THE DISPLACEMENT PROBLEM FOR ELASTIC CRYSTALS

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Abstract : In this paper we obtain necessary and sufficient conditions for the existence of Lipschitz minimizers of a functional of the type

$$J(u) := \int_\Omega h(\det \nabla u(x))\, dx - \int_\Omega f(x).u(x)\, dx,$$

where h is a convex function converging to zero at infinity and u is subjected to displacement boundary conditions. We provide examples of body forces f for which the infimum of $J(\cdot)$ is not attained.

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1.INTRODUCTION.

During the past few years, the stability properties of solid crystals have been discussed within the framework of a continuum theory proposed by ERICKSEN [7], [10]. In this model, thermoelasticity is introduced via the Cauchy-Born hypothesis (see ERICKSEN [9]), relating changes in atomic positions to macroscopic deformations. This assumption, together with molecular considerations, yields the invariance of the energy density W with respect to an infinite discrete group conjugate to a subgroup of $GL(\mathbb{Z}^3)$. As noticed by FONSECA [12] and KINDERLEHRER [14], the material symmetry renders the analysis of equilibria and stability problems quite complicated.

The stability of configurations held in a dead loading device and subjected to surface tractions was studied by FONSECA [12]. It was shown that only residual stresses can provide (global) minima of the total energy functional. CHIPOT & KINDERLEHRER [4] and FONSECA [13] analyzed the role played by the sub-energy function φ in the stability of unloaded crystals subjected to homogeneous boundary conditions; they proved that

$$\varphi^{**}(\det F) = \inf\left\{ \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega W(\nabla u(x))\, dx \mid u \in W^{1,\infty}(\Omega;\, \mathbb{R}^n),\ u(x) = Fx \text{ on } \partial\Omega \right\},$$

where F is a nxn real valued matrix with $\det F > 0$, $\Omega \subset \mathbb{R}^n$ is an open bounded strongly Lipschitz domain, φ^{**} is the lower convex envelope of sub-energy φ given by (see ERICKSEN [8] and FLORY [11])

$$\varphi(t) := \inf\{W(F) \mid \det F = t\}.$$

Moreover, FONSECA [13] shows that

$$QW(F) = \varphi^{**}(\det F) \tag{1.1}$$

for all F, where QW denotes the $W^{1,\infty}$-quasiconvex envelope of W, recovering the characterization of QW obtained by DACOROGNA [6] when W(F) is finite for all nxn real valued matrix F.

In this paper we are concerned with the existence of minimizers for the total energy functional

$$E(u) := \int_\Omega W(\nabla u(x))\, dx - \int_\Omega f(x).u(x)\, dx$$

when displacement boundary conditions are prescribed. In order to relax this problem, and according to (1.1), we introduce the functional

$$J(u) := \int_\Omega \varphi^{**}(\det \nabla u(x))\, dx - \int_\Omega f(x).u(x)\, dx.$$

In Section 2 we study the relation between

$$\inf\{E(u) \mid u \in u_0 + W^{1,\infty}(\Omega;\, \mathbb{R}^n)\}$$

and

$$\inf\{J(u) \mid u \in u_0 + W^{1,\infty}(\Omega;\, \mathbb{R}^n)\}.$$

In Theorem 3.1 we obtain necessary and sufficient conditions for a deformation u to be a minimizer of $J(\cdot)$ in $u_0 + W^{1,\infty}(\Omega; \mathbb{R}^n)$. Furthermore, the necessary conditions hold also for the mixed displacement - traction and pure traction problems (see Remark 3.16 (ii)). It turns out that the characterization thus obtained can be useful to detect body forces f for which inf $J(\cdot)$ is not attained; as an example, if

$$\text{meas}\{x \in \Omega \mid \det \nabla f(x) < 0\} > 0$$

then $J(\cdot)$ does not admit minima (see Corollary 3.15 and Remark 3.16 (i)).

Finally, in Section 4 we provide an example where there is existence of minimizers for strong materials in the presence of gravity-type forces. Also, we show that for "weak materials" minimizers may fail to exist if the amplitude k of the external force f exceeds some critical value.

2. ELASTIC CRYSTALS. RELAXATION OF THE DISPLACEMENT PROBLEM.

In the sequel, $O^+(n)$ is the proper orthogonal group, $M^{n\times n}$ denotes the set of real nxn matrices, $M_+^{n\times n} := \{F \in M^{n\times n} \mid \det F > 0\}$ and $G^+ := \{M \in M^{n\times n} \mid M_{ij} \in \mathbb{Z}, i,j = 1,\dots,n$ and $\det F = 1\}$.

According to ERICKSEN [7], [9], [10], the energy density per unit reference volume of a (pure) solid crystal under isothermal conditions is given by a function $W : M_+^{n\times n} \to \mathbb{R}$ invariant under change of lattice basis. This invariance is expressed by the relation

$$W(F) = W(FM) \tag{2.1}$$

for all $F \in M_+^{n\times n}$ and $M \in AG^+A^{-1}$, where $A \in M^{n\times n}$ is a fixed matrix describing the molecular symmetry of the undeformed configuration. Furthermore, due to frame indifference, we have

$$W(F) = W(RF) \tag{2.2}$$

for all $F \in M_+^{n\times n}$ and $R \in O^+(n)$. As it is usual in nonlinear elasticity, we assume,in addition, that

$$W : M^{n\times n} \to \mathbb{R} \cup \{+\infty\} \text{ is continuous and bounded below,} \tag{2.3}$$

$$W(F) \to +\infty \text{ when } \det F \to 0^+ \tag{2.4}$$

and

$$W(F) = +\infty \text{ if and only if } \det F \leq 0. \tag{2.5}$$

For a detailed description of this model we refer the reader to ERICKSEN [7]-[10], FONSECA [12], [13] and KINDERLEHRER [14].

Suppose that, in a fixed reference configuration, the crystal occupies an open bounded strongly Lipschitz domain $\Omega \subset \mathbb{R}^n$. Let $u : \Omega \to \mathbb{R}^n$ be a deformation of the lattice and let $f : \Omega \to \mathbb{R}^n$ represent the body force per unit volume in the undeformed configuration. The pure displacement boundary value problem consists in finding a solution $u \in \mathcal{Q}$ of

$$\begin{cases} -\operatorname{div} \dfrac{\partial W}{\partial F}(\nabla u) = f & \text{in } \Omega \\ u = Uo & \text{on } dQ, \end{cases}$$

where Q is a suitable class of admissible deformations and u_0: *dCl* -> IR^n is given. Here, we are interested in the stable solutions, i.c., minimizers in Q of the total energy

$$E(u):= f \ W(Vu(x))dx- \ f \ f(x).u(x)dx$$

when

UQ € C($\bar{£2}$; IR^n) is injective in *il*

and

Q :={ u € W^-OB; IR^n) | det Vu > 0 a.e. in Q and u = u_0 on dQ}.

As it was pointed out by ERICKSEN [10], FONSECA [12] and KINDERLEHRER [14], the minimization of functionals of this type escape the methods of the calculus of variations . In fact, due to the symmetry invariance (2.1), W remains bounded on some directions and the functional

$$u \to f \ W(Vu(x)) \, dx$$

is not sequentially weakly * lower semicontinuous (s.w.*l.s.c). Since W^-quasiconvexity is a necessary condition for s.w.*Ls. continuity (see BALL & MURAT [2] and MORREY [15]), in order to "relax" the problem we introduce the new functional

$$u \text{-»} \ f_{JQ} \ QW(Vu(x)) \, dx,$$

where QW denotes the lower W^-quasiconvex envelope of W. FONSECA [13] proved that QW(F) reduces to a function of the determinant of F, precisely

QW(F) = <p(detF),**

where <p is the lower convex envelope of the sub-energy <p given by**

q>(t):=inf{W(F)|detF = t}

for all t € IR. From (2.3) - (2.5) it follows that

<p : IR -»IR u {+«>} is convex and bounded below,**

<p(t) -*+oo when t -> 0+** (2.6)

and

<p(t) = +OP if and only if t £ 0.**

Hence we are led to study the following problem: findue Q such that

J(u) = inf{J(v)|v€ Q} (2.7)

where

$$J(u):= f \ <p^{**}(detVu(x))dx- \ f \ f(x).u(x)dx.$$

Naturally, we seek for relations between the solutions of (2.7) and the minimizers of $E(\cdot)$. In what follows, we use the notation

$$\alpha := \inf\{E(u) \mid u \in \mathcal{A}\},$$
$$\beta := \inf\{J(u) \mid u \in \mathcal{A}\},$$
$$\alpha' := \inf\{E(u) \mid u \in \mathcal{A}'\}$$
$$\beta' := \inf\{J(u) \mid u \in \mathcal{A}'\}$$

where

$$\mathcal{A}' := \{u \in \mathcal{A} \mid u \text{ is piecewise affine and } \inf \det \nabla u(x) > 0\}.$$

Proposition 2.8

Under the hypotheses (2.1) - (2.5), if $f \in L^1(\Omega;\mathbb{R}^n)$ then $\beta \leq \alpha \leq \alpha' = \beta'$.

