NORMAL VARIETIES OF COMBINATORS

by

Rick Statman
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

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Introduction

We adopt for the most part the terminology and notation of [1]. A combinator is a term with no free variables. A set of combinators which is both recursively enumerable and closed under β conversion is said to be Visseral ([5]). Given combinators F and G, the variety defined by $Fx \equiv Gx$ is the set of all combinators M such that $FM = GM$. Such a variety is said to be normal if both F and G are normal. In this note we shall be principally concerned with normal varieties..

Example 1 (Böhm and Dezani [2]): The normal variety defined by $x \circ K^n = I$ consists of the combinators $X =_\beta \lambda x.\ x\ \mathcal{H}_1 \ldots \mathcal{H}_n$ for some $\mathcal{H}_1 \ldots \mathcal{H}_n$.

Clearly, a variety of combinators is always Visseral. We recall the following theorem from [4].

Theorem 1. For Σ a set of combinators, the following are equivalent

(1) Σ is Visseral

(2) Σ is a variety

(3) Σ is the variety defined by $Fx = F$, for some combinator F.

For normal varieties the situation is quite different. For example, the β closure of $\{\Omega\}$ is Visseral but not a normal variety ([1] pg 445). More generally, if F and G are distinct normal forms then $F\Omega \neq_\beta G\Omega$. Combinators with this property are said to be transcendental. A transcendental is always order zero.

We shall proceed as follows. First, we shall discuss normal varieties of solvable combinators. In this context, the pattern matching equation $Fx = I$ plays a special role. Among the Visseral sets of solvable combinators are the binary languages. We shall characterize those binary languages which are normal varieties. From this characterization follows the result that the collection of normal varieties is Σ^0_3 complete. Next, we shall consider order zero solutions to normal equations. In this context, the fixed point equation $Fx = x$ plays a special role. We shall prove a number of results analogous to theorems from classical algebra and number theory; such as, Hilbert's Nullstelllensatz and Lindemann's theorem concerning the transcendence of e. We conclude with several open problems and applications.

It will be useful to have some special terminology for terms. A typical term $\mathcal{H}$ has the form

$$\underbrace{\lambda x_1 \ldots x_k}_{\text{prefix}}\ \underbrace{\underbrace{\underbrace{\overset{\overset{\underline{x}}{\text{variable}}}{\lambda x\ \mathcal{H}_0}}_{\text{abstraction}}\ \underbrace{\mathcal{H}_1}_{\text{arguement component}}}_{\text{head}}\ \Big\}\ \underbrace{\mathcal{H}_2 \ldots \mathcal{H}_\ell}_{\text{other componeents}}}_{\text{matrix}}$$

A head normal combinator of order one all whose components are combinators is called a <u>word.</u>

$$M \equiv \lambda x.\ x\, M_1 \ldots M_m$$

a typical word

A normal combinator of order one is called a <u>deed</u>

$$X \equiv \lambda x. x\mathcal{H}_1 \ldots \mathcal{H}_k$$

a typical deed

Words and deeds with at least one component are said to be <u>non–trivial.</u>

<u>Varieties</u> of <u>Solvable</u> <u>Combinators</u>

We begin with the following.

<u>Lemma</u> <u>1.</u> Suppose Σ is a normal variety. Then one of the following holds.

(1) $\Sigma = \emptyset$

(2) $K^{\infty} \in \Sigma$

(3) There exists a deed F such that Σ is contained in the variety defined by $Fx = I$

Proof. Suppose Σ is defined by $Gx = Hx$. The proof is by induction on the structure of G and H, and its routine.

Remark 1 It is easy to see that the deed F in lemma 1 be assumed to be non–trivial.

Theorem 1 and lemma 1. yield the following.

Proposition 1. Suppose that Σ is a set of solvable combinators. Then the following are equivalent.

