MULTIPHASE THERMOMECHANICS WITH INTERFACIAL STRUCTURE
2. EVOLUTION OF AN ISOTHERMAL INTERFACE

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Multiphase thermomechanics with interfacial structure. 2. Evolution of an isothermal interface.

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1. Introduction.

Paper 1 [1988g][1] of this series began an investigation whose goal is a thermomechanics of two-phase continua based on Gibbs's notion of a sharp phase-interface endowed with thermomechanical structure. In that paper a new balance law, balance of capillary forces, was introduced and then applied in conjunction with suitable statements of the first two laws of thermodynamics; the chief results are thermodynamic restrictions on constitutive equations, exact and approximate free-boundary conditions at the interface, and a heirarchy of free-boundary problems. The simplest versions of these problems (the Mullins-Sekerka problems) are essentially the classical Stefan problem with the free-boundary condition $u=0$ for the temperature replaced by the condition $u=hK$, where K is the curvature of the free-boundary and $h>0$ is a material constant. This dependence on curvature renders the problem difficult, and apart from numerical studies involving linearization-stability, there are almost no supporting theoretical results.

For perfect conductors the theory seems far more tractable;[2] there the temperature is constant, and the underlying free-boundary problem reduces to a single set of evolution equations for the

[1]See also [1986g,1988gg].

[2]The theory of perfect conductors might be applicable to small interfaces, where bulk effects are small, or to interfaces of arbitrary size in superconductors such as solid helium in which heat flow is insignificant (cf. Maris and Andreev [1987]). A mechanical theory of this type might also model the motion of grain boundaries (cf. Allen and Cahn [1979]).

interface.

In this paper we develop further the theory of perfect conductors, but to avoid the severe geometric complications associated with the motion of surfaces in $\mathbb{R}^3$, we restrict our attention to interfaces that evolve as curves in $\mathbb{R}^2$. For any such interface, we write $\mathbf{T}(\theta)$ for the unit tangent, $\mathbf{N}(\theta)$ for the unit normal, and θ for the angle from a fixed coordinate axis to $\mathbf{N}(\theta)$.

We begin with a fairly thorough description of the basic laws, which are balance of capillary forces and a mechanical version of the second law, and we derive corresponding thermodynamic restrictions on constitutive equations.[3] In particular, we show that the capillary force $\mathbf{C}(\theta)$ must be related to the interfacial energy[4] $f(\theta)$ through the relation

$$\mathbf{C}(\theta) = f(\theta)\mathbf{T}(\theta) + f'(\theta)\mathbf{N}(\theta). \tag{1.1}$$

Balance of capillary forces in conjunction with the thermodynamically-reduced constitutive equations lead to an evolution equation which relates the normal velocity V to the curvature K; this relation has the form[5]

$$\beta(\theta)V = [f(\theta) + f''(\theta)]K - F \tag{1.2}$$

with F the (constant) energy-difference between bulk phases, and $\beta(\theta) > 0$ a kinetic coefficient which measures the drag opposing interfacial motion. The relation (1.2), when combined with purely kinematical conditions for an evolving curve and applied on a convex

[3]The underlying proofs are more transparent in $\mathbb{R}^2$ than $\mathbb{R}^3$, and for that reason we rederive many results which could simply be taken from [1988g].

[4]Throughout we use the term "energy" as a synonym for "free energy".

[5]There is a large and growing literature concerning the evolution equation $V = K$ (cf., e.g., Brakke [1978], Gage [1986], Gage and Hamilton [1986], Grayson [1987], and Angenent [1988]).

section of the interface, results in a single partial differential equation for the velocity $V = V(B,t)$:

$$*(B)V_t = [V + *(B)]^2[V_{B0} + V], \qquad (1.3)$$

where

$$\Phi(B) = [f(B) + f^n(B)]/j3(8), \qquad *(B) = F/J3(B).$$

For $(B)>0 this equation is parabolic[6] and yields a theory which seems quite similar in structure to its isotropic counterpart based on $V = K\text{-}F$. There is, however, no compelling physical reason to exclude energies f(B) for which $f(B) + f''(B)<0$ over certain ranges of the angle 8;[7] for such ranges the equation (1.3) is backward-parabolic and corresponding evolution problems are generally not well posed. We show that a necessary condition for the statical stability of the interface is that $f(B) + f''(B) \geq 0$, and for that reason use the terms strictly stable, stable, or unstable according as $f(B) + f^M(B)>0$, $f(B) + f''(8) \geq 0$, or $f(B) + f''(B)<0$.