Remark 2.9

(i) It was shown by CHIPOT & KINDERLEHRER [4] and by FONSECA [13] that

$$\alpha = \beta$$

when $f = 0$ a. e. in Ω and u_0 is an affine deformation, i. e.

$$\inf\left\{ \int_\Omega \varphi^{**}(\det(F + \nabla\xi(x)))\,dx \mid \xi \in W_0^{1,\infty}(\Omega;\mathbb{R}^n)\right\}$$
$$= (\text{meas}\Omega)\,\varphi^{**}(\det F) \tag{2.10}$$
$$= \inf\left\{ \int_\Omega W(F + \nabla\xi(x))\,dx \mid \xi \in W_0^{1,\infty}(\Omega;\mathbb{R}^n)\right\}.$$

It can be verified easily that (2.10) is still valid when the infima are taken over piecewise affine functions.

(ii) In the general case, the equality $\alpha = \beta$ would follow from (2.10) if we could devise a density argument allowing us to conclude that $\beta' = \beta$. However, due to the behaviour of φ^{**} near zero (see (2.6)), this question remains open.

Proof of Proposition 2.8. Without loss of generality, we can assume that $W \geq 0$. Clearly, it suffices to show that $\alpha' \leq \beta'$. Let $\varepsilon > 0$ and consider $u \in \mathcal{A}'$ such that

$$u(x) = \sum_{i=1}^{\infty} \chi_{\Omega_i}(x)\,(F_i x + C_i),$$

where $\sup_i \|F_i\| < +\infty$ and $\inf_i \det F_i > 0$. Then, by (2.3) and (2.5), $\sup_i W(F_i) = M < +\infty$ and so

$$\sum_{i=1}^{\infty} \int_{\Omega_i} W(F_i)\,dx \leq M\,\text{meas}(\Omega).$$

Let i_0 be such that

$$\sum_{i > i_0} \text{meas}(\Omega_i)\, W(F_i) < \frac{\varepsilon}{2}.$$

By Remark 2.9 (i), for each $i \le i_0$ there exists $u_i \in W^{1,\infty}(\Omega_i; \mathbb{R}^n)$ such that u_i is piecewise affine,

$$u_i(x) = F_i x + C_i \text{ on } \partial\Omega_i$$

and

$$\int_{\Omega_i} W(\nabla u_i)\, dx \le \text{meas}(\Omega_i)\, \varphi^{**}(\det F_i) + \frac{\varepsilon}{4 i_0}.$$

Using Vitali's covering theorem, we can decompose Ω_i as a disjoint union of the type

$$\Omega_i = \bigcup_{j=1}^{\infty} (a_{i,j} + \varepsilon_{i,j}\, \Omega_i) \cup A_i, \text{ with } \quad \sum_{j=1}^{\infty} \varepsilon_{i,j}^3 = 1, \quad 0 \le \varepsilon_{i,j} \le \frac{\varepsilon}{4\, \|f\|_{L^1} \sup_{i \le i_0} \|u - u_i\|_{L^\infty(\Omega_i)}}$$

and $\text{meas}(A_i) = 0$. We define the function

$$v(x) := \begin{cases} u(x) + \varepsilon_{i,j}(u_i - u)\left(\dfrac{x - a_{i,j}}{\varepsilon_{i,j}}\right) & \text{if } x \in a_{i,j} + \varepsilon_{i,j}\Omega_i \text{ and } i \le i_0 \\ u(x) & \text{otherwise.} \end{cases}$$

It is clear that $v \in \mathcal{Q}'$; moreover we obtain

$$\int_\Omega W(\nabla v(x))\, dx - \int_\Omega f(x).v(x)\, dx \le \frac{\varepsilon}{2} + \sum_{i \le i_0} \sum_{j=1}^{\infty} \int_{a_{i,j} + \varepsilon_{i,j}\Omega_i} W\left(\nabla u_i\left(\frac{x - a_{i,j}}{\varepsilon_{i,j}}\right)\right) dx - \int_\Omega f(x).u(x)\, dx$$

$$+ \int_\Omega |f(x)|\, |u(x) - v(x)|\, dx$$

$$\le \frac{3\varepsilon}{4} + \int_\Omega \varphi^{**}(\det \nabla u(x))\, dx - \int_\Omega f(x).u(x)\, dx + \|f\|_{L^1} \|u - v\|_{L^\infty}$$

$$\le \varepsilon + \int_\Omega \varphi^{**}(\det \nabla u(x))\, dx - \int_\Omega f(x).u(x)\, dx.$$

Given the arbitrariness of $u \in \mathcal{Q}'$ and $\varepsilon > 0$ we conclude that $\alpha' \le \beta'$.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF MINIMIZERS.

Throughout this section we use the notation introduced in Section 2. The main result consists on a set of necessary and sufficient conditions for the existence of minimizers for $J(\cdot)$. In what follows, the functional $J(\cdot)$ is given by

$$J(u) := \int_\Omega h(\det \nabla u(x))\, dx - \int_\Omega f(x).u(x)\, dx,$$

where $f \in L^1(\Omega; \mathbb{R}^n)$, $h : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex, bounded below, $h(t) \to +\infty$ as $t \to 0^+$, and $h(t) = +\infty$ if and only if $t \le 0$.

Theorem 3.1

Let $u \in \mathcal{Q}$ be such that $J(u) < +\infty$.

(N) If $J(u) \leq J(v)$ for all $v \in \mathcal{Q}$ then

(a) $\nabla u^T f = \nabla \psi$ for some $\psi \in \mathcal{D}'(\Omega; \mathbb{R})$;

(b) *(cyclic monotonicity)*

$$\sum_{i=1}^{N} f(p_i).(u(p_{i+1}) - u(p_i)) \leq 0$$

for all $N \in \mathbb{N}$, $(p_1,...,p_N) \in (\Omega \setminus \Omega_f)^N$, where Ω_f is the Lebesgue set of f ($\text{meas}(\Omega_f) = 0$) and $p_{N+1} \equiv p_1$;

(c) there is a convex function G: convex hull $u_0(\Omega) \to \mathbb{R}$ such that

$$\int_\Omega G(u(x))\, dx \in \mathbb{R}$$

and

$$f(x) \in \partial G(u(x)) \quad \text{a.e. } x \in \Omega;$$

(d) if, in addition, there is a $\lambda \in (0,1)$ such that

$$\int_\Omega h'(\lambda \det \nabla u(x)) \det \nabla u(x)\, dx > -\infty$$

then

$$\psi(x) + C_1 = h(\det \nabla u(x)) - h'(\det \nabla u(x)) \det \nabla u(x) = G(u(x)) + C_2 \quad \text{a.e. } x \in \Omega$$

for some $C_1, C_2 \in \mathbb{R}$.

(S) If

$$\int_\Omega h'(\det \nabla u(x)) \det \nabla u(x)\, dx > -\infty$$

and if there is a convex function G: convex hull $u_0(\Omega) \to \mathbb{R}$ such that

$$G(u(x)) = h(\det \nabla u(x)) - h'(\det \nabla u(x)) \det \nabla u(x) \qquad \text{a.e. } x \in \Omega$$

with

$$f(x) \in \partial G(u(x)) \qquad \text{a.e. } x \in \Omega$$

then

$$J(u) \leq J(v) \qquad \text{for all } v \in \mathcal{Q}.$$

The cyclic monotonicity (N)(b) is a consequence of the following lemma.

