# MARTINGALE AND DUALITY METHODS FOR UTUILITY MAXIMIZATION IN AN INCOMPLETE MARKET

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# ABSTRACT

The problem of maximizing the expected utility from terminal wealth is well understood in the context of a complete financial market. This paper studies the same problem in an incomplete market containing a bond and a finite number of stocks, whose prices are driven by a multidimensional Brownian motion process $W$. The coefficients of the bond and stock processes are adapted to the filtration (history) of $W$, and incompleteness arises when the number of stocks is strictly smaller than the dimension of $W$. It is shown that there is a way to complete the market by introducing additional, "fictitious" stocks, so that the optimal portfolio for the thus completed market coincides with the optimal portfolio for the original incomplete market. The notion of a "least favorable" completion is introduced and is shown to be closely related to the existence question for an optimal portfolio in the incomplete market. This notion is expounded upon using martingale techniques; several equivalent characterizations are provided for it, examples are studied in detail, and a fairly general existence result for an optimal portfolio is established based on convex duality theory.

## 1. INTRODUCTION

This paper studies the problem of an agent who receives a deterministic initial capital, which he must then invest in an incomplete market so as to maximize the expected utility of his wealth at a prespecified final time. The market consists of a bond and $m$ stocks, the latter being driven by a d–dimensional Brownian motion. In such a model, incompleteness arises when $m$ is strictly smaller than $d$. The market coefficients, i.e., the interest rate, the rates of stock appreciation, and the stock volatility coefficients, are random processes adapted to the full d–dimensional Brownian motion. When $m < d$, it is typically not possible to construct a portfolio consisting of the bond and the $m$ available stocks, so as to completely hedge the risk associated with these coefficient processes.

In Sections 2 through 5, we define the utility maximization problem faced by the agent. In Section 6 we present the solution when the market is complete ($m = d$), and complete hedging is possible. This solution proceeds in three steps. First, on the underlying probability space one determines a new measure which discounts the growth inherent in the market; under this measure, the expected value of the final wealth attained by any reasonable portfolio is equal to the initial endowment. Secondly, among all random variables whose expectation under the new measure is equal to the initial endowment, a most desirable one is determined. Thirdly, it is shown that a portfolio can be constructed which attains this most desirable random variable as its terminal wealth; this portfolio is optimal. A complete market is one in which the agent can construct a portfolio which attains as final wealth any random variable with expectation under the new measure equal to the initial endowment. Because such a construction is possible, it is said that the agent can hedge against the risk associated with this market. Mathematically, the construction of a portfolio uses the fact that any martingale with respect to a Brownian filtration can be represented as a stochastic integral with respect to the Brownian motion; the integrand in this representation leads to the portfolio we are seeking. However, if there are fewer than $d$ stocks, this line of argument fails.

In Section 7 we introduce a convenient way of thinking about an incomplete market: fictitious completion. When there are fewer than $d$ stocks, then one augments the stocks with certain fictitious ones so as to create a complete market. If the fictitious stocks have a high appreciation rate, then under an optimal portfolio the agent will hold a long position in them, but if they have a low (even negative) appreciation rate, then he will hold a short position. Thus one would expect to be able to adjust the appreciation rates of the fictitious stocks so that the agent, by optimal choice, does not invest in them at all. These judiciously chosen fictitious stocks allow us to write down the complete market solution for the utility maximization problem but are superfluous in the actual implementation of the optimal portfolio, which must then also be optimal for the original incomplete market. The fictitious completion with the above property is the least advantageous to the agent, because the portfolio which is optimal under this completion is available to him under every other fictitious completion. We thus have the notion of a least favorable fictitious completion: for every fictitious completion we compute the portfolio which maximizes the expected utility of final wealth, and then we choose the completion which makes this maximum expected utility as small as possible.

As explained in Section 7, a convenient way to parametrize fictitious completions of an incomplete market is by a certain space of continuous local martingales, each local martingale being the Radon–Nikodym derivative process of the new measure alluded to in the earlier discussion of complete markets. This kind of parametrization is studied in Section 8, and several pertinent results are established. One would also like to be able to characterize the local martingale corresponding to the least favorable fictitious completion, and to show that it gives rise to an optimal portfolio in the original incomplete problem; this program is carried out in Section 9, in which various such equivalent characterizations are provided. Section 10 studies two examples in which the least favorable fictitious completion can be computed fairly explicitly. In the first example the utility function is logarithmic, and it is discovered that the fictitious stocks in the least favorable completion should have rates of appreciation equal to the interest rate of the bond. This is a very general result, insensitive to the nature of the dependence of market coefficients on the driving Brownian motion. In the second example it is assumed that the utility function is of the power form, and that the driving Brownian motion splits into two independent parts; the first part drives the stock processes, whose coefficients are adapted solely to the second part. The least favorable local martingale is exhibited as the solution to a martingale representation problem, and the optimal portfolio is found to be given by the formula already known to be correct for deterministic model coefficients.

In Section 11 we introduce an auxiliary optimization problem, involving the family of local martingales which characterize fictitious completions; this problem is "dual" to the "primal" utility maximization question of Section 5, in the sense of convex duality. We study the relation between the primal and dual problems, and explain how a solution to the latter induces one for the former. The question of existence in the dual problem is tantamount to the existence of a least favorable fictitious completion; it is dealt with in section 12, by the use of methods from convex analysis.

Our model for the financial market can be traced back to Merton [15,16] and Samuelson [19]. The modern mathematical approach to portfolio management in complete markets, built around the ideas of equivalent martingale measures and the creation of portfolios from martingale representation theorems, began with Harrison & Kreps [5] and was further developed by Harrison & Pliska [6,7], in the context of option pricing. Pliska [18], Cox & Huang [2,3] and Karatzas, Lehoczky & Shreve [12] adapted the martingale ideas to problems of utility maximization. Much of this development appears in Section 5.8 of Karatzas & Shreve [13], from which Section 6 of the present paper is drawn; see also the review article of Karatzas [11] for a survey of financial economics problems in complete markets. An extension of the above papers to infinite horizon problems is reported by Huang and Pages [10].

A first step toward a martingale analysis of incomplete markets was taken by Pages [17], who considered a Brownian model in which the number of stocks was strictly less than the dimension of the driving Brownian motion. However, the coefficients of the bond and stock prices in this model were allowed to depend on the underlying Brownian model only through the bond and stock prices themselves. Thus, the vector of bond and stock prices formed a Markov process. This specialization created an essentially complete market, and thus it avoided the more interesting case of a market with genuinely unhedgeable risk. However, Pagès did characterize the class of equivalent martingale measures which could arise in an incomplete model, and this laid the groundwork for further developments (e.g., Lemma 8.2 in this paper). A more substantive step was taken by He & Pearson [8] in a discrete–time, finite probability space model, where the authors proposed finding the optimal intermediate consumption and terminal wealth corresponding to each of the equivalent martingale measures, and then searching over those policies to find a pair yielding the minimum expected total utility. Using separating hyperplane arguments, they were able to show that the total utility obtained by this two–step "minimax" process is the optimal total value for the incomplete problem.

He & Pearson have also studied the incomplete problem in a continuous–time, Brownian model. In an early version of He & Pearson [9], the authors consider Pagès' characterization of the family of equivalent martingale measures and search over this family for a "minimax" equivalent martingale measure, which would lead them to the optimal consumption and portfolio processes just as in a complete market. The martingale associated with this measure would create the "Arrow–Debreu" state prices in the incomplete model. However, the continuous–time model is more subtle than one might expect, and although it is now clear that Arrow–Debreu state prices exist for the incomplete model under some assumptions, it is not clear that they are associated with a martingale.

The present paper uses local martingales rather than martingales to address the issue of market incompleteness in continuous–time models. This work was motivated by the aforementioned previous version of He & Pearson [9], and by the use of local martingale methods introduced by Xu [20] in the study of incompleteness induced by a prohibition on the short–selling of stocks. Using the stochastic duality thoery of Bismut [1], Xu formulated a dual problem whose solution could be shown to exist and could then be used to obtain existence and characterization of the solution of the original problem. As this paper shows, such duality methods can also be used in the traditional incomplete Brownian market model. While we still do not know if the minimax equivalent martingale measure sought by He & Pearson exists in any generality, we show here that the solution to Bismut's dual problem is a "least favorable local martingale" which can be used to generate a sequence of equivalent measures. The existence of this least favorable local martingale is sufficient for the study of many models. A notable exception is the incomplete markets model in which the agent's endowment is a stochastic process; we do not know how to obtain the existence and a characterization of the optimal policy for such a model in terms of a least favorable local martingale, unless it is actually a martingale.

He & Pearson [9] have incorporated Xu's local martingale techniques into their original work. He & Pearson [9] report the existence of an optimal portfolio for the terminal wealth utility maximization problem when the index of relative risk aversion is everywhere greater than or equal to one, and they report similar results for the problem with intermediate consumption and consumption at the terminal time when the index of relative risk aversion is everywhere less than or equal to one. Our paper deals only with the case of terminal wealth utility maximization when the index of relative risk aversion is everywhere less than or equal to one; the generalization to also allow for intermediate consumption is straight–forward. Whereas He & Pearson [9] assume that some augmentation of the market model will result in Markov prices, we allow general Itô price processes. He & Pearson [9] do not address the difficulties which necessitated our assumption (4.8) and the introduction of the set $K_1(\sigma)$ in Section 9, and consequently there is still some doubt whether their conditions are sufficient to justify the results they claim.

## 2. THE MARKET MODEL

We adopt a model for the financial market consisting of one bond with price $P_0(t)$ given by

$$dP_0(t) = r(t)P_0(t)dt, \quad P_0(0) = 1, \tag{2.1}$$

and $m$ stocks with prices per share $P_i(t)$, $i = 1,\ldots,m$, satisfying the equations

$$dP_i(t) = P_i(t)[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)], \quad i = 1,\ldots,m. \tag{2.2}$$

Here $W = (W_1,\ldots,W_d)^*$ is a d–dimensional Brownian motion on a probability space $(\Omega,\mathcal{F},P)$, and we denote by $\{\mathcal{F}_t\}$ the P–augmentation of the filtration generated by $W$. It is assumed throughout that $d \geq m$, i.e., the number of sources of uncertainty in the model is at least as large as the number of stocks available for investment.

The <u>interest rate</u> $r(t)$, the vector $b(t) = (b_1(t),\ldots,b_m(t))^*$ of <u>stock appreciation rates</u>, and the <u>volatility matrix</u> $\sigma(t) = \{\sigma_{ij}(t)\}_{\substack{1\leq i\leq m\\ 1\leq j\leq d}}$ are the <u>coefficients of the model</u>. They are taken to be progressively measurable with respect to $\{\mathcal{F}_t\}$; it is also assumed that

$$\int_0^T \|b(t)\|dt < \infty, \quad \int_0^T |r(t)|dt \leq L \tag{2.3}$$

hold almost surely, for some given real constant $L > 0$. The positive constant $T$ is the <u>terminal time</u> for the problem. All processes are defined on $[0,T]$.

We assume that the matrix $\sigma(t)$ has <u>full rank</u> for every $t$, so the matrix $(\sigma(t)\sigma^*(t))^{-1}$ and the <u>relative risk</u> process

$$\theta(t) \triangleq \sigma^*(t)(\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\underline{1}] \tag{2.4}$$

are defined. Throughout this paper, we denote by $\underline{1}$ a vector whose every component is 1 and whose dimension is appropriate for the context. It will be assumed that

$$\int_0^T \|\theta(t)\|^2 dt < \infty, \quad \text{a.s. } P. \tag{2.5}$$

We shall have occasion to use the so-called <u>discount process</u>

$$\beta(t) \triangleq \frac{1}{P_0(t)} = \exp\{-\int_0^t r(s)ds\}, \tag{2.6}$$

as well as the process

$$W_0(t) \triangleq W(t) + \int_0^t \theta(s)ds \tag{2.7}$$

and the exponential local martingale

$$Z_0(t) \triangleq \exp\{-\int_0^t \theta^*(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\}. \tag{2.8}$$

**2.1 Definition:** A financial market as above will be called <u>complete</u> if $m = d$, and <u>incomplete</u> if $m < d$.

## 3. PORTFOLIO AND WEALTH PROCESSES

A portfolio process $\pi(t) = (\pi_1(t),\ldots,\pi_m(t))^*$ is an $\mathbb{R}^m$–valued, $\{\mathcal{F}_t\}$–adapted process satisfying

$$\int_0^T \|\sigma^*(t)\pi(t)\|^2 dt < \infty, \quad \text{a.s. } P. \tag{3.1}$$

We regard $\pi_i(t)$ as the proportion of an agent's wealth invested in stock $i$ at time $t$; the remaining proportion $1 - \pi^*(t)\underline{1} = 1 - \sum_{i=1}^m \pi_i(t)$ is invested in the bond. We do not constrain these proportions to take values in the interval $[0,1]$; in other words, we allow both short–selling of stocks, and borrowing at the interest rate of the bond. For a given, nonrandom, initial wealth $x > 0$, let $X^{x,\pi}$ denote the wealth process corresponding to a portfolio $\pi$ defined by $X^{x,\pi}(0) = x$ and

$$\begin{aligned} dX^{x,\pi}(t) &= r(t)X^{x,\pi}(t)dt + X^{x,\pi}(t)\pi^*(t)[(b(t) - r(t)\underline{1})dt + \sigma(t)dW(t)] \\ &= r(t)X^{x,\pi}(t)dt + X^{x,\pi}(t)\pi^*(t)\sigma(t)dW_0(t). \end{aligned} \tag{3.2}$$

In other words,

$$\begin{aligned} \beta(t)X^{x,\pi}(t) &= x \exp\{\int_0^t \pi^*(s)\sigma(s)dW_0(s) - \tfrac{1}{2}\int_0^t \|\sigma^*(s)\pi(s)\|^2 ds\} \\ &= x + \int_0^t \beta(s)X^{x,\pi}(s)\pi^*(s)\sigma(s)dW_0(s), \quad 0 \le t \le T. \end{aligned} \tag{3.3}$$

**3.1 Remark:** An application of Itô's rule to the product of the processes $Z_0$ and $\beta X^{x,\pi}$ of

(2.8), (3.3) leads to

$$\beta(t)Z_0(t)X^{x,\pi}(t) = x + \int_0^t \beta(s)Z_0(s)X^{x,\pi}(s)(\sigma^*(s)\pi(s) - \theta(s))^* dW(s). \tag{3.4}$$

This shows, in particular, that the process $\beta Z_0 X^{x,\pi}$ is a nonnegative local martingale, hence a supermartingale, under the original measure P.