(1) Σ is a normal variety

(2) Σ is Visseral and there exists a non–trivial deed F such that Σ is contained in the variety defined by $Fx = I$

Proof. (1) $\Longrightarrow$ (2) by lemma 1 and remark 1. Suppose (2). By theorem 1 there is a combinator G such that $M \in \Sigma \Leftrightarrow GM =_\beta G$. Let $x\mathcal{F}_1 \ldots \mathcal{F}_k$ be the matrix of F. We "make G normal" by replacing each redex $(\lambda x\, \mathcal{H})\, \mathcal{Y}$ by $x\mathcal{F}_1 \ldots \mathcal{F}_k\, (\lambda x\, \mathcal{H})\, \mathcal{Y}$. Set $H_1 \equiv \lambda x\, [\,Gx, Fx]$ and $H_2 \equiv \lambda x\, [G, I]$. Then $M \in \Sigma \Leftrightarrow H_1\, M =_\beta H_2\, M$.

Example 2: Suppose k and l are given and Σ is a set of combinators such that whenever $M \in \Sigma$ there exist $m \le k$, $n \le l$, $i \le m$, and $\mathcal{H}_1, \ldots \mathcal{H}_n$ satisfying $M =_\beta \lambda x_1 \ldots x_m . x_i\, \mathcal{H}_1 \ldots \mathcal{H}_n$. Then Σ is contained in the variety defined by $Fx = I$ where $F \equiv \lambda x.$

$$x\ \underbrace{(K^{\ell} I) \ldots (K^{\ell} I)}_{k}\ \underbrace{I \ldots I}_{\ell}$$

A language is a Visseral set of combinators each of which β converts to a word. If Σ is a set of combinators then Σ^+ is the β closure of the set of all non–trivial words with components form Σ. Clearly, Σ is Visseral if and only if Σ^+ is a language.

Proposition 3. Suppose that Σ is a β closed set of solvable combinators. Then the following are equivalent.

(1) Σ is a normal variety

(2) Σ^+ is normal variety

The proof of proposition 3 requires two ideas that we shall need to refine for the proof of theorem 2 below.

First, we consider the symbolic action of a deed on a word under head reduction. Let 2£ = Xx. xuj. . . u_n. Recall that ||F|| is the number of symbols in F.

Lemma 2 Suppose that k and n are given with n > o. Suppose that F is a normal combinator with at least one component, and

(i) F has a prefix of length k+1, and

(ii) ||F|| ≤ n.

Then F U. u_1. . . u, has a normal form with an empty prefix and some u. at the head.

Proof. By induction F.

To apply this to proposition 3, suppose that J^+ is ∈ the variety defined by Fx = I for F a non—trivial deed. Then, for sufficiently large n, F(Xx x u . . . . u) has a normal form U with an empty prefix and u at the head. Thus, for the deed Xn U}

we have M , e S =* (Xn % M = F (Ax xM. . . M) = I.

Second we must take into consideration the fact that a combinator computes on a word sequentially from left to right. Memory is added by concatenating on the right. Suppose that F is a deed with k > o components and MeE=> FM = K. As in [1] 6.1 we can define a combinator G satisfying Gxyz = x (Xn IF Fu THEN G (Xv x(Kv) (K I^{k+1})) (Aab a(yab)) z ELSE zxy. Let "n" be the n^{th} Church numeral, and suppose M = Ax.x M_1 . . . M_m e S. We have GM V z ^ G(Xx. xM_2 . . . M_m (K I^{k+1})) "Wl"^{1} z M_2 . . . M_m = . . . ^ G(Ax . x(K^{k+1}I) . . . (K^{k+1} I)) ^{n}n+nT z

m-2

$$\underbrace{\underbrace{KI^{k+1}) \ldots (KI^{k+1})}_{m-1} M_m \underbrace{(KI^{k+1}) \ldots (KI^{k+1})}_{m-2} \ldots M_2 \ldots M_m}_{m \text{ blocks}} =_\beta$$

$$z\ (\lambda x\ x \underbrace{(K I^{k+1}) \ldots (K I^{k+1})}_{m})\ \ulcorner n+m \urcorner \underbrace{\ldots}_{(m+1)\,.\,(m-1)}$$