Lemma 3.2

Let $\Omega \subset \mathbb{R}^n$ be an open bounded strongly Lipschitz domain and let $(p_1,...,p_N) \in \Omega^N$, $N \in \mathbb{N}$. If $p_i \neq p_j$ for $i \neq j$, $i, j = 1,...,N$, then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there is a sequence $w_k \in C^\infty(\bar{\Omega}; \mathbb{R}^n)$ verifying

$$\begin{cases} \det \nabla w_k(x) = 1 & \text{in } \Omega \\ w_k(x) = x & \text{on } \partial\Omega \end{cases}$$

and

$$w_k(x) \to w(x) \ \text{ a.e. } x \in \Omega,$$

where

$$w(x) = x \quad \text{if } x \in \Omega \setminus \bigcup_{i=1}^{N} (p_i + [-\varepsilon, \varepsilon]^n),$$

and $w(p_i + (-\varepsilon, \varepsilon)^n) = p_{i+1} + (-\varepsilon, \varepsilon)^n$ for $1 \le i \le N$ and $p_1 \equiv p_{N+1}$.

BRENIER [3] obtained a similar result. The proof of Lemma 3.2 can be found in the Appendix.

For completeness, before proving Theorem 3.1, we state Theorem 1 of BALL [1] (see BALL [1], page 317).

Theorem 3.3

Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded connected strongly Lipschitz open set. Let $u_0 : \bar{\Omega} \to \mathbb{R}^n$ be continuous in $\bar{\Omega}$ and one–to–one in Ω. Let $p > n$ and let $u \in W^{1,p}(\Omega)$ take values in $\mathbb{R}^n$ and satisfy $u|_{\partial\Omega} = u_0|_{\partial\Omega}$, $\det\nabla u(x) > 0$ almost everywhere in Ω. Then
(i) $u(\bar{\Omega}) = u_0(\bar{\Omega})$;
(ii) u maps measurable sets in $\bar{\Omega}$ to measurable sets in $u_0(\bar{\Omega})$, and the change of variable formula

$$\int_A f(u(x)) \det \nabla u(x)\, dx = \int_{u(A)} f(v)\, dv \tag{3.4}$$

holds for any measurable $A \subset \bar{\Omega}$ and any measurable function $f : \mathbb{R}^n \to \mathbb{R}$ provided only that one of the the integrals in (3.4) exists;
(iii) u is one-to-one almost everywhere; i.e. the set

$$S := \{v \in u_0(\bar{\Omega}) : u^{-1}(v) \text{ contains more than one element}\}$$

has measure zero;
(iv) if $v \in u_0(\Omega)$ then $u^{-1}(v)$ is a continuum contained in Ω, while if $v \in \partial u_0(\Omega)$ then each connected component of $u^{-1}(v)$ intersects $\partial\Omega$.

Proof of Theorem 3.1. Without loss of generality, we can assume that $h \ge 0$. If $v \in \mathcal{Q}$, by Theorem 3.3 (i), it follows that

$$J(v) \ge -\int_\Omega f(x).v(x)\, dx \ge -\| f \|_{L^1} \| u_0 \|_\infty,$$

and so $J(\cdot)$ is bounded below.

(S) (Sufficient condition) According to Theorem 3.3 (iii), define the function

$$\hat{u}(y) := \begin{cases} u^{-1}(y) & \text{if } y \in u_0(\Omega) \setminus S \\ 0 & \text{otherwise,} \end{cases}$$

where $S := \{y \in u_0(\Omega) \mid \# u^{-1}(y) > 1\}$. Fix $v \in \mathcal{G}$ and let

$$w(x) := \hat{u}(v(x)).$$

By Theorem 3.3 we have

$$\text{meas } S = 0,\ \bar{\Omega} = v^{-1}(u_0(\Omega) \setminus S) \cup B_0 \text{ and } \bar{\Omega} = u^{-1}(v(v^{-1}(u_0(\Omega) \setminus S))) \cup B_1,$$

where meas $B_0 = 0$ and meas $B_1 = 0$. Due to the convexity of the functions h and G we obtain

$$\begin{aligned} J(v) &= \int_{v^{-1}(u_0(\Omega)\setminus S)} \{h(\det \nabla v(x)) - f(x).v(x)\}\, dx \\ &\geq \int_{v^{-1}(u_0(\Omega)\setminus S)} \{h(\det \nabla u(w(x))) + h'(\det \nabla u(w(x)))\,(\det \nabla v(x) - \det \nabla u(w(x))) - f(x).v(x)\}\, dx \\ &= \int_{v^{-1}(u_0(\Omega)\setminus S)} \{G(u(w(x))) - f(x).v(x) + h'(\det \nabla u(w(x))) \det \nabla v(x)\}\, dx \\ &\geq \int_{v^{-1}(u_0(\Omega)\setminus S)} \{G(u((x))) + f(x).(u(w(x)) - u(x) - v(x)) + h'(\det \nabla u(w(x))) \det \nabla v(x)\}\, dx \\ &= J(u) + \int_{v^{-1}(u_0(\Omega)\setminus S)} \{h'(\det \nabla u(w(x))) \det \nabla v(x) - h'(\det \nabla u(x)) \det \nabla u(x)\}\, dx. \qquad (3.5) \end{aligned}$$

Since $u \in W^{1,\infty}(\Omega; \mathbb{R}^n)$, h' is nondecreasing and

$$\int_\Omega h'(\det \nabla u(x)) \det \nabla u(x)\, dx > -\infty,$$

we conclude that

$$\int_\Omega h'(\det \nabla u(x)) \det \nabla u(x)\, dx \in \mathbb{R}.$$

Thus, by Theorem 3.3 (ii),

$$\begin{aligned} \int_{v^{-1}(u_0(\Omega)\setminus S)} h'(\det \nabla u(x)) \det \nabla u(x)\, dx &= \int_{u^{-1}(v(v^{-1}(u_0(\Omega)\setminus S)))} h'(\det \nabla u(x)) \det \nabla u(x) \\ &= \int_{v(v^{-1}(u_0(\Omega)\setminus S)))} h'(\det \nabla u(\hat{u}(y))\, dy \\ &= \int_{v^{-1}(u_0(\Omega)\setminus S)))} h'(\det \nabla u(w(x)) \det \nabla v(x)\, dx, \end{aligned}$$

which, together with (3.5), yields $J(v) \geq J(u)$.

(N) (Necessary conditions) Let u be a minimizer of $J(\cdot)$ in $\mathcal{G}$, $J(u) < +\infty$.

(a) It suffices to show that

$$\frac{\partial}{\partial x_m}\left(f_i \frac{\partial u_i}{\partial x_k}\right) = \frac{\partial}{\partial x_k}\left(f_i \frac{\partial u_i}{\partial x_m}\right) \text{ in } \mathcal{D}'(\Omega)$$

for all $k, m \in \{1, \ldots, n\}$, or, equivalently,

$$\int_\Omega f_i \frac{\partial u_i}{\partial x_k} \Lambda_{km} \frac{\partial \varphi}{\partial x_m} \, dx = 0 \tag{3.6}$$

for all $\varphi \in \mathcal{D}(\Omega; \mathbb{R})$ and for every $\Lambda = -\Lambda^T \in M^{n\times n}$. Fix $x_0 \in \Omega$ and $R > 0$ such that $B(x_0, R) \subset \subset \Omega$. Given $\lambda \in \mathbb{R}$, let $\theta_k \in \mathcal{D}([0,R])$ be a sequence of cut-off functions such that

$$\lim_{k\to\infty} \theta_k(t) = \begin{cases} \lambda & \text{if } t \in (0, R) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$w_k(x) := x_0 + e^{\theta_k(|x - x_0|)\,\Lambda} (x - x_0).$$

Clearly, $\det \nabla w_k = 1$ in Ω and $w_k = x$ on $\partial\Omega$, therefore

$$J(u) \le J(u \circ w_k)$$

i. e., by Theorem 3.3 (ii),

$$\int_\Omega f(x).\{u(w_k(x)) - u(x)\} \, dx \le 0.$$

Letting $k \to +\infty$, we obtain

$$\int_{B(x_0, R)} f(x).\{u(x_0 + e^{\lambda\Lambda}(x - x_0)) - u(x)\} \, dx \le 0$$

for all $\lambda \in \mathbb{R}$. Differentiating with respect to λ at $\lambda = 0$ we obtain

$$\int_{B(x_0, R)} f_i(x) \frac{\partial u_i}{\partial x_k}(x) \Lambda_{km} (x_m - x_{0m}) \, dx = 0. \tag{3.7}$$