## 4. UTILITY FUNCTIONS

We introduce a utility function $U : (0,\infty) \to \mathbb{R}$ which is strictly increasing, strictly concave, continuous and continuously differentiable, and satisfies

$$U'(0) \triangleq \lim_{x\downarrow 0} U'(x) = \infty, \quad U'(\infty) \triangleq \lim_{x\to\infty} U'(x) = 0. \tag{4.1}$$

The (continuous, strictly decreasing) inverse of the function $U'$ will be denoted by $I : (0,\infty) \to (0,\infty)$; by analogy with (4.1), it satisfies

$$I(0) \triangleq \lim_{y\downarrow 0} I(y) = \infty, \quad I(\infty) \triangleq \lim_{y\to\infty} I(y) = 0. \tag{4.2}$$

We introduce also the function

$$\tilde{U}(y) \triangleq \max_{x>0}[U(x) - xy] = U(I(y)) - yI(y), \quad 0 < y < \infty \tag{4.3}$$

which is the Legendre transform of $-U(-x)$, with U extended to be $-\infty$ on the negative real axis. The function $\tilde{U}$ is strictly decreasing, strictly convex, and satisfies

(4.4) $$\tilde{U}'(y) = -I(y), \quad 0 < y < \infty$$

(4.5) $$U(x) = \min_{y>0}[\tilde{U}(y) + xy] = \tilde{U}(U'(x)) + xU'(x), \quad 0 < x < \infty.$$

The useful inequalities

(4.6) $$U(I(y)) \geq U(x) + y[I(y) - x]; \; \forall\, x > 0, y > 0$$

(4.7) $$\tilde{U}(U'(x)) \leq \tilde{U}(y) - x[U'(x) - y]; \; \forall\, x > 0, y > 0$$

follow then directly from (4.3), (4.5).

The monotonicity of $U$ and $\tilde{U}$ guarantees that the limits

$$U(0) \triangleq \lim_{x \downarrow 0} U(x) \;, \quad U(\infty) \triangleq \lim_{x \to \infty} U(x)$$
$$\tilde{U}(0) \triangleq \lim_{y \downarrow 0} \tilde{U}(y) \;, \quad \tilde{U}(\infty) \triangleq \lim_{y \to \infty} \tilde{U}(y)$$

exist in the extended real number system.

**4.1 LEMMA:** $U(0) = \tilde{U}(\infty)$, $\tilde{U}(0) = U(\infty)$.

**PROOF:** It follows from (4.3) that $\tilde{U}(\infty) \leq \lim_{y \to \infty} U(I(y)) = U(0)$, as well as

$$\tilde{U}(\infty) \geq \lim_{y \to \infty} \left(U\left(\tfrac{\epsilon}{y}\right) - \epsilon\right) = U(0) - \epsilon, \quad \forall\, \epsilon > 0,$$

whence $\tilde{U}(\infty) = U(0)$. Similarly, it follows from (4.5) that $U(\infty) \geqq \lim_{x\to\infty} \tilde{U}(U'(x)) = \tilde{U}(0)$, as well as

$$U(\infty) \leqq \lim_{\xi\to\infty} [\tilde{U}(\tfrac{\epsilon}{\xi}) + \epsilon] = \tilde{U}(0) + \epsilon, \quad \forall\, \epsilon > 0,$$

whence $U(\infty) = \tilde{U}(0)$. □

We shall have occasion to impose the following condition on the utility function $U$:

$$\alpha\, U'(x) \geqq U'(\gamma x)\,; \quad \forall\ x \in (0,\infty) \tag{4.8}$$

for some $\alpha \in (0,1)$, $\gamma \in (1,\infty)$. Quite obviously, such a condition is satisfied by utility functions like $U(x) = \log x$ or $U(x) = \frac{1}{\delta} x^{\delta}$, with $\delta < 1$, $\delta \neq 0$.

Upon replacing $x$ by $I(y)$ in (4.8), and then applying $I$ to both sides of the resulting inequality, we see that (4.8) is equivalent to the condition:

$$I(\alpha y) \leq \gamma\, I(y)\,; \quad \forall\ \ y \in (0,\infty)\,, \tag{4.8$'$}$$

for some $\alpha \in (0,1)$, $\gamma \in (1,\infty)$. By iterating (4.8)$'$, one obtains the apparently stronger statement

$$\forall\, \alpha \in (0,1), \exists\, \gamma \in (1,\infty), \text{ such that: } I(\alpha y) \leq \gamma\, I(y)\ \ ;\ \forall\, y \in (0,\infty). \tag{4.9}$$

## 5. THE UTILITY MAXIMIZATION PROBLEM

For a given utility function U and a given initial capital $x > 0$, the stochastic control problem considered in this paper is the following: to maximize the expected utility from terminal wealth $EU(X^{x,\pi}(T)) < \infty$, over the class $\mathcal{A}(x)$ of portfolio processes $\pi$ that satisfy

$$(5.1) \qquad E(U(X^{x,\pi}(T)))^- < \infty.$$

The value function of this problem is denoted by

$$(5.2) \qquad V(x) \triangleq \sup_{\pi \in \mathcal{A}(x)} E\, U(X^{x,\pi}(T)),$$

and we shall assume throughout that it is finite:

$$(5.3) \qquad V(x) < \infty, \quad \forall\ x \in (0,\infty).$$

A portfolio process $\pi \in \mathcal{A}(x)$ which attains the supremum in (5.2) is called optimal. In sections 9, 11 and 12 we provide conditions that ensure the existence of optimal portfolios, as well as various characterizations of optimality. Some examples, in which optimal portfolios can be computed explicitly, appear in section 10.

**5.1 Remark:** In the case of a market model for which the relative risk process $\theta(\cdot)$ of (2.4) satisfies the condition

$$(5.4) \qquad \int_0^T \|\theta(t)\|^2 dt \le C, \quad \text{a.s.}$$

for some given real constant $C > 0$, a sufficient condition for (5.3) is

$$(5.5) \qquad U(x) \leqq k_1 + k_2 x^\delta, \ \forall \ x \in (0,\infty)$$

for some $k_1 > 0$, $k_2 > 0$, $\delta \in (0,1)$.

Indeed, then the process $Z_0$ of (2.8) is a martingale, and $W_0(\cdot)$ is a Brownian motion under the probability measure $P_0(A) = E[Z_0(T)1_A]$ on $\mathcal{F}_T$ (the Girsanov theorem; c.f. Karatzas & Shreve (1988), §3.5). For any $p \in (1, \frac{1}{\delta}]$ and suitable constants $c_1 > 0$, $c_2 > 0$, we have

$$(5.6) \qquad U_+^p(x) \leq c_1 + c_2 x^{\delta p}, \ \forall \ x \in (0,\infty)$$

from (5.5), where $U_+(x) \triangleq \max\{U(x),0\}$. Also, we have

$$(X^{x,\pi}(T))^{\delta p} = x^{\delta p} \cdot \exp[\delta p \int_0^T r(s)ds - \frac{\delta p(1-\delta p)}{2} \int_0^T \|\sigma^*(s)\pi(s)\|^2 ds]$$

$$(5.7) \qquad \cdot \exp[\delta p \int_0^T \pi^*(s)\sigma(s)dW_0(s) - \frac{\delta^2 p^2}{2} \int_0^T \|\sigma^*(s)\pi(s)\|^2 ds]$$

$$\leq (e^L x)^{\delta p} \cdot \exp[\delta p \int_0^T \pi^*(s)\sigma(s)dW_0(s) - \frac{1}{2}\delta^2 p^2 \int_0^T \|\sigma^*(s)\pi(s)\|^2 ds]$$

from (3.3), and

$$(5.8) \quad E_0 Z^{-q}(t) = E_0[\exp\{q \int_0^T \theta^*(s)dW_0(s) - \frac{q^2}{2} \int_0^T \|\theta(s)\|^2 ds\} \cdot \exp\{\frac{q(q-1)}{2} \int_0^T \|\theta(s)\|^2 ds\}]$$

$$\leqq \exp\{\frac{q(q-1)}{2} C\}$$

for (2.8) with $\frac{1}{p}+\frac{1}{q}=1$. Now (5.6)–(5.8), in conjunction with the Hölder inequality, give

$$E\,U(X^{x,\pi}(T)) = E_0[Z_0^{-1}(T)\,U(X^{x,\pi}(T))]$$

$$\leq (E_0 Z_0^{-q}(T))^{1/q}(E_0 U_+^p(X^{x,\pi}(T)))^{1/p}$$

$$\leq \exp\{\tfrac{(q-1)}{2}\,C\}\,(c_1 + c_2(e^L x)^{\delta p})^{1/p} < \infty$$

for every $\pi \in \mathscr{A}(x)$, justifying (5.3).

## 6. THE COMPLETE MARKET SOLUTION

The utility maximization problem of Section 5 admits a simple solution in the case $m = d$ of a complete market; this solution was derived by Karatzas, Lehoczky & Shreve [12] and independently by Cox & Huang [2,3]. We review briefly in this section the pertinent results, both for easy reference and for later usage in the treatment of the incomplete market case.

For every portfolio process $\pi \in \mathscr{A}(x)$, the supermartingale $\beta Z_0 X^{x,\pi}$ of (3.4) satisfies

$$(6.1) \qquad E[\beta(T)Z_0(T)X^{x,\pi}(T)] \leq x.$$

Let us assume now that

$$(6.2) \qquad E[\beta(T)Z_0(T)I(y\beta(T)Z_0(T))] < \infty \ , \ \forall \ y \in (0,\infty)$$

holds, so we may define a function $\mathscr{X}_0\colon (0,\infty) \to (0,\infty)$ by

$$(6.3) \qquad \mathscr{X}_0(y) \triangleq E[\beta(T)Z_0(T)I(y\beta(T)Z_0(T))].$$

The function $\mathscr{X}_0$ inherits from I the property of being a continuous, strictly decreasing mapping of $(0,\infty)$ onto itself, and so $\mathscr{X}_0$ has a (continuous, strictly decreasing) inverse $\mathscr{Y}_0$ from $(0,\infty)$ onto itself. We define

$$(6.4) \qquad \xi_0^x \triangleq I(\mathscr{Y}_0(x)\beta(T)Z_0(T)).$$

**6.1 Lemma.** The random variable $\xi_0^x$ satisfies

$$(6.5) \qquad E[\beta(T)Z_0(T)\xi_0^x] = x\,,$$

$$(6.6) \qquad E(U(\xi_0^x))^- < \infty\,,$$

and for every portfolio $\pi \in \mathscr{A}(x)$, we have

$$(6.7) \qquad EU(X^{x,\pi}(T)) \le EU(\xi_0^x).$$

Proof: Equation (6.5) follows directly from the definitions of $\xi_0^x$ and $\mathscr{Y}_0$. From (4.6) we have

$$U(\xi_0^x) \ge U(1) + \mathscr{Y}_0(x)\beta(T)Z_0(T)[\xi_0^x - 1]$$

$$\ge -|U(1)| - \mathscr{Y}_0(x)\beta(T)Z_0(T).$$

But $\beta$ is nonnegative and bounded a.s., and $Z_0$ is a nonnegative local martingale, thus a supermartingale. Therefore $E[\beta(T)Z_0(T)] < \infty$, and (6.6) follows. Now let $\pi$ be a portfolio satisfying (5.1). From (4.6), (6.1) and (6.5) we have

$$EU(\xi_0) \geq E\{U(X^{x,\pi}(T)) + /_o(x)/?(T)Zo(T)[ft - X^{x,\pi}(T)]\}$$

$$\geq E\, U(X^{x,\pi}(T)). \qquad \square$$

From Lemma 6.1 it develops that if there exists a portfolio $\hat{\pi}$ such that {$ = $X^{x,\hat{\pi}}(T)$, then $\hat{\pi}$ is optimal. We have so far not used the assumption of market completeness; this assumption is used only in the construction of the portfolio fr which finances f $^*_0$, a question that we now broach.