Now it is easy to construct a combinator P satisfying

$$P(\lambda x.\ \underbrace{x(K I^{k+1}) \ldots (K^{k+1} I)}_{m})\ \ulcorner n+m \urcorner = K I^{(n+m)\,.\,(m-1)}$$

since the combinators $\lambda x\, x \underbrace{(K I^{k+1}) \ldots (K I^{k+1})}_{m}$ form an adequate numeral system ([1] 6.4). We have that $GM \ulcorner 1 \urcorner P =_\beta I$, but $Gx \ulcorner 1 \urcorner P$ is not yet normal. We "make $Gy \ulcorner 2 \urcorner P$ normal" as follows. Let $\mathscr{F}$ be the matrix of F with head variable x. Replace each redex $(\lambda x\ \mathscr{H})\ \mathscr{Y}$ by $\mathscr{F}(\lambda x\ \mathscr{H})\ I\ \mathscr{Y}$. Let the result be $\mathscr{G}$. Final ly set $H \equiv \lambda y\ y(\lambda x\ \mathscr{G})$. For M as above we have $HM =_\beta M(\lambda x\ [^M|y]\,\mathscr{G}) =_\beta GM \ulcorner 2 \urcorner P\ M_2 \ldots M_m =_\beta I$. This completes the proof of proposition 3.

Thus, for our purposes, for sets of solvable combinators it suffices to study languages.

A language Σ is said to be <u>binary</u> if whenever

$\lambda x\,.\, xM_1 \ldots M_m \in \Sigma$, for each i=1 . . . m

$$M_i =_\beta \begin{cases} I \\ \text{unsolvable} \end{cases}$$

Examples 3: For each e and k define combinators M_k^e by

$$M_k^e = \begin{cases} I \text{ if } k \in W_e \\ \text{unsolvable} \quad \text{else} \end{cases}$$

as in [1] pg 179. Let Σ_e = the β closure of the set

$$\{\lambda x.\ \underbrace{xM_k^e \ldots M_k^e}_{k} : k=1, \ldots\}$$

A binary language Σ is said to be bounded away from $\perp$ if there is an infinite recursively enumerable set $\mathscr{S}$ of positive integers such that whenever $s \in \mathscr{S}$,

$\lambda x.xM_1 \ldots M_m \in \Sigma$ and $s \leq m$ we have $M_s =_\beta I$.

Theorem 3 Suppose that Σ is a binary language. Then the following equivalent

(1) Σ is a normal variety

(2) Σ is bounded away from $\perp$

To prove theorem 2 we need to refine the two ideas in the proof of proposition 3. First, we refine the "symbolic computation". For this we need a refinement of the standardization theorem ([1]) pg. 318). A cut of $\mathscr{H}$ is a maximal applicative subterm of $\mathscr{H}$ with a redex at its head. The cuts of $\mathscr{H}$ and the redexes of $\mathscr{H}$ are in one to one correspondence, and we shall use notions defined for one freely for the other. The following notions will be used exclusively when $\mathscr{H}$ has a head redex. The major cut of $\mathscr{H}$ is the leftmost cut whose abstraction term is in head normal form. The major variable of $\mathscr{H}$ is the head variable of the abstraction term of the major cut. The <u>base</u> of $\mathcal{H}$ is the matrix of the abstraction term of the major cut. When the major variable of $\mathcal{H}$ is bound in the prefix of the abstraction term of a cut, this cut is called the <u>minor cut</u>.

$$\lambda w_1 \ldots w_k .$$

$$(\lambda x_1 \ldots x_\ell (\ldots ((\lambda y_1 \ldots y_m (\ldots ((\lambda z_1 \ldots z_p$$

$$y_i \ \mathcal{W}_1 \ldots \mathcal{W}_q) \mathcal{Z}_1 \ldots \mathcal{Z}_r) \ldots)) \mathcal{Y}_1 \ldots \mathcal{Y}_s) \ldots)) \mathcal{H}_1 \ldots \mathcal{H}_t$$

base

major cut

minor cut

head cut

the major and minor cuts of a term

A reduction of $\mathcal{H}$ is said to be <u>solving</u> if the redex contracted corresponds to the minor cut if exists and the major cut otherwise.