Finally, let $\varphi \in \mathcal{D}(\Omega; \mathbb{R})$ with $\operatorname{supp} \varphi = \Omega^* \subset\subset \Omega$ and let $0 < \varepsilon < \text{distance}(\Omega^*; \partial\Omega)$. From (3.7) we have

$$\begin{aligned} 0 &= \lim_{\delta \to \varepsilon^+} \int_{\Omega^*} \varphi(y) \left\{ \frac{1}{\text{meas}(\{x \mid \varepsilon < |x - y| < \delta\})} \int_{\varepsilon<|x-y|<\delta} f_i(x) \frac{\partial u_i}{\partial x_k}(x) \Lambda_{km} (x_m - y_m) \, dx \right\} dy \\ &= \lim_{\delta \to \varepsilon^+} \int_\Omega f_i(x) \frac{\partial u_i}{\partial x_k}(x) \Lambda_{km} \left\{ \frac{1}{\text{meas}(\{x \mid \varepsilon < |x - y| < \delta\})} \int_{\varepsilon<|x-y|<\delta} \varphi(y)(x_m - y_m) \, dy \right\} dx \\ &= \int_\Omega f_i(x) \frac{\partial u_i}{\partial x_k}(x) \Lambda_{km} \left\{ \int_{|x-y|=\varepsilon} \varphi(y) (x_m - y_m) \, dS(y) \right\} dx. \end{aligned} \tag{3.8}$$

As

$$\begin{aligned} \int_{|x-y|=\varepsilon} \varphi(y) (x_m - y_m) \, dS(y) &= -\frac{\partial \varphi}{\partial x_j}(x) \int_{|y| = \varepsilon} y_j \, y_m \, dS(y) + O(\varepsilon^{n+2}) \\ &= -\frac{\partial \varphi}{\partial x_m}(x) \, \varepsilon \, \text{meas } B(0, \varepsilon) + O(\varepsilon^{n+2}), \end{aligned}$$

dividing (3.8) by ε^{n+1}, and letting $\varepsilon \to 0^+$, we obtain (3.6).

(b) Let Ω_f be the Lebesgue set of f (meas Ω_f=0) and let $p_1,\ldots, p_N \in \Omega \setminus \Omega_f$, $p_i \ne p_j$ for $i \ne j$. Let $\varepsilon_0 > 0$, $w_{k,\varepsilon}$ and w_ε, $0 < \varepsilon < \varepsilon_0$, satisfy the conditions of Lemma 3.2. As

$$J(u) \le J(u \circ w_{k,\varepsilon}),$$

by Theorem 3.3 (ii), it follows that

$$\int_{\Omega} f(x).\{u(x) - u(w_M(x))\}\, dx \le 0,$$

and so, by the dominated convergence theorem, we conclude that

$$\int f(x).\{u(x) - u(w_\varepsilon(x))\}\, dx \le 0,$$

where

$$w_\varepsilon(x) = x \quad \text{if } x \in \bigcup_{i=1} (p_i + [-\varepsilon, \varepsilon]^n)$$

and

$$w_\varepsilon|_{p_i + (-\varepsilon,\varepsilon)^n}(x) = p_{i+1} + T_{iu}(x) \quad \text{with } \| T_i \|_{L^\infty(p_i + (-\varepsilon,\varepsilon)^n)} \le \varepsilon$$

for all $i \in \{1,\ldots, N\}$, $p_{N+1} \equiv p_1$. Hence,

$$0 \le \lim_{\varepsilon \to 0^+} \frac{1}{(2\varepsilon)^n} \sum_{i=1}^{N} \int_{p_i + (-\varepsilon,\varepsilon)^n} f(x).\{u(x) - u(p_{i+1} + T_{iu}(x))\}\, dx$$

$$= \sum_{i=1}^{N} f(p_i).(u(p_i) - u(p_{i+1})).$$

(c) Let $\Omega_0 := \Omega \setminus S$ and, for $y \in \mathbb{R}^n$, define

$$G(y) := \sup \left\{ \sum_{i=1}^{N-1} f(p_i).(u(p_{i+1}) - u(p_i)) + f(p_N).(y - u(p_N)) \mid N \in \mathbb{N},\ (p_1, \ldots, p_N) \in (\Omega \setminus \Omega_0)^N \right\}.$$

G is convex and lower semicontinuous; moreover, if $x \in \Omega \setminus \Omega_0$ and if $(p_1,\ldots, p_N) \in (\Omega \setminus \Omega_0)^N$, by (b) we obtain

$$\sum_{i=1}^{N-1} f(p_i).(u(p_{i+1}) - u(p_i)) + f(p_N).(u(x) - u(p_N)) =$$

$$\sum_{i=1}^{N-1} f(p_i).(u(p_{i+1}) - u(p_i)) + f(p_N).(u(x) - u(p_N)) + f(x).(u(p_1) - u(x)) + f(x).(u(x)-u(p_1))$$

$$\le f(x).(u(x) - u(p_1)),$$

thus

$$0 \le G(u(x)) \le f(x).(u(x) - u(p_2)) \tag{3.9}$$

for all $x, p_2 \in \Omega \setminus \Omega_0$. As, by Theorem 3.3,

$$u(\Omega) \setminus u(\Omega_0) \subset u(\Omega) \setminus u(\Omega_0) \subset u(\Omega \setminus \Omega_0), \quad \text{meas } \% \approx 0$$

and, by the Theorem of Invariance of Domain, $u(\Omega)$ is an open, if $y \in u_0(\Omega)$ then there are $y_1, y_2 \in u(\Omega \setminus \Omega_0)$ and $a \in [0,1]$ such that $y = a y_1 + (1-a)y_2$. Therefore, by (3.9) and since G is convex, we conclude that

$$G(y) \in \mathbb{R} \text{ for all } y \in \text{convex hull } u_0(\Omega)$$

and

$$0 \le \int G(u(x))\, dx \le 2 \| u \|_\infty \| f \|_{L^1(\Omega)}.$$

Let $x \in \Omega \setminus \Omega_0$ and let $y \in \mathbb{R}^n$. If $(p_1,\dots, p_N) \in (\Omega \setminus \Omega_0)^N$ then

$$G(y) \geq \sum_{i=1}^{N-1} f(p_i).(u(p_{i+1}) - u(p_i)) + f(p_N).(u(x) - u(p_N)) + f(x).(y - u(x))$$

which implies

$$G(y) \geq G(u(x)) + f(x).(y - u(x)),$$

and so

$$f(x) \in \partial G(u(x)) \quad \text{for all } x \in \Omega \setminus \Omega_0.$$

(d) Let $\varphi \in \mathcal{D}(\Omega; \mathbb{R}^n)$ and let $\varepsilon_0 > 0$ be such that

$$0 < \lambda \leq \det(\mathbb{1} + \varepsilon \nabla\varphi(x)) \leq \lambda + 1$$

for all $x \in \Omega$, $|\varepsilon| < \varepsilon_0$. Setting

$$w_\varepsilon(x) := x + \varepsilon\varphi(x)$$

and

$$g_\varepsilon(x) := \det\nabla w_\varepsilon(w_\varepsilon^{-1}(x)),$$

we have

$$J(u(w_\varepsilon)) \geq J(u)$$

or, by Theorem 3.3 (ii), the function

$$\varepsilon \to J(u(w_\varepsilon)) = \int_\Omega h(g_\varepsilon(x) \det \nabla u(x)) \frac{1}{g_\varepsilon(x)} \, dx - \int_\Omega f(x).u(w_\varepsilon(x)) \, dx \tag{3.10}$$

has a minimum at $\varepsilon = 0$. In order to differentiate under the integral sign, we want to find a function $F \in L^1(\Omega)$ such that

$$\left|\frac{d}{d\varepsilon}\left\{ h(g_\varepsilon(x) \det \nabla u(x)) \frac{1}{g_\varepsilon(x)}\right\}\right| \leq F(x) \quad \text{a. e. } x \in \Omega \tag{3.11}$$

for all $|\varepsilon| < \varepsilon_0$. As

$$\frac{dg_\varepsilon}{d\varepsilon}(x) = \operatorname{div} \varphi(x) + O(\varepsilon),$$

we have that

$$\left|\frac{d}{d\varepsilon}\left\{ h(g_\varepsilon(x) \det \nabla u(x)) \frac{1}{g_\varepsilon(x)}\right\}\right| \leq \text{Const.}\{h(g_\varepsilon(x) \det \nabla u(x)) + |h'(g_\varepsilon(x) \det \nabla u(x))| \det \nabla u(x)\}, \tag{3.12}$$

with, due to the convexity of h,

$$\begin{aligned} h(g_\varepsilon(x) \det \nabla u(x)) &\leq h(\det \nabla u(x)) + h'(g_\varepsilon(x) \det \nabla u(x)) (g_\varepsilon(x) - 1) \det \nabla u(x) \\ &\leq h(\det \nabla u(x)) + |h'(g_\varepsilon(x) \det \nabla u(x))| \det \nabla u(x). \end{aligned} \tag{3.13}$$

