PHASE TRANSITIONS OF ELASTIC SOLID MATERIALS

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1. INTRODUCTION.

In recent years the analysis of phase transitions for mixtures of two or more non-interacting fluids has been successfully undertaken within the Van der Waals-Cahn-Hilliard gradient theory of phase transitions (see BALDO [1], FONSECA & TARTAR [5], GURTIN [9], KOHN & STERNBERG [11], MODICA [12], OWEN [13], STERNBERG [15]). In the case where the nonnegative Gibbs free energy W vanishes only at two points a and b, this theory permits to select among all the minimizers of

$$\int_\Omega W(v(x))\,dx$$

with prescribed total mass

$$m = \int_\Omega v(x)\,dx = \text{meas}(\Omega)(\theta a + (1-\theta)b), \text{ with } \theta \in (0, 1),$$

those that have minimal interfacial area, i. e. it singles out those solutions $v \in \{a, b\}$ a. e. such that the set $\{v = a\}$ minimizes $\text{Per}_\Omega(\omega)$ among all subsets ω of Ω with $\text{meas}(\omega) = \theta = \text{meas}(\{v = a\})$.

We consider an analogous situation in the context of nonlinear elasticity. Here the stored energy density W is nonnegative and, due to frame indifference, W has two orbits of minima $\{RA \mid R \text{ rotation}\}$ and $\{RB \mid R \text{ rotation}\}$ where A and B differ by a rank one matrix,

$$A = B + a\otimes n.$$

It is clear that the problem

(P_0) minimize

$$\int_\Omega W(\nabla u(x))\,dx,$$

for u such that

$$\int_\Omega u(x)\,dx = m \quad \text{and} \int_\Omega \nabla u(x)\,dx = \text{meas}(\Omega)(\theta A + (1-\theta)B)$$

where $\theta \in (0, 1)$ and $m \in \mathbb{R}^3$ are fixed, admits infinitely many solutions. In particular, if E is a subset of Ω layered normally to n and if $\text{meas}(E) = \theta\,\text{meas}(\Omega)$ then there exists a solution u of (P_0) with

$$\nabla u = B + \chi_E\, a\otimes n. \tag{1.1}$$

We search for a model that will select among all the solutions of (P_0) those of the form (1.1) for which E is a solution of

(P^*_0) minimize $\text{Per}_\Omega(E')$, where $E'\subset\Omega$ is layered normally to n and $\text{meas}(E') = \theta\,\text{meas}(\Omega)$.

In order to apply the gradient theory of phase transitions to this setting, we have to add to the former problem the constraint curl $v = 0$ which renders the analysis very difficult (see FONSECA & TARTAR [6]). In this paper we study a model in which the second deformation gradient is replaced by a Radon measure penalizing the formation of interfaces and where the spinodal region is removed as in GURTIN's theory for phase transitions for fluids (see GURTIN [8]). In Section 2 we discuss briefly some notions and results of the theory of functions of bounded variation. In Proposition 2.16 we prove the converse of a result due to BALL & JAMES [3] characterizing the Lipschitz deformations satisfying (1.1) (see Theorem 2.14). In Section 3 we introduce a model accomodating the constraint curl $v = 0$. If we disregard the hypothesis of frame indifference, it can be shown that a sequence of minimizers of the regularized problems admits a subsequence converging weakly to a solution of the problem (P_0) of the form (1.1) with minimal interfacial area (see Theorem 3.2). In Section 4 we adapt a model proposed by GURTIN [8] for the analysis of fluid phase transitions. Here the spinodal region is removed and we penalize directly the interface. In Theorem 4.5 we obtain the analog of Theorem 3.2 for the sequence of penalized problems. In order to handle the frame indifference hypothesis, we combine the models of Sections 3 and 4. In Theorem 5.10 we show that a sequence of minimizers of the approximation problems admits a subsequence converging weakly to a solution of the problem (P_0) of the form

$$\nabla u = R(x)(B + \chi_E \; a \otimes n), \tag{1.2}$$

where $R(x)$ is a rotation, ∇u, R and $\chi_E \in BV(\Omega) \cap L^\infty(\Omega)$. We conjecture that R = identity a. e. and that E is a solution of (P^*_0). We prove that the conjecture is confirmed if the set E is reasonably smooth, precisely if E determines a partition of Ω into countably many open, strongly Lipschitz, connected domains (see Theorem 5.8). The rest of Section 5 is dedicated to finding results asserting the required smoothness of E. In Proposition 5.19 we show that if a is parallel to $B^{-T}n$ then the outward unit normal to $\partial E \cap \Omega$ is parallel to n. In this case the set E is layered normally to n and the conjecture is valid (see Corollary 5.20). If a is not parallel to $B^{-T}n$, we prove in Proposition 5.19 that the normal to $\partial E \cap \Omega$ is parallel either to n or to a known vector m, where m is a linear combination of a and n. Moreover, it is possible to prove that $\partial E \cap \Omega$ cannot have "corners" (see Lemma 5.5). We do not know if these properties imply the smoothness of E required by Theorem 5.8, in which case the conjecture would be confirmed.

2. STATEMENT OF THE PROBLEM AND PRELIMINARIES.

We discuss briefly some results of the theory of functions of bounded variation (see EVANS & GARIEPY[4], GIUSTI [7]).

Let Ω be an open bounded strongly Lipschitz domain of $\mathbb{R}^n$.

Definition 2.1.

A function $u \in L^1(\Omega)$ is said to be a *function of bounded variation* ($u \in BV(\Omega)$) if

$$\int_\Omega |\nabla u(x)|\, dx := \sup\left\{ \int_\Omega u(x).\, \operatorname{div} \varphi(x)\, dx \mid \varphi \in C_0^1(\Omega;\, \mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\} < +\infty.$$

It follows immediatly that if $u_\varepsilon \to u$ in $L^1(\Omega)$ then

$$\int_\Omega |\nabla u(x)|\, dx \leq \liminf_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon(x)|\, dx. \tag{2.2}$$

It can be shown that the sets

$$\left\{ u \in L^1(\Omega) \mid \int_\Omega |u(x)| + |\nabla u(x)|\, dx \leq C < +\infty \right\} \tag{2.3}$$

are compact in $L^1(\Omega)$.

Proposition 2.4.

There exists a constant $C > 0$ such that for all $f \in BV(\Omega)$

$$\int_\Omega f(x)\, dx = 0 \Rightarrow \left(\int_\Omega |f(x)|^{n/n-1}\, dx \right)^{\frac{n-1}{n}} \leq C \int_\Omega |Df(x)|\, dx.$$

Definition 2.5.

If A is a subset of $\mathbb{R}^n$ then the *perimeter of A in Ω* is defined by

$$Per_\Omega(A) := \int_\Omega |\nabla \chi_A(x)|\, dx = \sup\left\{ \int_A \operatorname{div} \varphi(x)\, dx \mid \varphi \in C_0^1(\Omega;\, \mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\},$$

where χ_A denotes the characteristic function of A.

Proposition 2.6.

If $f, g \in L^\infty(\Omega) \cap BV(\Omega)$ then $f.g \in L^\infty(\Omega) \cap BV(\Omega)$.

Clearly, if $A \subset \Omega$ and if

$$u(x) = \begin{cases} a & \text{if } x \in A \\ b & \text{if } x \in \Omega \setminus A \end{cases}$$

then $u \in BV(\Omega)$ if and only if $Per_{\Omega}(A) < +\infty$. Suppose that E is a set of finite perimeter in $\mathbb{R}^n$. There exist a Radon measure $\|\partial E\|$ and a $\|\partial E\|$ - measurable function

$$\nu_E : \mathbb{R}^n \to \mathbb{R}^n, \| \nu_E \| = 1 \quad \|\partial E\| \text{ a. e.},$$

such that

$$\int_E \operatorname{div} \varphi(x)\, dx = \int_{\mathbb{R}^n} \varphi(x).\nu_E(x)\, d\|\partial E\| \text{ for all } \varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^n).$$

Definition 2.7.

Let $x \in \mathbb{R}^n$. We say that $x \in \partial^* E$, the *reduced boundary of E*, if
$\|\partial E\| (B(x,r)) > 0$ for all $r > 0$,

$$\| \nu_E (x) \| = 1$$

and

$$\lim_{\varepsilon \to 0} \frac{1}{\|\partial E\|(B(x,\varepsilon))} \int_{B(x,\varepsilon)} \nu_E(y)\, d\|\partial E\| = \nu_E(x).$$

Theorem 2.8. (Blow-up of the reduced boundary)

If $x \in \partial^* E$ then

$$\chi_{x + \frac{E-x}{\varepsilon}} \to \chi_{H^-(x)} \quad \text{in } L^1_{loc} \text{ as } \varepsilon \to 0^+,$$

where

$$H^-(x) := \{y \in \mathbb{R}^n | \nu_E(x). (y - x) < 0\}.$$

Theorem 2.9. (Generalized Gauss-Greeen Theorem)

$$\int_E \operatorname{div} \varphi\, dx = \int_{\partial^* E} \varphi.\nu_E\, dH_{n-1}$$

for all $\varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$.

Given $f \in BV(\Omega)$, we define

Li(x):=aphmsupf(y) = inf-{t | hm meas(B(x,e)n{f>t» / en = 0}
y-»x E-»O

and

X(x):=apluninff(y) = sup-{ t| lun meas(B(x,e)n{f<t» / g^n = 0}.
y-» x e-» 0

Let

$$J := \{x \in \Omega \mid \lambda(x) < \mu(x)\}$$

denote the set of points at which f is not approximately continuous.

