ON FORMATION OF SINGULARITIES IN ONE-DIMENSIONAL NONLINEAR THERMOELASTICITY

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Dedicated to Bernard D. Coleman on the occasion of his 60^{th} *birthday*

1. Introduction.

It is well known that smooth motions of nonlinear elastic bodies generally will break down in finite time due to the formation of shock waves. On the other hand, for thermoelastic materials, the conduction of heat provides dissipation that competes with the destabilizing effects of nonlinearity in the elastic response. The work of Coleman and Gurtin [2] on the growth and decay of acceleration waves provides a great deal of insight concerning the interplay between dissipation and nonlinearity in one-dimensional nonlinear thermoelastic bodies. Assuming that the elastic modulus, specific heat, and thermal conductivity are strictly positive, the stress-temperature modulus is nonzero, and that the elastic response is genuinely nonlinear they show that acceleration waves of small initial amplitude decay but waves of large initial amplitude can explode in finite time. In other words, thermal diffusion manages to restrain waves of small amplitude but nonlinearity in the elastic response is dominant for waves of large amplitude.

For smooth initial data that are close to equilibrium (in appropriate Sobolev norms) global existence and decay of classical solutions to the field equations of one-dimensional nonlinear thermoelasticity has been established by Slemrod [11], Zheng and Shen [13, 14], Jiang [8], and Hrusa and Tarabek [7]. Further, assuming that the constitutive relations are of a special form, Dafermos and Hsiao [5] have shown that for certain smooth initial data with large gradients the solution of the Cauchy problem will develop singularities in finite time. However, the equations of motion studied in [5] are partially uncoupled and consequently do not satisfy the assumptions used to establish global existence when the data are close to equilibrium. In particular, for the equations of [5], if the initial temperature is spatially homogeneous then the temperature remains constant in time while the strain ε and the velocity v satisfy the system

$$\varepsilon_t(x,t) = v_x(x,t)$$

$$v_t(x,t) = p(\varepsilon(x,t))_x, \qquad x \in \mathbb{R},\ t \geq 0, \tag{1.1}$$

of nonlinear elasticity. If the elastic response function p is nonlinear then solutions of (1.1) generally will develop singularities in finite time even if the initial data are very smooth and small (cf. Lax [9] and MacCamy and Mizel [10]). Moreover, as Dafermos and Hsiao point out, their constitutive relations for the stress and internal energy are not consistent with the existence of a free energy.

In this paper we study a special class of nonlinear thermoelastic materials. Our constitutive relations are similar to those used by Dafermos and Hsiao [5]. Although we do not claim that our constitutive equations are appropriate to any real material, they do satisfy the assumptions used to establish global existence of smooth solutions when the data are close to the equilibrium. Moreover, they are fully compatible with the second law of thermodynamics; in particular there is a free energy. We consider the Cauchy problem in which the body occupies the entire real line and the initial values of the strain, velocity, and temperature are prescribed. As our main result, we show that there are smooth initial data for which the solution will develop singularities in finite time. Our proof follows the argument of Dafermos and Hsiao [5] very closely. The primary difference is that our equations of evolution contain an additional term that prevents us from using the maximum principle to obtain an important a priori bound. This difficulty is overcome by exploiting some relations associated with the second law of thermodynamics.

2. Balance of Laws and Constitutive Equations.

Consider a homogeneous one-dimensional body that occupies the interval B in a (fixed) reference configuration and has unit reference density. We denote by

8 the strain,

v the velocity,

a the stress,

e the internal energy,

q the heat flux, and

T the absolute temperature.

Each of the above fields is assumed to be a smooth function of the reference position x and the time t; the strain and the temperature are required to satisfy

(2.1) $£ > - \wedge \qquad T > a$

In the absence of external heat supply and body force, the laws of balance of momentum and energy can be written as

$$
\begin{aligned}
&\varepsilon_t = v_x \\
(2.2) \qquad &v_t = \sigma_x \\
&e_t + q_x = \sigma\varepsilon_t, \qquad x \in B, \ t \geq 0.
\end{aligned}
$$

A thermoelastic material is described by constitutive relations of the form

$$
\begin{aligned}
&\sigma(x,t) = \hat{\sigma}(\varepsilon(x,t),\ T(x,t)) \\
(2.3) \qquad &e(x,t) = \hat{e}(\varepsilon(x,t),\ T(x,t)) \\
&q(x,t) = \hat{q}(\varepsilon(x,t),\ T(x,t),\ T_x(x,t)).
\end{aligned}
$$

The second law of thermodynamics imposes restrictions [1] on $\hat{\sigma}$, $\hat{e}$, and $\hat{q}$. In particular, $\hat{\sigma}$ and $\hat{e}$ are required to satisfy the compatibility relation

$$(2.4) \qquad \hat{e}_\varepsilon(\varepsilon,T) = \hat{\sigma}(\varepsilon,T) - T\hat{\sigma}_T(\varepsilon,T), \quad \forall\ \varepsilon > -1, \ \ T > 0$$

and $\hat{q}$ must obey the heat conduction inequality

$$(2.5) \qquad g \cdot \hat{q}(\varepsilon,T,g) \leq 0, \qquad\qquad \forall\ \varepsilon > -1, \ \ T > 0, \ \ g \in \mathbb{R}.$$

We note that (2.4) is equivalent to the existence of a free energy,

$$(2.6) \qquad \psi = \hat{\psi}(\varepsilon,T),$$

and a corresponding entropy,

---

1. The idea of using the second law of thermodynamics to obtain restrictions on constitutive relations is due to Coleman and Noll [4]. The restrictions (2.4), (2.5) follow from the results of [4]. Coleman and Noll assume that the stress and internal energy are independent of the temperature gradient. A subsequent argument of Coleman and Mizel [3] shows that compatibility with the second law requires that stress and internal energy (as well as the entropy and free energy) be independent of the temperature gradient.

(2.7) $\eta = \hat{\eta}(\varepsilon, T),$

which satisfy

(2.8) $\psi = e - T\eta,$

and

(2.9) $\hat{\sigma} = \hat{\psi}_{\varepsilon}, \quad \hat{\eta} = -\hat{\psi}_T.$

An important consequence of (2.6), (2.7), (2.9) is the identity

(2.10) $\psi_t + \eta T_t - \sigma\varepsilon_t = 0,$

which is equivalent to Gibbs's relation

(2.11) $e_t - T\eta_t - \sigma\varepsilon_t = 0,$

by virtue of (2.8). In view of (2.11), the equation of balance of energy $(2.2)_3$ can be rewritten as

(2.12) $T\eta_t + q_x = 0.$

We refer to the recent book of Day [6] for more information on one-dimensional nonlinear thermoelasticity.