We begin with the martingale

$$M(t) \triangleq E[\beta(T)Z_0(T)\xi_0 | \mathcal{F}_t]. \qquad (6\text{-}8)$$

Being adapted to the Brovmian filtration {^A M admits the stochastic integral representation

$$M(t) = x + \int_0^t \psi^*(s)dW(s) \qquad (6.9)$$

for some {^.j—adapted process ^ satisfying $\int_0^T \|\psi(s)\|^2 ds < \infty$ a.s. According to Itô's lemma

$$d\left(\frac{M(t)}{Z_0(t)}\right) = \frac{1}{Z_0(t)}(\psi(t) + M(t)\theta(t))^* dW_0(t),$$

and thus:

$$\beta(T)\xi_0 = \frac{M(T)}{Z_0(T)} = x + \int_0^T \frac{1}{Z_0(t)}(\psi(t) + M(t)\theta(t))^* dW_0(t).$$

We define

$$(6.10) \qquad \hat{X}(t) \triangleq \frac{1}{\beta(t)}[x + \int_0^t \frac{1}{Z_0(s)}(\psi(s) + M(s)\theta(s))^* dW_0(s)],$$

$$(6.11) \qquad \hat{\pi}(t) \triangleq \frac{1}{\beta(t)Z_0(t)\hat{X}(t)}(\sigma^*(t))^{-1}(\psi(t) + M(t)\theta(t)),$$

and verify that $\hat{X}(0) = x$, $\hat{X}(T) = \xi\mathfrak{z}$ as well as $d(\beta(t)\hat{X}(t)) = \beta(t)\hat{X}(t)\hat{\pi}^*(t)\sigma(t)dW_0(t)$ hold. A comparison with (3.3) shows that $\hat{X}(\cdot)$ is the wealth process corresponding to the portfolio $\hat{\pi} : \hat{X}(\cdot) \equiv X^{x,\pi}(\cdot)$.

We have proved the following result:

**6.2 Theorem.** Let an initial wealth $x > 0$ be given. In a complete market $(d = m)$ under the assumption (6.2), the portfolio $\hat{\pi}$ given by (6.11) is optimal. The resulting optimal terminal wealth is given by (6.4).

**6.3 Example.** (*Logarithmic utility function*).

Suppose $U(x) = \log x$. Then $\mathcal{Y}_0(x) = \frac{1}{x}$ and

$$(6.12) \qquad \xi\mathfrak{z} = x \exp\{\int_0^T (r(t) + \tfrac{1}{2}\|\theta(t)\|^2)dt + \int_0^T \theta^*(t)dW(t)\}.$$

Let $\hat{\pi}$ be given by

$$(6.13) \qquad \hat{\pi}(t) \triangleq (\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\,\underline{1}].$$

From (3.4) we have: $X^{x,\hat{\pi}}(T) = \dfrac{x}{\beta(T)Z_0(T)} = \xi_\delta^{\mathfrak{z}}$, so $\hat{\pi}$ is optimal and

$$(6.14)\qquad V(x) = E[\log X^{x,\hat{\pi}}(T)] = \log x + E\int_0^T (r(t) + \tfrac{1}{2}\|\theta(t)\|^2)dt,$$

provided that this last expectation is finite (cf. Karatzas [11], §9.3 and §9.6).

**6.4 Example.** (*Power utility function and deterministic model coefficients*).

Suppose that $U(x) = \frac{1}{\delta}x^\delta$, where $\delta < 1$, $\delta \neq 0$, and suppose that the processes $r$ and $\theta$ are deterministic. Then $\exp\{\frac{\delta}{1-\delta}\int_0^t \theta^*(s)dW(s) - \frac{\delta^2}{2(1-\delta)^2}\int_0^t \|\theta(s)\|^2 ds\}$ is a martingale with expectation equal to one (Karatzas & Shreve [13], p. 199, Corollary 5.13), and from (6.3):

$$\mathscr{X}_0(y) = y^{\frac{1}{\delta-1}} \exp\{\tfrac{\delta}{1-\delta}\int_0^T (r(s) + \tfrac{1}{2}\|\theta(s)\|^2)ds \cdot E\exp\{\tfrac{\delta}{1-\delta}\int_0^T \theta^*(s)dW(s)\}$$

$$= y^{\frac{1}{\delta-1}} \exp\{\tfrac{\delta}{1-\delta}\int_0^T m(s)ds\},$$

where

$$(6.15)\qquad m(t) \triangleq r(t) + \frac{1}{2(1-\delta)}\|\theta(t)\|^2.$$

It follows that $\mathscr{Y}_0(x) = x^{\delta-1}\exp\{\delta\int_0^T m(s)ds\}$, and

$$(6.16)\qquad \xi_\delta^{\mathfrak{z}} = x\exp\{\int_0^T (r(t) + \frac{1-2\delta}{2(1-\delta)^2}\|\theta(t)\|^2)dt + \frac{1}{1-\delta}\int_0^T \theta^*(t)dW(t)\}.$$

Taking

$$\hat{\pi}(t) \triangleq \frac{1}{1-\delta}(\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\,\underline{1}] \tag{6.17}$$

in (3.4), we obtain

$$\beta(t)Z_0(t)X^{x,\hat{\pi}}(t) = x\exp\{-\frac{\delta^2}{2(1-\delta)^2}\int_0^t \|\theta(s)\|^2 ds + \frac{\delta}{1-\delta}\int_0^t \theta^*(s)dW(s)\},$$

from which follows $X^{x,\hat{\pi}}(T) = \xi_\delta^x$ and thereby the optimality of $\hat{\pi}$.

## 7. FICTITIOUS COMPLETIONS OF AN INCOMPLETE MARKET

The utility maximization problem of section 5 for an incomplete market ($d > m$) will be studied by the method of "fictitious completion". We shall perform, in other words, the thought–experiment of introducing $d-m$ additional stocks driven by the d–dimensional Brownian motion $W$, thus creating a fictitious complete market in which the utility maximization problem can be solved as in section 6. We will then try to determine appreciation rates for these additional stocks, so that the optimal portfolio in the resulting complete market does not invest in the additional stocks at all.

We expect that appreciation rates with this, rather special, property will exist, based on the following heuristic grounds. If the additional stocks in the fictitious completion have appreciation rates that are too high, then the resulting optimal portfolio will hold a long position in them; if they have appreciation rates that are too low, the optimal portfolio will hold a short position in them. Somewhere in between these two extremes, one expects that there should be a choice of appreciation rates for which the optimal portfolio does not invest in the additional stocks at all. In the remainder of the paper we shall try to place this intuition on firm mathematical ground.

Following the above program, we introduce an $\{\mathcal{F}_t\}$–progressively measurable, uniformly bounded, $(d-m) \times d$ maxtrix–valued process $\rho(t)$ whose rows, thought of as vectors in $\mathbb{R}^d$, are orthonormal and in the kernel of $\sigma(t)$, i.e., $\sigma(t)\rho^*(t) = 0$. We also introduce an $\{\mathcal{F}_t\}$–progressively measurable, $(d-m)$–dimensional vector process $a$ satisfying

$$\int_0^T \|a(t)\|dt < \infty, \quad \text{a.s.} \tag{7.1}$$

We create fictitious stocks with prices $S_i(t)$ governed by

$$dS_i(t) = S_i(t)[a_i(t)dt + \sum_{j=1}^{d} \rho_{ij}(t)dW_j(t)], \quad i = 1,\dots,d-m. \tag{7.2}$$

The matrix–valued process $\rho$ will be held fixed throughout the remainder of the paper, but the process $a$ will be considered as a parameter.

For the augmented stock appreciation rate vector $\tilde{b} \triangleq \begin{bmatrix} b \\ a \end{bmatrix}$ and the augmented volatility matrix $\tilde{\sigma} \triangleq \begin{bmatrix} \sigma \\ \rho \end{bmatrix}$ we can define an augmented <u>relative risk</u> process

$$\tilde{\theta}(t) \triangleq \tilde{\sigma}^*(t)(\tilde{\sigma}(t)\tilde{\sigma}^*(t))^{-1}[\tilde{b}(t) - r(t)\,\underset{\sim}{1}] = \theta(t) + \nu(t) \tag{7.3}$$

by analogy with (2.4), where

$$\nu(t) \triangleq \rho^*(t)[a(t) - r(t)\,\underset{\sim}{1}]. \tag{7.4}$$

Notice that $\theta^*(t)\nu(t) = 0$, and thus $\|\tilde{\theta}(t)\|^2 = \|\theta(t)\|^2 + \|\nu(t)\|^2$. It will be assumed that

(7.5) $$\int_0^T \|\nu(t)\|^2 dt < \infty$$

holds almost surely, so that (by analogy with (2.8) and (6.3)) we may define the exponential local martingale

(7.6) $$Z_\nu(t) \triangleq \exp\{-\int_0^t (\theta^*(s) + \nu^*(s))dW(s) - \frac{1}{2}\int_0^t (\|\theta(s)\|^2 + \|\nu(s)\|^2)ds\}$$

$$= 1 - \int_0^t Z_\nu(s)(\theta(s) + \nu(s))^* dW(s)$$

and the function

(7.7) $$\mathcal{X}_\nu(y) \triangleq E[\beta(T)Z_\nu(T)I(y\beta(T)Z_\nu(T))], \qquad 0 < y < \infty.$$

If the condition

(7.8) $$\mathcal{X}_\nu(y) < \infty, \quad \forall\ y \in (0,\infty)$$

prevails, we may define $\mathcal{Y}_\nu$ to be the inverse of $\mathcal{X}_\nu$ and set

(7.9) $$\xi_\nu^x \triangleq I(\mathcal{Y}_\nu(x)\beta(T)Z_\nu(T))$$

by analogy with (6.4).

**7.1 Remark:** If the fictitious stocks introduced in this section were really available, then

$EU(\xi_\nu^x)$ would be the maximal expected utility of final wealth (Theorem 6.2). Since these stocks are not available, we have

(7.10) $$V(x) \triangleq \sup_{\pi \in \mathscr{A}(x)} EU(X^{x,\pi}(T)) \le EU(\xi_\nu^x),$$

and equality holds if there exists a portfolio process $\pi$ such that

(7.11) $$X^{x,\pi}(T) = \xi_\nu^x, \quad \text{a.s.},$$

i.e., if the terminal wealth $\xi_\nu^x$ can be financed without investment in the fictitious stocks. In light of (7.10), such a $\pi$ would be optimal for the problem of utility maximization in the incomplete market. In Section 9 we shall discuss properties which $\pi$ and $\nu$ must have in order to be related by (7.11).

## 8. A FAMILY OF EXPONENTIAL LOCAL MARTINGALES

Let us denote by $L^2[0,T]$ the class of $\{\mathscr{F}_t\}$–adapted, $\mathbb{R}^d$–valued processes $\psi$ satisfying

(8.1) $$\int_0^T \|\psi(t)\|^2 dt < \infty$$

almost surely, and decompose $L^2[0,T]$ into the orthogonal subspaces

(8.2) $$K(\sigma) \triangleq \{\nu \in L^2[0,T];\ \sigma(t)\nu(t) = 0\,,\ \forall\, t \in [0,T],\ \text{a.s.}\},$$

(8.3) $$K^\perp(\sigma) \triangleq \{\varphi \in L^2[0,T];\ \varphi(t) \in \text{Range}(\sigma^*(t)),\ \forall\, t \in [0,T],\ \text{a.s.}\}.$$

**8.1 Remark:** The process $0$ of (2.4) belongs to $K^A(a)$, whereas the process $v$ of (7.4) belongs to $K(<r)$. On the other hand, if $v \in K\{a)$ is given, then (7.4) can be solved for the appreciation rate vector $a$ of the fictitious stocks, by taking this vector equal to

(8.4) $$a_w(t) \ 4\ ,(tMt) + r(t)\ \underline{1}.$$

Thus, the class $K(a)$ provides a parameter space for fictitious completions of the incomplete market.

We shall denote by $ffl_v$ the fictitious completion of the financial market by the additional stocks of (7.2), with $/?(\bullet)$ fixed and $a(\cdot) \equiv a_v(\cdot)$, $v \in K(a)$.

D

The associated family of exponential local martingales $K\wedge v\dot{i}y\wedge YXaV$ $6*^{ven}$ $\wedge y$ $(^7$-6), will play a fundamental role in what follows.

**8.2 LEMMA:** Consider the discounted stock price processes

$$Qi(t) \wedge /?(t)Pi(t); \quad i = 1,...,m.$$

Then for every $v \in K(a)$, the processes $Z_vQi$ are local martingales under $P$.

PiOOF: It is seen from (2.2), (2.6) that

$$dQi(t) = Qi(t)[(bi(t) - r(t))dt + (Ti(t)dW(t)]$$

where $ai(t)$ is the $i^{th}$ row vector of the matrix $o(t)$. It follows from this, (7.6), Itô's rule, and $a\{i)u\{t) = 0$, that

$$d(Z_\nu(t)Q_i(t)) = Z_\nu(t)Q_i(t)[\sigma_i(t) - (\theta(t) + \nu(t))^*]dW(t).$$

**8.3 Proposition:** For any given $\pi \in \mathcal{A}(x)$, $\beta Z_\nu X^{x,\pi}$ is a local martingale under $P$ for every $\nu \in K(\sigma)$; in particular,

$$E[\beta(T)Z_\nu(T)X^{x,\pi}(T)] \leqq x, \ \forall \ \nu \in K(\sigma). \tag{8.5}$$

**Proof:** From (3.3), (2.7) and (7.6) follows the analogue

$$\beta(t)Z_\nu(t)X(t) = x + \int_0^t \beta(s)Z_\nu(s)X(s)[\sigma^*(s)\pi(s) - (\theta(s) + \nu(s))]^* dW(s) \tag{8.6}$$

of (3.4) for the process $X \equiv X^{x,\pi}$. This representation shows that $\beta Z_\nu X^{x,\pi}$ is a positive local martingale, hence a supermartingale, and (8.5) follows.