Theorem 2.10.

Assume that f e BV(£2). Then *X* and \i are Borcl measurable and

(i) meas(J) = 0;

(ii) -oo ^ X(x) ≤, n(x) ^ +oo in Q;

(in) hm ——— ^T—rr · |f(y)——— TT—— | dy = O **(2.11)**
£ _»0+ meas (B(x, e)) JB(x, *t*) *2*

for HJJ^ a. e. x € *Q* \ J;

(iv) for HJJJ a. e. x e J there exists a unit vector v such that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\text{meas}\,(B(x,\varepsilon)\cap H_v^+)} \int_{B(x,\varepsilon)\cap H_v^+} |f(y) - \mu(x)|^{n/n-1}\, dy = 0$$

and **(2.12)**

$$\lim_{\varepsilon \to 0^+} \frac{1}{\text{meas}\,(B(x,\varepsilon)\cap H_v^-)} \int_{B(x,\varepsilon)\cap H_v^-} |f(y) - \lambda(x)|^{n/n-1}\, dy = 0.$$

Consider a hyj>erelastic body that occupies in a reference configuration a bounded, simply connected, strongly Lipschitz domain Q C K^3, with meas (Q) = 1. Let W : M^{3x3} -» [0, +«>] denote the stored energy density, and assume that

(HI) W(F) = +00 if and only if detF≤0;

(H2)W(F) = 0ifandonlyifFe {RA,RB| Re O$^+$(IR^3)}, where A = B + a®n, ||n|| = 1, a*0. Here, and in what follows, M^{3x3} is the set of real 3x3 matrices and O$^+$(IR^3) denotes the set of rotations of IR^3, Let 6 *e* (0, 1) and *me* IR^3 be fixed, and define the class of admissible deformations

$$\mathcal{Q}_0 := \left\{ u \in W^{1,1}(\Omega; \mathbb{R}^3) \mid \int_\Omega u(x)\, dx = m, \int_\Omega \nabla u(x)\, dx = \theta A + (1-\theta) B \right\}.$$

In this paper, we study the variational problem

(P_0) Minimize

$$\int_\Omega W(\nabla u(x))\, dx,$$

for $u \in \mathcal{Q}_0$.

Remark 2.13.

(P_0) admits infinitely many solutions u such that $\nabla u \in BV(\Omega)$. In fact, let α be such that

$$\text{meas}\,\{x \in \Omega \mid x.n > \alpha\} = \theta,$$

and define

$$C := m - \int_{\{x \in \Omega \mid x.n > \alpha\}} Ax\, dx - \int_{\{x \in \Omega \mid x.n < \alpha\}} (Bx + \alpha a)\, dx.$$

Setting

$$u^*(x) := \begin{cases} Ax + C & \text{if } x.n > \alpha \\ Bx + \alpha a + C & \text{if } x.n < \alpha, \end{cases}$$

it follows immediatly that u^* is a solution of (P_0). Similarly, an infinit set of Lipischitz solutions with gradients taking only the values A and B can be found by layering Ω by finitely many parallel planes with normal n, in such a way that meas($\{x \in \Omega \mid \nabla u(x) = A\}$) = θ (see Fig. 1).

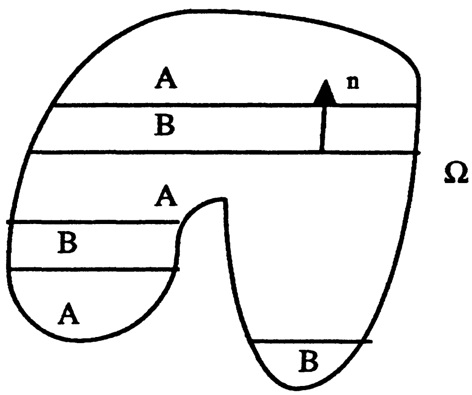

Fig. 1

The following result is due to BALL & JAMES [3].

Theorem 2.14.

Let $E \subset \Omega$, $0 < \text{meas}(E) < \text{meas}(\Omega)$, and let χ_E denote the characteristic function of E. If u

$\in W^{1,\infty}(\Omega; \mathbb{R}^3)$ satisfies

$$\nabla u = \chi_E\, C + (1 - \chi_E) D,$$

then

$$C = D + b \otimes m$$

for some $b \in \mathbb{R}^3$, $m \in \mathbb{R}^3$, $\|m\| = 1$. Moreover, for all convex set $\Gamma \subset \Omega$, $E \cap \Gamma$ and $(\Omega \setminus E) \cap \Gamma$ consist of parallel layers normal to m; precisely, there exists a Lipschitz function f with $f' \in \{0, 1\}$ a. e. such that

$$E \cap \Gamma = \{x \in \Gamma \mid f'(x.n) = 1\}.$$

We define the set

$$\mathcal{S} := \{ E \subset \Omega \mid \text{meas}(E) = \theta \text{ and for all convex set } \Gamma \subset \Omega \text{ there exists a Lipschitz function f with } f' \in \{0, 1\} \text{ a. e. such that } E \cap \Gamma = \{x \in \Gamma \mid f'(x.n) = 1\}\}.$$

Remark 2.15.
It is clear that the solutions u exhibited in Remark 2.13 verify

$$\nabla u = B + \chi_E\, a \otimes n, \text{ with } E \in \mathcal{S} \text{ and } \nabla u \in BV(\Omega). \tag{2.15}$$

We prove that the converse of Theorem 2.14 is also true.

Proposition 2.16.
If $E \in \mathcal{S}$ then there exists $u \in W^{1,\infty}(\Omega; \mathbb{R}^3) \cap \mathcal{Q}_0$ such that

$$\nabla u = B + \chi_E\, a \otimes n.$$

Proof. Suppose for simplicity that $n = e_3$. Let $x_0 \in \Omega$ be fixed, and define for $x \in \Omega$ the function

$$\theta(x) := \int_{\gamma_x} \chi_E\, dx_3 = \int_0^1 \chi_E(\alpha(s))\, \alpha_3'(s)\, ds$$

where $\gamma_x = \{\alpha(t) \mid t \in [0, 1], \alpha(0) = x_0, \alpha(1) = x\}$ is a piecewise C^1 curve in Ω joining x_0 to x.

(i) θ does not depend on the choice of the curve γ_x. Let $\alpha, \beta: [0, 1] \to \Omega$ be piecewise C^1 functions such that $\alpha(0) = \beta(0) = x_0$, $\alpha(1) = \beta(1) = x$. As Ω is a simply connected domain, there exists a C^1 function $H : [0, 1] \times [0, 1] \to \Omega$ such that $H(0, t) = \alpha(t)$, $H(1, t) = \beta(t)$. Define

$$F(s) := \int_0^1 \chi_E(H(s, t)) \frac{\partial}{\partial t} H_3(s, t)\, dt.$$

We want to show that $F(0) = F(1)$. Clearly, it suffices to prove that for all $s_0 \in [0, 1]$ there exists $\varepsilon > 0$ such that $F(s) = F(s_0)$ for all $s \in [0, 1]$, $|s - s_0| < \varepsilon$. Let $s_0 \in [0, 1]$ and let

$$H(s_0, t) \in \bigcup_{i=0}^{N} B(x_i, \varepsilon_i) \subset \Omega \quad \text{for all } t \in [0, 1], \quad \text{where } x_N \equiv x.$$

As H is smooth, there exists $\varepsilon > 0$ such that

$$H(s,t) \in \bigcup_{i=0}^{N} B(x_i, \varepsilon_i) \quad \text{for } |s - s_0| < \varepsilon \text{ and for all } t \in [0, 1].$$

Let f_i be such that

$$\chi_E(x) = f_i'(x.n) \quad \text{for } x \in B(x_i, \varepsilon_i). \tag{2.17}$$

Define the open sets

$$\omega_i := \{t \in \mathbb{R} \mid \exists\, x \in B(x_i,\varepsilon_i) \cap B(x_{i+1},\varepsilon_{i+1}) \text{ such that } x.n = t\},\ i = 0, \ldots, N\text{-}1.$$