We begin our analysis of (1.3) by restricting attention to interfacial energies that are strictly stable. We deduce steady solutions of (1.3) for which the interface is convex and infinite, in the shape of a bump. The bump recedes in one solution and advances in the other; for the receding bump the kinetic coefficient can be arbitrary, but the advancing bump requires a nonconvex polar diagram for]3(B).

We next analyze the global behavior of a smooth interface as measured by its perimeter L(t) and enclosed area A(t). Our main result, based on the asumption of a stable interfacial energy, is most easily stated in terms of a bounded solid in an infinite

[6]Aside from the trivial degeneracy (v=-U^(B)) which occurs at inflection points.

[7]Material scientists often consider such models (cf., e.g., Gjostein [1963], Cahn and Hoffman [1974]).

liquid bath.

(1.4) If the bath is <u>not</u> supercooled, then $A(t) \to 0$; if the bath is supercooled, then initially small interfaces have $L(t) \to 0$, initially large interfaces have $A(t) \to \infty$.

We also show that, for the case in which $A(t) \to \infty$, the isoperimetric ratio $L(t)^2/4\pi A(t)$ remains bounded as $t \to \infty$. We show further that if (for a nonconvex interface) one defines a finger as a section of the interface between inflection points, then the total number of fingers as well as the total curvature of each finger cannot increase with time. These results presume the existence of a smooth, simple (non self-intersecting) interface. In this regard, it is clear that in certain circumstances the interface can pierce itself as it evolves.

We next consider energies $f(\theta)$ which are unstable for certain values of θ. Here we find it convenient to introduce a global definition of stability which is based on ideas of Wulff [1901], Herring [1951b], and Frank [1963]. We define global stability in terms of the convexity of the Frank diagram, which is the polar diagram of the reciprocal function $f(\theta)^{-1}$; we refer to the convex sections of this diagram as the globally-stable (GS) sections, to the remaining sections as the globally-unstable sections. These definitions are consistent: $f(\theta)$ is stable on GS sections; $f(\theta)$ is unstable somewhere within each globally-unstable section.

One way of treating unstable energies is to allow the interface to be <u>nonsmooth</u> with corners which correspond to jumps in θ across the globally-unstable sections. Balance of capillary forces for corresponding "weak solutions" of the evolution equations leads to the requirement that $C(\theta)$ be continuous across each such corner; interestingly, this requirement is automatically met.

In contrast to standard results for a strictly stable energy, the presence of corners leads to the possibility of <u>facets</u> (flat sections); in fact, to the presence of wrinklings, where a wrinkling is a series of facets with normals that oscillate between two fixed values. We show that such wrinklings are dynamically stable: the lengths of the individual facets do not increase with time.

The use of corners leads to free-boundary problems for the evolution of the interface, as the positions of the corners are not generally known a-priori. We discuss these problems in some detail.

Material scientists often consider interfacial energies that are continuous but have derivatives which suffer jump discontinuities.[8] We study such interfaces; as before, we use corners to remove the globally unstable sections. We show that, in agreement with statical results,[9] discontinuities in $f'(\theta)$ lead to facets in the evolving interface. We show further that the result (1.4) remains valid for nonsmooth, nonstable energies.

Following Taylor's [1978] statical treatment of crystal shapes, we consider a particular class of nonsmooth energies, called crystalline, for which the GS sections are isolated points (that is, for which the Frank diagram touches the boundary of its convex hull only at discrete points). An interesting property of crystalline energies is that their evolution is governed by a system of ordinary differential equations. Moreover these equations are of a particularly simple form, involving only nearest-neighbor interactions. We solve these equations for a rectangular crystal; the corresponding solution shows that, in situations for which the crystal shrinks (cf. (1.4)), the corresponding isoperimetric ratio generally tends to infinity. This is in sharp contrast to an isotropic interface, which shrinks to a round point.[10]

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[8] Cf., e.g., Herring [1951ab], Cahn and Hoffman [1974].

[9] Cf., e.g., Taylor [1978].

[10] Gage [1984], Gage and Hamilton [1986], Grayson [1987].