**8.4 Remark**: Suppose that $\pi$ is a portfolio process, and that $X$ is a continuous, $\{\mathcal{F}_t\}$–adapted process which satisfies (8.6) almost surely, for some $\nu \in K(\sigma)$. Then $X$ is the wealth process corresponding to the initial endowment $x$ and the portfolio process $\pi$, i.e., $X = X^{x,\pi}$. Indeed, apply Itô's rule to the product of the processes $\beta Z_\nu X$ and $\Lambda_\nu$, where $\Lambda_\nu = Z_\nu^{-1}$ is easily seen from (7.6) to satisfy

$$d\Lambda_\nu(t) = \Lambda_\nu(t)[(\theta(t) + \nu(t))^* dW(t) + (\|\theta(t)\|^2 + \|\nu(t)\|^2)dt],$$

and obtain (3.3).

The following result provides a kind of "converse" to Proposition 8.3.

**8.5 PROPOSITION:** Consider a positive, $\mathcal{F}_T$–measurable random variable B, for which there exists a process $\lambda \in K(\sigma)$ with

$$E[\beta(T)Z_\nu(T)B] \le x = E[\beta(T)Z_\lambda(T)B]; \quad \forall\, \nu \in K(\sigma). \tag{8.7}$$

Then there exists a portfolio $\pi \in \mathcal{A}(x)$, such that $X^{x,\pi}(T) = B$, a.s.

**PROOF:** Define a positive, $\{\mathcal{F}_t\}$–adapted process X via

$$\beta(t)Z_\lambda(t)X(t) = M(t) \triangleq E[\beta(T)Z_\lambda(T)B \mid \mathcal{F}_t]; \quad 0 \le t \le T. \tag{8.8}$$

Certainly $X(0) = x$, $X(T) = B$ a.s., and the positive martingale M in (8.8) has $M(0) = x$. From the martingale representation theorem (Karatzas & Shreve [13], Problem 3.4.16, p. 184), $M(t) = x + \int_0^t \varphi(s)dW(s)$ for some $\{\mathcal{F}(t)\}$–adapted process $\varphi$ satisfying $\int_0^T \|\varphi(t)\|^2 dt < \infty$ almost surely. Since M is continuous and $M(t) > 0$ for all $t \in [0,T]$, we may define $\psi \in L^2[0,T]$ by $\psi(t) = -\frac{\varphi(t)}{M(t)}$. Then

$$M(t) = x \exp\{-\int_0^t \psi^*(s)dW(s) - \frac{1}{2}\int_0^t \|\psi(s)\|^2 ds\} \tag{8.9}$$

$$= x - \int_0^t M(s)\, \psi^*(s)dW(s); \quad 0 \le t \le T.$$

Decomposing $\psi$ as $\psi = \psi_1 + \psi_2$ with $\psi_1 \in K^\perp(\sigma)$, $\psi_2 \in K(\sigma)$ and comparing (8.8), (8.9) with (8.6), it transpires that proving the Proposition amounts to finding a portfolio $\pi$ such that

$$(8.10) \qquad -M(t)(\psi_1(t) + \psi_2(t)) = \beta(t)Z_\lambda(t)X(t)[\sigma^*(t)\pi(t) - (\theta(t) + \lambda(t))].$$

This will certainly be possible, provided that

$$(8.11) \qquad \psi_2(t) = \lambda(t)$$

holds $dt \otimes dP$ – a.e. on $[0,T] \times \Omega$, because we can take then $\pi$ to satisfy $\sigma^*\pi = \theta - \psi_1 \in K^\perp(\sigma)$. Consequently, *we have to show that (8.7) implies (8.11).*

To this end, consider an arbitrary but fixed $\nu \in K(\sigma)$ and introduce the sequence of stopping times $\{\tau_n\}_{n=1}^\infty$ given by

$$(8.12) \qquad \tau_n \triangleq T \wedge \inf\{t \in [0,T];\ M(t) \geq n, \text{ or } \int_0^t (\|\psi_1(s)\|^2 + \|\psi_2(s)\|^2 + \|\lambda(s)\|^2)ds \geq n,$$

$$\text{or } \int_0^t \|\nu(s)\|^2 ds \geq n\ , \text{ or } \left|\int_0^t \nu^*(s)dW(s)\right| \geq n\}$$

for every $n \geq 1$. Obviously, $\lim_{n\to\infty} \tau_n = T$ almost surely, and we denote $\nu_n(t) \triangleq \nu(t)1_{[0,\tau_n]}(t)$. Clearly $\lambda + \epsilon\nu_n \in K(\sigma)$ and

$$(8.13) \qquad Z_{\lambda+\epsilon\nu_n}(t) = Z_\lambda(t)\exp\{-\epsilon\int_0^{t\wedge\tau_n} \nu^*(s)(dW(s) + \lambda(s)ds) - \frac{\epsilon^2}{2}\int_0^{t\wedge\tau_n} \|\nu(s)\|^2 ds\}\ ,$$

for every $\epsilon \in (-1,1)$, $n \geq 1$. On the other hand, the definition of $\tau_n$ in (8.12) gives

$$(8.14) \qquad e^{-3n|\epsilon|} \leq \frac{Z_{\lambda+\epsilon\nu_n}(T)}{Z_\lambda(T)} \leq e^{3n|\epsilon|}\ , \quad -1 < \epsilon < 1.$$

It follows then quite easily (from (8.12) – (8.14) and the Dominated Convergence Theorem) that (8.7) implies:

$$(8.15) \qquad 0 = \frac{\partial}{\partial\epsilon} E[\beta(T) Z_{\lambda+\epsilon\nu_n}(T) B]\Big|_{\epsilon=0} = E[\beta(T) \cdot \frac{\partial}{\partial\epsilon} Z_{\lambda+\epsilon\nu_n}(T)\Big|_{\epsilon=0} \cdot B]$$

$$= - E[\beta(T) Z_\lambda(T) B \int_0^{\tau_n} \nu^*(s)(dW(s) + \lambda(s)ds)] ,$$

or equivalently, in the notation of (8.8):

$$(8.16) \qquad E[M(\tau_n) \int_0^{\tau_n} \nu^*(s)(dW(s) + \lambda(s)ds)] = 0 , \quad \forall\, n \geq 1.$$

Now Itô's rule, in conjunction with (8.9), gives

$$M(\tau_n) \int_0^{\tau_n} \nu^*(s)(dW(s) + \lambda(s)ds) = \int_0^{\tau_n} M(t)\nu^*(t)(\lambda(t) - \psi_2(t))dt + \int_0^{\tau_n} M(t)\nu^*(t)dW(t)$$

$$(8.17) \qquad - \int_0^{\tau_n} M(t)\{\int_0^t \nu^*(s)(dW(s) + \lambda(s)ds)\}(\psi_1(t) + \psi_2(t))^* dW(t).$$

From the definition of $\tau_n$ in (8.12) we see that the expectation of the two stochastic integrals in (8.17) are equal to zero. Substituting back into (8.16), we obtain

$$(8.18) \qquad E \int_0^{\tau_n} M(t)\nu^*(t)(\lambda(t) - \psi_2(t))dt = 0, \quad \forall\; n \geq 1.$$

The arbitrariness of $\nu \in K(\sigma)$ in (8.18) leads to (8.11).

$\square$

## 9. EQUIVALENT OPTIMALITY CONDITIONS IN AN INCOMPLETE MARKET

The conclusions of section 7 were predicated on the assumption (7.8), but this condition will often not hold for all $\nu \in K(\sigma)$; cf. section 13 (Appendix). Accordingly, we restrict ourselves to the class

$$(9.1) \qquad K_1(\sigma) \triangleq \{\nu \in K(\sigma);\ \nu \text{ satisfies } (7.8)\}$$

in what follows.

**9.1 Remark**: If (4.8) holds, and $\nu \in K(\sigma)$ satisfies $\mathscr{X}_\nu(y) < \infty$ for <u>some</u> $y \in (0,\infty)$, then $\nu \in K_1(\sigma)$. This can be verified easily, using (4.9).

For a fixed initial capital $x > 0$, let $\hat{\pi} \in \mathscr{A}(x)$ be given, and consider the statement that $\hat{\pi}$ is optimal for the incomplete market maximization problem of section 5:

(A) OPTIMALITY OF $\hat{\pi}$: $EU(X^{x,\pi}(T)) \le EU(X^{x,\hat{\pi}}(T)),\ \forall\, \pi \in \mathscr{A}(x)$.

We shall characterize condition (A) with the help of the following conditions (B)–(E). For a given $\lambda \in K_1(\sigma)$ recall the notation of (4.3), (7.9) and consider the following statements.

(B) FINANCIBILITY OF $\xi_\lambda^x$: There exists a portfolio $\hat{\pi} \in \mathscr{A}(x)$ such that $X^{x,\hat{\pi}}(T) \equiv \xi_\lambda^x$, a.s.

(C) LEAST–FAVORABILITY OF $\lambda$: $EU(\xi_\lambda^x) \leq EU(\xi_\nu^x)$, $\forall\ \nu \in K_1(\sigma)$.

(D) DUAL OPTIMALITY OF $\lambda$: For all $\nu \in K_1(\sigma)$,

$$E\tilde{U}(\mathscr{Y}_\lambda(x)\beta(T)Z_\lambda(T)) \leqq E\tilde{U}(\mathscr{Y}_\lambda(x)\beta(T)Z_\nu(T)).$$

(E) PARSIMONY OF $\lambda$: $E[\beta(T)Z_\nu(T)\xi_\lambda^x] \leq x$, $\forall\ \nu \in K_1(\sigma)$.

Our principal result of this section, Theorem 9.4, states that conditions (A)–(E) are <u>equivalent</u>, provided that (4.8) and $U(0) > -\infty$ hold. This latter restriction is rather severe, for it excludes the important special case of the logarithmic utility function $U(x) = \log x$. For this reason we develop also a somewhat more modest result, Theorem 9.3, which suffices for a complete treatment of the logarithmic case (Example 10.1).

But first, let us try to motivate the developments that follow by discussing the significance of conditions (B)–(E). While we do not present any proofs for the claimed equivalences in the discussion that follows, we offer some plausible arguments to the effect that conditions (A)–(E) are connected to one another.

**9.2 Discussion:** For any given $\lambda \in K_1(\sigma)$, $\xi_\lambda^x$ is the optimal level of terminal wealth in the fictitiously completed market $\mathfrak{M}_\lambda$. When will it also be optimal in the original, incomplete market? Presumably, only when there exists a portfolio $\hat{\pi} \in \mathscr{A}(x)$ <u>which invests in the original m stocks only</u>, such that $X^{x,\hat{\pi}} = \xi_\lambda^x$. In other words, condition (B) has then to hold, and condition (E) follows directly from (8.5). In particular, (E) says that the value of the contingent claim $\xi_\lambda^x$ is at least as large in the fictitiously completed market $\mathfrak{M}_\lambda$ as in any other market $\mathfrak{M}_\nu$, $\nu \in K_1(\sigma)$. Note in this connection that, according to the definitions, $E[\beta(T)Z_\lambda(T)\xi_\lambda^x] = x$, $\forall\ \lambda \in K_1(\sigma)$.

Furthermore, the terminal wealth *ft* can be financed by investing in the stocks of any other market ST^ (since, in fact, it can be financed by investing in the original m stocks). Thus we obtain the condition (C), which captures the "least favorable" character of A.

Let us derive finally the condition (D), at least in the case U(0) > *-w* (in which $\tilde{U}$ is bounded from below, thanks to Lemma 4.1, and thus the expectations in (D) are well-defined). Indeed, by writing (4.7) with x replaced by *ft* and y replaced by $\wedge_x(x)0(T)Z_v(T)$ and taking expectations, we obtain

$$E\tilde{U}(\mathcal{Y}_\lambda(x)\beta(T)Z_\nu(T)) \geq E\tilde{U}(\wedge_x(x)/?(T)Z_x(T)) + \mathcal{Y}_\lambda(x)\cdot$$

$$\{E[\beta(T)Z_\lambda(T)\xi_\lambda^x] - E[\beta(T)Z_\nu(T)\xi_\lambda^x]\}$$

$$\geq E\tilde{U}(\wedge_x(x)/?(T)Z_x(T))$$

from condition (E).

9.3 TIEOIXM: Conditions (B) and (E) are equivalent, and imply both (C) and (A) with the same $\hat{r}$ as in (B).

PEOOF: *(B) =* (E):* Follows from Proposition 8.3.

*(E) =* (B):* Follows by letting B ≡ *ft* in Proposition 8.5. Notice that this Proposition remains valid if in it *K(a)* is replaced by Ki(<r). In order to see this, it suffices to observe that the processes A + $ei/_n$ (appearing in (8.13)) belongs to *Ki(a)* for every c € (—1,1), a >_1> because from (8.14) and the fact that A € Ki(a):

$$\mathcal{X}_{\lambda+\epsilon\nu_n}(y) \leqq e^{3n|\epsilon|}\, \mathcal{X}_\lambda(y\, e^{-3n|\epsilon|}) < \infty,\ \forall\ y \in (0,\infty).$$

*(B)* $\Longrightarrow$ *(A)*: Has already been shown in Remark 7.1.