<u>Lemma 3</u> If $\mathcal{H}$ is solvable then the solving reduction sequence beginning with $\mathcal{H}$ achieves a head normal form.

Proof. Each solving reduction reduces the length of a head reduction to head normal form. Suppose F is a deed. We symbolically calculate F $\mathcal{U}_m$ as follows. Perform solving reductions until some u_i is the major variable. Next, substitute I for this occurrence of u_i and repeat the process. The calculation terminates in I if and only if there is a binary word $M \equiv \lambda x . xM_1 \ldots \ldots M_m$ s.t. $FM =_\beta I$. Let $1 \le k < m$, and suppose that $\mathcal{U}_m$ is the abstraction term of the minor (= major) cut at some stage, say

$$\mathscr{U}_m \mathscr{H}_1 \mathscr{H}_2 \ldots \mathscr{H}_\ell \to$$

$$\cfrac{\cfrac{\mathscr{H}_1\ u_1 \cdots u_k}{\mathscr{H}_0}\ u_{k+1} \cdots u_m\ \mathscr{H}_2 \cdots \mathscr{H}_\ell}{\mathscr{H}}$$

From this stage on we trace the decendants ([3]) pg. 18) of $\mathscr{H}_0$ so long as they exist and maintain the form $\mathscr{H}' = \mathscr{H}_0' u_{k+1} \cdots u_m \mathscr{H}_2 \cdots \mathscr{H}_\ell$. Such an $\mathscr{H}_0'$ is either

(a) an initial segment of the base with the major variable at the head of $\mathscr{H}_0'$, or

(b) an initial segment of some cut with the abstraction term of the major cut contained in $\mathscr{H}_0'$.

Now suppose for some binary word $M = \lambda x.\, x M_1 \ldots M_m$, $FM = I$. Since $\mathscr{H}_0'$ has no decendant in I, the form of $\mathscr{H}_0'$ must change. Thus there is some stage at which $\mathscr{H}_0'$ coincides with the abstraction term of the minor cut if one exists or the major cut, with major variable not one of the u_i, otherwise. At this stage we have the following

(*) If $m-k$ exceeds the length of the prefix of $\mathscr{H}_0'$ then some u_i, for $k < i \leq m$, is the major variable at some later stage in the computation.

Since, if $\mathscr{H}_0'$ is the abstraction term of the major cut with major variable not one of the u_i, no minor cut exists, and m-k exceeds the number of A's in the prefix of $\mathscr{H}_0'$ then u_m is a component of the last term in the computation. The property (*) is just a refinement of lemma 2.

To prove $(1) \Rightarrow (2)$, suppose that the computation beginning with F 2^ terminates in I. Let k be larger than the number of symbols in the computation. Recall that we can assume that F is non—trivial so $\mathscr{U}_{m+k}$ is the abstraction term of the minor cut at least once in the computation beginning with F $\mathcal{U}_{m+k}$. Consider the first time any one of the $\mathcal{H}$ changes form. By choice of k the hypothesis of (*) is satisfied. Thus if the computation of F $\mathcal{U}_{m+k}$ terminates in I then there is some u_i for $m < i \leq m+k$ which is the major variable at some stage in the computation. Now it is easy to see that (1) $\Longrightarrow$ (2).