As

$$f'_i = f'_{i+1} \quad \text{in } \omega_i,$$

we can assume without loss of generality that

$$f_i = f_{i+1} \quad \text{in } \omega_i. \tag{2.18}$$

Choose points $\sigma(t_i)$, $\rho(t_i)$ such that $0 < t_1 < \ldots < t_N < 1$ and

$$\sigma(t_i), \beta(t_i) \in B(x_{i-1},\varepsilon_{i-1}) \cap B(x_i,\varepsilon_i), \text{ for } i = 1, \ldots, N,$$

where $\sigma(t) := H(s_0, t)$ and $\rho(t) := H(s, t)$. By (2.17) and (2.18) we have

$$\begin{aligned} F(s_0) &= \int_0^1 \chi_E(H(s_0, t)) \frac{\partial}{\partial t} H_3(s_0, t)\, dt \\ &= \int_0^{t_1} f'_0(\sigma(t).n)\, \sigma'(t).n\, dt + \sum_{i=1}^{N-1} \int_{t_i}^{t_{i+1}} f'_i(\sigma(t).n)\, \sigma'(t).n\, dt + \int_{t_N}^{1} f'_N(\sigma(t).n)\, \sigma'(t).n\, dt \\ &= f_0(\sigma(t_1).n) - f_0(x_0.n) + \sum_{i=1}^{N-1} [f_i(\sigma(t_{i+1}).n) - f_i(\sigma(t_i).n)] + f_N(x.n) - f_N(\sigma(t_N).n) \end{aligned}$$

$$= f_N(x.n) - f_0(x_0.n).$$

In a similar way, we obtain

$$F(s) = f_N(x.n) - f_0(x_0.n).$$

(ii) θ is a locally Lipschitz function. In fact, let $B(x, \varepsilon) \subset \Omega$ and let $y \in B(x, \varepsilon)$. By (i) we have

$$|\theta(x) - \theta(y)| \le \left|\int_0^1 \chi_E((1-t)x + ty)\,(y-x).n\ dt\right|$$

$$\le \|x - y\|.$$

(iii) We prove that $\nabla\theta = \chi_E\, n$ in $\mathcal{D}'(\Omega)$. Let $B(x, \varepsilon) \subset\subset \Omega$ and let

$$E \cap B(x, \varepsilon) = \{y \in B(x, \varepsilon) \mid f'(y.n) = 1\}.$$

By (i) we deduce that

$$\theta(y) = \int_0^1 \chi_E((1-t)x + ty)\,(y-x).n\ dt$$

$$= \int_0^1 f'((1-t)x + ty)\,(y-x).n\ dt$$

$$= f(y.n) - f(x.n).$$

Therefore, if $\phi \in \mathcal{D}(B(x, \varepsilon))$ and if $i = 1, 2$ we have

$$<\frac{\partial\theta}{\partial x_i}, \varphi> = \int_{B(x,\varepsilon)} [f(x.n) - f(y.n)]\, \frac{\partial\varphi}{\partial y_i}(y)\, dy$$

$$= 0$$

and

$$<\frac{\partial\theta}{\partial x_3}, \varphi> = \int_{B(x,\varepsilon)} [f(x.n) - f(y.n)]\, \frac{\partial\varphi}{\partial y_3}(y)\, dy$$

$$= \int_{B(x,\varepsilon)} f'(y.n)\, \varphi(y)\, dy$$

$$= \int_E \varphi(y)\, dy.$$

(iv) Finally, set

$u(x) := u_0 + Bx + \theta(x)a$, with u_0 such that

$$\int_\Omega u(x)\, dx = m.$$

Clearly, $u \in W^{1,\infty}(\Omega; \mathbb{R}^3) \cap \mathcal{Q}_0$ and

$$\nabla u = B + \chi_E\, a \otimes n.$$

We search for a model that will select among all solutions of (P_0) those of the form (2.15) with minimal interfacial area, i. e. where E is a solution of the geometric variational problem:

(P^*_0) Minimize $Per_\Omega(E')$ with $E' \in \mathcal{E}$.

In Sections 3 and 4 we consider simplified models where W does not satisfy the hypothesis of frame indifference.

3. REGULARIZATION.

Assume that W is continuous and verifies (H1) and

(H2') $W(F) = 0$ if and only if $F \in \{A, B\}$.

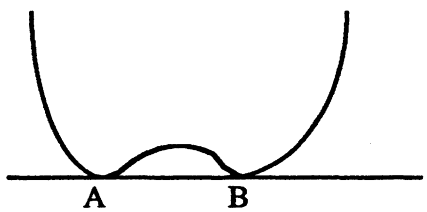

Fig. 2

In view of Remark 2.15, we define the class of admissible deformations

$$\mathcal{A}_1 := \left\{ u \in W^{1,1}(\Omega; \mathbb{R}^3) \mid \nabla u \in BV(\Omega), \int_\Omega u(x)\, dx = m, \int_\Omega \nabla u(x)\, dx = \theta A + (1-\theta) B \right\}$$

and for $\varepsilon > 0$ we introduce the "regularized" problem:

(P_ε) Minimize in $\mathcal{A}_1$

$$E_\varepsilon(u) := \int_\Omega W(\nabla u(x))\, dx + \varepsilon \int_\Omega |D\nabla u(x)|\, dx.$$

Here and in what follows we use the notation

$$\|D\nabla u\| = \int_\Omega |D\nabla u(x)|\, dx := \sum_{i,j=1}^{3} \int_\Omega |D\frac{\partial u_i}{\partial x_j}(x)|\, dx.$$

Proposition 3.1.

For all $\varepsilon > 0$, (P_ε) admits a solution.

Proof. Let u^* be as defined in Remark 2.13. Then

$E_\varepsilon(u^*) = \varepsilon C^*$ where $C^* :=$ area $\{x \in \Omega \mid x.n = \alpha\}$.

Let $\{u_j\}$be a minimizing sequence with j large enough so that

$E_\varepsilon(u_j) \le \varepsilon C^* + 1.$

Thus

$\{\|D\nabla u_j\|\}$ is a bounded sequence

and so, by Proposition 2.4 and Poincaré's inequality, we deduce that

$\{u_j\}$is a bounded sequence in $W^{1,3/2}(\Omega)$

and

$\{\nabla u_j\}$ is a bounded sequence in $BV(\Omega)$.

By (2.3) we conclude that there exist $u \in \mathcal{G}_1 \cap W^{1,3/2}(\Omega)$ and a subsequence $\{u_m\}$ such that

$u_m \to u$ weakly in $W^{1,3/2}(\Omega)$

and

$\nabla u_m \to \nabla u$ strongly in $L^1(\Omega)$.

By Fatou's Lemma and (2.2) it follows that

$$E_\varepsilon(u) \le \liminf_{m\to\infty} E_\varepsilon(u_m) = \inf_{\mathcal{G}} E_\varepsilon(.).$$

Theorem 3.2.

Let v_ε be a sequence of minimizers of E_ε in $\mathcal{G}_1$. There exist a solution u of (P_0) and a subsequence u_ε such that

$u_\varepsilon \to u$ in $W^{1,3/2}(\Omega; \mathbb{R}^3)$ weak

and

$\nabla u = B + \chi_E\, a \otimes n$ a. e. in Ω,

where $E \in \mathcal{S}$ is a solution of (P^*_0).

Proof. Considering u* as in the proof of Proposition 3.1, we have

$E_\varepsilon(v_\varepsilon) \le E_\varepsilon(u^*) = \varepsilon C^*$

and so

$$\int_\Omega |D\nabla v_\varepsilon(x)|\, dx \le C^*.$$

Thus, by Proposition 2.4 and Poincaré's inequality, we conclude that

$\{v_\varepsilon\}$ is a bounded sequence in $W^{1,3/2}(\Omega)$

and

$\{\nabla v_\varepsilon\}$ is a bounded sequence in $BV(\Omega)$.

Hence, by (2.3) there exist $u \in \bigcap W^{1,p}(\Omega)$ and a subsequence $\{u_\varepsilon\}$ such that

$u_\varepsilon \rightharpoonup u$ weakly in $W^{1,3/2}(\Omega)$

and

$\nabla u_\varepsilon \to \nabla u$ strongly in $L^1(\Omega)$.

By Fatou's Lemma it follows that

$$
\begin{aligned}
\int_\Omega W(\nabla u(x))\, dx &\leq \liminf_{\varepsilon \to 0} \int_\Omega W(\nabla u_\varepsilon(x))\, dx \\
&\leq \lim \varepsilon C^* \\
&= 0.
\end{aligned}
$$

Therefore

$\nabla u(x) \in \{A, B\}$ a. e. $x \in \Omega$

and so, by Theorem 2.14, we deduce that

$\nabla u(x) = B + \chi_E(x)\, a \otimes n$, where $E \in \mathcal{D}$.

Let $E' \in \mathcal{D}$ and, by Proposition 2.16, let $v \in W^{1,\infty}(\Omega; \mathbb{R}^3) \cap \mathcal{Q}_\Omega$ be such that

$\nabla v = B + \chi_{E'}\, a \otimes n.$

If $Per_\Omega(E') < +\infty$ then $v \in \mathcal{Q}_\varepsilon$ and so

$$
\begin{aligned}
E_\varepsilon(v) &= \varepsilon \int |D\nabla v(x)|\, dx \\
&\geq E_\varepsilon(u_\varepsilon) \\
&\geq \varepsilon \int_\Omega |D\nabla u_\varepsilon(x)|\, dx.
\end{aligned}
$$

Hence, by (2.2) we obtain

$$
\begin{aligned}
\int |D\nabla v(x)|\, dx &\geq \liminf \int |D\nabla u_\varepsilon(x)|\, dx \\
&\geq \int |D\nabla u(x)|\, dx,
\end{aligned}
$$

and so we conclude that

$$C\,Per_{\Omega}(E') \geq C\,Per_{\Omega}(E)$$

where

$$C = \sum_{i,j=1}^{3} |a_i|\,|b_j|.$$

4. DIRECT PENALIZATION OF THE INTERFACE. REMOVAL OF THE SPINODAL REGION.

Here, we adapt to our present setting a model proposed by GURTIN [8] for phase transitions in the case of a mixture of two fluids. Let D_1 and D_2 be closed, convex, bounded subsets of $\{F \in M^{3x3} | \det F \geq 0\}$, with $A \in \operatorname{int} D_1$, $B \in \operatorname{int} D_2$ and $D_1 \cap D_2 = \emptyset$. Assume that W satisfies (H1), (H2') and

(H3) (polyconvexity) there exist convex functions $G_i : M^{3x3} \times M^{3x3} \times (0, +\infty) \to [0, +\infty)$, i = 1, 2, such that

$$W(F) = G_i\,(F, \operatorname{adj} F, \det F) \text{ for all } F \in D_i, \det F > 0, i = 1, 2.$$