*(B)* $\Longrightarrow$ *(C)*: From the previous implication and (7.10)

$$EU(\xi_\lambda^x) = V(x) \le EU(\xi_\nu^x),\ \forall\ \nu \in K_1(\sigma)$$

□

**9.4 Theorem**: Assume that $U(0) > -\infty$ holds. Then

(i) the conditions (B)–(E) are equivalent, and imply (A) with the same portfolio $\hat{\pi}$ as in condition (B); and

(ii) conversely, if $\hat{\pi} \in \mathcal{A}(x)$ satisfies (A), then there exists a $\lambda \in K_1(\sigma)$ for which (B)–(E) hold, provided that the condition (4.8) is also in force.

**Proof**: In view of Theorem 9.2, we need discuss only the implications (C) $\Longrightarrow$ (D) $\Longrightarrow$ (B) and (A) $\Longrightarrow$ (B), under the appropriate conditions.

*(C)* $\Longrightarrow$ *(D)*: For any given $y > 0$ and $\nu \in K_1(\sigma)$, the convexity of $\tilde{U}$ yields

$$(9.2)\qquad \frac{1}{|\epsilon|}\,|\,\tilde{U}((y+\epsilon)\beta(T)Z_\nu(T)) - \tilde{U}(y\beta(T)Z_\nu(T))\,| \leqq \beta(T)Z_\nu(T)\, I(\tfrac{y}{2}\,\beta(T)Z_\nu(T))$$

in conjunction with (4.4), for $\epsilon > -y/2$, $\epsilon \neq 0$. From the assumption $\nu \in K_1(\sigma)$, the random variable on the right–hand side of (9.2) has expectation equal to $\mathcal{X}_n(y/2) < \infty$, and the Dominated Convergence Theorem shows that

$$(9.3)\qquad \frac{d}{dy}\, E\tilde{U}(y\beta(T)Z_\nu(T)) = -\,E[\beta(T)Z_\nu(T)I(y\beta(T)Z_\nu(T))] = -\,\mathcal{X}_\nu(y).$$

Therefore, for any given $x > 0$, $\nu \in K_1(\sigma)$, the convex function

$$(9.4)\qquad f_\nu(y) \triangleq E\tilde{U}(y\beta(T)Z_\nu(T)) + xy\,; \quad 0 < y < \infty,$$

attains its minimum at $\mathcal{Y}_\nu(x)$, since $f'_\nu(y) = x - \mathcal{X}_\nu(y)$. But thanks to (C) and (4.3), we now have for any $y > 0$ that

$$(9.5)\quad f_\nu(y) \geq f_\nu(\mathcal{Y}_\nu(x)) = E[\tilde{U}(\mathcal{Y}_\nu(x)\beta(T)Z_\nu(T)) + \mathcal{Y}_\nu(x)\beta(T)Z_\nu(T)\cdot I(\mathcal{Y}_\nu(x)\beta(T)Z_\nu(T))]$$

$$= EU(I(\mathcal{Y}_\nu(x)\beta(T)Z_\nu(T))) = EU(\xi^x_\nu) \geq EU(\xi^x_\lambda) =$$

$$\ldots = f_\lambda(\mathcal{Y}_\lambda(x))\,, \quad \forall\, \nu \in K_1(\sigma)$$

and thus,

$$E\tilde{U}(\mathcal{Y}_\lambda(x)\beta(T)Z_\lambda(T)) = E[U(\xi^x_\lambda) - \mathcal{Y}_\lambda(x)\beta(T)Z_\lambda(T)\xi^x_\lambda]$$

$$= f_\lambda(\mathcal{Y}_\lambda(x)) - x\mathcal{Y}_\lambda(x) \leq f_\nu(\mathcal{Y}_\lambda(x)) - x\mathcal{Y}_\lambda(x) = E\tilde{U}(\mathcal{Y}_\lambda(x)\beta(T)Z_\nu(T)).$$

*(D)* $\Longrightarrow$ *(B)*: Repeat the proof of Proposition 8.5 up to (8.14), with $K(\sigma)$ replaced by $K_1(\sigma)$, (8.7) by (D), and B by $\xi^x_\lambda$. Everything then boils down to showing that the analogue

$$(9.6)\qquad E[\beta(T)Z_\lambda(T)\xi^x_\lambda \int_0^{\tau_n} \nu^*(s)(dW(s) + \lambda(s)ds)] = 0$$

of (8.15) can be obtained from the consequence of D

$$\frac{\partial}{\partial \epsilon} E[\tilde{U}(\mathcal{Y}_\lambda(x)\beta(T)Z_{\lambda+\epsilon\nu_n}(T))]\Big|_{\epsilon=0} = 0, \tag{9.7}$$

since $\lambda + \epsilon\nu_n \in K_1(\sigma)$ for every $\epsilon \in (-1,1)$, $n \geq 1$ (recall argument in the proof of implication (E) $\Rightarrow$ (B) in Theorem 9.3). Indeed, (9.6) follows formally from (9.7) by differentiating inside the expectation sign, and using (4.4), (7.9), (8.13).

For a rigorous justification, recall (8.12) and use the convexity of $\tilde{U}$ to obtain, for any given $y > 0$:

$$\begin{aligned} &|\tilde{U}(y\beta(T)Z_{\lambda+\epsilon\nu_n}(T)) - \tilde{U}(y\beta(T)Z_\lambda(T))| \\ &\leq y\beta(T)\, I(y\beta(T)\min\{Z_\lambda(T), Z_{\lambda+\epsilon\nu_n}(T)\})\,|Z_{\lambda+\epsilon\nu_n}(T) - Z_\lambda(T)| \\ &\leq y\beta(T)(e^{3n|\epsilon|} - 1)\, Z_\lambda(T)\, I(y\beta(T)e^{-3n}Z_\lambda(T)) \\ &\leq K_n|\epsilon| \cdot y\beta(T)Z_\lambda(T)\, I(y\beta(T)\, e^{-3n}Z_\lambda(T)), \end{aligned} \tag{9.8}$$

where $K_n \triangleq \sup\limits_{0<\epsilon<1} \dfrac{e^{3n\epsilon} - 1}{\epsilon}$. The expectation of the right–hand side of (9.8) is equal to $yK_n|\epsilon|$ times $E[\beta(T)Z_\lambda(T)I(ye^{-3n}\beta(T)Z_\lambda(T)] = \mathcal{X}_\lambda(ye^{-3n})$, a finite quantity by the assumption $\lambda \in K_1(\sigma)$.

On the other hand, the Mean–Value Theorem implies that for each $\epsilon \in (-1,1)\backslash\{0\}$ there is a random variable $\gamma_\epsilon$ with values in [0,1], such that

$$\frac{1}{\epsilon}[\tilde{U}(y\beta(T)Z_{\lambda+\epsilon\nu_n}(T)) - \tilde{U}(y\beta(T)Z_\lambda(T))] = y\beta(T)\,\frac{Z_{\lambda+\epsilon\nu_n}(T) - Z_\lambda(T)}{\epsilon}$$

$$\cdot \, (\tilde{U}'(y\beta(T)\{Z_\lambda(T) + \gamma_\epsilon(Z_{\lambda+\epsilon\nu_n}(T) - Z_\lambda(T))\})$$

$$= - I(y\beta(T)\{Z_\lambda(T) + \gamma_\epsilon(Z_{\lambda+\epsilon\nu_n}(T) - Z_\lambda(T))\}) \cdot y\beta(T)Z_\lambda(T)$$

$$\cdot \, \frac{1}{\epsilon}[\exp\{-\epsilon \int_0^{\tau_n} \nu^*(s)(dW(s) + \lambda(s)ds) - \frac{\epsilon^2}{2} \int_0^{\tau_n} \|\nu(s)\|^2 ds\} - 1].$$

From this and (9.7), the conclusion (9.6) follows, thanks to the Dominated Convergence Theorem, by letting $\epsilon \downarrow 0$.

*(A)* $\Longrightarrow$ *(B)*. Step (i): Let $\hat{X}$ be the wealth process corresponding to the optimal portfolio $\hat{\pi}$. We have from (3.3):

$$\beta(t)\hat{X}(t) = x + \int_0^t \beta(s)\hat{X}(s)(\sigma^*(s)\hat{\pi}(s))^* dW_0(s) \tag{9.9}$$

$$= x \exp\{\int_0^t \hat{\pi}^*(s)\sigma(s)\, dW_0(s) - \frac{1}{2}\int_0^t \|\sigma^*(s)\hat{\pi}(s)\|^2 ds\}.$$

Now take a bounded, $\{\mathcal{F}_t\}$—progressively measurable portfolio process $\eta$ with values in $\mathbb{R}^m$, and perform a small random perturbation of $\hat{\pi}$ according to

$$\pi_\epsilon(t) \triangleq \hat{\pi}(t) + \epsilon\eta(t)1_{\{t \leq \tau_n\}}, \tag{9.10}$$

where $-1 < \epsilon < 1$, $\epsilon \neq 0$, and

$$\tau_n = T \wedge \inf\{t \in [0,T];\ |\int_0^t \eta(s)\sigma(s)dW_0(s)| \geq n, \text{ or } \int_0^t \|\sigma^*(s)\hat{\pi}(s)\|^2 ds \geq n, \text{ or}$$

$$(9.11) \qquad \int_0^t \|\sigma^*(s)\eta(s)\|^2 ds \geq n, \text{ or } \int_0^t \|\theta(s)\|^2 ds \geq n, \text{ or } N(t) \geq n, \text{ or}$$

$$|A(t)| \geq n, \text{ or } \int_0^t \|\psi(s)\|^2 ds \geq n\}$$

(see (9.19), (9.20) below for the definitions of the processes $N$, $A$ and $\psi$). We define also the process $X_\epsilon(\cdot)$ via

$$\beta(t)X_\epsilon(t) \triangleq x \exp\{\int_0^t \pi_\epsilon^*(s)\sigma(s)dW_0(s) - \frac{1}{2}\int_0^t \|\sigma^*(s)\pi_\epsilon(s)\|^2 ds\}$$

$$(9.12) \qquad = x + \int_0^t \beta(s)X_\epsilon(s)\pi_\epsilon^*\sigma(s)dW_0(s)\ ,$$

and notice that $X_\epsilon(\cdot) \equiv X^{x,\pi_\epsilon}(\cdot)$. Consequently, (A) gives

$$(9.13) \qquad \frac{\partial}{\partial \epsilon} EU(X_\epsilon(T))\Big|_{\epsilon=0} = 0.$$

A comparison of (9.9), (9.12) yields

$$(9.14) \qquad X_\epsilon(t) = \hat{X}(t) \exp\{\epsilon \int_0^{t\wedge\tau_n} \eta^*(s)\sigma(s)d\hat{W}(s) - \frac{\epsilon^2}{2}\int_0^{t\wedge\tau_n} \|\sigma^*(s)\eta(s)\|^2 ds\},$$

where

$$(9.15) \qquad \hat{W}(t) \triangleq W_0(t) - \int_0^t \sigma^*(s)\hat{\pi}(s)ds = W(t) + \int_0^t (\theta(s) - \sigma^*(s)\hat{\pi}(s))ds.$$

Then, at least formally, (9.13) and (9.14) lead to

$$\text{(9.16)} \qquad E[U'(\hat{X}(T))\hat{X}(T)\int_0^{\tau_n} \eta^*(s)\sigma(s)d\hat{W}(s)] = 0, \quad \forall\, n \geq 1.$$

Step (ii): In order to justify (9.16) rigorously, observe from (9.14) and (9.11) that $e^{-3n|\epsilon|} \leq \dfrac{X_\epsilon(T)}{\hat{X}(T)} \leq e^{3n|\epsilon|}$, a.s., and from the concavity of $U$:

$$\frac{1}{|\epsilon|}\,|U(X_\epsilon(T)) - U(\hat{X}(T))| \leq U'(\min\{X_\epsilon(T),\hat{X}(T)\})\left|\frac{X_\epsilon(T) - \hat{X}(T)}{\epsilon}\right|$$

$$\leq U'(e^{-3n|\epsilon|}\hat{X}(T))\hat{X}(T)\,\frac{e^{3n|\epsilon|} - 1}{|\epsilon|}$$

$$\text{(9.17)} \qquad \leqq [U(e^{-3n|\epsilon|}\,\hat{X}(T)) - U(0)]e^{3n}K_n$$

$$\leqq e^{3n}K_n \cdot [U(\hat{X}(T)) - U(0)]\,,$$

with $K_n$ as in (9.8). The right–hand side of (9.17) has finite expectation, namely $e^{3n}K_n(V(x) - U(0))$. On the other hand, the Mean–Value Theorem implies the existence of a random variable $\gamma_\epsilon$ with values in [0,1], such that

$$\frac{1}{\epsilon}[U(X_\epsilon(T)) - U(\hat{X}(T))] = \frac{1}{\epsilon}(X_\epsilon(T) - \hat{X}(T)) \cdot U'(\hat{X}(T) + \gamma_\epsilon\{X_\epsilon(T) - \hat{X}(T)\})$$

$$= U'(\hat{X}(T) + \gamma_\epsilon\{X_\epsilon(T) - \hat{X}(T)\})\,\hat{X}(T)\,\frac{1}{\epsilon}\Big[\exp\Big\{\epsilon\int_0^{\tau_n}\eta^*(s)\sigma(s)d\hat{W}(s) - \frac{\epsilon^2}{2}\int_0^{\tau_n}\|\sigma^*(s)\eta(s)\|^2ds\Big\} - 1\Big].$$

It is dear now that (9.16) follows from this expansion, (9.13), and the Dominated Convergence Theorem, by letting e [ 0.