Next, we need to refine the construction in the second part of the proof of proposition 3. Changes are needed for the following reason. Not every M_i is solvable, so in cycling through them some must be skipped; this is where $\mathcal{S}$ is used. We sketch only the construction of G; the rest is routine. Suppose that h is a total recursive function which enumerates an infinite subset of $\mathcal{S}$ in increasing order, and H is combinator which represents ([1] 6.3) h on the Church numerals. Suppose that P is a combinator which satisfies P

$$(\lambda x \ \underbrace{x\ K_* \ . \ . \ . \ K_*}_{m}) \ulcorner n \urcorner =_\beta K^{n.(m-1)} I$$

Now construct G satisfying $G\ u\ v\ w\ x =_\beta$

IF $(\text{Zero}_c\ x)$ THEN $(\ u(\lambda y$ IF $y\ K^2\ K_*$ THEN

$G\ (\lambda z\ \ u\ (Kz)\ K_*\ )\ \ (S_c^+\ v)\ (S_c^+\ w)\ (\text{Minus}_c\ (H(J_c^+\ w))\ (Hn))$

ELSE $P\ u\ v)$ ELSE $G\ (\lambda z\ \ u(Kz)\ K_*\ )\ (S_c^+\ v)\ n$

$P_c^- x)$ (see [1] pg. 135). To understand the action of G it suffices to understand the function of v,w, and x. v counts the total number of moves made by G, w is the number of the next integer in $\mathcal{S}$, according to h, and x is the number of moves needed to get to $M_{h(w)}$. This completes our sketch of the proof of theorems 3.

We obtain the following

<u>Corollary 1.</u> The collection of normal varieties is

Σ_3^0 complete.

Proof. Σ_e is a normal variety $\Leftrightarrow$ W_e is confinite.

Example 3 (continued): Let Δ_e = the β closure of $\{\lambda x.\ xM_1^e \ldots M_k^e : k = 1 \ldots\}$. Then Δ_e is a normal variety if and only if W_e is infinite.

Order Zero Solutions

If F is a deed we write F for its matrix; we shall always assume that x is the head variable. For deed F, G, and H the relation $F \underset{G}{\succ\!\!\rightarrow} H$ holds if H is obtained from F by replacing one occurrence of x by G. If Δ is a set of deeds the relations $\underset{\Delta}{\succ\!\!\rightarrow}$, $\underset{\Delta}{\succ\!\!\twoheadrightarrow}$, and $\underset{\Delta}{=}$ are defined in the obvious way.

Example 4: $F \underset{G}{\succ\!\!\twoheadrightarrow}$ the normal form of F∘G. It is easy to see that $\underset{G}{\succ\!\!\twoheadrightarrow}$ is Church–Rosser, upward Church–Rosser, and has unique upward normal forms (see [1] 3.5). If $F \underset{G}{=} H$ and M is any fixed point of G then $FM \underset{\beta}{=} HM$. A set Δ of deeds is called a behavior if whenever $F \underset{G}{\succ\!\!\rightarrow} H$ and two of F, G, and H belongs to Δ then the third of F, G, and H belongs to Δ

Example 5: The powers of F is the set $\{ H : F \underset{F}{\succ\!\!\twoheadrightarrow} H\} \cup \{I\}$. The powers of F form a behavior

Suppose $F \equiv \lambda x_1 \ldots x_k . x_i\, \mathcal{F}_1 \ldots \mathcal{F}_\ell$ and $G \equiv \lambda x_1 \ldots x_m . x_j\, \mathcal{G}_1 \ldots \mathcal{G}_n$ are normal combinators and there is an order zero combinator α s.t. $F\,\alpha \underset{\beta}{=} G\,\alpha$. Note that k=m, i=j, and $i \neq 1 \Rightarrow l = n$. Symmetrically, assume $l \leq n$. We define a set $\delta(F,G)$ of deeds as follows.