I. The thermomechanics of evolving curves.

2. Kinematics.

This chapter discusses the kinematics of smooth curves which evolve smoothly in time, and forms the basis of our theory of the motion of phase interfaces in $\mathbb{R}^2$.

2.1. Curves.

A **curve** is a smooth map $p \mapsto \mathbf{r}(p)$ from an interval of $\mathbb{R}$ into $\mathbb{R}^2$ such that:

(i) $\mathbf{r}_p$ never vanishes;

(ii) the domain of $\mathbf{r}$ is either all of $\mathbb{R}$ or a bounded interval $[P,Q]$;

(iii) if the domain is $\mathbb{R}$, either $\mathbf{r}$ is periodic[11] or $|\mathbf{r}(p)| \to \infty$ as $|p| \to \infty$.

The set Range($\mathbf{r}$) is then called the **trace** of $\mathbf{r}$.

We will classify curves $\mathbf{r}$ as follows: $\mathbf{r}$ is **bounded** or **unbounded** according as its trace is bounded or unbounded; $\mathbf{r}$ is **closed** if the domain is $\mathbb{R}$ and $\mathbf{r}$ is <u>periodic</u>, $\mathbf{r}$ has **endpoints** if its domain is a bounded interval. A nonclosed curve is **simple** if it is one-to-one; a closed curve is **simple** if given any $p,q \in \mathbb{R}$, $\mathbf{r}(p) = \mathbf{r}(q)$ only when $p-q$ is a multiple of the minimal period of $\mathbf{r}$.

Let $\mathbf{r}$ be a curve. An **arc-length map** for $\mathbf{r}$ is a smooth mapping $s(p)$ from the domain of $\mathbf{r}$ into $\mathbb{R}$ such that

$$s_p = |\mathbf{r}_p|. \tag{2.1}$$

We assume henceforth that an arc-length map is prescribed. Since the **arc length** $s = s(p)$ is an invertible function of p, any function $\phi(p)$ may be considered a function $\phi(s)$, and vice versa.

[11]A function ϕ on $\mathbb{R}$ is **periodic** if there is a $\lambda > 0$ such that $\phi(p) = \phi(p+\lambda)$ for all $p \in \mathbb{R}$; λ is then a **period** of ϕ and the infimum of all periods is the **minimal period** of ϕ. (The minimal period of a curve $\mathbf{r}$ is strictly positive since $|\mathbf{r}_p| \neq 0$.)

The vector

$$\mathbf{T}(s) = \mathbf{r}_s(s) \tag{2.2}$$

defines a (unit) **tangent** to the curve in the direction of increasing p. We define a corresponding (unit) **normal** $\mathbf{N}(s)$ through the requirement that $\{\mathbf{T},\mathbf{N}\}$ be a positively-oriented orthonormal basis of $\mathbb{R}^2$, and we define the **angle** $\theta(s)$, as a smooth function of s, through[12]

$$\mathbf{N} = (\cos\theta, \sin\theta), \qquad \mathbf{T} = (\sin\theta, -\cos\theta). \tag{2.3}$$

We will refer to the range of the function $s \mapsto \theta(s)$ as the **angle range** (Figure 2A). Note that, $\mathbf{N}$ and $\mathbf{T}$ may be considered as functions of θ, in which case

$$\mathbf{N}_\theta = -\mathbf{T}, \qquad \mathbf{T}_\theta = \mathbf{N}. \tag{2.4}$$

The function

$$K(s) = \theta_s(s) \tag{2.5}$$

is the **curvature**; by (2.4), $K(s)$ obeys the Frenet formulas:

$$\mathbf{N}_s = -K\mathbf{T}, \qquad \mathbf{T}_s = K\mathbf{N}. \tag{2.6}$$

Let $\mathbf{r}$ be a curve with trace $\mathfrak{z}$ and normal $\mathbf{N}$. Then $\mathbf{r}$ is a **boundary curve** if $\mathbf{r}$ is simple and either closed or unbounded. By the Jordan-curve theorem, $\mathfrak{z}$ then divides $\mathbb{R}^2$ into two regions,[13] and one of these regions, Ω, say, will have $\mathbf{N}$ as <u>outward</u> normal; we will refer to Ω as the **reference region**.