Step (iii): Now proving (B) amounts to finding A 6 Ki(a) such that
$\hat{X}(T)$ = I(^$_x$(x)/?(T)Z$_X$(T)), or equivalent^

(9.18) $$U'(\hat{X}(T)) = \mathcal{Y}_\lambda(x)\beta(T)Z_\lambda(T).$$

We shall show that (9.16) leads to a "natural" candidate process A € K(a) (Step (iv)), which is actually in Kj(a) and for which (9.18) is then shown to hold (Step (v)).

Step (iv): Consider the process

(9.19) A(t)4 f$^t_{Jo}$ 7?*(s)o(s)d$\hat{W}$(s) = f$^t_{JO}$ *7*(s)cr(s)dW(s) + f$^t_{\wedge o}$ v*{$)a(s)[$\theta$(s) − $\sigma$(s)$\hat{\pi}$(s)]ds ,

as well as the positive martingale

(9.20) N(t) 4 E[U'($\hat{X}$(T))$\hat{X}$(T)|^$_*$] = $y_0$ + f$^t_{J0}$ N(s)/(s)dW(s),

where $y_0$ = EN(T) and ^ is some process in L*[O,TJ, constructed by **the** argument preceding (8.9). Obviously (9.16) amounts to E[N($r_n$)A($r_n$)] = 0; on the other hand, we have from (9.19) and (9.20):

$$N(\tau_n)A(\tau_n) = \int_0^{\tau_n} N(t)\eta^*(t)\sigma(t)[\psi(t) + \theta(t) - \sigma^*(t)\hat{\pi}(t)]\,dt$$

$$+ \int_0^{\tau_n} N(t)[\sigma^*(t)\eta(t) + A(t)\psi(t)]^* dW(t).$$

From the definition of $\tau_n$ in (9.11), the stochastic integral has zero expectation, and thus $E[N(\tau_n)A(\tau_n)] = 0$ leads to

$$E\int_0^{\tau_n} N(t)\eta^*(t)\sigma(t)[\psi(t) + \theta(t) - \sigma^*(t)\hat{\pi}(t)]dt = 0, \quad \forall\, n \geq 1 \tag{9.21}$$

for arbitrary $\eta$ as described above. Because $\tau_n \nearrow T$ almost surely as $n \to \infty$, we obtain that

$$\lambda \triangleq \sigma^*\hat{\pi} - (\psi + \theta) \tag{9.22}$$

belongs to $K(\sigma)$. For this choice of $\lambda$, the exponential local martingale $Z_\lambda$ of (7.6) becomes:

$$Z_\lambda(t) = \exp\{-\int_0^t (\theta(s) + \lambda(s))^* dW(s) - \tfrac{1}{2}\int_0^t \|\theta(s) + \lambda(s)\|^2 ds\}$$

$$= \exp\{\int_0^t (\psi(s) - \sigma^*(s)\hat{\pi}(s))^* dW(s) - \tfrac{1}{2}\int_0^t \|\psi(s) - \sigma^*(s)\hat{\pi}(s)\|^2 ds\}. \tag{9.23}$$

<u>Step (v)</u>: Finally, we justify $\lambda \in K_1(\sigma)$ and (9.18). From (9.20) and (9.9) and $\sigma\lambda = 0$, it follows that

(9.24) $$U'(\hat{X}(T)) = \beta(T)\frac{N \quad (T)}{\beta( \quad T)\hat{X}(T)} =$$

$$= \beta(T)\,\frac{y_0}{x}\,\frac{\exp\{\int_0^T \quad \psi^*(s)dW(s) - \frac{1}{2}\int_0^T \|\psi(s)\|^2 ds\}}{\exp\{\int_0^T \quad (\sigma^*(s)\,\hat{\pi}(s))^* dW_0(s) - \frac{1}{2}\int_0^T \|\sigma^*(s)\hat{\pi}(s)\|^2 ds\}}$$

$$= \frac{y_0}{x}\,\beta(T)Z_\lambda(T),$$

thanks to (9.23) and (9.22). It remains to show that $\lambda \in K_1(\sigma)$ and $y_0 = x\mathscr{Y}_\lambda(x)$. In order t[o] see this, apply $I(\cdot)$ to both sides of (9.24), take expectations, and use (9.24) again to obtain

$$\mathscr{X}_\lambda\left(\frac{y_0}{x}\right) = E[\beta(T \quad )Z_\lambda(T)I(\frac{y_0}{x}\,\beta(T)Z_\lambda(T))] = E[\beta(T)Z_\lambda(T)\hat{X}(T)]$$

$$= \frac{x}{y_0}\,E[U'(\hat{X}(T))\hat{X}(T)] = \frac{x}{y_0}\,EN(T) = x < \infty.$$

From Remark 9.1 we have $\lambda \in K_1(\sigma)$, and $y_0 = x\mathscr{Y}_\lambda(x)$ follows. □

## 10. EXAMPLES IN AN INCOMPLETE MARKET

In the examples of this section, we assume $m < d$ and produce the optimal portfolio and the process $\lambda \in K_1(\sigma)$ satisfying conditions (B), (C) and (E). In Example 10.2, $\lambda$ also satisfies (D).

**10.1 Example.** (*Logarithmic utility function*).

Suppose $U(x) = \log x$. Then $\mathscr{X}_\nu(y) = \frac{1}{y}$, $\mathscr{Y}_\nu(x) = \frac{1}{x}$, $\forall\, \nu \in K(\sigma)$, and $\xi_\nu^x = \dfrac{x}{\beta(T)Z_\nu(T)}$. The process $\lambda \equiv 0$ satisfies (E), because

$$E[\beta(T)Z_\nu(T)\xi_0^x] = x\, E[\exp\{-\int_0^T \nu^*(s)dW(s) - \frac{1}{2}\int_0^T \|\nu(s)\|^2 ds\}] \le x, \;\; \forall\, \nu \in K(\sigma).$$

The last inequality follows from the fact that $\exp\{-\int_0^t \nu^*(s)dW(s) - \frac{1}{2}\int_0^t \|\nu(s)\|^2 ds\}$, being a nonnegative local martingale, must be a supermartingale. According to Theorem 9.3, the optimal portfolio process $\hat{\pi}$ must satisfy $X^{x,\hat{\pi}}(T) = \xi_0^x$, and this $\hat{\pi}$ was determined in (6.13) of Example 6.3. The value function $V(x)$ is given by the expression (6.14) and it is finite for every $x \in (0,\infty)$ if $E\int_0^T \|\theta(t)\|^2 dt < \infty$ (recall condition (5.3)). From (7.4) we see that $\lambda \equiv 0$ corresponds to completion of the market by stocks whose appreciation rates are equal to the interest rate. With a logarithmic utility function, the agent will not use such stocks even for hedging purposes.

10.2 Example (*Power utility function and totally unhedgeable market coefficients*).

Suppose $U(x) = \frac{1}{\delta}x^\delta$, where $\delta < 1$, $\delta \neq 0$. Suppose that the volatility matrix $\sigma(t)$ has the form $\sigma(t) = [\check{\sigma}(t),0]$, where $\check{\sigma}(t)$ is an $m \times m$, nonsingular matrix for all $t \in [0,T]$, almost surely. Decompose $W$ into $\check{W}(t) = (W_1(t),\ldots,W_m(t))^*$ and $\overset{o}{W}(t) = (W_{m+1}(t),\ldots,W_d(t))^*$, and let $\{\check{\mathcal{F}}_t\}$ and $\{\overset{o}{\mathcal{F}}_t\}$ be the augmentations under $P$ of the (independent) filtrations generated by $\check{W}$ and $\overset{o}{W}$, respectively. Assume that the processes $r$, $b$ and $\check{\sigma}$ are adapted to $\{\overset{o}{\mathcal{F}}_t\}$, a situation we refer to as totally unhedgeable market coefficients because the stock prices are driven solely by $\check{W}$:

$$dP_i(t) = P_i(t)[b_i(t)dt + \sum_{j=1}^m \check{\sigma}_{ij}(t)d\check{W}_j(t)], \quad i = 1,\ldots,m.$$

We show that under these conditions, the portfolio process given by (6.17) is optimal. In the present context, this process is random and $\{\overset{o}{\mathcal{F}}_t\}$–adapted, rather than deterministic as in Example 6.4.

To verify the above assertion, we note first that $\theta^*(t) = [\check{\theta}^*(t),0]$, where

$$\check{\theta}(t) \triangleq \check{\sigma}^*(t)(\check{\sigma}(t)\check{\sigma}^*(t))^{-1}[b(t) - r(t)\underline{1}].$$

We note also that the processes $\lambda \in K(\sigma)$ are of the form $\lambda^*(t) = [0,\overset{o}{\lambda}{}^*(t)]$, where $\overset{o}{\lambda}(t)$ is (d–m)–dimensional. With $m(\cdot)$ given by (6.15), we define

$$A \triangleq E \exp \{\delta \int_0^T m(t)dt\}.$$

The differential of the positive $\{\overset{o}{\mathcal{F}}_t\}$–martingale $N(t) \triangleq E[\exp\{\delta \int_0^T m(s)ds\} | \overset{o}{\mathcal{F}}_t\}$ has a representation as $dN(t) = -N(t)\, \overset{o}{\lambda}{}^*(t)d\overset{o}{W}(t)$, where $\lambda = \begin{bmatrix} 0 \\ \overset{o}{\lambda} \end{bmatrix} \in K(\sigma)$ (see the argument leading to (8.9) for a justification). Therefore

$$\exp\{\delta \int_0^T m(t)dt\} = N(T) = A \exp\{-\int_0^T \overset{o}{\lambda}{}^*(t)d\overset{o}{W}(t) - \frac{1}{2}\int_0^T \|\overset{o}{\lambda}(t)\|^2 dt\}. \tag{10.1}$$

We may assume without loss of generality that $\check{W}$ is the coordinate mapping process $\check{W}(t,\check{\omega}) = \check{\omega}(t)$ defined on $\check{\Omega} \triangleq C([0,T], \mathbb{R}^m)$, the space of continuous functions from $[0,T]$ to $\mathbb{R}^m$, and $\overset{o}{W}$ is the coordinate mapping process on $\overset{o}{\Omega} \triangleq C([0,T], \mathbb{R}^{d-m})$. Then $\Omega = \check{\Omega} \times \overset{o}{\Omega}$, and P is the product of m–dimensional Wiener measure $\check{P}$ on $\check{\Omega}$ and (d–m)–dimensional Wiener measure $\overset{o}{P}$ on $\overset{o}{\Omega}$. Abusing notation slightly, we regard the $\{\overset{o}{\mathcal{F}}_t\}$–adapted process $\check{\theta}$ as a process on $\overset{\circ}{\Omega}$. For $\overset{\circ}{P}$ – almost every $\overset{\circ}{\omega} \in \overset{\circ}{\Omega}$, we have $\int_0^T \|\check{\theta}^*(s,\overset{\circ}{\omega})\|^2 ds < \infty$, and thus the process

$$\check{w} \mapsto \exp\{\frac{\delta}{1-\delta}\int_0^t \check{\theta}(s,\overset{\circ}{\omega})d\check{W}(s,\check{\omega}) - \frac{\delta^2}{2(1-\delta)^2}\int_0^t \|\check{\theta}(s,\overset{\circ}{\omega})\|^2 ds\}$$

is an $\{\check{\mathscr{F}}_t\}$–martingale on $\check{\Omega}$ under $\check{P}$, with expectation equal to one (Karatzas & Shreve [13], p. 199, Corollary 5.13). Consequently,

$$(10.2) \qquad E[\exp\{\frac{\delta}{1-\delta}\int_0^T \check{\theta}^*(s)d\check{W}(s)\} \,|\, \overset{\circ}{\mathscr{F}}_T](\check{\omega},\overset{\circ}{\omega})$$

$$= \int_{\check{\Omega}} \exp\{\frac{\delta}{1-\delta}\int_0^T \check{\theta}^*(s,\overset{\circ}{\omega})d\check{W}(s,\check{\omega})\}P(d\check{\omega})$$

$$= \exp\{\frac{\delta^2}{2(1-\delta)^2}\int_0^T \|\check{\theta}(s,\overset{\circ}{\omega})\|^2 ds\}.$$

From (10.2) and (10.1) we have

$$\mathscr{X}_\lambda(y) = y^{\frac{1}{\delta-1}} E[\exp\{\frac{\delta}{1-\delta}\int_0^T (r(s) + \frac{1}{2}\|\check{\theta}(s)\|^2 + \frac{1}{2}\|\overset{\circ}{\lambda}(s)\|^2)ds + \frac{\delta}{1-\delta}\int_0^T \overset{\circ}{\lambda}{}^*(s)d\overset{\circ}{W}(s)\}$$

$$\cdot E[\exp\{\frac{\delta}{1-\delta}\int_0^T \check{\theta}^*(s)d\check{W}(s)\} \,|\, \overset{\circ}{\mathscr{F}}_T]]$$

$$= y^{\frac{1}{1-\delta}} E[\exp\{\frac{\delta}{1-\delta}\int_0^T (m(s)ds + \frac{1}{2}\|\overset{\circ}{\lambda}(s)\|^2)ds + \frac{\delta}{1-\delta}\int_0^T \overset{\circ}{\lambda}{}^*(s)d\overset{\circ}{W}(s)\}]$$

$$= A^{\frac{\delta}{1-\delta}} y^{\frac{1}{\delta-1}} E[\exp\{\delta\int_0^T m(s)ds\}] = (\tfrac{y}{A})^{\frac{1}{\delta-1}}.$$

It follows that $\lambda \in K_1(\sigma)$, $\mathcal{Y}_\lambda(x) = Ax^{\delta-1}$, and using (10.1) we obtain

$$\xi_\lambda^x = (\mathcal{Y}_\lambda(x)\beta(T)Z_\lambda(T))^{\frac{1}{\delta-1}} = x\exp\{\int_0^T (r(t) + \tfrac{1-2\delta}{2(\delta-1)^2}\|\theta(t)\|^2)dt + \tfrac{1}{1-\delta}\int_0^T \theta^*(t)dW(t)\}.$$

Just as in Example 6.4, we conclude from (3.4) that $X^{x,\hat{\pi}}(T) = \xi_\lambda^x$, where $\hat{\pi}$ is given by (6.17).