Moreover, since h' is increasing,

$$h'(\lambda \det \nabla u(x)) \det \nabla u(x) \leq h'(g_\varepsilon(x) \det \nabla u(x)) \det \nabla u(x) \leq \text{Const.}$$

with, by hypothesis,

$$h'(\lambda \det \nabla u(x)) \det \nabla u(x) \in L^1(\Omega).$$

Therefore, by (3.12), (3.13) and since $J(u) < +\infty$, we obtain (3.11) which, together with (3.10), implies that

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}\, J(u(w_\varepsilon)) = 0,$$

i. e.,

$$\int_\Omega \{h'(\det \nabla u(x)) \det \nabla u(x) - h(\det \nabla u(x))\} \operatorname{div} \varphi(x)\, dx = \int_\Omega f_i(x) \frac{\partial u_i}{\partial x_k}(x)\, \varphi_k(x)\, dx.$$

Given the arbitrariness of φ, we obtain

$$\frac{\partial}{\partial x_k} \{h(\det \nabla u(x)) - h'(\det \nabla u(x)) \det \nabla u(x)\} = f_i(x) \frac{\partial u_i}{\partial x_k}(x) \text{ in } \mathcal{D}'(\Omega;\ \mathbb{R}^n).$$

Thus, by (a) there is a constant C_1 such that

$$h(\det \nabla u(x)) - h'(\det \nabla u(x)) \det \nabla u(x) = \psi(x) + C_1.$$

Also, by (c) we have

$$\int_\Omega \{G(u(w_\varepsilon(x))) - G(u(x))\}\, dx \geq \int_\Omega f(x).(u(w_\varepsilon(x)) - u(x))\, dx$$

which implies

$$0 = \frac{d}{d\varepsilon}|_{\varepsilon=0} \left\{ \int_\Omega \left(G(u(x)) \frac{1}{g_\varepsilon(x)} - f(x).u(w_\varepsilon(x)) \right) dx \right.$$

$$= \int_\Omega \left(-G(u(x)) \operatorname{div} \varphi(x)\, dx - f_i(x) \frac{\partial u_i}{\partial x_k}(x)\, \varphi_k(x) \right) dx,$$

and so, there is a constant C_2 such that

$$\psi(x) + C_1 = G(u(x)) + C_2 \quad \text{a.e. } x \in \Omega.$$

Corollary 3.14

Let $f \in L^1(\Omega;\ \mathbb{R}^n)$ and let $u \in \mathcal{Q}$ be such that $J(u) < +\infty$ and

$$\int_\Omega h'(\lambda \det \nabla u(x)) \det \nabla u(x)\, dx > -\infty$$

for some $\lambda \in (0, 1)$. Then $J(u) \leq J(v)$ for all $v \in \mathcal{Q}$ if and only if there is a convex function G: convex hull $u_0(\Omega) \to \mathbb{R}$ such that

$$\int_\Omega G(u(x))\, dx \in \mathbb{R}$$

and $f(x) \in \partial G(u(x))$ a.e. $x \in \Omega$.

Proof. It follows immediatly from Theorem 3.1. Note that, because h' is increasing and $\lambda \in (0,1)$,

$$\int_\Omega h'(\det \nabla u(x)) \det \nabla u(x)\, dx > -\infty.$$

Corollary 3.15

Let $f \in L^1(\Omega;\ \mathbb{R}^n) \cap W^{1,1}_{loc}(\Omega;\ \mathbb{R}^n)$. If $J(\cdot)$ admits a minimizer in $\mathcal{Q}$ then $\det \nabla f(x) \geq 0$ a.e. in Ω.

Proof. Let u be a minimizer of $J(\cdot)$ in $\mathcal{Q}$. From Theorem 3.1 (N)(a) we have

$$\frac{\partial}{\partial x_m}\left(f_i \frac{\partial u_i}{\partial x_k}\right) = \frac{\partial}{\partial x_k}\left(f_i \frac{\partial u_i}{\partial x_m}\right) \text{ in } \mathcal{D}'(\Omega)$$

for all $k, m \in \{1, ..., n\}$; as $f \in W^{1,1}{}_{loc}(\Omega; \mathbb{R}^n)$ we deduce that

$\nabla f^T \nabla u$ is a symmetric matrix.

Furthermore, by Theorem 3.1 (N)(b),

$$(f(x) - f(y)) \,.\, (u(x) - u(y)) \geq 0 \qquad \text{a.e. } x, y \in \Omega$$

which implies

$$\frac{\partial f_i}{\partial x_m}(x) \frac{\partial u_i}{\partial x_k}(x)\, \xi_m\, \xi_k \geq 0$$

for all $\xi \in \mathbb{R}^n$, a. e. $x \in \Omega$. Hence, $\nabla f^T \nabla u$ is a symmetric nonnegative matrix and so

$$\det \nabla f(x) \det \nabla u(x) \geq 0$$

a.e. in Ω. Finally, since $\det \nabla u(x) > 0$ a.e. $x \in \Omega$, we conclude that $\det \nabla f(x) \geq 0$ a.e. $x \in \Omega$.

Remark 3.16

(i) Corollary 3.15 may be useful to detect body forces f for wich $J(\cdot)$ does not admit a minimizer. As an example, if n is odd and if f is a "compressive force" of the type

$$f(x) = -\varepsilon x, \quad \varepsilon > 0,$$

then $\inf J(\cdot)$ is not attained.

(ii) The necessary conditions of Theorem 3.1 (N) were obtained regardless of the boundary conditions. In fact, it is possible to generalize them to the case of mixed displacement - traction boundary conditions as follows: let Ω be an open, bounded, strongly Lipschitz domain and let $h :(0,+\infty) \to \mathbb{R}$ be convex, bounded below, and such that $h(t) \to +\infty$ when $t \to 0^+$. Let $n < p \leq +\infty$, $1/p + 1/q = 1$ and let $f \in L^q(\Omega; \mathbb{R}^n)$, $t \in W^{1-1/q,q}(\partial\Omega_2; \mathbb{R}^n)$, where $\partial\Omega_1$ and $\partial\Omega_2$ form a partition of $\partial\Omega$. Given $u_0 :\partial\Omega_1 \to \mathbb{R}^n$, set

$$\mathcal{Q}_p := \{u \in W^{1,p}(\Omega; \mathbb{R}^n) \mid \det \nabla u > 0 \text{ a. e. in } \Omega \text{ and } u|_{\partial\Omega_1} = u_0\}$$

and

$$J_p(v) := \int_\Omega h(\det \nabla v(x))\, dx - \int_\Omega f(x).v(x)\, dx - \int_{\partial\Omega_2} t(x).v(x)\, dS(x).$$

Let $u \in \mathcal{Q}_p$ be such that $J_p(u) < +\infty$ and $J_p(u) \leq J_p(v)$ whenever $v \in \mathcal{Q}_p$ and $\|u - v\|_p < \varepsilon$. Suppose further that there is $u_1 \in C(\bar{\Omega}; \mathbb{R}^n)$ one–to–one in Ω such that $u|_{\partial\Omega} = u_1|_{\partial\Omega}$. Then

(a) $\nabla u^T f = \nabla\psi$ for some $\psi \in \mathcal{D}'(\Omega; \mathbb{R})$;

(b) there is a convex function G: convex hull $u_0(\Omega) \to \mathbb{R}$ such that

$$\int_\Omega G(u(x))\, dx \in \mathbb{R}$$

and

$$f(x) \in \partial G(u(x)) \text{ a.e. } x \in \Omega;$$

(c) if, in addition,

$$-\infty < \int_\Omega h'(\lambda \det \nabla u(x)) \det \nabla u(x)\, dx \leq \int_\Omega h'(\mu \det \nabla u(x)) \det \nabla u(x)\, dx < +\infty$$

for some $0 < \lambda < 1 < \mu$, then there are constants C_1 and C_2 such that

$$\psi(x) + C_1 = h(\det \nabla u(x)) - h'(\det \nabla u(x)) \det \nabla u(x) = G(u(x)) + C_2 \quad \text{a.e. } x \in \Omega.$$

4. AN EXAMPLE: GRAVITY–TYPE EXTERNAL FORCES.

In this section we establish existence of minimizers of $J(\cdot)$ when the body force f is of the gravity type and h grows sufficiently fast at infinity. Here $\{e_1, ..., e_n\}$ is the canonical basis of $\mathbb{R}^n$.