A curve is **convex** if K never vanishes In view of (2.5), the

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[12]This defines the function $\theta(s)$ up to a multiple of 2π.

[13]We use the term **region** as a synonym for connected open set.

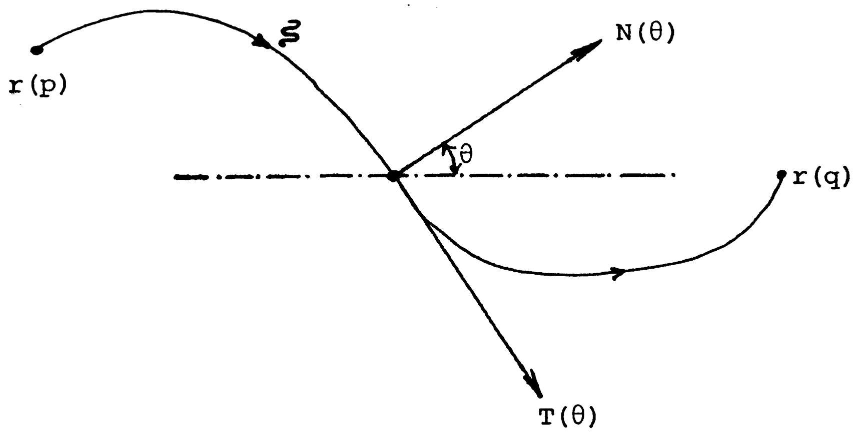

Figure 2A. Sign conventions for curves.

mapping $s \mapsto \theta(s)$ is then invertible and we may use θ in place of s or p as independent variable. In particular, we may parametrize the curve itself by θ, giving a function $\mathbf{r}(\theta)$; granted this, (2.2) and (2.5) yield

$$\mathbf{r}_\theta = K^{-1}\mathbf{T}. \tag{2.7}$$

Note that, because of our sign curvature for curvature, $K<0$ for a boundary curve whose reference region is bounded and strictly convex.

Useful in the study of convex curves is the **support function**

$$p = \mathbf{r}\cdot\mathbf{N}; \tag{2.8}$$

by (2.4) and (2.7),

$$\mathbf{r} = p\mathbf{N} - p_\theta\mathbf{T}, \qquad p_{\theta\theta} + p = -K^{-1}. \tag{2.9}$$

We now give three useful lemmas concerning convex curves.

Lemma 1. Two convex curves with the same angle range and the same curvature are equal modulo a translation.

Proof. By $(2.9)_2$, the difference between the support functions of the two curves $\mathbf{r}_1(\theta)$ and $\mathbf{r}_2(\theta)$ must have the form $\mathbf{a}\cdot\mathbf{N}(\theta)$, and this with $(2.9)_1$ implies that $\mathbf{r}_1(\theta)-\mathbf{r}_2(\theta)=\mathbf{a}$. ■

Lemma 2. Consider a curve whose curvature is not identically zero, and suppose that the curvature has the form $K(\theta(s))$ with $K(\theta)$ a smooth function on the angle range. Then the curve is convex.

Proof. Let Υ denote the angle range, and let Γ be a connected component of the set $\{\theta\in\Upsilon : K(\theta)\neq 0\}$. We must show that $\Gamma=\Upsilon$. Assume that $\Gamma\neq\Upsilon$. Then there is a boundary point θ_0 of Γ in $\mathbb{R}$ with $\theta_0\in\Upsilon$ and $K(\theta_0)=0$. Since $K(\theta)$ is smooth up to θ_0, $|K(\theta)| \leq C|\theta - \theta_0|$ near θ_0. But on Γ, $s = s(\theta)$ and $s_\theta(\theta) = K(\theta)^{-1}$, so that $|s(\theta)| \to \infty$ as $\theta \to \theta_0$, a contradiction. ∎

Lemma 3. <u>Let</u> $K(\theta)$ <u>be a smooth, 2π-periodic function on $\mathbb{R}$. Then the restriction</u> of $K(\theta)$ <u>to an open interval</u> $\mathcal{O}$ <u>is the curvature of a convex boundary curve if and only if either</u> (a) <u>or</u> (b) <u>is satisfied</u>:

(a) $\mathcal{O} = \mathbb{R}$, $K(\theta)$ <u>is nonvanishing, and</u>

$$\int_0^{2\pi} K(\theta)^{-1} e^{i\theta}\, d\theta = 0; \tag{2.10}$$

(b) $\mathcal{O}$ <u>is a bounded interval</u> (θ_1, θ_2) <u>with</u> $\theta_1 - \theta_2 \leq \pi$; $K(\theta)$ <u>is nonvanishing on</u> (θ_1, θ_2); $K(\theta_1) = K(\theta_2) = 0$.