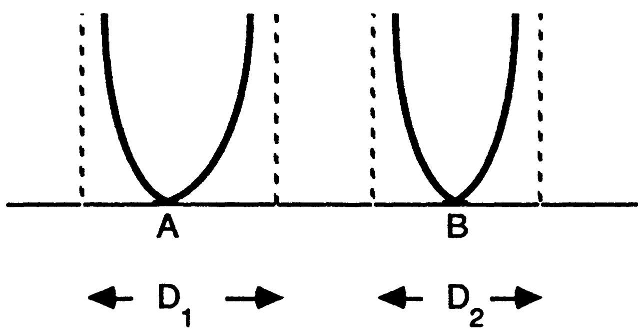

Fig. 3

Let

$$\mathcal{G}_2 := \left\{ u \in W^{1,1}(\Omega; \mathbb{R}^3) \mid \nabla u \in D_1 \cup D_2 \text{ a. e.}, \int_{\Omega} u(x)\,dx = m, \int_{\Omega} \nabla u(x)\,dx = \theta A + (1-\theta) B \right\},$$

and consider the problem:

(P_0) Minimize in $\mathcal{G}_2$

$$\int_{\Omega} W(\nabla u(x))\,dx.$$

If $u \in \mathcal{Q}_2$ we define

$$\Omega_1(u) := \{x \in \Omega | \nabla u(x) \in D_1\}$$

and

$$I(u) := \mathrm{Per}_\Omega(\Omega_1(u)).$$

We introduce the family of penalized problems:

(P_ε) Minimize in $\mathcal{Q}_2$

$$E_\varepsilon(u) := \int_\Omega W(\nabla u(x))\, dx + \varepsilon\, I(u).$$

Proposition 4.1.

For all $\varepsilon > 0$, (P_ε) admits a solution.

Proof. Let $\{u_j\}$ be a minimizing sequence. By Poincaré's inequality we have

$$\|u_j - m\|_{L^2(\Omega)} \le \text{Const.}\ \|\nabla u_j\|_{L^2(\Omega)} \le \text{Const.}\ \|\nabla u_j\|_\infty,$$

and so, as $\{\|\nabla u_j\|_\infty\}$ is a bounded sequence, there exist $u \in H^1(\Omega)$ and a subsequence $\{u_m\}$such that

$$u_m \to u \quad \text{weakly in } H^1(\Omega),$$

$$u_m \to u \quad \text{strongly in } L^2(\Omega)$$

and (see BALL [2])

$$(\nabla u_m, \operatorname{adj} \nabla u_m, \det \nabla u_m) \to (\nabla u, \operatorname{adj} \nabla u, \det \nabla u) \text{ in } L^\infty(\Omega) \text{ weak } *.$$

Moreover, with u^* as in Proposition 3.1, for m sufficiently large we have

$$\varepsilon I(u_m) \le E_\varepsilon(u^*) + 1 = \varepsilon \mathrm{Per}_\Omega(\{x \in \Omega \,|\, x.n \ge \alpha\}) + 1,$$

thus the sequence

$$\{\chi_{\Omega_1(u_m)}\} \quad \text{is bounded in } BV(\Omega).$$

By (2.2) and (2.3) there exists a subsequence $\{u_k\}$ and a subset ϖ of Ω such that

$$\chi_{\Omega_1(u_k)} \to \chi_\omega \quad \text{in } L^1(\Omega) \text{ strong}$$

and

$$\mathrm{Per}_\Omega(\omega) \le \liminf_{k \to \infty} I(u_k). \tag{4.2}$$

Hence, as D_1 is a closed convex set, and since

$$\chi_{\Omega_1(u_k)}\nabla u_k \in D_1 \text{ a. e.}$$

and

$$\chi_{\Omega_1(u_k)}\nabla u_k \to \chi_{\omega}\nabla u \text{ in } L^\infty(\Omega) \text{ weak } *,$$

it follows that

$$\chi_{\varpi} \nabla u \in D_1 \text{ a. e.}$$

and, in a similar way,

$$(1 - \chi_{\varpi}) \nabla u \in D_2 \text{ a. e.}$$

Therefore we conclude that

$$u \in \mathcal{Q}_2.$$

Let

$$g_1(F, H, \delta) := \begin{cases} G_1(F, H, \delta) & \text{if } (F, H, \delta) \in M^{3x3} \times M^{3x3} \times (0, +\infty) \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

As g_1 is a nonnegative, convex and lower semicontinuous function, there exists an increasing sequence of piecewise affine convex nonnegative functions $g_{1,n}$ such that

$$g_1 = \sup_n g_{1,n}.$$

Since

$$0 \le \chi_{\Omega_1(u_k)}\, g_{1,n}(\nabla u_k, \text{adj } \nabla u_k, \det \nabla u_k) \le C_n < +\infty \quad \text{a. e.}$$

there is a subsequence $\{u_{k,n}\}$ such that

$$\chi_{\Omega_1(u_{k,n})}\, g_{1,n}(\nabla u_{k,n}, \text{adj } \nabla u_{k,n}, \det \nabla u_{k,n}) \to h_n \text{ in } L^\infty(\Omega) \text{ weak } *$$

and so

$$\chi_{\omega}\, g_{1,n}(\nabla u_{k,n}, \text{adj } \nabla u_{k,n}, \det \nabla u_{k,n}) \to h_n \text{ in } L^\infty(\Omega) \text{ weak } *.$$

On the other hand, since

$$(\nabla u_{k,n}, \text{adj } \nabla u_{k,n}, \det \nabla u_{k,n}) \to (\nabla u, \text{adj } \nabla u, \det \nabla u) \text{ in } L^\infty(\Omega) \text{ weak } *$$

and $g_{1,n}$ is a convex nonnegative function, we have that, after extracting a subsequence of $\{u_{k,n}\}$,

$$g_{1,n}(\nabla u_{k,n}, \text{adj } \nabla u_{k,n}, \det \nabla u_{k,n}) \to L_n \ge g_{1,n}(\nabla u, \text{adj } \nabla u, \det \nabla u) \quad \text{in } L^\infty(\Omega) \text{ weak } *.$$

Therefore

$$h_n = \chi_{\varpi} L_n \ge \chi_{\varpi}\, g_{1,n}(\nabla u, \text{adj } \nabla u, \det \nabla u) \text{ a. e. in } \Omega.$$

We conclude that

$$\int_\Omega \chi_\omega\, g_{1,n}(\nabla u, \text{adj}\, \nabla u, \det \nabla u)\, dx \le \liminf \int_{\Omega_1(u_k)} W(\nabla u_k)\, dx$$

and so, by Lebesgue's monotone convergence theorem,

$$\int_\omega g_1(\nabla u, \text{adj}\, \nabla u, \det \nabla u)\, dx \le \liminf \int_{\Omega_1(u_k)} W(\nabla u_k)\, dx \tag{4.3}$$

and, in a similar way,

$$\int_{\Omega\setminus\omega} g_2(\nabla u, \text{adj}\, \nabla u, \det \nabla u)\, dx \le \liminf \int_{\Omega\setminus\Omega_1(u_k)} W(\nabla u_k)\, dx. \tag{4.4}$$

Therefore, $\det \nabla u > 0$ a. e. in Ω and (4.2), (4.3) and (4.4) yield

$$E_\varepsilon(u) \le \liminf E_\varepsilon(u_k).$$

Theorem 4.5.

Let v_ε be a sequence of minimizers of E_ε in $\mathcal{Q}_2$. There exist a solution u of (P_0) and a subsequence u_ε such that

$$u_\varepsilon \to u \text{ in } W^{1,\infty}(\Omega; \mathbb{R}^3) \text{ weak } *,$$

and

$$\nabla u = B + \chi_E\, a \otimes n \text{ a. e. in } \Omega,$$

where $E \in \mathcal{S}$ is a solution of (P^*_0).

Proof. Let u* be as in Remark 2.13. Since $\nabla v_\varepsilon \in D_1 \cup D_2$ a. e. and

$$\varepsilon\, I(v_\varepsilon) \le E_\varepsilon(v_\varepsilon) \le E_\varepsilon(u^*) = \varepsilon \text{ Const.}, \tag{4.6}$$

as in the proof of Proposition 4.1 we deduce that there is a function $u \in H^1(\Omega)$ a subsequence $\{u_\varepsilon\}$ such that

$$u_\varepsilon \to u \quad \text{weakly in } H^1(\Omega),$$

$$u_\varepsilon \to u \quad \text{strongly in } L^2(\Omega),$$

$$\nabla u_\varepsilon \to \nabla u \quad \text{weakly * in } L^\infty(\Omega),$$

$$\chi_{\Omega_1(u_\varepsilon)} \to \chi_\omega \quad \text{in } L^1(\Omega) \text{ strong},$$

$$\text{Per}_\Omega(\omega) \le \liminf I(u_\varepsilon) \tag{4.7}$$

and

$$\int_\Omega W(\nabla u(x))\,dx \le \liminf \int_\Omega W(\nabla u_\varepsilon(x))\,dx, \tag{4.8}$$

where

$u \in \mathcal{A}_2$, $\chi_{\varpi} \nabla u \in D_1$ a. e. and $(1 - \chi_{\varpi}) \nabla u \in D_2$ a. e.