Here we consider special constitutive equations of the form

$$\hat{\sigma}(\varepsilon, T) = p(\varepsilon) + \beta(T - \overline{T})$$

(2.13) $$\hat{e}(\varepsilon, T) = P(\varepsilon) + c\,(T - \overline{T}) - \beta\overline{T}\varepsilon$$

$$\hat{q}(\varepsilon, T, T_x) = -\kappa T_x,$$

where $p : (-1, \infty) \to \mathbb{R}$ is a given smooth function,

(2.14) $P(\varepsilon) := \int_0^{\varepsilon} p(\xi)d\xi, \qquad \forall\ \varepsilon > -1,$

and $\beta, \overline{T}, c, \kappa$ are constants with

$$\overline{T}, c, \kappa > 0. \tag{2.15}$$

It is easy to check that these constitutive equations satisfy (2.4) and (2.5) and that

$$\hat{\psi}(\varepsilon,T) = P(\varepsilon) + \beta\varepsilon\,(T - \overline{T}) + cT\log(\overline{T}/T) + c\,(T - \overline{T}) \tag{2.16}$$

is a free energy; moreover the corresponding entropy is given by

$$\hat{\eta}(\varepsilon,T) = -\beta\varepsilon - c\log(\overline{T}/T). \tag{2.17}$$

Remark 2.1 : Dafermos and Hsiao [5] assume that the stress and the heat flux are given by $(2.13)_1$, $(2.13)_3$ and that

$$\hat{e}(\varepsilon,T) = P(\varepsilon) + c\,(T - \overline{T}).$$

As they point out , their constitutive relations comply with (2.4) only if $\overline{T} = 0$.

3. Formation of Singularities.

We assume that the body occupies the entire real line in its reference configuration (i.e. $B = \mathbb{R}$) and that the stress, internal energy, and heat flux are given by (2.3), where $\hat{\sigma}$, $\hat{e}$, and $\hat{q}$ have the special forms (2.13). We assume further that

$$p \in C^4(-1, \infty), \qquad p(0) = 0, \qquad p'(\xi) > 0 \qquad \forall\, \xi > -1, \tag{3.1}$$

and that

$$\overline{T},\ c,\ \kappa > 0. \tag{3.2}$$

We seek a smooth solution of (2.2) when the initial values of the strain, velocity, and temperature are prescribed. Thus we consider the initial value problem

$$
(3.3)\qquad
\begin{aligned}
&\varepsilon_t(x,t) = v_x(x,t),\\
&v_t(x,t) = p'(\varepsilon(x,t))\, \varepsilon_x(x,t) + \beta\, \theta_x(x,t),\\
&c\theta_t(x,t) = \kappa\, \theta_{xx}(x,t) + \beta\, (\theta + \overline{T})\, v_x(x,t), \qquad x \in \mathbb{R},\ t \geq 0,\\
&\varepsilon(x,0) = \varepsilon_0(x), \quad v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \qquad x \in \mathbb{R},
\end{aligned}
$$

where ε_0, v_0, and θ_0 are given functions and

$$
(3.4)\qquad \theta(x,t) := T(x,t) - \overline{T}\,.
$$

The existence of a local solution can be established by a standard contraction-mapping argument and we omit the details. The relevant result is recorded below.

Proposition : *Assume that (3.1), (3.2) hold,*

$$
(3.5)\qquad \varepsilon_0,\ v_0,\ \theta_0 \in H^3(\mathbb{R}),
$$

and that

$$
(3.6)\qquad \varepsilon_0(x) > -1, \qquad \theta_0(x) > -\overline{T} \qquad \forall\, x \in \mathbb{R}.
$$

Then the initial-value problem (3.3) has a unique local solution (ε, v, θ) *on a maximal time interval* $[0, t_0)$, $t_0 > 0$, *with*

$$
(3.7)\qquad
\begin{aligned}
&\varepsilon, v \in C([0, t_0); H^3(\mathbb{R})) \cap C^1([0, t_0); H^2(\mathbb{R})),\\
&\theta \in C([0, t_0); H^3(\mathbb{R})) \cap C^1([0, t_0); H^1(\mathbb{R}))
\end{aligned}
$$

$$
(3.8)\qquad \varepsilon(x,t) > -1, \quad \theta(x,t) > -\overline{T} \quad \forall\, x \in \mathbb{R},\ t \in [0, t_0).
$$

Moreover, if

$$
(3.9)\qquad \sup_{\substack{-\infty < x < \infty \\ 0 \leq t < t_0}} \Big(\{|\varepsilon_x| + |\varepsilon_t| + |\theta_x| + |\theta|\}(x,t)\Big) < \infty,
$$

$$(3.10) \qquad \inf_{\substack{-\infty < x < \infty \\ 0 \le t < t_0}} \varepsilon(x,t) > -1, \qquad \inf_{\substack{-\infty < x < \infty \\ 0 \le t < t_0}} \theta(x,t) > -\overline{T},$$

then $t_0 = \infty$.

Remark 3.1 : If (3.1), (3.2), (3.5), (3.6) hold and $\varepsilon_0, v_0, \theta_0 \in W^{2,1}(\mathbb{R})$ then the local solution (ε, v, θ) of (3.3) also satisfies

$$(3.11) \qquad \varepsilon, v, \theta \in C([0, t_0); W^{2,1}(\mathbb{R})).$$

(See Theorem 2 of [12] and Section 2 of [13].)

Remark 3.2: Under suitable assumptions on $\hat{\sigma}$, $\hat{e}$, and $\hat{q}$, an analogous result on local existence holds for general constitutive equations of the form (2.3). In this case, a bound stronger than (3.9) is needed to continue the solution globally in time.

Remark 3.3: It follows from the results of [7], [13] that if $\beta \neq 0$, (3.6) holds, and

$$\| \varepsilon_0 \|_{H^3} + \| v_0 \|_{H^3} + \| \theta_0 \|_{H^3}$$

is sufficiently small, then the initial value problem (3.3) has a globally defined smooth solution. (See Remark 2 of [7].)