$$\delta(F,G) = \begin{cases} \{I\} \cup \bigcup_{p=1}^{\ell} \delta(\lambda x_1 \ldots x_k\, \mathcal{F}_p, \lambda x_1 \ldots x_m\, \mathcal{G}_\rho) & \text{if } i \neq 1 \\ \{\lambda x_1 .\ x_1 \mathcal{G}_1 \ldots \mathcal{G}_{n-\ell}\} \cup \bigcup_{p=1}^{\ell} \alpha(\lambda x_1 \ldots x_k\ \mathcal{F}_p,\ \lambda x_1 \ldots x_m\, \mathcal{G}_{n-\ell+p}) & \text{if } i=1. \end{cases}$$

Note that $I \in \delta(F,G)$ and whenever $H \in \delta(F,G)$ $\check{H}$ is a subterm of F or G.

<u>Lemma 4.</u> If F and G are normal and α is an order zero combinator then $F\ \alpha = G\ \alpha \Leftrightarrow \forall$ $H \in \delta(F,G)\ \alpha =_{\beta} H\ \alpha$.

Proof. By induction on F and G.

<u>Example 6:</u> If a is a set of order zero combinators let $\delta(a) = \{F\colon \forall\, \alpha \in a\ \ \alpha =_{\beta} F\ \alpha$ and F is a deed$\}$. Then $\delta(a)$ is a behavior, since whenever $F, H \in \delta(a)$ and $F \mathrel{>\!\!\xrightarrow[G]{}} H$ we have $G \in \delta(F,H)$.

A behavior Δ is said to be <u>non–trivial</u> if there are order zero α and β such that $\Delta \subseteq \delta(\alpha)$ but $\Delta \nsubseteq \delta(\beta)$

The following is an analogue of Hilbert's Nullstellensatz.

<u>Proposition 4.</u> If a is a non empty set of order zero combinators then $\delta(a)$ is the set of all powers of one of its members.

Proof. Let F be a shortest member of $\delta(a) - \{I\}$. We show by induction on $G \in \delta(a) - \{I\}$, that $F \mathrel{>\!\!\xrightarrow[G]{}\!\!>} G$. If $F \nsubseteq G$ then there is some $H \in \delta(F,G) - \{I\}$. By lemma 4 $H \in \delta(a)$. Since F is shortest H is a proper subterm of G; so by induction hypothesis $F \mathrel{>\!\!\xrightarrow[F]{}\!\!>} H$. There is some deed L s.t. $L \mathrel{>\!\!\xrightarrow[H]{}} G$ so $L \in \delta(a) - \{I\}$. Thus by induction hypothesis $F \mathrel{>\!\!\xrightarrow[F]{}\!\!>} L$. Hence $F \mathrel{>\!\!\xrightarrow[F]{}\!\!>} G$. This completes the proof.

<u>Corollary 2.</u> For any normal combinators F and G there exists a normal H such that for all order zero combinators α

$$F\ \alpha =_{\beta} G\ \alpha \Leftrightarrow \alpha = H\ \alpha$$

Proof. Let $a = \{\alpha\colon F\ \alpha =_{\beta} G\ \alpha$ and α is order zero.$\}$. If $a = \emptyset$ put $H \equiv K$. Otherwise $\delta(a)$ is the set of all powers of some $H \in \delta(a)$. We claim that this is the desired H. For suppose $F\ \alpha =_{\beta} G\ \alpha$. Then $\alpha \in a$ so $\alpha =_{\beta} H\ \alpha$. Conversely, if $\alpha =_{\beta} H\ \alpha$ then, since $\delta(F,G) \subseteq \delta(a)$, whenever $L \in \delta(F,G)$ $\alpha = L\ \alpha$. Thus, by lemma 4, $F\ \alpha \underset{\beta}{=} G\alpha$.

Corollary 3. Every non trivial behavior is contained in the set of powers of some deed.

Remark 2.. We say G is an atom if whenever G is a power of F then $F \equiv G$. It can be proved that each deed is a power of a unique atom. This analogous to the fact from formal language theory that whenever $u^n = v^m$ both u and v are power of a common atom. We shall not give the proof here because the result will not be used below.