By (4.6) and (4.8) we have

$W(\nabla u(x)) = 0$ a. e. in Ω

and so

$\nabla u(x) = B + \chi_{\varpi}(x)\ a \otimes n$ a. e. $x \in \Omega$

and, by Theorem 2.14, $\varpi \in \mathcal{S}$. Let $E' \in \mathcal{S}$ and, by Proposition 2.16, let $v \in W^{1,\infty}(\Omega; \mathbb{R}^3) \cap \mathcal{A}_0$ be such that

$\nabla v = B + \chi_{E'}\, a \otimes n.$

If $Per_\Omega(E') < +\infty$ then $v \in \mathcal{A}_2$ and

$E_\varepsilon(v) = \varepsilon\, Per_\Omega(E') \ge E_\varepsilon(u_\varepsilon) \ge \varepsilon\, I(u_\varepsilon).$

Therefore, by (4.7) we conclude that

$Per_\Omega(E') \ge \liminf I(u_\varepsilon) \ge Per_\Omega(\omega).$

5. REGULARIZATION AND DIRECT PENALIZATION OF THE INTERFACES.

Suppose that W satisfies the hypotheses (H1) and (H2) and consider (P_0) and (P_ε) as in Section 3. In a similar way, we obtain existence of solutions v_ε of the problem (P_ε), with

$u_\varepsilon \to u$ in $W^{1,3/2}(\Omega; \mathbb{R}^3)$ weak,

where $\{u_\varepsilon\}$is a subsequence of $\{v_\varepsilon\}$ and u is a solution (P_0). Thus, $u \in \mathcal{A}_1$, $\nabla u \in BV(\Omega)$ and, by (H2),

$\nabla u(x) = R(x)\,(B + \chi_E(x)\, a \otimes n)$ a. e. in Ω, where $R(x) \in O^+(\mathbb{R}^3)$.

We search for a model implying that

$$\nabla u \in \{A, B\} \text{ a. e. in } \Omega \text{ and } E \text{ is a solution of } (P_0{}^*). \tag{5.1}$$

We show that (5.1) is valid if E is a sufficiently smmoth set. Assume that E determines a partition of Ω into countably many strongly Lipschitz connected subdomains Ω_i. Then

$$\nabla u = \sum_i R_i (B + \chi_{\Omega_i \cap E}\, a \otimes n), \text{ with } R_i \text{ constant rotations.}$$

Here, we used the following result (see RESHETNYAK [14] and Appendix Proposition A.1).

Proposition 5.2.

Let Ω be an open bounded connected domain of $\mathbb{R}^n$ and let $u \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ be such that $\nabla u \in O^+(\mathbb{R}^n)$ a. e. in Ω. Then u is an affine function.

By Theorem 2.14, if Ω_i and Ω_j are such that $\mathrm{int}\,(\Omega_i \cup \Omega_j \cup (\partial \Omega_i \cap \partial \Omega_j))$ is a connected set, then

$$\nabla u = \chi_{\Omega i} R_i B + \chi_{\Omega j} R_j (B + a \otimes n) \text{ in } \Omega_i \cup \Omega_j$$

with the interface between Ω_i and Ω_j planar and

$$R_j (B + a \otimes n) = R_i B + b \otimes m, \tag{5.3}$$

for some $b \in \mathbb{R}^3$, $m \in \mathbb{R}^3$, $||m|| = 1$.

JAMES [10] studied the condition (5.3) in the context of elastic crystals. As a consequence of his analysis, we have the following result (see Appendix Lemma A.5)

Lemma 5.4 (JAMES [10])

Given a and $n \in \mathbb{R}^3$, $a.n > -1$, $a \neq 0$ and $||n|| = 1$, let $R \in O^+(\mathbb{R}^3)$, $b \in \mathbb{R}^3$, $m \in \mathbb{R}^3$, $||m|| = 1$, satisfy the equation

$$R(\mathbb{1} + a \otimes n) = \mathbb{1} + b \otimes m.$$

Then

(i) $R = \mathbb{1}$ and m is parallel to n if a is parallel to n;

(ii) If a is not parallel to n, either $R = \mathbb{1}$ and m is parallel to n, or $R^2 \neq \mathbb{1}$ and $R = (\mathbb{1} + b \otimes m)(\mathbb{1} + a \otimes n)^{-1}$ where $b \otimes m$ is uniquely defined, $b \in \mathrm{Span}\,\{a, n\}$ and

$$m = \pm \frac{2a + ||a||^2 n}{||2a + ||a||^2 n||}.$$

Clearly, (5.3) is equivalent to

$R_i^T R_j($ H + a ®B"^{T}n) = H + R_i^T b ® B~^{T}m.

First case: a is parallel to B~^{T}n.

By Lemma 5.4 we conclude that m is parallel to n and Rj = Rj. Thus, there is a fixed rotation R such that

Vu(x) = R (B + %$_E$(x) a ®n) a. e. in 12,

and so, by Theorem 2.14 and due to the constraint

f Vu(x)dx = eA + (l-e)B,

we conclude that E **e** *J&* and R = 1. Hence

Vu € {A, B} a. e. in *Q*,

and (see Theorem 3.2) E is a solution of (P_o*).

Second case: a is not parallel to B^{-}n.

By Lemma 5.4, either Rj = R_j and m is parallel to n, or

VRiQ,

where

Q := (1 + b^f <*8*> B~^{T}m) (11 + a ^ B ^ n) - ^{1}

with b^f € Span {a, B~^{T}n} uniquely defined and

$$m = \frac{2B^T a - f»a|l^2 n}{||2B^T a + ||a||^2 n||}.$$

Hence, locally eitherE is layered normally to n or E is layered normally to m* In the next lemma we prove that the boundary of E in *Cl* cannot have "corners".

Lemma 5.5.

dEnQ = $_m$u$_{i=1}^{co}$ *Ki* u *K*,

where, for some p_i, x_i **e** IR, TC_i is a connected component of *il* n {x.n = p_i}, *%* is a connected

component of Q n {x.m = x_i} and H_x($ic._i$n TC^) = 0.

Proof. Suppose that there exist XQ **e** ft, e > 0, p and x such that either

(i) $B(x_0, \varepsilon) \cap E = \{x \in B(x_0, \varepsilon) | (x - x_0).n < \rho \text{ and } (x - x_0).m < \tau\}$ (see Fig. 4)

or

(ii) $B(x_0, \varepsilon) \cap E = \{x \in B(x_0, \varepsilon) | [(x - x_0).n - \rho] . [(x - x_0).m - \tau] > 0\}$ (see Fig. 5).

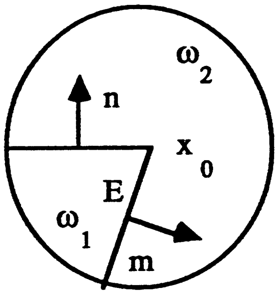

Fig. 4

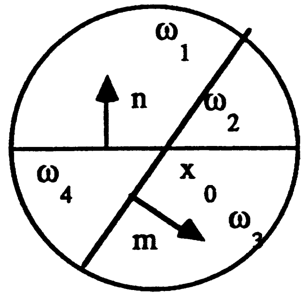

Fig. 5

In case (i) we have, using the notation of Fig. 4,

$$\nabla u = \begin{cases} R_1(B + a\otimes n) & \text{in } \omega_1 \\ R_2 B & \text{in } \omega_2 \end{cases}$$

and so, there exist $b, b' \in \mathbb{R}^3$ such that

$$R_2 + b\otimes B^{-T}n = R_1(\mathbb{1} + a\otimes B^{-T}n) \tag{5.6$_1$}$$

and

$$R_2 + b'\otimes B^{-T}m = R_1(\mathbb{1} + a\otimes B^{-T}n) \tag{5.6$_2$}$$

Since two rotations cannot differ by a nonzero rank one matrix, $(5.6)_1$ implies that $R_1 = R_2$ and so by $(5.6)_2$ we have that m is parallel to n, in which case there are no corners. If (ii) holds then

$$\nabla u = \begin{cases} R_1 B & \text{in } \omega_1 \\ R_2(B + a\otimes n) & \text{in } \omega_2 \\ R_3 B & \text{in } \omega_3 \\ R_4(B + a\otimes n) & \text{in } \omega_4. \end{cases}$$

The necessary compatibility conditions imply that there exist $b, b', c, c' \in \mathbb{R}^3$ such that

$$R_1 + b\otimes B^{-T}m = R_2(\mathbb{1} + a\otimes B^{-T}n), \tag{5.7$_1$}$$

$$R_3 + c\otimes B^{-T}n = R_2(\mathbb{1} + a\otimes B^{-T}n), \tag{5.7$_2$}$$

$$R_3 + b'\otimes B^{-T}m = R_4(\mathbb{1} + a\otimes B^{-T}n) \tag{5.7$_3$}$$

and

$$R_1 + c'\otimes B^{-T}n = R_4(\mathbb{1} + a\otimes B^{-T}n). \tag{5.7$_4$}$$

From $(5.7)_2$ and $(5.7)_4$ we have

$R_2 = R_3$ and $R_1 = R_4$,

which, together with $(5.7)_1$ and $(5.7)_3$ and Lemma 5.4, yield

$$R_1^T R_2 \text{ and } R_2^T R_1 \in \{\mathbb{1}, Q\}.$$

If m is not parallel to n then $R_1 \neq R_2$ and so

$$R_1^T R_2 = R_2^T R_1 = Q$$

which implies that

$$Q^2 = \mathbb{1},$$

contradicting Lemma 5.4 (ii). We deduce that, as in case (i), m and n are parallel and $\partial E \cap \Omega$ does not have any "corners".