Theorem : *Assume that (3.1), (3.2), (3.11) hold and that* $p''(0) > 0$. *Let* $\gamma, L, J > 0$ *be given. Then there exist* $\delta, M > 0$ *(depending on* γ, L, J*) with the following property : For each* $\varepsilon_0, v_0, \theta_0 \in H^3(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$ *satisfying (3.6),*

$$(3.12) \qquad |\varepsilon_0(x)| \le \delta^{1/2}, \qquad |v_0(x)| \le \delta^{1/2}, \qquad |\theta_0(x)| \le \delta, \quad \forall x \in \mathbb{R},$$

$$(3.13) \qquad \int_{-\infty}^{\infty} \left(\varepsilon_0^2(x) + v_0^2(x) + \theta_0^2(x) + |\theta_0(x)| \right) dx \le \delta^2,$$

(3.14)
$$\min_{-\infty < x < \infty} \{v_0'(x) + p'(\varepsilon_0(x))^{1/2}\, \varepsilon_0'(x)\} + \min_{-\infty < x < \infty} \{v_0'(x) - p'(\varepsilon_0(x))^{1/2}\, \varepsilon_0'(x)\} \geq -J,$$

and

(3.15)
$$\max_{-\infty < x < \infty} \{v_0'(x) + p'(\varepsilon_0(x))^{1/2}\, \varepsilon_0'(x)\} + \max_{-\infty < x < \infty} \{v_0'(x) - p'(\varepsilon_0(x))^{1/2}\, \varepsilon_0'(x)\} \geq M,$$

the length t_0 of the maximal interval of existence of a smooth solution (ε, v, θ) *of (3.3) is less than (or equal to) L; moreover the local solution satisfies*

(3.16)
$$|\varepsilon(x,t)| \leq \gamma, \quad |v(x,t)| \leq \gamma, \quad |\theta(x,t)| \leq \gamma, \quad \forall\, x \in \mathbb{R}, \quad t \in [0, t_0).$$

Remark 3.4 : It is not difficult to show that for every $\delta, J, M > 0$ there exist $\varepsilon_0, v_0, \theta_0 \in H^3(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$ satisfying (3.12) through (3.15).

Remark 3.5 : An analogous theorem holds if the assumption $p''(0) > 0$ is replaced by $p''(0) < 0$.

4. Proof of the Theorem.

The first part of the proof consists of establishing a priori estimates for the local solution of (3.3). After the estimates are obtained, we establish the formation of singularities in finite time by analyzing a differential inequality.

Let $t^* \in (0, 1]$ and $\delta, J > 0$ be given. Assume that (3.12), (3.13), (3.14) hold and that (3.3) has a solution (ε,v,θ) on $\mathbb{R} \times [0, t^*]$ satisfying

(4.1)
$$\varepsilon, v \in C([0, t^*]; H^3(\mathbb{R}) \cap W^{2,1}(\mathbb{R})) \cap C^1([0, t^*]; H^2(\mathbb{R}))$$
$$\theta \in C([0, t^*]; H^3(\mathbb{R}) \cap W^{2,1}(\mathbb{R})) \cap C^1([0, t^*]; H^1(\mathbb{R}))$$

and

(4.2) $\quad | e(x,t) | \leq \frac{1}{2}, \quad 19(x,t) \backslash \leq \} \quad \text{V x e I R}, \ t e [O, t^*].$

We shall derive several a priori bounds for (e, v, 6) in terms of the constant 5 appearing in (3.12), (3.13). Throughout this section we use T to denote a generic positive constant that depends only on p*, c, K, $\bar{T}$, J, and properties of the function p.

Lemma 4.1 : *There exists a constant* $\tilde{Y}$ *(which depends only on* c, $\bar{T}$, K, *and the maximum and the minimum ofp' on*$[-\frac{1}{2}, \frac{1}{2}]$ *) such that*

(4.3) $\quad \mathbf{I} \ \{e^2 + v^2 + 6^2\}(x,t)\ dx + \mathbf{I} \ \ \mathbf{I} \ 9_Y(x,s)\ dx\ ds \leq T\ 6^2 \quad \text{V t e } [0, t^*].$

Proof. It follows from (2.12) that

(4.4) $\quad 0Ti_t = - - = \frac{\theta}{(T+9)} q_x -$

Using $(2.2)_{19}$ $(2.2)_2$, (2.10), and (4.4) we conclude that

(4.5) $\quad \frac{\partial}{\partial t} (\psi + \theta\eta + \frac{1}{2} v^2) - \frac{\bar{T}}{(T+9)} \wedge q \frac{\theta}{x} = \frac{\partial}{dx} \ \sigma v - \frac{9}{(T+9)} q \ ).$

We integrate (4.5) over IR x [0, t] and use (2.13), (2.16), and (2.17) to obtain

(4.6) $\quad J_{-\infty} \ \{P(e) + \backslash v^2 - cf \log(l + \frac{\sigma}{T}) + c9\}(x,t)\ dx +$

$$\kappa \int_0 \int \frac{}{(\bar{T}H\text{-}9)} r\ 9_x(x,s)\ dx\ ds = \int_{-\infty} \{P(e_0) + \frac{1}{2} v_0^2 - cT\log(l + \frac{9_0}{T}) + c9_0\}(x)\ dx.$$

It is straightforward to verify that

(4.7) $\quad \frac{1}{2} m_1 \lambda^2 \leq P(\lambda) \leq \frac{1}{2} m_2 \lambda^2 \qquad \forall\ \lambda \in [-\frac{1}{2}, \frac{1}{2}],$

(4.8) $$\frac{2c}{9\overline{T}}\xi^2 \le -c\overline{T}\log(1+\frac{\xi}{\overline{T}}) + c\xi \le \frac{2c}{\overline{T}}\xi^2 \qquad \forall\ \xi \in [-\frac{\overline{T}}{2}, \frac{\overline{T}}{2}],$$

where

(4.9) $$m_1 := \min_{\lambda \in [-\frac{1}{2}, \frac{1}{2}]} p'(\lambda), \qquad m_2 := \max_{\lambda \in [-\frac{1}{2}, \frac{1}{2}]} p'(\lambda).$$

Consequently (4.6) yields the inequality (4.3). □

Following Dafermos and Hsiao [5] we use the method of Alikakos [1] to obtain a pointwise bound for θ in terms of δ.