<u>In case</u> (a) <u>the boundary curve is closed; in case</u> (b) <u>the boundary curve is unbounded. In either case a boundary curve with angle range</u> $\mathcal{O}$ <u>and curvature</u> $K(\theta)$ <u>is generated by</u> $(2.9)_1$ <u>with</u> p <u>any solution of</u> $(2.9)_2$.

<u>Proof</u>. Note first that if $g(\theta)$ is a smooth 2π-periodic function on $\mathbb{R}$, then

$$\int_0^{2\pi} [g(\theta) + g''(\theta)] e^{i\theta}\, d\theta = 0, \tag{2.11}$$

an assertion which follows immediately upon integrating the term $g''(\theta)e^{i\theta}$ twice by parts.

Suppose that the restriction of $K(\theta)$ to an open interval $\mathcal{O}$ is the curvature of a convex boundary curve $\mathbf{r}$. Then $\mathbf{r}$ is simple and either closed or unbounded. Assume that $\mathbf{r}$ is closed. Then $\mathcal{O} = \mathbb{R}$ and (2.10) is a consequence of (2.11) and $(2.9)_2$. Thus (a) is satisfied.

Assume that $\mathbf{r}$ is unbounded. Then $|\mathbf{r}(p)| \to \infty$ as $p \to \pm\infty$, so that

$$|s(p)| \to \infty \quad \text{as} \quad p \to \pm\infty. \tag{2.12}$$

Further, since $\mathbf{r}$ is simple, Θ is a bounded interval (θ_1,θ_2) with $\theta_1-\theta_2 \le \pi$. Trivially, $K(\theta)$ is nonzero on (θ_1,θ_2). Further, $s_\theta(\theta)=K(\theta)^{-1}$ and, by (2.12), $|s(\theta)|\to\infty$ as $\theta\to\theta_1,\theta_2$; thus, since $K(\theta)$ is smooth up to θ_1 and θ_2, $K(\theta_1)=K(\theta_2)=0$. Therefore (b) is satisfied.

Conversely, suppose that Θ and $K(\theta)$ satisfy either (a) or (b). Let $p(\theta)$ be any solution of $(2.9)_2$ on Θ and define $\mathbf{r}(\theta)$ by $(2.9)_1$.

Case (a). By (2.10), $\mathbf{r}(\theta)$ is 2π-periodic and defines a closed curve parametrized by θ. Further, K is the curvature of $\mathbf{r}$, so that $\mathbf{r}$ is convex. Moreover, θ is the angle and $\mathbf{r}(0)=\mathbf{r}(2\pi)$, so that $\mathbf{r}$ is simple. Thus $\mathbf{r}$ is a convex boundary curve.

Case (b). Let $s(\theta)$ be any solution of $s_\theta(\theta)=K(\theta)^{-1}$ on Θ. Since $K(\theta)$ is smooth up to θ_1 and θ_2, $|K(\theta)|\le C|\theta-\theta_0|$ for all θ near θ_1 and θ_2. Thus, since $K(\theta_1)=K(\theta_2)=0$, $|s(\theta)|\to\infty$ as $\theta\to\theta_1,\theta_2$ and, by $(2.9)_1$, $|\mathbf{r}(\theta)|\to\infty$ as $\theta\to\theta_1,\theta_2$. The mapping $\theta\mapsto s(\theta)$ is therefore a smooth bijection of (θ_1,θ_2) onto $\mathbb{R}$. Writing $\theta=\theta(s)$, $|\mathbf{r}(s)|\to\infty$ as $|s|\to\infty$; hence $\mathbf{r}(s)$ is a convex, unbounded curve, parametrized by arc length, with $\theta(s)$ the angle function and $K(\theta)$ the curvature. Since $\theta_1-\theta_2 \le \pi$, $\mathbf{r}$ is simple; hence $\mathbf{r}$ is a convex boundary curve.