An order zero combinator α is said to be algebraic if there are normal F and G such that $F \not\equiv G$ and $F\ \alpha \underset{\beta}{=} G\ \alpha$.. Otherwise α is said to be transcendental. A set $\{\alpha_1, \ldots .\ \alpha_k\}$ of order zero combinators is said to be algebraically dependent if there exist normal combinators F and G with $F \not\equiv G$ such that $F\ \alpha_1 \ldots \alpha_k \underset{\beta}{=} G\ \alpha_1 \ldots \alpha_k$

Lemma 5 Suppose $\{\alpha_1 \ldots \alpha_k\}$ is set of algebraically dependent combinators. Then there exists an i, $1 \leq i \leq k$, and a normal $H \not\equiv U_i^k$ such that $\alpha_i \underset{\beta}{=} H\ \alpha_1 \ldots \alpha_k$.

Proof. Suppose F and G are distinct normal combinators such that $F\ \alpha_1 \ldots \alpha_k \underset{\beta}{=} G\ \alpha_1 \ldots \alpha_k$. The proof is a routine induction on F and G.

Remark 3 In lemma 5, since each α_i is order zero, H can be put in the form $\lambda x_1 \ldots x_k . x_j (H_1 x_1 \ldots x_k) \ldots (H_\ell x_1 \ldots x_k)$ with each H_p normal.

Proposition 5 Suppose $\{\alpha_1, \ldots, \alpha_k\}$ is a set of algebraically independent combinators. Then there is an order zero combinator α_{k+1} such that $\{\alpha_1, \ldots, \alpha_k, \alpha_{k+1}\}$ is algebraically independent.

Proof. Consider the equations:

(1) $M \underset{\beta}{=} M(H_1 \alpha_1 \ldots \alpha_k M) \ldots (H_\ell \alpha_1 \ldots \alpha_k M),\ \ell \geq 1$

$(2)_i$ $M \underset{\beta}{=} \alpha_i (H_1 \alpha_1 \ldots \alpha_k M) \ldots (H_\ell \alpha_i \ldots \alpha_k M)$

$(3)_i$ $\alpha_i \underset{\beta}{=} M(H_1 \alpha_1 \ldots \alpha_k M) \ldots (H_\ell \alpha_1 \ldots \alpha_k M)$

$(4)_{ij}$ $\alpha_i \underset{\beta}{=} \alpha_j (H_1 \alpha_1 \ldots \alpha_k M) \ldots (H_\ell \alpha_1 \ldots \alpha_k M)$

where if i=j then $L \geq 1$. For each of the $k^2 + 2k + 1$

equations $E(H_1, \ldots, H_\ell, M)$ the set $\{M : \exists H_1 \ldots H_\ell$ normal s.t. $E(H_1, \ldots, H_\ell, M)\}$ is Visseral. For each of these sets the following are not members: (1) Ω, $(2)_i$ K^∞, $(3)_i$ K^∞, $(4)_{ij}$ α_i. Thus each of the complements is co–Visseral and non–empty. In addition, the set of order zero combinators is co–Visseral and non–empty. Hence, by [5] 2.5, all these co–Visseral sets intersect, and any member of the intersection is the desired α_{k+1}. This completes the proof.

Proposition 5 allows the construction of many algebraically independent transcendentals. However, a more direct construction of transcendentals can be achieved using proposition 4. The following is an analogue of Lindemann's theorem.

Theorem 4 Suppose α is an order zero algebraic combinator. Then there are only finitely many algebraic combinators β, up to β conversion, of the form $\beta \underset{\beta}{=} F\ \alpha$, for F a deed.