Lemma 4.2 : *The temperature* θ *obeys the estimate*

(4.10) $$|\,\theta(x,t)\,| \le \Gamma\,\delta \qquad \forall\ x \in \mathbb{R},\ \ t \in [0,\ t^*].$$

Proof. The idea of the proof is to put

(4.11) $$A_n := \max_{0 \le t \le t^*} \int_{-\infty}^{\infty} \theta^{2^n}(x,t)\,dx, \qquad n = 1,2,3,...$$

and show that

(4.12) $$A_n \le (\,\Gamma\,\delta)^{2^n}, \qquad n = 1,2,3,... \quad .$$

The desired conclusion then follows by raising (4.12) to the 2^{-n} power and letting n tend to infinity.

To establish (4.12), we multiply $(3.3)_3$ by $2^n\,\theta^{2^n-1}$, and integrate with respect to x, using integration by parts, to obtain

(4.13) $$\frac{d}{dt}\int_{-\infty}^{\infty} \theta^{2^n}(x,t)\,dx = -\,4\,(1-2^{-n})\,\frac{\kappa}{c}\int_{-\infty}^{\infty} [\partial_x\theta^{2^{n-1}}(x,t)]^2\,dx$$

$$- 2\cdot 2^n \frac{\beta}{c}\int_{-\infty}^{\infty} \{\partial_x\theta^{2^{n-1}}(x,t)\, \theta^{2^{n-1}}(x,t)\, v(x,t)\}\, dx + 2^n\,\overline{T}\,\frac{\beta}{c}\int_{-\infty}^{\infty} \theta^{2^n-1}(x,t)\, v_x(x,t)\, dx\,.$$

We note that (4.13) is similar to (2.9) of [5] except for the term

$\int_{-\infty}^{\infty} \theta^{2^n-1}(x,t)\, v_x(x,t)\, dx$ which can be handled as follows :

$$(4.14)\qquad \left|\int_{-\infty}^{\infty} \theta^{2^n-1}(x,t)\, v_x(x,t)\, dx\right| = (2^n-1)\left|\int_{-\infty}^{\infty} \theta_x(x,t)\, \theta^{2^n-2}(x,t)\, v(x,t)\, dx\right|$$

$$\le (2^n-1)\int_{|\theta|\le\delta} |\theta_x(x,t)\, \theta^{2^n-2}(x,t)\, v(x,t)|\, dx + (2^n-1)\int_{|\theta|\ge\delta} |\theta_x(x,t)\, \theta^{2^n-2}(x,t)\, v(x,t)|\, dx$$

$$\le (2^n-1)\,\delta^{2^n-2}\int_{-\infty}^{\infty} |\theta_x(x,t)\, v(x,t)|\, dx + (2^n-1)\,\delta^{-1}\int_{-\infty}^{\infty} |\theta_x(x,t)\, \theta^{2^n-1}(x,t)\, v(x,t)|\, dx$$

$$\le (2^n-1)\,\delta^{2^n-2}\left\{\frac{1}{2}\int_{-\infty}^{\infty} \theta_x(x,t)^2 dx + \frac{1}{2}\int_{-\infty}^{\infty} v(x,t)^2\, dx\right\}$$

$$+ (2^n-1)\,\delta^{-1}\,2^{-(n-1)}\int_{-\infty}^{\infty} |\partial_x\theta^{2^{n-1}}(x,t)\, \theta^{2^{n-1}}(x,t)\, v(x,t)|\, dx$$

$$\le \frac{1}{2}\,(2^n-1)\,\delta^{2^n-2}\int_{-\infty}^{\infty} \theta_x(x,t)^2\, dx + \frac{1}{2}\,\widetilde{\Gamma}\,(2^n-1)\,\delta^{2^n}$$

$$+ (2^n-1)\,\delta^{-1}\;2^{1-n}\left\{\frac{\kappa}{\beta\overline{T}}\,\frac{\delta}{2.2^n}\int_{-\infty}^{\infty} (\partial_x\theta^{2^{n-1}}(x,t))^2\, dx\right.$$

$$\left. + \frac{1}{2}\,\frac{\beta\overline{T}}{\kappa}\,\frac{2^n}{\delta}\,\max\theta^{2^n}(x,t)\int_{-\infty}^{\infty} v^2(x,t)\, dx\right\}$$

$$\le \frac{1}{2}\,(2^n-1)\,\delta^{2^n-2}\int_{-\infty}^{\infty} \theta_x(x,t)^2\, dx + \frac{1}{2}\,\widetilde{\Gamma}\,(2^n-1)\,\delta^{2^n}$$

$$+ \frac{\kappa}{\beta\overline{T}}\,2^{-n}\,(1-2^{-n})\int_{-\infty}^{\infty} (\partial_x\theta^{2^{n-1}}(x,t))^2\, dx + \frac{\beta\overline{T}}{\kappa}\,\widetilde{\Gamma}\,(2^n-1)\,\max\theta^{2^n}(x,t),$$

where $\widetilde{\Gamma}$ is the constant in (4.3).

Next we observe that

(4.15)
$$\left|\int_{-\infty}^{\infty} \partial_x\theta^{2^{n-1}}(x,t)\,\theta^{2^{n-1}}(x,t)\,v(x,t)\,dx\right|$$

$$\leq \frac{1}{2}\frac{\kappa}{\beta 2^n}\int_{-\infty}^{\infty}(\partial_x\theta^{2^{n-1}}(x,t))^2\,dx + \frac{1}{2}\frac{\beta 2^n}{\kappa}\int_{-\infty}^{\infty}(v(x,t)\,\theta^{2^{n-1}}(x,t))^2\,dx$$

$$\leq \frac{1}{2}\frac{\kappa}{\beta 2^n}\int_{-\infty}^{\infty}(\partial_x\theta^{2^{n-1}}(x,t))^2\,dx + \frac{1}{2}\frac{\beta 2^n}{\kappa}\max\theta^{2^n}(x,t)\int_{-\infty}^{\infty}v^2(x,t)\,dx.$$

By combining (4.13) through (4.15) we find that

(4.16)
$$\frac{d}{dt}\int_{-\infty}^{\infty}\theta^{2^n}(x,t)\,dx \leq -\frac{\kappa}{c}\int_{-\infty}^{\infty}(\partial_x\theta^{2^{n-1}}(x,t))^2\,dx$$

$$+\ 2\cdot 2^{n}\frac{1}{2}\frac{\beta^2 2^n}{\kappa c}\max\theta^{2^n}(x,t)\int_{-\infty}^{\infty}v^2(x,t)\,dx$$

$$+\ 2^n\,\overline{T}\frac{\beta}{c}\Big\{\frac{1}{2}(2^n-1)\,\delta^{2^n-2}\int_{-\infty}^{\infty}\theta_x(x,t)^2\,dx + \frac{1}{2}\widetilde{\Gamma}(2^n-1)\,\delta^{2^n} + \frac{\beta\overline{T}}{\kappa}\widetilde{\Gamma}(2^n-1)\max\theta^{2^n}(x,t)\Big\}.$$