We conclude the proof of the second case (a is not parallel to $B^{-T}n$). By Lemma 5.5 we deduce that there is a constant rotation R such that

$$\nabla u \in \{RQ^k B, RQ^k(B + a\otimes n), RQ^{kT}B, RQ^{kT}(B + a\otimes n) \mid k \in \mathbb{N}_0\},$$

with

$$\sum_{k=0}^{\infty} \alpha_k RQ^k B + \beta_k RQ^k(B + a\otimes n) + \gamma_k RQ^{kT}B + \eta_k RQ^{kT}(B + a\otimes n) = B + \theta a\otimes n,$$

where

$$\alpha_k, \beta_k, \gamma_k, \eta_k \geq 0, \sum_{k=0}^{\infty}(\alpha_k + \beta_k + \gamma_k + \eta_k) = 1,$$

$$\alpha_i + \beta_i = 0 \Rightarrow \alpha_j + \beta_j = 0 \text{ for all } j \geq i$$

and

$$\gamma_i + \eta_i = 0 \Rightarrow \gamma_j + \eta_j = 0 \text{ for all } j \geq i.$$

Suppose that $\alpha_1 + \beta_1 > 0$ and let $x.n = 0$, $x \neq 0$. Then

$$\|Bx\| = \|\sum_{k=0}^{\infty} (\alpha_k + \beta_k) Q^k Bx + (\gamma_k + \eta_k) Q^{kT}Bx\| \leq \sum_{k=0}^{\infty} (\alpha_k + \beta_k + \gamma_k + \eta_k) \|Bx\| = \|Bx\|$$

and so, as $\alpha_0 + \beta_0$, $\alpha_1 + \beta_1 > 0$, we have

$$QBx = Bx \text{ for all } x \text{ such that } x.n = 0.$$

As $\det Q = 1$ we conclude that $Q = \mathbb{1}$ which contradicts Lemma 5.4. Therefore we have

$$\alpha_i + \beta_i = 0 \text{ for all } i \geq 1$$

and, in a similar way,

$\gamma_i + \eta_i = 0$ for all $i \geq 1$.

Finally, we conclude that

$$\nabla u(x) = R\,(B + \chi_E(x)\, a \otimes n) \text{ a. e. in } \Omega,$$

reducing this case to the first case.

The result that we proved can be stated as follows.

Theorem 5.8.

Let $u \in \mathcal{Q}_1$ be such that

$$\nabla u(x) = R(x)\,(B + \chi_E(x)\, a \otimes n) \text{ a. e. in } \Omega, \text{ where } R(x) \in O^+(\mathbb{R}^3).$$

If the set E determines a partition of Ω into countably many strongly Lipschitz connected subdomains then

$\nabla u \in \{A, B\}$ a. e. in Ω and $E \in \mathcal{S}$ is a solution of $(P_0{}^*)$.

Next, we try the same approach as in Section 4, with the hypothesis (H2) replaced by

(H2") if $F \in D_1$ then $W(F) = 0$ if and only if $F \in \{RA \mid R \in O^+(\mathbb{R}^3)\} \cap D_1$,

if $F \in D_2$ then $W(F) = 0$ if and only if $F \in \{RB \mid R \in O^+(\mathbb{R}^3)\} \cap D_2$.

As in Theorem 4.5 we obtain the existence of a subsequence $\{u_\varepsilon\}$ of solutions of (P_ε) such that

$$u_\varepsilon \to u \text{ in } W^{1,\infty}(\Omega; \mathbb{R}^3) \text{ weak } *$$

and

$$I(u) \leq \liminf_{\varepsilon \to 0^+} I(u_\varepsilon).$$

where u is a solution (P_0). By (H2") we have

$$\nabla u(x) = R(x)\,(B + \chi_E(x)\, a \otimes n) \text{ a. e. in } \Omega, \text{ where } R(x) \in O^+(\mathbb{R}^3) \text{ and } \chi_E \in BV(\Omega).$$

In order to be able to conclude (5.1) using Theorem 5.8, we neeed to obtain more information regarding the set E. As we will see, this is possible if we consider a model that combines the approaches undertaken in Sections 3 and 4.

Let D_1 and D_2 be closed subsets of $\{F \in M^{3x3} \mid \det F \geq 0\}$, with $A \in \operatorname{int} D_1$, $B \in \operatorname{int} D_2$ and

$D_1 \cap D_2 = \emptyset$. Assume that W is continuous and satisfies (H1) and (H2"). We define the class of admissible deformations

$$\mathcal{A} := \left\{ u \in W^{1,1}(\Omega; \mathbb{R}^3) \mid \nabla u \in D_1 \cup D_2 \text{ a. e.}, \nabla u \in BV(\Omega), \int_\Omega u \, dx = m, \int_\Omega \nabla u \, dx = B + \theta a \otimes n \right\}.$$

Consider the approximating problems

(P_ε) minimize in $\mathcal{A}$

$$E_\varepsilon(u) := \int_\Omega W(\nabla u(x)) \, dx + \varepsilon \int_\Omega |D\nabla u(x)| \, dx + \varepsilon \, I(u).$$

Proposition 5.9.

(P_ε) admits a solution.

Proof. Let $\{u_j\}$ be a minimizing sequence such that

$$E_\varepsilon(u_j) \le E_\varepsilon(u^*) + 1 = \varepsilon C^* + 1 \quad \text{for all } j,$$

where u^* is the particular deformation introduced in Remark 2.13. As

$$\int_\Omega |D\nabla u_j| \, dx \quad \text{and } Per_\Omega(\Omega_1(u_j)) \quad \text{are bounded}$$

by Proposition 2.4 we have

$\{u_j\}$ is bounded in $W^{1,3/2}$,

$\{\nabla u_j\}$ is bounded in $BV(\Omega)$

and

$\{\chi_{\Omega_1(u_j)}\}$ is bounded in $BV(\Omega)$.

Hence, there is a subsequence $\{u_m\}$ and there exists $u \in W^{1,3/2}$ such that

$u_m \to u$ weakly in $W^{1,3/2}$,

$u_m \to u$ strongly in $L^{3/2}$,

$\nabla u_m \to \nabla u$ strongly in L^1, $\nabla u \in BV(\Omega)$

and

$\chi_{\Omega_1(u_m)} \to \chi_\varpi$ strongly in L^1, with $\chi_\varpi \in BV(\Omega)$.

Therefore, and since

$$\int_\Omega |\chi_{\Omega_1(u_m)} \nabla u_m - \chi_\omega \nabla u| \, dx \le \int_\Omega \chi_\omega |\nabla u_m - \nabla u| \, dx + \int_\Omega |\chi_{\Omega_1(u_m)} - \chi_\omega| \, |\nabla u_m| \, dx$$

$$\leq \int_\Omega |\nabla u_m - \nabla u|\, dx + \left(\int_\Omega |\chi_{\Omega_1(u_m)} - \chi_\omega|\, dx\right)^{1/3} \left(\int_\Omega |\nabla u_m|^{3/2}\, dx\right)^{2/3},$$

it follows that

$$\chi_{\Omega_1(u_m)} \nabla u_j \to \chi_\varpi \nabla u \text{ strongly in } L^1$$

which implies that

$$\chi_\omega \nabla u \in D_1 \text{ a. e.}$$

and, in a similar way,

$$(1 - \chi_\omega)\nabla u \in D_2 \text{ a. e.}$$

We conclude that $u \in \mathcal{Q}$ and, by (2.2) and Fatou's Lemma,
$E_\varepsilon(u) \leq \liminf E_\varepsilon(u_m)$.

Theorem 5.10.

Let v_ε be a sequence of minimizers of E_ε in $\mathcal{Q}$. There is a subsequence u_ε converging weakly in $W^{1,3/2}(\Omega; \mathbb{R}^3)$ to a solution u of (P_0) such that

$$\nabla u(x) = R(x)\,(B + \chi_E(x)\, a \otimes n) \text{ a. e. in } \Omega.$$

Moreover, $u \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ and $\nabla u, R, \chi_E \in BV(\Omega)$.

Proof. Since

$$E_\varepsilon(v_\varepsilon) \leq E_\varepsilon(u^*) = \varepsilon C^*,$$

as in the proof of Proposition 5.9 we can extract a subsequence $\{u_\varepsilon\}$ and there exists $u \in \mathcal{Q}$ such that

$$u_\varepsilon \to u \text{ weakly in } W^{1,3/2},$$

$$u_\varepsilon \to u \text{ strongly in } L^{3/2},$$

$$\nabla u_\varepsilon \to \nabla u \text{ strongly in } L^1,\ \nabla u \in BV(\Omega),$$
$$\chi_{\Omega_1(u_\varepsilon)} \to \chi_E \text{ strongly in } L^1, \text{ with } \chi_E \in BV(\Omega)$$

$$\chi_E \nabla u \in D_1 \text{ a. e. and } (1 - \chi_E)\nabla u \in D_2 \text{ a. e.}$$

Since, by Fatou's Lemma,

$$\int_\Omega W(\nabla u(x))\,dx \le \liminf \int_\Omega W(\nabla u_\varepsilon(x))\,dx = 0,$$

we deduce that

$$\nabla u(x) = R(x)\,(B + \chi_E(x)\, a \otimes n) \quad \text{a. e. in } \Omega.$$

Moreover, as

$$\nabla u \text{ and } \chi_E \in L^\infty \cap BV(\Omega),$$

by Proposition 2.6 we conclude that

$$R \in L^\infty \cap BV(\Omega).$$

Note that, since Ω is a strongly Lipschitz domain and as $Per_\Omega(E) < +\infty$, E is a set of finite perimeter in $\mathbb{R}^3$. Also, by Theorem 2.10, to each entrie R_{ij} of the rotation R it correspond a set J_{ij} and functions $\lambda_{ij}, \mu_{ij}, \nu_{ij}$ with $\| \nu_{ij} \| = 1$, for which (2.11) and (2.12) hold.