**10.3 Remark.** An important unresolved question is whether there are simple, widely applicable conditions which guarantee that for the process $\lambda$ satisfying conditions (B)–(E) of Section 9, the nonnegative local martingale $Z_\lambda$ is actually a martingale. In Example 10.1 we have

$$Z_\lambda(t) = Z_0(t) = \exp\{-\int_0^t \theta^*(s)dW(s) - \tfrac{1}{2}\int_0^t \|\theta(s)\|^2 ds\},$$

so we must assume at least that $Z_0$ is a martingale in order to conclude that $Z_\lambda$ is. In Example 10.2 a computation similar to (10.2) reveals that

$$E\,Z_\lambda(T) = E\Big[\exp\Big\{-\int_0^T \tilde{\lambda}^*(t)\,d\mathring{W}(t) - \frac{1}{2}\int_0^T \big(\|\tilde{\theta}(t)\|^2 + \|\mathring{\lambda}(t)\|^2\big)dt\Big\}$$

$$\cdot\, E\Big[\exp\Big\{-\int_0^T \mathring{\lambda}^*(t)\,d\mathring{W}(t)\Big\} \,\Big|\, \mathring{\mathcal{F}}_T\Big]\Big]$$

$$= E\Big[\exp\Big\{-\int_0^T \mathring{\lambda}^*(t)\,d\mathring{W}(t) - \frac{1}{2}\int_0^T \|\mathring{\lambda}(t)\|^2 dt\Big\}\Big].$$

Taking expectations in (10.1) and recalling the definition of $\mathring{\lambda}$, we see that $E\,Z_\lambda(T) = 1$. This is enough to ensure that $Z_\lambda$ is a martingale (Karatzas & Shreve [13], p. 198). We have so far been unable to produce an example in which $Z_0$ is a martingale but $Z_\lambda$ is not.

## 11. DUALITY

Let us assume from now on that

(11.1) $$U(0) > -\infty.$$

In addition to the original, or "primal", optimization problem

(11.2) $$V(x) = \sup_{\pi\in\mathcal{A}(x)} J(x,\pi); \quad J(x,\pi) \triangleq EU(X^{x,\pi}(T))$$

of section 5, we shall consider in what follows the <u>dual optimization problem</u> for $y \in (0,\infty)$, namely

(11.3) $$\tilde{V}(y) = \inf_{\nu\in K(\sigma)} \tilde{J}(y,\nu); \quad \tilde{J}(y,\nu) \triangleq E\tilde{U}(y\gamma(T)Z_\nu(T)).$$

This problem will have a convex value function $\tilde{V} : (0,\infty) \to \mathbb{R}$ under the assumption which we now make, that

$$\forall\, y \in (0,\infty),\ \exists\, \nu \in K(\sigma) \text{ such that } \tilde{J}(y,\nu) < \infty. \tag{11.4}$$

See Remark 11.7 in connection with (11.4).

For arbitrary $x > 0$, $y > 0$, $\pi \in \mathscr{A}(x)$ and $\nu \in K(\sigma)$ it follows from (4.3):

$$U(X^{x,\pi}(T)) \le \tilde{U}(y\beta(T)Z_\nu(T)) + y\beta(T)Z_\nu(T)X^{x,\pi}(T), \tag{11.5}$$

with equality if and only if

$$X^{x,\pi}(T) = I(y\beta(T)Z_\nu(T)). \tag{11.6}$$

By taking expectations in (11.5) and recalling Proposition 8.3, we obtain

$$J(x,\pi) \le \tilde{J}(y,\nu) + xy\ , \tag{11.7}$$

with equality prevailing if and only if (11.6) and $x = \mathscr{X}_\nu(y)$ hold. In particular, it follows from (11.7) that

$$V(x) \le \tilde{V}(y) + xy\ ;\quad \forall\, x > 0,\ y > 0. \tag{11.8}$$

**11.1 Remark**: Suppose that for some given $x > 0$ and $y > 0$, there exist $\hat{\pi}_x \in \mathscr{A}(x)$ and $\lambda_y \in K(\sigma)$ such that

$$J(x;\hat{\pi}_x) = \tilde{J}(y;\lambda_y) + xy. \tag{11.9}$$

Then $\hat{\pi}_x$ achieves the supremum in (11.2), and $\lambda_y$ achieves the infimum in (11.3).

**11.2 Proposition:** Assume (11.1), (11.4) hold and suppose that, for a given $y > 0$, there is an optimal process $\lambda_y \in K_1(\sigma)$ for the dual problem of (11.3). Then there exists an optimal portfolio $\hat{\pi}_x \in \mathscr{A}(x)$ for the primal problem of (11.2) with $x = \mathscr{X}_{\lambda_y}(y)$, and we have

$$\tilde{V}(y) = \sup_{\xi>0} [V(\xi) - y\xi]. \tag{11.10}$$

**Proof:** The optimality of $\lambda_y$ gives

$$E[\tilde{U}(\mathscr{Y}_{\lambda_y}(x)\beta(T)Z_{\lambda_y}(T))] \le E[\tilde{U}(\mathscr{Y}_{\lambda_y}(x)\beta(T)Z_\nu(T))], \quad \forall\, \nu \in K(\sigma)$$

for this particular $x = \mathscr{X}_{\lambda_y}(y)$. Then the implications (D) $\Longrightarrow$ (B) $\Longrightarrow$ (A) in Theorem 9.4(i) show the existence of a portfolio $\hat{\pi}_x \in \mathscr{A}(x)$, which is optimal for the primal problem and satisfies

$$X^{x,\hat{\pi}_x}(T) = I(y\beta(T)Z_{\lambda_y}(T)).$$

We conclude that (11.9) prevails (i.e., (11.7) holds as an equality with $\pi = \hat{\pi}_x$, $\nu = \lambda_y$), and thus

$$\tilde{V}(y) = \tilde{J}(y;\lambda_y) = J(x;\hat{\pi}_x) - xy = V(x) - xy \le \sup_{\xi>0} [V(\xi) - y\xi].$$

The inequality in the opposite direction follows directly from (11.8), and the duality relationship (11.10) is established.

□

**11.3 Assumption:** Suppose that the dual problem of (11.3) admits an optimal process $\lambda_y \in K_1(\sigma)$, for every $y > 0$.

A sufficient condition (Theorem 12.1) for (11.3) will be given in the next section. Under the Assumption 11.3, (11.10) holds for all $y > 0$, and the following question arises: Under what conditions can we guarantee that, for every given $x > 0$, there exists an optimal portfolio $\hat{\pi}_x$ for (11.2)?

According to Proposition 11.2, this will happen if for every $x > 0$ we can find a real number $y(x) > 0$ such that

$$x = \mathscr{X}_{\lambda_{y(x)}}(y(x)). \tag{11.11}$$

**11.4 Proposition:** Suppose that the conditions of Proposition 11.2 hold, as well as Assumption 11.3, (4.8) and $U(\infty) = \infty$. Then for every $x > 0$, there exists a real number $y(x) > 0$ that achieves $\inf_{y>0}[\tilde{V}(y) + xy]$; this number satisfies (11.11) as well.

**Proof:** From (11.3), Jensen's inequality, the supermartingale property of $Z_\nu$, and the decrease of $\tilde{U}$, we have

$$\tilde{J}(y;\nu) \geq \tilde{U}(y\ E[\beta(T)Z_\nu(T)]) \geq \tilde{U}(ye^L EZ_\nu(T)) \geq \tilde{U}(ye^L),\ \ \forall\ \nu \in K(\sigma) \tag{11.12}$$

for the constant $L > 0$ of (2.3). Therefore, $\tilde{V}(y) \geq \tilde{U}(ye^L)$ holds for every $y \in (0,\infty)$, and $\tilde{V}(0)$

$\triangleq \lim_{y \downarrow 0} \tilde{V}(y) \geqq \tilde{U}(0) = U(\infty) = \infty$ (Lemma 4.1).

Consequently, for any given $x > 0$, the convex function $f_x(y) = \tilde{V}(y) + xy$, $0 < y < \infty$ satisfies $f_x(0+) = f_x(\infty) = \infty$, and thus attains its infimum on $(0,\infty)$ at some point $y(x) > 0$. Now by the Assumption 11.3, there exists a process $\lambda_{y(x)} \in K_1(\sigma)$ such that $\tilde{V}(y(x)) = \tilde{J}(y(x); \lambda_{y(x)})$, and we have

$$\inf_{\eta>0} [\eta y(x)x + \tilde{J}(\eta y(x); \lambda_{y(x)})] = \inf_{y>0} [xy + \tilde{J}(y;\lambda_{y(x)})]$$

$$\geqq \inf_{y>0} [xy + \tilde{V}(y)] = xy(x) + \tilde{V}(y(x)).$$

In other words, with the notation

$$G_y(u) \triangleq \tilde{J}(uy;\lambda_y) = E\tilde{U}(uy\beta(T)Z_{\lambda_y}(T)), \quad 0 < u < \infty, \tag{11.13}$$

the function

$$D_x(u) \triangleq uxy(x) + G_{y(x)}(u), \quad 0 < u < \infty \tag{11.14}$$

achieves its infimum at $u = 1$.

From these considerations and Lemma 11.5 below, it transpires that

$$D_x'(1) = xy(x) + G_{y(x)}'(1) = xy(x) - y(x)\mathcal{X}_{\lambda_{y(x)}}(y(x))$$

is equal to zero, and thus (11.11) holds.

□

**11.5 LEMMA:** Under the conditions of Proposition 11.4, the function $G_y(\cdot)$ of (11.13) is well–defined and finite on $(0,\infty)$ for any given $0 < y < \infty$, and satisfies

$$G'_y(1) = -y\mathscr{X}_{\lambda_y}(y). \tag{11.15}$$

**PROOF:** Since $\tilde{U}(\infty) = U(0) > -\infty$ by assumption and by Lemma 4.1, we have from (4.4):

$$\tilde{U}(y) - \tilde{U}(\infty) = -\int_y^\infty \tilde{U}'(\xi)d\xi = \int_y^\infty I(\xi)d\xi, \quad 0 < y < \infty.$$

Thus, for any given $\alpha \in (0,1)$, it follows with the help of (4.9):

$$\tilde{U}(\alpha y) - \tilde{U}(\infty) = \int_{\alpha y}^\infty I(\xi)d\xi = \alpha\int_y^\infty I(\alpha\eta)d\eta$$

$$\leqq \alpha\gamma\int_y^\infty I(\eta)d\eta = \alpha\gamma\,[\tilde{U}(y) - \tilde{U}(\infty)], \quad 0 < y < \infty$$

for a suitable constant $\gamma \in (1,\infty)$. Consequently, for any given $y \in (0,\infty)$,

$$E\tilde{U}(\alpha y\beta(T)Z_{\lambda_y}(T)) \leqq \alpha\gamma\, E\tilde{U}(y\beta(T)Z_{\lambda_y}(T)) + (1-\alpha\gamma)\tilde{U}(\infty)$$

$$= \alpha\gamma\,\tilde{J}(y;\lambda_y) + (1-\alpha\gamma)U(0) < \infty\,.$$

Since $\alpha \in (0,1)$ is arbitrary,

$$E\tilde{U}(uy\beta(T)Z_{\lambda_y}(T)) < \infty \tag{11.16}$$

holds for every $u \in (0,1]$. But the function $\tilde{U}(\cdot)$ is decreasing, so (11.16) holds also for every $u > 1$.

Now use the convexity of $\tilde{U}$, the Dominated Convergence Theorem, (4.4) and the fact that $\lambda_y \in K_1(\sigma)$, to justify the computations

$$G'_y(1) = y\, E[\beta(T)Z_{\lambda_y}(T)\tilde{U}'(y\beta(T)Z_{\lambda_y}(T))]$$

$$= -y\, E[\beta(T)Z_{\lambda_y}(T)I(y\beta(T)Z_{\lambda_y}(T))] = -y\, \mathscr{X}_{\lambda_y}(y),$$

which leads to (11.15). □

It just remains now to put Propositions 11.2 and 11.4 together, in order to obtain the following existence result for the primal problem (11.2).

**11.6 Theorem**: Suppose that the Assumption 11.3 holds, as well as (11.1), (11.4), (4.8) and $U(\infty) = \infty$. Then for any given level $x > 0$ of initial capital, there exists an optimal portfolio $\hat{\pi}_x \in \mathscr{A}(x)$ for the utility maximization problem of section 5.

□

In other words, under appropriate conditions, in order to obtain the existence of an optimal portfolio it is sufficient to deal with the existence of a solution $\lambda_y \in K_1(\sigma)$ for the dual problem (11.3). We shall do that in the next section.