For every $\upsilon > 0$ we have

(4.17)
$$\max\theta^{2^n}(x,t) \leq 2\int_{-\infty}^{\infty}\theta^{2^{n-1}}(x,t)\,|\partial_x\theta^{2^{n-1}}(x,t)|\,dx$$

$$\leq \upsilon\int_{-\infty}^{\infty}(\partial_x\theta^{2^{n-1}}(x,t))^2dx + \upsilon^{-1}\int_{-\infty}^{\infty}\theta^{2^n}(x,t)\,dx$$

$$\leq \upsilon\int_{-\infty}^{\infty}(\partial_x\theta^{2^{n-1}}(x,t))^2dx + \upsilon^{-1}\max\theta^{2^{n-1}}(x,t)\int_{-\infty}^{\infty}\theta^{2^{n-1}}(x,t)\,dx$$

$$\leq \upsilon \int_{-\infty}^{\infty} (\partial_x \theta^{2^{n-1}}(x,t))^2 dx + \frac{1}{2} \max \theta^{2^n}(x,t) + \frac{1}{2} \upsilon^{-2} \Big(\int_{-\infty}^{\infty} \theta^{2^{n-1}}(x,t)\, dx \Big)^2,$$

and consequently

$$\text{(4.18)} \qquad \max \theta^{2^n}(x,t) \leq 2\upsilon \int_{-\infty}^{\infty} (\partial_x \theta^{2^{n-1}}(x,t))^2 dx + \upsilon^{-2} \Big(\int_{-\infty}^{\infty} \theta^{2^{n-1}}(x,t)\, dx \Big)^2.$$

By using (4.3), (4.16), and (4.18) (with υ sufficiently small) we conclude that

$$\text{(4.19)} \qquad \frac{d}{dt} \int_{-\infty}^{\infty} \theta^{2^n}(x,t)\, dx \leq 2^{6n} \bar{\Gamma} \Big(\int_{-\infty}^{\infty} \theta^{2^{n-1}}(x,t)\, dx \Big)^2 + \tilde{\Gamma}\, 2^{2n-1} \delta^{2^n}$$

$$+ 2^{2n-1} \delta^{2^n - 2} \int_{-\infty}^{\infty} \theta_x(x,t)^2\, dx,$$

where $\bar{\Gamma}$ is a constant that depends on β, c, κ, and $\tilde{\Gamma}$. Integrating (4.19) with respect to time and using (3.12), (3.13), and (4.3) we find that

$$\text{(4.20)} \qquad A_n \leq \delta^{2^n} + 2^{6n} \bar{\Gamma} (A_{n-1})^2 + \tilde{\Gamma}\, 2^{2n-1} \delta^{2^n} + \tilde{\Gamma}\, 2^{2n-1} \delta^{2^n}.$$

We choose Γ such that $\Gamma^{1/2} \geq \max \{2^{14} \bar{\Gamma},\ 16,\ \tilde{\Gamma} \}$. Using Lemma 4.1 and (4.21) we obtain

$$\text{(4.21)} \qquad A_n \leq 2^{-6n} \Gamma^{-1/2} (\Gamma \delta)^{2^n}, \quad n = 1, 2, 3, \ldots$$

by induction. In particular (4.12) holds and the proof is complete. □

We now proceed to obtain pointwise bounds for ε and v. For this purpose we introduce

$$\text{(4.22)} \qquad r(x,t) := v(x,t) + \int_0^{\varepsilon(x,t)} p'(\xi)^{1/2} d\xi, \quad s(x,t) := v(x,t) - \int_0^{\varepsilon(x,t)} p'(\xi)^{1/2} d\xi$$

and the differential operators

$$(4.23)\qquad \grave{} := \frac{\partial}{\partial t} - p'(\varepsilon(x,t))^{1/2}\frac{\partial}{\partial x}, \qquad {}' := \frac{\partial}{\partial t} + p'(\varepsilon(x,t))^{1/2}\frac{\partial}{\partial x}.$$

A straightforward calculation yields

$$(4.24)\qquad \grave{r} = \beta\theta_x, \qquad s' = \beta\theta_x .$$

In order to express θ_x in a convinient form, we define

$$(4.25)\qquad \chi(x,t) := \int_{-\infty}^{x} \{P(\varepsilon(y,t)) + c\theta(y,t) + \frac{1}{2}v^2(y,t)\}dy.$$

We note that by virtue of $(3.3)_3$, θ admits the representation

$$(4.26)\qquad \theta(x,t) = \frac{c}{(4\pi\kappa t)^{1/2}}\int_{-\infty}^{\infty} \exp\{-\frac{c}{4\kappa}\frac{(x-y)^2}{t}\}\theta_0(y)dy$$

$$+ \frac{\beta}{(4\pi\kappa c)^{1/2}}\int_0^t\int_{-\infty}^{\infty}(t-\tau)^{-1/2}\exp\{-\frac{c}{4\kappa}\frac{(x-y)^2}{t-\tau}\}[\theta(y,\tau)+\overline{T}]\, v_y(y,\tau)\, dy\, d\tau.$$

Integrating (4.26) with respect to x, using Lemma 4.1, and integrating by parts, we conclude that

$$(4.27)\qquad \left|\int_{-\infty}^{x}\theta(y,t)\, dy\right| \le \Gamma\delta, \qquad \forall\, x \in \mathbb{R},\ t\in[0,\ t^*]$$

and consequently

$$(4.28)\qquad |\chi(x,t)| \le \Gamma\delta, \qquad \forall\, x\in\mathbb{R},\ t\in[0,\ t^*].$$

By virtue of $(2.2)_1$ and $(2.2)_2$, the equation of balance of energy, $(2.2)_3$, can be expressed in the form

$$(4.29)\qquad (e + \frac{1}{2}v^2)_t + q_x = (\sigma v)_x.$$

We integrate (4.29) with respect to x over ($-\infty$, x) and use (2.13), (4.25) to obtain

$$\kappa\,\theta_x = \chi_t - (p(\varepsilon) + \beta\theta + \beta\,\overline{T})\,v, \tag{4.30}$$

$$= \chi^{\backprime} + p'(\varepsilon)^{1/2}\,(P(\varepsilon) + c\theta + \frac{1}{2}\,v^2) - (p(\varepsilon) + \beta\theta + \beta\,\overline{T})\,v,$$

$$= \chi^{/} - p'(\varepsilon)^{1/2}\,(P(\varepsilon) + c\theta + \frac{1}{2}\,v^2) - (p(\varepsilon) + \beta\theta + \beta\,\overline{T})\,v.$$