Proposition 4.1

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, strongly Lipschitz domain and let $f(x) = ke_n$, $k \in \mathbb{R}$. If u_0 = identity and if $h : (0,+\infty) \to [0,+\infty)$ is a C^2 convex function satisfying $h(t) \to +\infty$ when $t \to 0^+$, $h(t)/t \to +\infty$ when $t \to +\infty$, and $h'' > 0$, then there exists $u \in \mathcal{Q}$ such that $J(u) = \inf\{J(v) | v \in \mathcal{Q}\}$. Moreover, u^* is another minimizer of J in $\mathcal{Q}$ if and only if there exists $w \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ such that $u^* = u \circ w$, where

$$\begin{cases} \det \nabla w(x) = 1 & \text{in } \Omega \\ w(x) = x & \text{on } \partial\Omega. \end{cases}$$

We will use the following theorem due to BALL (see BALL [1], Theorem 2, page 320).

Theorem 4.2

Let the hypotheses of Theorem 3.3 hold, let $u_0(\Omega)$ satisfy the cone condition, and suppose that for some $q>n$,

$$\int_\Omega |\nabla u^{-1}(x)|^q \det \nabla u(x)\, dx < +\infty.$$

Then u is a homeomorphism of Ω onto $u_0(\Omega)$, and the inverse function x(u) belongs to $W^{1,q}(u_0(\Omega))$. The matrix of weak derivatives of $x(\cdot)$ is given by

$$\nabla x(v) = \nabla u^{-1}(x(v)) \quad \text{almost everywhere in } u_0(\Omega).$$

If, further, $u_0(\Omega)$ is strongly Lipschitz, then u is a homeomorphism of $\bar{\Omega}$ onto $u_0(\bar{\Omega})$.

Proof of Proposition 4.1. Define

$$H(t) := h\left(\frac{1}{t}\right) - \frac{1}{t}h'\left(\frac{1}{t}\right) \quad \text{for } t > 0.$$

Clearly $H' \geq 0$; moreover, if $t > 1$ then

$$H(t) \geq h\left(\frac{1}{t}\right) - \frac{1}{t}h'(1)$$

and so

$$\lim_{t \to +\infty} H(t) = +\infty.$$

Also,

$$\lim_{t \to 0^+} H(t) = -\infty.$$

In fact, if $\inf H = \alpha > -\infty$, then

$$\frac{d}{dt}\left(\frac{h(t) - \alpha}{t}\right) = \frac{\alpha - H\left(\frac{1}{t}\right)}{t^2} \le 0$$

and so

$$\lim_{t \to +\infty} \frac{h(t) - \alpha}{t} = \inf_{t > 0} \frac{h(t) - \alpha}{t} < +\infty$$

which contradicts the hypothesis on the growth of h at infinity. Hence, it is possible to find an increasing function $F: \mathbb{R} \to (0,+\infty)$ such that $F(t) \to 0$ when $t \to -\infty$, $F(t) \to +\infty$ when $t \to +\infty$, and $H \circ F =$ identity. Set

$$\rho(y) := F(ky_n + C)$$

and

$$G(y) := ky_n + C$$

where the constant C is determined by

$$\int_\Omega F(ky_n + C)\, dy = \text{meas}(\Omega).$$

Clearly

$$G \text{ is convex and } \nabla G = f. \tag{4.3}$$

Let $v \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ be a solution of (see DACOROGNA [5], MOSER [16] and TARTAR [17])

$$\begin{cases} \det \nabla v(y) = \rho(y) & \text{in } \Omega \\ v(y) = y & \text{on } \partial\Omega, \end{cases}$$

and set

$$\beta := \inf\{ F(ky_n + C) |\, y \in \Omega\}, \gamma := \sup\{F(ky_n + C) |\, y \in \Omega\}.$$

For all $q > n$ we obtain

$$\int_\Omega |\nabla v^{-1}(y)|^q \det \nabla v(y)\, dy = \int_\Omega |\text{adj } \nabla v(y)|^q \det \nabla v(y)^{1-q}\, dy$$
$$\le \text{Const. } \beta^{1-q} \|\nabla v\|_\infty^{q(n-1)}.$$

Thus, by Theorems 3.3 and 4.2,

$$u := v^{-1} : \bar{\Omega} \to \bar{\Omega}$$

satisfies

$$u = u_0 \text{ on } \partial\Omega \text{ and } \|\nabla u\|_\infty = \left\|\frac{\text{adj } \nabla v^T}{\rho}\right\|_\infty \le \text{Const. } \frac{\|\nabla v\|_\infty^{n-1}}{\beta}.$$

We conclude that $u \in \mathcal{G}$ and

$$h(\det \nabla u(x)) - h'(\det \nabla u(x)) \det \nabla u(x) = H(\rho(u(x)))$$
$$= H(F(ku_n(x) + C))$$
$$= G(u(x)). \qquad (4.4)$$

Moreover, by Theorem 3.3,

$$\int_\Omega h(\det \nabla u(x))\, dx = \int_\Omega h\left(\frac{1}{\rho(y)}\right) \rho(y)\, dy$$
$$\le \gamma \, \text{meas}(\Omega) \max_{t \in [1/\gamma, 1/\beta]} h(t)$$

and so,

and

$$\int_\Omega h'(\det \nabla u(x)) \det \nabla u(x)\, dx = \int_\Omega h'(\rho(y))\, dy$$
$$\ge \text{meas}(\Omega) \min_{t \in [1/\gamma, 1/\beta]} h'(t) > -\infty.$$

Thus, $J(u) < +\infty$ and by Theorem 3.1 (S), (4.3) and (4.4) it follows that u is a minimizer of $J(\cdot)$.

Finally, let $u^* = u \circ w$ where

$$\begin{cases} \det \nabla w(x) = 1 & \text{in } \Omega \\ w(x) = x & \text{on } \partial\Omega. \end{cases}$$

By Theorem 3.3 we have

$$J(u^*) = \int_\Omega h(\det \nabla u(w(x)))\, dx - \int_\Omega k e_n . u(w(x))\, dx$$
$$= \int_\Omega h(\det \nabla u(x))\, dx - \int_\Omega k u_n(w(x))\, dx$$
$$= J(u)$$

and so, u^* is another minimizer for J. Conversely, let $u^* \in \mathcal{Q}$ and define

$$w := v \circ u^*.$$

It is clear that $u^* = u \circ w$ in Ω, and $w(x) = x$ on $\partial\Omega$. If $\det \nabla w(x) \neq 1$ in a set of positive measure, then, by Theorem 3.3 and (4.4), we have

$$J(u^*) = \int h(\det \nabla u(w(x)) \det \nabla w(x))\, dx - \int k u_n(w(x))\, dx$$
$$> \int_\Omega \{h(\det \nabla u(w(x))) + h'(\det \nabla u(w(x))) \det \nabla u(w(x)) (\det \nabla w(x) - 1) - k u_n(w(x))\}\, dx$$
$$= \int_\Omega \{h'(\det \nabla u(w(x))) \det \nabla u(w(x)) \det \nabla w(x) + C\}\, dx$$
$$= \int_\Omega \{h'(\det \nabla u(x)) \det \nabla u(x) + C\}\, dx$$
$$= \int_\Omega \{h(\det \nabla u(x)) - k u_n(x)\}\, dx$$
$$= J(u).$$

We conclude that, if u^* is a minimizer of $J(\cdot)$, then $\det \nabla w = 1$ a.e. in Ω.