The final assertions of the lemma are clear from the preceding analysis. ∎

2.2. Evolving curves.

Roughly speaking, an evolving curve is a smooth family of curves $p\mapsto\mathbf{r}(p,t)$, where t, the **time**, ranges in a half-open interval $[0,T)$, called the **duration** of $\mathbf{r}$. We will use evolving curves to model the motion of phase interfaces in $\mathbb{R}^2$. For a given motion $\mathbf{r}$, the underlying physics must be independent of the choice of **parameter** p, and hence can involve $\mathbf{r}$ only through <u>intrinsic</u> quantities such as curvature and normal velocity which are independent of parametrization. On the other hand, this invariance allows us to use any parametrization we wish. In fact, we shall restrict our attention to parametrizations with $\mathbf{r}_t(p,t) \perp \mathbf{r}_p(p,t)$; such parametrizations greatly simplify the analysis, chiefly because the <u>velocity</u> $\mathbf{r}_t(p,t)$ is equal to the normal time-derivative $\mathbf{r}^\circ(p,t)$, an intrinsic quantity.

Precisely, an **evolving curve**[14] is a smooth mapping $(p,t) \mapsto \mathbf{r}(p,t)$ with the following properties:

(i) the domain of $\mathbf{r}$ is either $\mathbb{R} \times [0,T)$ or a set of the form

$$\{(p,t): \quad p \in [P(t),Q(t)], \quad t \in [0,T)\}, \tag{2.13}$$

where $P,Q : [0,T) \to \mathbb{R}$ $(P < Q)$ are smooth functions;

(ii) $\mathbf{r}(\cdot,t)$ is a <u>curve</u> for each $t \in [0,T)$;

(iii) $\mathbf{r}_t(p,t) \perp \mathbf{r}_p(p,t)$ for all (p,t) (**orthogonality**).

We will refer to $\mathbf{r}(P(t),t)$ and $\mathbf{r}(Q(t),t)$ as the **initial** and **terminal points** (or collectively, as the **endpoints**) of $\mathbf{r}$, and to the interval $[P(t),Q(t)]$ (or $\mathbb{R}$) as the **parameter interval** at time t

Let $\mathbf{r}$ be an evolving curve: $\mathbf{r}$ is bounded, unbounded, closed, simple, convex, or has endpoints, according as $\mathbf{r}(\cdot,t)$ has that property for each $t \in [0,T)$; a <u>restriction</u> $\mathbf{r}_0$ of $\mathbf{r}$ is an **evolving subcurve** of $\mathbf{r}$ if, modulo a translation of time, $\mathbf{r}_0$ is a <u>bounded</u> evolving curve; $\mathbf{r}$ is an **evolving facet** if its trace $\mathfrak{z}(t)$ is a segment of a straight line at each t.

Let an evolving curve $\mathbf{r}$ be given. An **arc-length map** for $\mathbf{r}$ is a smooth mapping $s(p,t)$ such that $s(\cdot,t)$ is an arc-length map for the curve $\mathbf{r}(\cdot,t)$ at each t. It is not difficult to construct an arc-length map for $\mathbf{r}$, and any two such maps differ by a smooth function of time. We assume henceforth that an arc-length map is prescribed. Since $s = s(p,t)$ is an invertible function of p, any function $\phi(p,t)$ may be considered a function $\phi(s,t)$, and vice-versa. We will refer to $\phi(s,t)$ as the **arc-length**

[14]We use the term "normally evolving curve" in Appendix B.

description of ϕ.

We write[15] ϕ° for the **normal time-derivative** of ϕ, the time derivative holding p fixed. In particular, we define the **normal velocity** $V(s,t)$ through the identity

$$\mathbf{r}^\circ = V\mathbf{N}, \tag{2.14}$$

the **arc velocity** $v(s,t)$ through

$$v = s^\circ. \tag{2.15}$$

Given a function $\phi(s,t)$,

$$\phi^\circ = \phi_t + v\phi_s, \tag{2.16}$$

with ϕ_t the time derivative holding s fixed; thus

$$(\phi^\circ)_s = (\phi_t + v\phi_s)_s = \phi_{st} + v\phi_{ss} + v_s\phi_s = (\phi_s)^\circ + v_s\phi_s. \tag{2.17}$$

Transport identities.

