^2) = \lambda^2(dx^2 + dy^2)$. In particular, F maps geodesies of g onto straight lines and the lengths are preserved. It follows $F(z) \to \infty$ as $z \to \partial\mathcal{U}$ but this is not possible since this implies that $F = \infty$ on $\mathcal{U}\setminus \bar{B}_r(0)$.

Case 2: $\int_{S_r} e^{f(z)}\,dz \neq 0$.

Let $\beta = \frac{1}{2\pi i}\int_{S_r} e^{f(z)}\,dz$.

Let $F: \mathcal{U}\setminus \bar{B}_r(0) \to \mathbb{C}$ be a holomorphic function such that $F'(z) = e^{f(z)} + \beta/z$. We will show that $F(z)$ is unbounded in $\mathcal{U}\setminus \bar{B}_r(0)$. If $p \in S_r$, let σ be the radial line through p and let σ meet S_1 at q. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points on σ in $\mathcal{U}\setminus \bar{B}_r(0)$ which converge to q. The completeness of g implies that $\{d(p,x_n)\}_{n=1}^{\infty}$ is unbounded where d is the topological metric associated with g. Let c be the diameter of $B_r(0)$ with respect to d. Clearly $c < \infty$. Now suppose $L < \infty$ is such that $|F(z)| < L$ for all $z \in \mathcal{U}\setminus \bar{B}_r(0)$. Let $y \in \sigma \cap (\mathcal{U}\setminus \bar{B}_r(0))$ satisfy $d(p,y) = 3L + c$. Clearly $\mathcal{B}_{3L}(y) \cap \bar{B}_r(0) = \emptyset$. $\partial\mathcal{B}_{3L}(y)$ meets σ at one or more points between y and q. Let z be the point on $\sigma \cap \partial\mathcal{B}_{3L}(y)$ which is between y and q and such that $(y,z) \in \mathcal{B}_{3L}(y)$. If u is a geodesic with respect to g in $\mathcal{B}_{3L}(y)$ joining y to z, then

$$F(z) - F(y) = \int_{\mu} F'(\omega)d\omega = \int_{\mu} (e^{f(\omega)} + \frac{\beta}{\omega})d\omega$$

$$= \int_{\mu} e^{f(\omega)}d\omega + \int_{\hat{\sigma}} \frac{\beta}{\omega}\, d\omega\ ,$$

where $\hat{\sigma}$ denotes the portion of σ which is between y and z.

But $\int_{\mu} e^{f(\omega)}d\omega = 3L$, since this is the length of μ and $\int_{\hat{\sigma}} \frac{\beta}{\omega} = \beta\{\ln|z| - \ln|y|\}$. So

$$\left|\int_{\hat{\sigma}} \frac{\beta}{\omega}\, d\omega\right| < |\beta|\ \ln\ (\tfrac{1}{r}).$$

By increasing L, if necessary we ensure that $L > |\beta|\ \ln\ (\frac{1}{r})$. Then $|F(z) - F(g)| > 2L$, which contradicts the assumption that $|F| < L$. Thus F is unbounded on $\mathcal{U}\backslash B_r(0)$. Since $\lim_{z\to\partial\mathcal{U}} F(z) \neq \infty$ (since that implies that $F \equiv \infty$), F has a Julia geodesic. But the argument used in the case $\int_{\mathbb{R}^2} \kappa dA = 0$ for $\mathcal{U} = \mathbb{R}^2$ now applies, and we get a contradiction. Therefore $\mathcal{U} = \mathbb{R}^2$.

This completes the proof of Theorem 2.2. □

Proof of Corollary 2.2: It follows from Theorem 2.1 that if g is complete, then there exists a constant $a > 0$ such that $\lambda(\sqrt{x^2+y^2}) \geq \frac{1}{a}$.

Let $\bar{\lambda}$ be a C^{∞} function such that $\bar{\lambda}(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ for $x^2 + y^2 \geq r^2$. Then it is easy to verify that $\bar{g}: = (\bar{\lambda})^{-2}(dx^2 + dy^2)$ is complete. Let $\bar{d}$ be the topological metric corresponding to $\bar{g}$. By making a smaller, if necessary, we conclude that $g \geq a\bar{g}$. Let $t \in [0,1]$. $\lambda^t \sqrt{x^2+y^2} \geq (\frac{1}{a})^t$ for $x^2 = y^2 \geq r^2$; then $\lambda^{-2t}(dx^2 + dy^2) \geq a^t\bar{g}$. But since $\bar{g}$ is complete, $\lambda^{-2t}(dx^2 + dy^2)$ is complete.

Q.E.D.

References

[Sa] S. Samelson, Global Tchebychev nets on two dimensional Riemannian surfaces with compactly supported Gauss curvature, under preparation.

[BT] R. Bott and L. Tu, Differential Forms in Algebraic Topology, Springer Verlag, 1985.

[Hi] M.W. Hirsch, Differential Topology, Springer Verlag, 1983.