The following proposition provides an example where, if the material is not "strong enough", the infimum of $J(\cdot)$ is not attained. Suppose, for simplicity, that $n = 3$ and define

$$g = \text{center of gravity} := \frac{1}{\text{meas}(\Omega)} \int_\Omega x \, dx,$$

$$\theta^+ := \sup \{x.e_3 | \, x \in \Omega\},$$

$$\theta^- := \inf \{x.e_3 | \, x \in \Omega\}.$$

As in Theorem 4.1, u_0 = identity and $f = ke_3$, $k \in \mathbb{R}$.

Proposition 4.5

Let $h(t) = \alpha t + \beta/t$, $\alpha, \beta > 0$ and $t \in (0,+\infty)$. There exists $u \in \mathcal{G}$ such that $J(u) \leq J(v)$ for all $v \in \mathcal{G}$ if and only if

$$k \in \left(\frac{2\beta}{g_3 - \theta^+}, \frac{2\beta}{g_3 - \theta^-} \right).$$

Moreover, u is unique up to composition with a function $w \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ such that

$$\begin{cases} \det \nabla w(x) = 1 & \text{in } \Omega \\ w(x) = x & \text{on } \partial\Omega. \end{cases}$$

Proof. We divide the proof into two parts.

(i) Assume that

$$k \in \left(\frac{2\beta}{g_3 - \theta^+}, \frac{2\beta}{g_3 - \theta^-} \right)$$

and let

$$C := 2\beta - kg_3.$$

Clearly,

$$\int_\Omega \frac{ky_3 + C}{2\beta} dy = \text{meas}(\Omega) \quad \text{and} \quad \frac{ky_3 + C}{2\beta} > 0 \ \text{ in } \Omega.$$

Therefore, (see DACOROGNA [5], MOSER [16] and TARTAR [17]), there exists $v \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ such that

$$\begin{cases} \det \nabla v(y) = \dfrac{ky_3 + C}{2\beta} & \text{in } \Omega \\ v(y) = y & \text{on } \partial\Omega. \end{cases}$$

Setting $u := v^{-1}$ and $G(y) := ky_3 + C$, it can be shown that u is a (unique) minimizer of $J(\cdot)$ using Theorem 3.1(S) and following the proof of Theorem 4.1.

(ii) Suppose that

$$k \notin \left(\frac{2\beta}{g_3 - \theta^+}, \frac{2\beta}{g_3 - \theta^-} \right)$$

and assume that $J(\cdot)$ admits a minimizer u in $\mathcal{G}$.

Since $J(u) <+\infty$, we have

$$\int_\Omega \frac{1}{\det \nabla u(x)}\, dx < +\infty$$

which implies that

$$\int_\Omega h'(\lambda \det \nabla u(x)) \det \nabla u(x)\, dx = \int_\Omega \left\{ \alpha \det \nabla u(x) - \frac{\beta}{\lambda^2 \det \nabla u(x)} \right\} dx > -\infty$$

for all $\lambda \in (0,1]$. Hence, by Theorem 3.1 (N) (d) we obtain

$$\frac{2\beta}{\det \nabla u(x)} = k u_3(x) + C \quad \text{a. e. in } \Omega \tag{4.6}$$

for some constant C. Therefore, by Theorem 3.3 we deduce that

$$\begin{aligned} \text{meas}(\Omega) &= \int_\Omega \det \nabla u(x) \frac{k u_3(x) + C}{2\beta}\, dx \\ &= \int_\Omega \frac{k y_3 + C}{2\beta}\, dy \end{aligned}$$

and so,

$$C = 2\beta - k g_3$$

which, together with (4.6) implies that

$$\det \nabla u(x) = \frac{2\beta}{2\beta + k(u_3(x) - g_3)} \quad \text{a. e. in } \Omega \tag{4.7}$$

and

$$2\beta + k(y_3 - g_3) > 0 \quad \text{a.e. } y \in \Omega. \tag{4.8}$$

Finally, (4.8) and the assumption on k imply that

$$k \in \left\{ \frac{2\beta}{g_3 - \theta^+}, \frac{2\beta}{g_3 - \theta^-} \right\}$$

which, by (4.7), yields

$$\| \det \nabla u \|_\infty = +\infty$$

contradicting $u \in \mathcal{Q}$.

Remark 4.9

According to Proposition 4.5, $\inf\{J(v) | \ v \in \mathcal{Q}\}$ is not attained if $|k|$ is too big. In fact, it is possible to show that for

$$k \geq \frac{2\beta}{g_3 - \theta^-} \quad \left(\text{resp. } k \leq \frac{2\beta}{g_3 - \theta^+} \right)$$

the infimum is reached when some upper (resp. lower) slice of Ω of the form $\Omega_0 = \{x \in \Omega | \ \theta_C < x.e_3 < \theta^+\}$ (resp. $\Omega_0 = \{x \in \Omega | \ \theta^- < x.e_3 < \theta_C\}$) is "crushed down": the limit configuration reduces to

$$\{x \in \Omega | \ x.e_3 < \theta_C\} \quad (\text{resp. } \{x \in \Omega | \ x.e_3 > \theta_C\}).$$

APPENDIX.

Lemma 3.2

Let $\Omega \subset \mathbb{R}^n$ be an open bounded strongly Lipschitz domain and let $(p_1,...,p_N) \in \Omega^N$, $N \in \mathbb{N}$. If $p_i \neq p_j$ for $i \neq j$, $i, j = 1,...,N$, then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there is a sequence $w_k \in C^\infty(\bar{\Omega}; \mathbb{R}^n)$ verifying

$$\begin{cases} \det \nabla w_k(x) = 1 & \text{in } \Omega \\ w_k(x) = x & \text{on } \partial\Omega \end{cases}$$

and

$$w_k(x) \to w(x) \text{ a.e. } x \in \Omega,$$

where

$$w(x) = x \quad \text{if } x \in \Omega \setminus \bigcup_{i=1}^{N} (p_i + [-\varepsilon, \varepsilon]^n),$$

and $w(p_i + (-\varepsilon, \varepsilon)^n) = p_{i+1} + (-\varepsilon, \varepsilon)^n$ for $1 \leq i \leq N$ and $p_1 \equiv p_{N+1}$.

Proof. We divide this proof into three steps. For convenience of notation, we say that a sequence w_k satisfies the property $P(\Omega; Q_1, ..., Q_N)$, with $Q_i \subset\subset \Omega$ mutually disjoint parallelepipeds, if

$$\begin{cases} \det \nabla w_k(x) = 1 & \text{in } \Omega \\ w_k(x) = x & \text{on } \partial\Omega \end{cases}$$

and

$$w_k(x) \to w(x) \text{ a.e. } x \in \Omega,$$

where

$$w(x) = x \quad \text{if } x \in \Omega \setminus \bigcup_{i=1}^{N} \bar{Q}_i$$

and $w(Q_i) = Q_{i+1}$ for $1 \leq i \leq N$ and $p_1 \equiv p_{N+1}$.

1. Suppose that the lemma is true for $N = 2$ and let $(p_1, \dots, p_N) \in \Omega^N$, $p_{N+1} \equiv p_1$, $N \geq 3$. Then there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and for every $i \in \{2, ..., N\}$ there exist $w_k{}^i$ satisfying $P(\Omega; p_1 + (-\varepsilon, \varepsilon)^n, p_i + (-\varepsilon, \varepsilon)^n)$ (see Fig. 1). It is clear that the sequence

$$w_k := w_k{}^N \circ \dots \circ w_k{}^3 \circ w_k{}^2$$

verifies $P(\Omega; p_1 + (-\varepsilon, \varepsilon)^n, ..., p_N + (-\varepsilon, \varepsilon)^n)$.