Therefore, by setting

$$\Phi := r - \frac{\beta}{\kappa}(\chi - \chi_0), \qquad \Psi := s - \frac{\beta}{\kappa}(\chi - \chi_0), \qquad \chi_0(x) := \chi(x,0), \tag{4.31}$$

we get

$$\Phi^{\backprime} = \frac{\beta}{\kappa}\,p'(\varepsilon)^{1/2}\,(P(\varepsilon) + c\theta + \frac{1}{2}\,v^2) - (p(\varepsilon) + \beta\theta + \beta\,\overline{T})\,v + \frac{\beta}{\kappa}\,p'(\varepsilon)^{1/2}\,\partial_x\chi_0, \tag{4.32}$$

$$\Psi^{/} = -\frac{\beta}{\kappa}\,p'(\varepsilon)^{1/2}\,(P(\varepsilon) + c\theta + \frac{1}{2}\,v^2) - (p(\varepsilon) + \beta\theta + \beta\,\overline{T})\,v + \frac{\beta}{\kappa}\,p'(\varepsilon)^{1/2}\,\partial_x\chi_0. \tag{4.33}$$

The right-hand sides of (4.32), (4.33) can be expressed in terms of Φ and Ψ through the relations

$$\varepsilon = E^{-1}(\Phi - \Psi), \qquad v = \frac{1}{2}\,(\Phi + \Psi) + \frac{\beta}{\kappa}\,(\chi - \chi_0), \tag{4.34}$$

where

$$E(z) := \int_0^z p'(\xi)^{1/2}\,d\xi \qquad \forall\, z \in \mathbb{R}. \tag{4.35}$$

Lemma 4.3 : *For every* $\lambda > 0$ *there is* a $\delta^* > 0$ *such that if* $\delta < \delta^*$ *then*

$$|\,\varepsilon(x,t)\,| < \lambda, \qquad |\,v(x,t)\,| < \lambda, \qquad \forall\, x \in \mathbb{R},\ t \in [0,\ t^*]. \tag{4.36}$$

Proof. By virtue of (4.34), (4.35), and (4.28), it suffices to show that given $\overline{\lambda} > 0$ there is a $\overline{\delta} > 0$ such that if $\delta < \overline{\delta}$ then

$$|\,\Phi(x,t)\,| < \overline{\lambda}, \qquad |\,\Psi(x,t)\,| < \overline{\lambda}, \qquad \forall\, x \in \mathbb{R},\ t \in [0,\ t^*]. \tag{4.37}$$

Following Dafermos and Hsiao [5], we define the nonnegative Lipschitz functions O* and *F* by

(4.38) <D*(t): = max | O(x,t) |, *F*(t): = max | ¥(x,t) | V t e [0, t*].
x x

(The maxima in (4.38) exist because O(x,t) and *F(x,t) tend to zero as x tends to ±©o.)

We fix t e (0, t*] and choose $\hat{x}$ and $\check{x}$ such that

(4.39) <D*(t) = | <D($\hat{x}$,t) |, *F*(t) = | *F($\check{x}$,t) |.

Then for every h e (0, t], we have

(4.40)
<D*(t - h) ≥ | O($\hat{x}$ + h p'(e($\hat{x}$,t))$^{1/2}$, t - h) |,

¥*(t - h) ≥ | >F($\check{x}$ - h p'(e($\hat{x}$,t))$^{1/2}$, t - h) |.

We subtract (4.40) from (4.39), divide the resulting inequalities by h, and let h *X* 0, to conclude that

(4.41) D" <D*(t) <| <D ($\hat{x}$,t) |, D" *F*(t) ≤ | ^ ^$\check{x}$.t) |.

It follows from (2.14), (4.34), (4.35), (4.10) and (4.28) that

(4.42) | ± $\frac{\beta}{K}$ p'(e)$^{1/2}$ (P(e) + c0 + $\frac{1}{y}$v 2) - (p(e) +pe +p T) v + -2- $\frac{}{K}$p'(e)$^{1/2}$ *dxr* \
o

<:r{[|<D|+ |^|]^2+[|<D|+ |^|] + 8}.

By combining (4.41), (4.32), (4.33), (4.38) and (4.42) we find that

(4.43) ^ [O*(t) + *W*(t)*] <T { [<D*(t) + T*(t)]^2 + [O*(t) + T*(t)] + 5 },

for almost all t € [0, t*]. Moreover, by virtue of (4.38), (4.31), (4.28), (4.22), and (3.13), we have

$$(4.44) \qquad \Phi^*(0) \le \Gamma\, \delta^{1/2}, \qquad\qquad \Phi^*(0) \le \Gamma\, \delta^{1/2}.$$

Consequently if δ is sufficiently small, (4.43), (4.44) yield

$$(4.45) \qquad \Phi^*(t) + \Psi^*(t) \le \bar{\lambda}, \qquad \forall\, t \in [0,\ t^*],$$

which implies (4.37). □

Our next task is to estimate the partial derivatives of ε, v, and θ. For this purpose we define

$$(4.46) \qquad w := r_x, \qquad\qquad \omega := s_x,$$

and note that

$$(4.47) \qquad \varepsilon_x = \frac{1}{2}\, p'(\varepsilon)^{-1/2}\, (w - \omega), \qquad\qquad v_x = \frac{1}{2}\,(w + \omega).$$

A simple computation yields

$$(4.48) \qquad \varepsilon^{\backslash} = \omega, \qquad \varepsilon^{/} = w.$$

Differentiation of (4.24) with respect to x gives

$$(4.49) \qquad \partial_t\, r_x - p'(\varepsilon)^{1/2}\, r_{xx} - \frac{1}{2}\, p'(\varepsilon)^{-1/2}\, p''(\varepsilon)\, \varepsilon_x\, r_x = \beta\, \theta_{xx},$$

$$(4.50) \qquad \partial_t\, s_x + p'(\varepsilon)^{1/2}\, s_{xx} + \frac{1}{2}\, p'(\varepsilon)^{-1/2}\, p''(\varepsilon)\, \varepsilon_x\, s_x = \beta\, \theta_{xx}.$$