Proposition 5.11.
For H_2 a. e. $x \in (\partial^* E \setminus \bigcup_{i,j=1}^{3} J_{ij}) \cap \Omega$, $\nu_E(x)$ is parallel to n.

Proof. By (2.11), for H_2 a. e. $x_0 \in (\partial^* E \setminus \bigcup_{i,j=1}^{3} J_{ij}) \cap \Omega$ we have

$$\lim_{\varepsilon \to 0} \frac{1}{\text{meas}(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |R(x) - F(x_0)|^{3/2}\,dx = 0 \tag{5.12}$$

where $F(x_0)$ is the matrix with entries

$$F(x_0)_{ij} = \frac{\lambda_{ij} + \mu_{ij}}{2}.$$

For $\varepsilon > 0$ sufficiently small, define the difference quotient

$$v_\varepsilon(x) := \frac{u(x_0 + \varepsilon(x - x_0)) - u(x_0)}{\varepsilon} \quad \text{for } y \in B(x_0, 1).$$

As

$$\nabla v_\varepsilon(x) = R(x_0 + \varepsilon(x - x_0))\,(B + \chi_{x_0 + \frac{E - x_0}{\varepsilon}}(x)\, a \otimes n),$$

it follows that $\{v_\varepsilon\}$ is bounded in $W^{1,\infty}(B(x_0, 1))$, and so it admits a subsequence converging in $W^{1,\infty}(B(x_0, 1))$ weak * to a function v. By (5.12) and Theorem 2.8, we have

$$\nabla v(x) = F(x_0)\,(B + \chi_{H^-(x_0)}(x) a \otimes n) \tag{5.13}$$

where

$H^-(x_0) = \{x \in IR^3 | v_E(x_0)\cdot(x - x_0) < 0\}$.

We claim that

$$F(x_0) \text{ is a rotation.} \tag{5.14}$$

Indeed, by (5.12)

$$0 = \lim \int |R(x_0+\varepsilon(x-x_0)) - F(x_0)| dx$$

$$= \lim \int |1 - R(x_0 + \varepsilon(x - x_0))^T F(x_0)| dx$$

and so

$$\left(\lim \int R(x_0+\varepsilon(x - x_0))\, dx \right)^T F(x_0) = \text{meas}(B(x_0, 1)) 1$$

which implies that

$$F^T(x_0)F(x_0)=1.$$

Finally, by (5.13) and (5.14) we conclude that n is parallel to $v_E(x_0)$.

Proposition 5.15.

For H^ a. e. x e 3*E n J^, $v_{ij}(x)$ is parallel to $v_E(x)$.

Proof. By (2.12), for H^ a. e. $x_0 \in$ 9*E n $J_{\{-j}$ and for all k, 1 e {1, 2, 3} we have

$R_{kl}(x_0 + \varepsilon(x - x_0))$ -> X H ^ <x> Mki^) + XiQUo) W ^ICXQ) strongly in L^1

where

$$H^+_{kl}(x_0) = \{x \in IR^3 | v_{kl}(x_0).(x - x_0) > 0\}$$

and

$$H^-_{kl}(x_0) = \{x \in IR^3 | v_{kl}(x_0)\cdot(x - x_0) < 0\}.$$

Suppose that $v_{ij}(x_0)$ is not parallel to $v_E(x_0)$. As $B(x_0, 1) \cap H^+(x_0)$ is sliced by the planes through x_0 with normal $v_{kl}(x_0)$, let CD be the union of the two adjacent slices $G>^+$ and CD' with common boundary the portion of the plane with normal $v_{ij}(x_0)$. For e > 0 sufficiently small consider the difference quotient

$$v_\varepsilon(x) := \frac{u(x_0+\varepsilon(x-x_0))-u(x_0)}{\varepsilon} \text{ for } x \in B(x_0, 1).$$

As in the proof of Proposition 5.11, $\{v_\varepsilon\}$admits a subsequence converging in $W^{1,\infty}(B(x_0, 1))$ weak * to a function v, where, by Theorem 2.8 and (2.12),

$$\nabla v(x) = S(x)\,(B + \chi_{H^-(x_0)}(x) a\otimes n) \text{ in } B(x_0, 1)$$

and

$$S_{kl}(x) = \chi_{H^+_{kl}(x_0)}(x)\, \mu_{kl}(x_0) + \chi_{H^-_{kl}(x_0)}(x)\, \lambda_{kl}(x_0).$$

Therefore

$$\nabla v(x) = \begin{cases} S^+B & \text{in } \omega^+ \\ S^-B & \text{in } \omega^- \end{cases} \tag{5.16}$$

where

$$S^+_{kl} = \mu_{kl}(x_0), \quad S^-_{kl} = \lambda_{kl}(x_0) \text{ and } S^-_{ij} < S^+_{ij}. \tag{5.17}$$

We claim that

$$S^+ \text{ and } S^- \text{ are rotations.} \tag{5.18}$$

Indeed

$$0 = \lim_{\varepsilon \to 0} \int_{B(x_0,1)\cap\omega^+} |R(x_0 + \varepsilon(x - x_0)) - S^+|\, dx$$

$$= \lim_{\varepsilon \to 0} \int_{B(x_0,1)\cap\omega^+} |\mathbb{1} - R^T(x_0 + \varepsilon(x - x_0))\, S^+|\, dx$$

and so

$$\left(\lim_{\varepsilon \to 0} \int_{B(x_0,1)\cap\omega^+} R(x_0+ \varepsilon(x - x_0))\, dx \right)^T S^+ = \text{meas}(B(x_0, 1\cap\varpi^+)\mathbb{1}$$

i. e.

$$(S^+)^T S^+ = \mathbb{1} \text{ and, in a similar way } (S^-)^T S^- = \mathbb{1}.$$

Finally, by (5.16) and (5.17) we conclude that S^+ and S^- differ by a nonzero rank one matrix which contradicts (5.18).

Proposition 5.19.

(i) If a is parallel to $B^{-T}n$ then $v_E(x)$ is parallel to n for H_2 a. e. $x \in \partial^*E \cap \Omega$.

(ii) If a is not parallel to $B^{-T}n$, then $v_E(x)$ is either parallel to n or to m for H_2 a. e. $x \in \partial^*E \cap \Omega$.

Proof. By Proposition 5.11, for H_2 a. e. $x_0 \in (\partial^*E \setminus \bigcup_{i,j=1}^{3} J_{ij}) \cap \Omega$ we have $v_E(x_0)$ parallel to n. On the other hand, by Proposition 5.15 and (5.18), for H_2 a. e. $x_0 \in \partial^*E \cap (\bigcup_{i,j=1}^{3} J_{ij})$ there exist rotations S^+ and S^- such that $S^+ \neq S^-$ and

$$R(x_0 + \varepsilon(x - x_0)) \to S^+ \text{ strongly in } L^1(B(x_0, 1) \cap H^+(x_0))$$

and

$$R(x_0 + \varepsilon(x - x_0)) \to S^- \text{ strongly in } L^1(B(x_0, 1) \cap H^-(x_0)).$$

Therefore, setting

$$v_\varepsilon(x) := \frac{u(x_0 + \varepsilon(x - x_0)) - u(x_0)}{\varepsilon} \quad \text{for } x \in B(x_0, 1),$$

$\{v_\varepsilon\}$admits a subsequence converging in $W^{1,\infty}(B(x_0, 1))$ weak * to a function v such that

$$\nabla v(x) = \begin{cases} S^+B & \text{if } v_E(x_0).\ (x - x_0) > 0 \\ \\ S^-(B + a \otimes n) & \text{if } v_E(x_0).\ (x - x_0) > 0. \end{cases}$$

Thus, there exists $b \in \mathbb{R}^3$ such that

$$S^-(\mathbb{1} + a \otimes B^{-T} n) = S^+ + b \otimes B^{-T} v_E(x_0)$$

and so, by Lemma 5.4. either a is parallel to $B^{-T}n$ and then $v_E(x)$ is parallel to n for H_2 a. e. $x \in \partial^* E \cap \Omega$ and $S^+ = S^-$ (i. e. H_2 a. e. $x \in \partial^* E$ is a point of approximate continuity of R) or a is not parallel to $B^{-T}n$ and then $v_E(x)$ is either parallel to n or to m for H_2 a. e. $x \in \partial^* E \cap \Omega$.

Corollary 5.20.
If a is parallel to $B^{-T}n$, then

$$\nabla u = B + \chi_E\, a \otimes n \quad \text{a. e. in } \Omega,$$

where $E \in \mathcal{S}$ is a solution of $(P_0{}^*)$.