**11.7 Remark**: *The condition (11.4) is satisfied if (5.4) and (5.5) hold.* Indeed, it is not hard to check that condition (5.4) and definition (4.3) lead to

$$\tilde{U}(y) \leqq k_1 + k_3 y^{-\alpha}, \quad \forall\, 0 < y < \infty \tag{11.17}$$

with $a = \frac{1}{1-\delta}$, $k_\delta = (1-\delta)(k_1\delta)^{\frac{\delta}{1-\delta}}$, and thus

(11.18) $$\mathrm{Ifo"}) \wedge k_1 + k_\delta\, r\, V * {}^1\, E\, Z_\nu^{-\alpha}(T)$$

where L is the constant of (2.3). But now

$$Z_\nu^{-\alpha}(T) = \exp\{\alpha\int_0^T(\theta(t)+\nu(t))^* dW(t) - \frac{\alpha^2}{2}\int_0^T(\|\theta(t)\|^2 + \|\nu(t)\|^2)dt\}$$

$$\cdot \exp[\frac{\alpha(\alpha+1)}{2}\int_0^T(\|\theta(s)\|^2 + \|\nu(s)\|^2)ds],$$

and if we take $\nu \in K(a)$ to satisfy

$$\int_0^T \|\nu(s)\|^2 ds \le C, \quad M.$$

(by analogy with (5.4)) we obtain $E\, Z_\nu^{-\alpha}(T) \le e^{\alpha(1+\alpha)C}$. Back into (11.18), this estimate shows that (11.4) is satisfied.

## 12. EXISTENCE IN THE DUAL PROBLEM

We shall establish here the following existence result for the dual optimization problem of (11.3).

**12.1 THEOREM:** Suppose that the conditions (4.8), (11.1), (11.4),

$$(12.1) \qquad E\int_0^T \|\theta(t)\|^2 dt < \infty ,$$

and

$$(12.2) \qquad x \mapsto xU'(x) \text{ is nondecreasing on } (0,\infty)$$

hold. Then for every $y \in (0,\infty)$, there exists a process $\lambda_y \in K_1(\sigma)$ which achieves the infimum in (11.3).

□

**12.2 Remark:** The condition (12.2) is equivalent to

$$(12.2)' \qquad y \mapsto yI(y) \text{ is nonincreasing on } (0,\infty).$$

If $U$ is of class $C^2(0,\infty)$, (12.2) amounts to the statement that the Arrow–Pratt measure of relative risk aversion does not exceed one:

$$(12.2)'' \qquad -\frac{xU''(x)}{U'(x)} \leqq 1, \quad \forall\, x \in (0,\infty).$$

On the other hand, it follows from (12.2) that $U'(x) \geqq \frac{U'(1)}{x}$, $\forall\, x \geq 1$, whence $U(x) \geq U(1) + U'(1) \log x$. Consequently,

$$(12.3) \qquad U(\infty) = \infty.$$

From this Remark and Theorems 12.1, 11.6 we deduce then the fundamental result of sections 11 & 12:

**12.3 Theorem:** Under the assumptions of Theorem 12.1, corresponding to every $x > 0$, there exists an optimal portfolio $\hat{\pi}_x \in \mathscr{A}(x)$ for the utility maximization problem of section 5.

We carry out the proof of Theorem 12.1 in a series of Lemmas, that take up much of the remainder of the section. Let us start with a rather simple observation.

**12.4 Lemma**: Under (11.1) and (12.2), we have $\tilde{U}(0) = \infty$, $\tilde{U}(\infty) > -\infty$ and

$$z \mapsto \tilde{U}(e^z) \quad \text{is convex on } \mathbb{R}. \tag{12.4}$$

**Proof**: The first two claims follow directly from Lemma 4.1 and (12.3). As for (12.4), observe from (4.4) and (12.2)′ that $\frac{d}{dz}\tilde{U}(e^z) = -e^z I(e^z)$ is a nondecreasing function of $z$. □

Introduce now the Hilbert space

$$K_2(\sigma) \triangleq \{\nu \in K(\sigma)\,;\;\; E\int_0^T \|\nu(s)\|^2 ds < \infty\} \tag{12.5}$$

with inner product $<\mu,\nu> \triangleq E\int_0^T \mu^*(s)\nu(s)ds$ and norm $[\nu] \triangleq \sqrt{<\nu,\nu>}$. For a fixed $y > 0$, we consider the functional $\tilde{J}_y : K_2(\sigma) \to \mathbb{R} \cup \{+\infty\}$ given by (11.3), namely

$$\tilde{J}_y(\nu) \triangleq E\tilde{U}(y\beta(T)e^{-\zeta_\nu(T)}) \tag{12.6}$$

with the notation

$$\zeta_\nu(t) \triangleq \int_0^t (\theta(s) + \nu(s))^* dW(s) + \frac{1}{2}\int_0^t (\|\theta(s)\|^2 + \|\nu(s)\|^2)ds\,;\quad \nu \in K(\sigma). \tag{12.7}$$

**12.5 Lemma:** Under (11.1), (12.1) and (12.2), $\tilde{J}_y(\cdot)$ is a convex functional on $K_2(\sigma)$, which satisfies

$$\lim_{[\nu]\to\infty} J_y(\nu) = \infty , \tag{12.8}$$

for every $y \in (0,\infty)$.

**Proof:** From the convexity of the Euclidean norm in $\mathbb{R}^d$, the decrease of $\tilde{U}$ and (12.4), we have

$$\tilde{J}_y(\lambda_1\nu_1 + \lambda_2\nu_2) \leq E\tilde{U}(\exp[\log y - \int_0^T r(s)ds - \lambda_1\zeta_{\nu_1}(T) - \lambda_2\zeta_{\nu_2}(T)]) \tag{12.9}$$

$$= E\tilde{U}(\exp\{\lambda_1[\log y - \int_0^T r(s)ds - \zeta_{\nu_1}(T)] + \lambda_2[\log y - \int_0^T r(s)ds - \zeta_{\nu_2}(T)]\})$$

$$\leqq \lambda_1 \tilde{J}_y(\nu_1) + \lambda_2 \tilde{J}_y(\nu_2)$$

for any $\nu_1,\nu_2$ in $K_2(\sigma)$ and $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$ with $\lambda_1 + \lambda_2 = 1$. On the other hand, with $L$ as in (2.3), we obtain from (12.4) and Jensen's inequality

$$\tilde{J}_y(\nu) \geqq E\tilde{U}(\exp[\log y + L - \zeta_\nu(T)]) \tag{12.10}$$

$$\geqq \tilde{U}(\exp[\log y + L - E\zeta_\nu(T)])$$

$$= \tilde{U}(\exp[\log y + L - \tfrac{1}{2}[\theta]^2 - \tfrac{1}{2}[\nu]^2]) \xrightarrow[{[\nu]\to\infty}]{} \infty.$$

**12.6 Lemma:** Under (11.1), (12.1) and (12.2) we have $\tilde{J}_y(\nu) = \infty$, for every $\nu \in K(\sigma)\backslash K_2(\sigma)$ and $y \in (0,\infty)$.

**Proof:** Fix $y \in (0,\infty)$, $\nu \in K(\sigma)\backslash K_2(\sigma)$ and define stopping times

$$\tau_n \triangleq T \wedge \inf\{t \in [0,T]; \int_0^t \|\nu(s)\|^2 ds = n\}$$

and processes $\nu_n \in K_2(\sigma)$ with $\lim_{n\to\infty} [\nu_n] = \infty$ by

$$\nu_n(t) \triangleq \nu(t) 1_{\{t \le \tau_n\}}; \quad 0 \le t \le T,$$

for $n = 1,2,\ldots$. With $L$ as in (2.3) it follows from Jensen's inequality, the supermartingale property of $Z_\nu$, and the decrease of $\tilde{U}$, that

$$\tilde{J}_y(\nu) = E\tilde{U}(y\beta(T)Z_\nu(T)) = E[E\{\tilde{U}(y\beta(T)Z_\nu(T)) \mid \mathcal{F}_{\tau_n}\}]$$

$$\geqq E[\tilde{U}(ye^L E\{Z_\nu(T) \mid \mathcal{F}_{\tau_n}\})] \geqq E\tilde{U}(ye^L Z_\nu(\tau_n)) \geqq \tilde{J}_{y_0}(\nu_n),$$

with $y_0 = ye^L$, for every $n \ge 1$. The conclusion follows from (12.8) by letting $n \to \infty$.

□

**Proof of Theorem 12.1:** Fix $y \in (0,\infty)$. The convex functional $\tilde{J}_y(\cdot)$ of (12.6) is lower-semicontinuous in the strong topology of $K_2(\sigma)$, by Fatou's Lemma. Therefore, it is also lower-semicontinuous in the weak topology (Ekeland & Temam [4], Chapter 1, Corollary 2.2). Thanks to the coercivity property (12.8) of Lemma 12.5, $\tilde{J}_y(\cdot)$ attains its infimum over

$K_2(\sigma)$ at some $\lambda_y \in K_2(\sigma)$ (ibid., Chapter 2, Proposition 1.2). In light of Lemma 12.6 and condition (11.4),

$$\inf_{y \in K(\sigma)} \tilde{J}_y(\nu) = \tilde{J}_y(\lambda_y) < \infty. \tag{12.11}$$

It remains to show that $\lambda_y \in K_1(\sigma)$. From the decrease of $\tilde{U}$ and I, (4.4) and (4.8)$'$ we obtain

$$\tilde{U}(\xi) - \tilde{U}(\infty) \geqq \tilde{U}(\xi) - \tilde{U}(\tfrac{\xi}{\alpha}) = \int_{\xi}^{\xi/\alpha} I(u)du \geqq \xi(\tfrac{1}{\alpha} - 1)\, I(\tfrac{\xi}{\alpha}) \tag{12.12}$$

$$\geqq \frac{1-\alpha}{\alpha\gamma}\, \xi\, I(\xi)\,, \quad \forall\, \xi \in (0,\infty).$$

Replacing $\xi$ by $y\beta(T)Z_{\lambda_y}(T)$ in (12.12) and taking expectations, we obtain

$$\mathcal{X}_{\lambda_y}(y) = E[\beta(T)Z_{\lambda_y}(T)I(y\beta(T)Z_{\lambda_y}(T))] \leqq \frac{\alpha\gamma}{y(1-\alpha)}[E\tilde{U}(y\beta(T)Z_{\lambda_y}(T)) - \tilde{U}(\infty)] < \infty,$$

thanks to (12.11), and thus $\lambda_y \in K_1(\sigma)$ by Remark 9.1.

□

## 13. APPENDIX

We provide in this section a counterexample, which shows that we <u>cannot</u> have $\mathcal{X}_\nu(y) < \infty$, $\forall\, y \in (0,\infty)$, $\forall\, \nu \in K(\sigma)$, even for a well–behaved utility function U. This fact necessitates the introduction of the set $K_1(\sigma)$ in section 9.

In the setting of section 2, take $m = 1$, $d = 2$, $\sigma(t) \equiv (0,1)$, $r(t) \equiv 0$, $b(t) \equiv 0$, $T = 1$, $B = W_1$, and define the stopping time $\tau \triangleq \inf\{t \in [0,1];\ t + B^2(t) = 1\}$ and the process

(13.1)
$$\varphi(t) \triangleq \begin{cases} -\frac{2B(t)}{(1-t)^2} 1_{\{t \le \tau\}} & ; \; 0 \le t < 1 \\ 0 & ; \; t = 1 \end{cases},$$

$\nu(t) \triangleq (\varphi(t),0)^*$. For the utility function $U(x) = 2\sqrt{x}$ we have $I(y) = y^{-2}$, and (7.6), (7.7) give

(13.2)
$$Z_\nu(t) = \exp\{-\int_0^t \varphi(s)dB(s) - \frac{1}{2}\int_0^t \varphi^2(s)ds\}$$

(13.3)
$$\mathscr{X}_\nu(y) = y^{-2} \, E[\exp\{\int_0^1 \varphi(s)dB(s) + \frac{1}{2}\int_0^1 \varphi^2(s)ds\}].$$

It is shown in Liptser & Shiryaev [14], p. 224 (or Karatzas & Shreve [13]), p. 201) that the process $Z_\nu$ of (13.2) is <u>not</u> a martingale; in fact, the construction (13.1) is made with this property in mind. This implies, in particular, that

(13.4)
$$E[\exp\{\frac{1}{2}\int_0^1 \varphi^2(s)ds] = \infty \; ;$$

for otherwise $Z_\nu$ would be a martingale, by Novikov's theorem (Karatzas & Shreve [13]), p. 199). According to Liptser & Shiryaev [14]), p. 225:

$$\int_0^1 \varphi(s)dB(s) - \frac{1}{2}\int_0^1 \varphi^2(s)ds = -1 - 2\int_0^\tau [(1-t)^{-4} - (1-t)^{-3}]B^2(t)dt,$$

whence

$$\mathscr{X}_\nu(y) = \frac{1}{ey^2} E[\exp\{2 \int_0^T [(1-t)^{-4} + (1-t)^{-8}] B^2(t)dt\}].$$

If this last expectation were finite, then so would be

$$E[\exp\{\tfrac{1}{2} \int_0^1 \varphi^2(s)ds\}] = E[\exp\{2 \int_0^T (1-t)^{-4} B^2(t)dt\}],$$

contradicting (13.4).

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