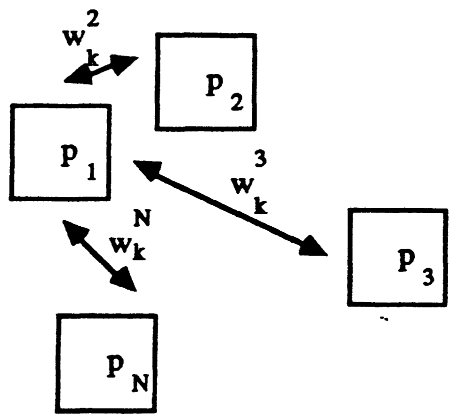

Fig. 1

2. In order to prove the lemma for $n = N = 2$, we start by considering the following auxiliary result:
(i) Let $Q := [-2, 2] \times [-1, 1]$ and define $Q^- := (-2,0) \times (-1,1)$, $Q^+ := (0,2) \times (-1,1)$. Let $U \supset\supset Q$ be an open bounded strongly Lipschitz domain. We claim that there exist $v_k \in C^\infty(U; \mathbb{R}^2)$ verifying $P(U; Q^+, Q^-)$ such that (see Fig. 2)

$$v(x) := \lim v_k(x) = -x \ \text{ if } x \in Q^- \cup Q^+.$$

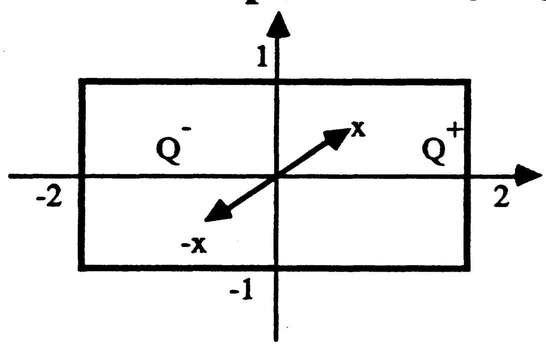

Fig. 2

Consider a sequence $\{U_k\}$ such that $Q \subset\subset U_k \subset\subset U$ and $U_k \downarrow Q$. Choose $\varphi_k \in C^\infty(U; \mathbb{R})$ verifying

$$\varphi_k(x) = 1 \ \text{ if } \ x \notin U_k, \ \nabla \varphi_k(x) \neq 0 \text{ for all } x \in U_k \text{ and } \varphi_k(x) = \varphi_k(-x) \text{ if } x \in Q.$$

Let $W_k = W_k(t,x)$ be the solution of

$$\begin{cases} \dfrac{dW_k}{dt} = F_k(W_k(t, x)) \\ \\ W_k(0, x) = x, \end{cases}$$

where

$$F_k := \left(-\frac{\partial \varphi_k}{\partial x_2}, \frac{\partial \varphi_k}{\partial x_1}\right).$$

As

$$\frac{d}{dt}\varphi_k(W_k) = 0,$$

we conclude that the trajectory of W_k is contained in the level set $L_k := \{\varphi_k = \varphi_k(x)\}$. Let $t_k(x)$

denote the period of $W_k(\cdot, x)$. Then t_k is a smooth function in U_k, t_k is constant on L_k and

$$t_k \to +\infty \text{ on } \partial U_K.$$

Thus

$$t_k(W_k(\cdot, x)) \equiv t_k(x) \quad \text{in } U_K, \tag{A.1}$$

$$W_k(t_k(x)/2, x) = -x \quad \text{if } x \in Q \tag{A.2}$$

and

$$W_k(t, x) \equiv x \quad \text{if } x \notin U_k. \tag{A.3}$$

Moreover, as

$$\text{div } F_k = 0,$$

we obtain

$$\begin{aligned} \frac{d}{dt} \det \nabla_x W_k &= (\det \nabla_x W_k)\, (\nabla_x W_k)^{-T} . \nabla_x (F_k(W_k)) \\ &= (\det \nabla_x W_k)\, (\nabla_x W_k)^{-T} . \nabla F_k \nabla_x W_k \\ &= (\det \nabla_x W_k) \text{ div } F_k \\ &= 0. \end{aligned}$$

Therefore

$$\det \nabla_x W_k = 1 \tag{A.4}$$

for all t. Let η_k be a smooth function such that

$$\eta_k(t) = \begin{cases} \dfrac{t}{2} & \text{if } t \le k \\ 0 & \text{if } t \ge 2k, \end{cases}$$

and set

$$v_k(x) := W_k(\eta_k(t_k(x)), x).$$

By (A.3) and (A.4) we have

$$v_k(x) = x \quad \text{if } x \notin U_k$$

and

$$v_k(x) \to -x \quad \text{if } x \in Q.$$

Finally, due to (A.1) we obtain

$$0 = \frac{\partial}{\partial t} t_k(W_k(t, x)) = \frac{\partial W_k}{\partial t} . \nabla t_k$$

which, together with (A.4), yields

$$\begin{aligned} \det \nabla v_k &= \det \left(\frac{\partial W_k}{\partial t} . \eta'_k(t_k) \nabla t_k + \nabla_x W_k \right) \\ &= 1. \end{aligned}$$

(ii) It follows immediatly from (i) that, if $Q^+_\theta := \{(x, -x(\tan \theta)/2 + y) \mid (x, y) \in Q^+\}$, $-\pi/2 < \theta < \pi/2$, obtained by shearing Q^+ (see Fig. 3), and if $Q^- \cup Q^+_\theta \subset\subset U$, then there exist $v_k \in C^\infty(U; \mathbb{R}^2)$ satisfying $P(U; Q^-, Q^+_\theta)$ such that

$$v \circ v\, (x) = x \quad \text{if } x \notin \partial Q^- \cup Q^+_\theta, \tag{A.5}$$

where

$v := \lim v_k$.

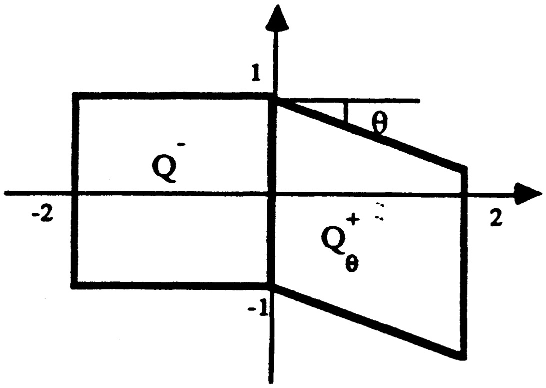

Fig. 3

(iii) We prove the assertion of the lemma in the case $N = n = 2$. Given $p, q \in \Omega$ with $p \neq q$, it is always possible to find an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there is a piecewise linear path $\mathcal{C}$ in Ω joining p to q that can be covered by a justaposition of m parallelograms $Q_i \subset\subset \Omega$ of the form $r_i + \varepsilon Q^+_{\theta i}$, where $\{r_1 = p - (\varepsilon, 0), r_2, \dots, r_m = q - (\varepsilon, 0)\} \subset \mathcal{C}$, $\theta_1 = \theta_m = 0$. The case $m = 2$ reduces to parts (i), (ii). Suppose that $m = 3$. Let v^1_k, v^2_k satisfy $P(\Omega; Q_1, Q_2)$ and $P(\Omega; Q_2, Q_3)$ respectively (see Fig. 4).

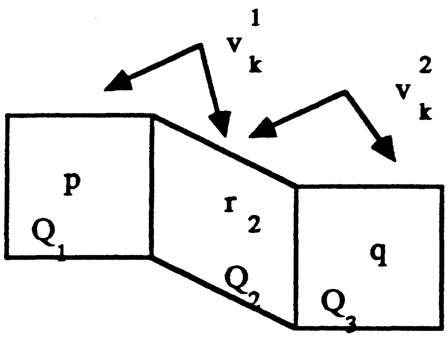

Fig. 4

Due to (A.5), we conclude that

$w_k := v_k^1 \circ v_k^2 \circ v_k^1$

verifies $P(\Omega; Q_1, Q_3)$ and

$$w \circ w(x) = x \quad \text{a. e.} \tag{A.6}$$

where

$w := \lim w_k$. The proof is similar for arbitrary m.

3. Let $N = 2$ and $n \geq 3$. Given two distinct points in Ω, p and q, it is possible to find an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there exist points $p_1 = p, \dots, p_m = q$, such that for all $1 \leq i, j \leq m$

p_i and p_{i+1} differ on at most two coordinates,

$p_i + [-\varepsilon, \varepsilon]^n \subset \Omega$

and

$$P_i + [-e,e]^n \cap P_j + [-e,e]^n = 0 \quad \text{if } i * j.$$

By 2 (iii), we can construct sequences w_k^* verifying P(£2; $p_{\{}$ + (-e, e)^n, p_{i+1} + (-e, e)^n), 1 £ i < m - 1 . It is clear that, by (A.6), the sequence

$$w_k := w£ \bullet w£ \ldots .wj^{1"2} \, 0 \, wj^{*"1} \, 0wj^{1"2} \ldots .w_k^2 \, 0w^\wedge$$

satisfies P(Q; p + (-e, e)^n, **q** + (-e, e)^n).

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