Proof. By Proposition 5.19, $v_E(x)$ is parallel to n for H_2 a. e. $x \in \partial^* E \cap \Omega$. We can suppose, without loss of generality, that $n = e_3$. Hence, by the Generalized Gauss-Green Theorem 2.9,

$$\frac{\partial \chi_E}{\partial x_1} = 0 = \frac{\partial \chi_E}{\partial x_2} \quad \text{in } \mathcal{D}'(\Omega)$$

and so,

$$\frac{\partial}{\partial x_j}(\delta_{ki} + \chi_E\, a_k n_i) = \frac{\partial}{\partial x_i}(\delta_{kj} + \chi_E\, a_k n_j) \quad \text{for all } k, i, j \in \{1, 2, 3\}.$$

Therefore, there exists $v \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ such that

$$\nabla v = \mathbb{1} + \chi_E\, a \otimes n \quad \text{a. e. in } \Omega$$

and, by Theorem 2.14, E is layered normally to n. As $Per_{\Omega}(E) < +\infty$ and Ω is a strongly Lipschitz domain, if we set

$$\Omega_k := \{x \in \Omega \mid dist(x, \partial\Omega) > 1/k\},$$

then $E \cap \Omega_k$ is formed by finitely many slices normal to n. By Proposition 5.2 in each one of these slices either

$$\nabla u = RB$$

or

$$\nabla u = R(B + a \otimes n),$$

with R a constant rotation. Due to the necessary jump condition of ∇u across an interface, for any two adjoint slices we have

$$R(B + a \otimes n) = R'B + b \otimes n$$

for some $b \in \mathbb{R}^3$. As a is parallel to $B^{-T}n$, from Lemma 5.4 we obtain

$$R = R'.$$

Using induction in k, with $k \to +\infty$, we obtain

$$\nabla u(x) = R\,(B + \chi_E(x)\, a \otimes n) \text{ a. e. in } \Omega,$$

for some fixed rotation R, and so (see proof of Theorem 5.8, first case) we conclude that $E \in \mathcal{S}$ is a solution of $(P_0{}^*)$ and $R = \mathbb{1}$.

Final comments. We conjecture that the solutions obtained in Theorem 5.10 satisfy (5.1). However, we were able to confirm the conjecture only in the case where a is parallel to $B^{-T}n$ (see Corollary 5.20). If a is not parallel to $B^{-T}n$, by Theorem 5.8 the conjecture remains valid if E is sufficiently smmoth. By Lemma 5.5, Theorem 5.10 and Proposition 5.19 we know that the set E has finite perimeter, the direction of the normal to ∂^*E is either n or m, and, due to kinematic compatibility conditions, there cannot be either "corners" or "intersections" (see Fig. 3 and 4). Is it possible to infer that E is under the hypothesis of Theorem 5.8, namely that E determines a partition of Ω into countably many strongly Lipschitz connected domains?

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APPENDIX.

For completeness, in this appendix we prove Proposition 5.2 (see also RESHETNYAK [14]) and Lemma 5.4 (see JAMES [10]).

Proposition A.1.

Let Ω be an open bounded connected domain of $\mathbb{R}^n$ and let $u \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ be such that $\nabla u \in O^+(\mathbb{R}^n)$ a. e. in Ω. Then u is an affine function.

Proof. Let

$$R := \nabla u \in O^+(\mathbb{R}^n) \text{ a. e. in } \Omega \text{ and } w_i := \nabla u_i,\ i = 1, ..., n.$$

Since

$$R = \operatorname{adj} R$$

we have

$$\begin{aligned} \Delta u_i &= \operatorname{div} w_i \\ &= (\operatorname{adj} \nabla u)_{ij,j} \\ &= 0 \quad \text{in } \mathcal{D}'(\Omega). \end{aligned}$$

Therefore $u \in C^\infty(\Omega; \mathbb{R}^n)$ and so

$$R \in C^\infty(\Omega; \mathbb{R}^n). \tag{A.2}$$

Set

$$C^k := R^T \frac{\partial R}{\partial x_k} \quad \text{for } k = 1, ..., n.$$

As

$$R^T R = \mathbb{1},$$

by (A.2) we have

$$C^{k^T} + C^k = 0. \tag{A.3}$$

Since

$$\frac{\partial^2 u_i}{\partial x_j\, \partial x_k} = \frac{\partial^2 u_i}{\partial x_k\, \partial x_j},$$

we obtain

$$R_{i,jk} = R_{i,kj}$$

i. e.

$$R_{im}\, C^k_{mj} = R_{is}\, C^j_{sk}, \quad \text{for all } i, j, k \in \{1, ..., n\}$$

and so

$$C^k_{mj} = C^j_{mk}. \tag{A.4}$$

Finally, (A.3) and (A.4) yield

$$C^k_{mj} = C^j_{mk} = -C^j_{km} = -C^m_{kj} = C^m_{jk} = C^k_{jm} = -C^k_{mj}$$

therefore

$$C^k = 0 \quad \text{for all } k \in \{1, ..., n\}$$

and we conclude that

R(x) is constant in Ω.

Lemma A.5 (JAMES [10])

Given a and n $\in \mathbb{R}^3$, a.n > -1, a $\neq$ 0 and $||n|| = 1$, let R $\in O^+(\mathbb{R}^3)$, b $\in \mathbb{R}^3$, m $\in \mathbb{R}^3$, $||m|| = 1$, satisfy the equation

$$R(\mathbb{1} + a \otimes n) = \mathbb{1} + b \otimes m.$$

Then

(i) $R = \mathbb{1}$ and m is parallel to n if a is parallel to n;

(ii) If a is not parallel to n, either $R = \mathbb{1}$ and m is parallel to n, or $R^2 \neq \mathbb{1}$ and $R = (\mathbb{1} + b \otimes m)(\mathbb{1} + a \otimes n)^{-1}$ where $b \otimes m$ is uniquely defined, b $\in$ Span {a, n} and

$$m = \pm \frac{2a + ||a||^2 n}{||2a + ||a||^2 n||}.$$

Proof. Suppose that

$$R(\mathbb{1} + a \otimes n) = \mathbb{1} + b \otimes m. \tag{A.6}$$

(i) If $a = \alpha n$ for some $\alpha \in \mathbb{R}$, by (A.6) we have

$$(2\alpha + \alpha^2)\, n \otimes n = b \otimes m + m \otimes b + ||b||^2 m \otimes m.$$

Therefore

m is parallel to n

and since a rotation cannot differ from the identity by a nonzero rank one matrix, we conclude that

$$R = \mathbb{1}.$$

(ii) Assume that $R \neq \mathbb{1}$. Then

m is not parallel to n and Ra is not parallel to b, (A.7)

and so, if e is a unit vector on the axis of rotation of R, i. e.

$$Re = e = R^T e,$$

we have

$$Ra(n.e) - b(m.e) = 0$$

and

$$n(a.e) - m(b.e) = 0,$$

which, together with (A.7) imply

$$m.e = 0, n.e = 0, a.e = 0 \text{ and } b.e = 0. \tag{A.8}$$

By part (i), a is not parallel to n and b is not parallel to m, thus we may set

$$e := \frac{a \wedge n}{\|a \wedge n\|}. \tag{A.9}$$

Since

$$a \otimes n + n \otimes a + \|a\|^2 n \otimes n = b \otimes m + m \otimes b + \|b\|^2 m \otimes m,$$

we obtain

$$a + (a.n)\, n + \|a\|^2 n = b(m.n) + m(b.n) + \|b\|^2 (m.n)m$$

and

$$n(a.\, n \wedge e) = b\,(m.\, n \wedge e) + m\,(b.\, n \wedge e) + \|b\|^2 m\,(m.\, n \wedge e).$$

Thus, we deduce that

$$2\, a.n + \|a\|^2 = 2(b.n)(m.n) + \|b\|^2 (m.n)^2, \tag{A.10$_1$}$$

$$a.\, n \wedge e = (b\, .n \wedge e)\,(m.n) + (m\, .n \wedge e)\,(b.n) + \|b\|^2 (m.n)\,(m\, .n \wedge e) \tag{A.10$_2$}$$

and

$$0 = 2\,(b\, .n \wedge e)\,(m\, .n \wedge e) + \|b\|^2 (m\, .n \wedge e)^2. \tag{A.10$_3$}$$

By (A.7) and (A.8),

$$m\, .n \wedge e \neq 0$$

and so, by $(A.10)_1$ - $(A.10)_3$ we have

$$m.n = (2a.n + \|a\|^2)\, \frac{(m.\, n \wedge e)}{2a.\, n \wedge e}$$

which, together with (A.8), implies that m has the direction of the vector

$$\frac{2a.n + \|a\|^2}{2a.\, n \wedge e}\, n + \frac{n \wedge e}{\|n \wedge e\|}.$$

Finally, by (A.9) we conclude that

$$m = \pm \frac{2a + \|a\|^2 n}{\|2a + \|a\|^2 n\|}.$$

Suppose that $R^2 = \mathbb{1}$ and let R' be the restriction of R to Span $\{n, n \wedge e\}$. Then R' is a rotation on a two dimensional vector space and $R^{f2} = 1$, therefore

$R' = \pm$ II.

As we are assuming that R * 1, we have

R ' - l .

By (A.7) and (A.8) we have

n £ Span{m, e},

thus it is possible to choose x € DR^3 such that

m.x = 0, x.e = 0 and n.x * 0.

As

R = -n®n - nAe 0nAe + e®e,

by (A.6) and (A.8) we obtain

-x - (n.x) a = x

which implies that

m.a = 0

or

a.n = -2,

contradicting the hypothesis

a.n > -1.

We conclude that

$R^2 \neq 1$.

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