We substitute ε_x from (4.47) into (4.49), (4.50), use (4.48) to obtain

$$(4.51) \qquad w^{\backslash} + \frac{1}{4}\, p'(\varepsilon)^{-1}\, p''(\varepsilon)\, \varepsilon^{\backslash}\, w - \frac{1}{4}\, p'(\varepsilon)^{-1}\, p''(\varepsilon)\, w^2 = \beta\, \theta_{xx},$$

$$(4.52) \qquad \omega^{/} + \frac{1}{4}\, p'(\varepsilon)^{-1}\, p''(\varepsilon)\, \varepsilon^{/}\, w - \frac{1}{4}\, p'(\varepsilon)^{-1}\, p''(\varepsilon)\, \omega^2 = \beta\, \theta_{xx}.$$

and multiply (4.51), (4.52) by $p'(\varepsilon)^{1/4}$ to arrive at

$$(4.53) \qquad (p'(\varepsilon)^{1/4}\, w)^{\backslash} - \frac{1}{4}\, p'(\varepsilon)^{-5/4}\, p''(\varepsilon)\, (p'(\varepsilon)^{1/4} w)^2 = \beta\, p'(\varepsilon)^{1/4}\, \theta_{xx},$$

(4.54) $$(p'(\varepsilon)^{1/4}\omega)' - \frac{1}{4}p'(\varepsilon)^{-5/4}p''(\varepsilon)(p'(\varepsilon)^{1/4}\omega)^2 = \beta p'(\varepsilon)^{1/4}\theta_{xx}.$$

We then define

(4.55) $$f := p'(\varepsilon)^{1/4}(w - \frac{\beta c}{\kappa}\theta), \qquad g := p'(\varepsilon)^{1/4}(\omega - \frac{\beta c}{\kappa}\theta).$$

By using (4.47) through (4.54) and

(4.56) $$\theta_{xx} = \frac{c}{\kappa}\theta_t - \frac{\beta}{\kappa}(\theta + \overline{T})v_x$$

$$= \frac{c}{\kappa}\theta^{\grave{}} + \frac{c}{\kappa}p'(\varepsilon)^{1/2}\theta_x - \frac{\beta}{2\kappa}(\theta + \overline{T})(w + \omega)$$

$$= \frac{c}{\kappa}\theta' - \frac{c}{\kappa}p'(\varepsilon)^{1/2}\theta_x - \frac{\beta}{2\kappa}(\theta + \overline{T})(w + \omega)$$

we find that

(4.57) $$f^{\grave{}} = \frac{1}{4}p'(\varepsilon)^{-5/4}p''(\varepsilon)f^2 + \frac{\beta c}{4\kappa}p'(\varepsilon)^{-1}p''(\varepsilon)\theta(2f - g) - \frac{\beta^2}{2\kappa}\theta(f + g) +$$

$$\frac{\beta c}{\kappa}p'(\varepsilon)^{3/4}\theta_x - \frac{c\beta^3}{2\kappa^2}p'(\varepsilon)^{1/4}\theta^2 - \frac{\beta^2}{2\kappa}\overline{T}\,p'(\varepsilon)^{1/4}(f + g) - \frac{c\beta^3}{2\kappa^2}\overline{T}\theta,$$

(4.58) $$g' = \frac{1}{4}p'(\varepsilon)^{-5/4}p''(\varepsilon)g^2 + \frac{\beta c}{4\kappa}p'(\varepsilon)^{-1}p''(\varepsilon)\theta(2g - f) - \frac{\beta^2}{2\kappa}\theta(f + g) -$$

$$\frac{\beta c}{\kappa}p'(\varepsilon)^{3/4}\theta_x - \frac{\beta^3 c}{\kappa^2}p'(\varepsilon)^{1/4}\theta^2 - \frac{\beta^2}{2\kappa}\overline{T}\,p'(\varepsilon)^{1/4}(f + g) - \frac{c\beta^3}{2\kappa^2}\overline{T}\theta.$$

In order to state our final estimate it is convinient to introduce

(4.59) $$F(t) := \max_x |f(x,t)|, \qquad G(t) := \max_x |g(x,t)| \quad \forall\, t \in [0,\ t^*].$$

Lemma 4.4 : *The temperature* θ *obeys the estimate*

$$|\theta_x(x,t)| \le \Gamma + \Gamma(\delta + \overline{T}) \int_0^t \frac{F(\tau) + G(\tau)}{(t-\tau)^{1/2}} d\tau \quad \forall\, t \in [0,\ t^*]. \tag{4.60}$$

Proof. By using (4.26) and substituting for v_x in terms of f, g, and θ, differentiation of (4.26) with respect to x yields

$$\theta_x(x,t) = \frac{c}{(4\pi\kappa t)^{1/2}} \int_{-\infty}^{\infty} \exp\{-\frac{c}{4\kappa}\frac{(x-y)^2}{t}\}\theta_0'(y)\, dy \tag{4.61}$$

$$- \frac{\beta}{(4\pi\kappa c)^{1/2}} \int_0^t \int_{-\infty}^{\infty} (t-\tau)^{-1\backslash 2} \exp\{-\frac{c}{4\kappa}\frac{(x-y)^2}{t-\tau}\}(\theta(y,\tau) + \overline{T})\,(p'^{\,-1\backslash 4}(f+g) + 2\frac{\beta c}{\kappa}\theta)\, dy\, d\tau.$$

The assertion of the lemma follows easily from (4.61). □

We are now ready to establish the formation of singularity in finite time. Let $L \in (0,\ 1]$, $M > 0$, and ε_0, v_0, $\theta_0 \in H^3(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$ satisfying (3.6) and (3.12) through (3.15) be given. Consider the solution (ε,v,θ) given by the proposition of Section 3. We shall show that if δ is sufficiently small and M is sufficiently large then $t_0 \le L$, where t_0 is the length of the maximal interval. For the purpose of obtaining a contradiction, we assume that $t_0 > L$. By virtue of Lemmas 4.2 and 4.3 we may choose δ small enough so that for every $t^* \in (0,\ L]$ such that (4.2) holds, the sharper bound

$$|\varepsilon(x,t)| \le \frac{1}{4}, \qquad |\theta(x,t)| \le \frac{\overline{T}}{4} \qquad \forall\ x \in \mathbb{R},\, t \in [0,\ t^*] \tag{4.62}$$

also holds. Consequently, by continuity, if δ is sufficiently small then

$$|\varepsilon(x,t)| \le \frac{1}{2}, \qquad |\theta(x,t)| \le \frac{\overline{T}}{2} \qquad \forall\ x \in \mathbb{R},\, t \in [0,\ L]. \tag{4.63}$$