Proof. First suppose α is order zero and $\alpha \underset{\beta}{=} \alpha\,(F_1\alpha) \ldots (F_n\alpha)$, $\beta \underset{\beta}{=} \alpha(G_1\alpha) \ldots (G_m\alpha)$, and $\beta \underset{\beta}{=} \beta\,(H_1\beta) \ldots (H_k\beta)$, where the F_i, G_j, and H_ℓ are all normal and $n \geq 1$, $m \geq o$, and $k \geq 1$ are as small as possible. Then $\alpha \underset{\beta}{=} \beta(H_1\beta) \ldots (H_{k-m}\beta)$, so, setting $G \equiv \lambda x.\ x(G_1x) \ldots (G_mx)$ and $H^- \equiv \lambda x.\ x(H_1x) \ldots (H_{k-m}x)$, we have $\alpha \underset{\beta}{=} (H^- \circ G)\ \alpha$. Now if F is a deed such that $\delta(\alpha)$ coincides with the powers of F and $F \equiv \lambda x\ \ x\mathcal{F}_1 \ldots \mathcal{F}_n$ let $F_i \equiv \lambda x\ \mathcal{F}_i$. If α is algebraic, we have $n \geq 1$ and n is as small as possible. Let J be the normal form of $H^- \circ G$. If $\beta \underset{\beta}{\neq} \alpha$ then $J \underset{\beta}{\neq} I$ and $F \underset{F}{\gg} J$. By inspection, there exists some $r \leq n$ such that $\lambda x.\ \ x\mathcal{F}_1 \ldots \mathcal{F}_r \underset{F}{\gg} G$ so $\beta \underset{\beta}{=} (\lambda x.\ \ x\mathcal{F}_1 \ldots \mathcal{F}_r)\ \alpha$. This completes the proof.

Applications and Open Problems

First we solve a problem of Böhm & Dezani ([2]) pg. 185)

Proposition 6 It is undecidable whether $Fx = I$ has a (normal) solution, for normal combinators F.

Proof. For any combinator M, "make M normal" by replacing each redex $(\lambda x\ \mathcal{H})\ \mathcal{Y}$

by $x (\lambda x \mathscr{H}) \mathscr{Y}$ and replace the result with $M^{\#} \equiv \lambda x.\ x \circ M \circ x$. Observe that $M^{\#} I =_{\beta} M$. Now, if $M^{\#} N =_{\beta} I$ then N is both right and left β invertible, so by [1] 21.4.8, $N =_{\beta} I$. Hence $M_e^{\#} x =_{\beta} I$ has a (normal, solution if and only if $e \in W_e$.

Remark 4 A similar arguement works for $\beta\eta$ conversion.

Below we gave a stronger result for general normal equations.

Our next application concerns Hilbert's 10th problem. Suppose d is an adequate numeral system ([1] pg. 139) with normal test for zero.

Proposition 7 Suppose Σ is the β closure of a recursively enumerable set of d_n. Then Σ is normal variety.

Proof. It is easy to see that there is deed F s.t. for all n $F\, d_n =_{\beta} I$

The next example shows that the condition of normality on the zero test is necessary.

Example 7 Let $d_n^o \equiv \lambda x.\ \ x(K^{n^2+1} \ulcorner n \urcorner)$ and $d_n^{m+1} \equiv \lambda x.\ \ x d_n^m \underbrace{\Omega \ . \ . \ . \ \Omega}_{n}$. We write d_n for d_n^n. Construct F satisfying $F =_{\beta} (\lambda xy.\ y\, x)\, F$. We have $F\, d_n^o =_{\beta} F\, (K^{n^2+1} \ulcorner n \urcorner) =_{\beta} K^{n^2} \ulcorner n \urcorner$. By induction, $F\, d_n^m =_{\beta} K^{n^2 - m.n} \ulcorner n \urcorner$, for $m \leq n$. Thus $F\, d_n =_{\beta} \ulcorner n \urcorner$, and d is an adequate numeral system. By the method of the proof of lemma 2, d is not a normal variety.

Despite Corollary 1 it may be possible to give a coherent solution to the following.

Open Problem Characterize the normal varieties

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