For the remainder of the proof, we assume that δ is small enough so that (4.63) holds. We define, as in [5], the nonnegative Lipschitz functions

$$(4.64)\qquad \begin{aligned} F^+(t) &= \max_x f(x,t), & G^+(t) &= \max_x g(x,t), \\ F^-(t) &= -\min_x f(x,t), & G^-(t) &= -\min_x g(x,t) \qquad \forall\ t \in [0, L], \end{aligned}$$

where f and g are given by (4.55), and note that

$$(4.65)\qquad F(t) \le F^+(t) + F^-(t), \qquad G(t) \le G^+(t) + G^-(t) \qquad \forall\ t \in [0, L].$$

We fix $t \in [0, L)$ with $F^+(t) > 0$ and / or $G^+(t) > 0$ and choose $\hat{x}$ and $\check{x}$ such that

$$(4.66)\qquad F^+(t) = f(\hat{x},t) \quad \text{and / or} \quad G^+(t) = g(\check{x},t).$$

For every $h \in (0, L - t]$ we then have

$$(4.67)\qquad \begin{aligned} F^+(t+h) &\ge f(\hat{x} - h\, p'(\varepsilon(\hat{x},t))^{1/2}, t + h), \\ G^+(t+h) &\ge g(\check{x} + h\, p'(\varepsilon(\check{x},t))^{1/2}, t + h). \end{aligned}$$

Subtracting (4.66) from (4.67), dividing through by h, and letting $h \downarrow 0$, we obtain

$$(4.68)\qquad D^+F^+(t) \ge \grave{f}(\hat{x},t) \qquad \text{and / or} \qquad D^+G^+(t) \ge g'(\check{x},t).$$

Next we fix $t \in [0, L)$ with $F^-(t) > 0$ and / or $G^-(t) > 0$ and choose $\hat{y}$ and $\check{y}$ such that

$$(4.69)\qquad F^-(t) = -f(\hat{y},t) \quad \text{and / or} \quad G^-(t) = -g(\check{y},t).$$

For every $h \in (0, t]$ we then have

$$(4.70)\qquad \begin{aligned} F^-(t+h) &\ge -f(\hat{y} + h\, p'(\varepsilon(\hat{y},t))^{1/2}, t - h), \\ G^-(t+h) &\ge -g(\check{y} - h\, p'(\varepsilon(\hat{y},t))^{1/2}, t - h). \end{aligned}$$

Subtracting (4.70) from (4.69), dividing through by h, and letting $h \downarrow 0$, we obtain

(4.71) $\qquad D^- F^-(t) \le -\grave{f}\,(\hat{y},t)$ and / or $\qquad D^- G^-(t) \le -\acute{g}\,(\check{y},t).$

Since $p''(0) > 0$ we may choose constants α_1, α_2 such that

(4.72) $$p''(\xi) \ge \alpha_1 \quad \forall\ \xi \in [-\alpha_2,\ \alpha_2].$$

By virtue of Lemma 4.3 we may choose δ small enough so that

(4.73) $$|\,\varepsilon(x,t)\,| \le \alpha_2 \quad \forall\ x \in \mathbb{R},\ t \in [0,\ L].$$

Thus there is a constant α such that

(4.74) $$\frac{1}{4}\, p'(\varepsilon(x,t))^{-5/4}\, p''(\varepsilon(x,t)) \ge 2\alpha > 0 \quad \forall\ x \in \mathbb{R},\ t \in [0,\ L].$$

Therefore, by combining the inequalities above, we get estimates of the form :

(4.75) $$\frac{d}{dt}\,[\,F^+(t) + G^+(t)] \ge \alpha\,[\,F^+(t) + G^+(t)]^2 - \Gamma\,(\delta + \overline{T}\,)[\,F^+(t) + G^+(t)]$$

$$-\,\Gamma\,(\delta + \overline{T}\,)[\,F^-(t) + G^-(t)] - \Gamma\,(\delta + \overline{T}\,)\int_0^t \frac{F^+(\tau) + G^+(\tau)}{(t-\tau)^{1/2}}\,d\tau - \Gamma\,(\delta + \overline{T}\,)\int_0^t \frac{F^-(\tau) + G^-(\tau)}{(t-\tau)^{1/2}}\,d\tau - \Gamma,$$

(4.76) $$\frac{d}{dt}\,[F^-(t) + G^-(t)] \le \Gamma\,(\delta + \overline{T}\,)[\,F^+(t) + G^+(t)] + \Gamma\,(\delta + \overline{T}\,)[F^-(t) + G^-(t)]$$

$$+\,\Gamma\,(\delta + \overline{T}\,)\int_0^t \frac{F^+(\tau) + G^+(\tau)}{(t-\tau)^{1/2}}\,d\tau + \Gamma\,(\delta + \overline{T}\,)\int_0^t \frac{F^-(\tau) + G^-(\tau)}{(t-\tau)^{1/2}}\,d\tau + \Gamma,$$

for almost all $t \in [0, L]$.

In view of (4.55), (4.64), (4.10), (3.14), and Lemma 4.3, (4.76) yields

(4.77) $$[F^-(t) + G^-(t)] \le \Gamma + \Gamma\,(\delta + \overline{T}\,)\int_0^t K\,(t-\tau\,)\,[\,F^+(t) + G^+(t)]\,d\tau,$$

where $K \in L^1(0, L)$. Finally by combining (4.75) with (4.77), we obtain the inequality

(4.78) $$\frac{d}{dt}\,[\,F^+(t) + G^+(t)] \ge \alpha\,[\,F^+(t) + G^+(t)]^2 - \Gamma\,(\delta + \overline{T}\,)[\,F^+(t) + G^+(t)]$$

$$-\,\Gamma\,(\delta + \overline{T}\,)\int_0^t Z\,(t-\tau\,)\,[\,F^+(t) + G^+(t)]\,d\tau - \Gamma,$$

where $Z \in L^1(0, L)$. By virtue of Lemmas 4.2 and 4.3 it follows that when M (in (3.15)) is sufficiently large then $[F^+ + G^+]$ blows up in a finite time $t^+ < L$, which contradicts the assumption that $t_0 > L$. This completes the proof. □

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