$$v_s = -KV, \qquad \theta^\circ = V_s, \qquad K^\circ = V_{ss} + K^2V. \tag{2.18}$$

Proof. By (2.1) and (2.2), $\mathbf{T} = |\mathbf{r}_p|^{-1}\mathbf{r}_p$. Thus $J = s_p = |\mathbf{r}_p|$ satisfies

$$J^\circ = |\mathbf{r}_p|^{-1}\mathbf{r}_p \cdot (\mathbf{r}_p)^\circ = \mathbf{T}\cdot(V\mathbf{N})_s J = \mathbf{T}\cdot(\mathbf{N}_s)VJ,$$

and, in view of (2.6), $J^\circ = -JKV$. On the other hand, (2.15) yields $J^\circ = v_p = v_sJ$ and $(2.18)_1$ follows.

Let $\mathbf{e}$ be a fixed unit vector. Then, by (2.2), (2.3), and (2.17),

$$(\mathbf{r}_s)^\circ \cdot \mathbf{e} = (\mathbf{T}\cdot\mathbf{e})^\circ = (\mathbf{N}\cdot\mathbf{e})\theta^\circ,$$

[15] Cf. Appendix B. We will also write $\phi^\circ(t)$ for the derivative of a function $\phi(t)$ of time alone.

$$(\mathbf{r}_s)^\circ \cdot \mathbf{e} = (\mathbf{r}^\circ)_s \cdot \mathbf{e} - v_s \mathbf{T} \cdot \mathbf{e},$$

while (2.6), (2.14), and $(2.18)_1$ imply

$$(\mathbf{r}^\circ)_s = V_s \mathbf{N} + v_s \mathbf{T}.$$

The last three relations yield $(2.18)_2$, since $\mathbf{e}$ is arbitrary (cf. (2.16)). Finally, $(2.18)_3$ follows from (2.17) with $\phi = \theta$, (2.5), and $(2.18)_{1,2}$. ■

Note that, trivially, for an evolving facet (cf. (2.5) and $(2.18)_1$),

$$K = \theta_s = 0, \qquad v_s = 0. \tag{2.19}$$

An endpoint $\mathbf{R}(t) = \mathbf{r}(P(t),t)$ is a **normal trajectory** if $\mathbf{R}^\circ(t) \cdot \mathbf{T}(P(t),t) = 0$. Trivially, since $\mathbf{R}^\circ(t) = \mathbf{r}_p(P(t),t)P^\circ(t) + \mathbf{r}^\circ(P(t),t)$, $\mathbf{r}(P(t),t)$ is a normal trajectory if and only if P is independent of t.

Proposition. <u>Let</u> $\mathbf{r}$ <u>have endpoints, and let</u> $S(t)$ <u>denote the arc length and</u> $\Theta(t)$ <u>the angle of an endpoint</u> $\mathbf{R}(t)$. <u>Then</u>

$$\begin{aligned} \mathbf{R}^\circ(t) &= V(S(t),t)\mathbf{N}(S(t),t) + [S^\circ(t) - v(S(t),t)]\mathbf{T}(S(t),t), \\ \Theta^\circ(t) &= V_s(S(t),t) + [S^\circ(t) - v(S(t),t)]K(S(t),t), \end{aligned} \tag{2.20}$$

<u>and, if the endpoint</u> $\mathbf{R}(t)$ <u>is a normal trajectory,</u>

$$S^\circ(t) = v(S(t),t), \qquad \Theta^\circ(t) = V_s(S(t),t). \tag{2.21}$$

<u>Proof</u>. Since $\mathbf{R}(t) = \mathbf{r}(S(t),t)$, $\Theta(t) = \theta(S(t),t)$, the identities (2.20) follow from (2.16), (2.14), (2.5), and $(2.18)_2$. If $\mathbf{R}(t)$ is a normal trajectory, then $\mathbf{R}(t) = \mathbf{r}(P,t)$ with P constant; thus, since $S(t) = s(P,t)$, $(2.21)_1$ follows from (2.15), and, in view of $(2.20)_2$, this yields $(2.21)_2$. ■

For a convex evolving curve the mapping $s \mapsto \theta(s,t)$ is invertible and we may use θ and t in place of s and t as independent variables. Then

$$K^\circ = K_t + K_\theta \theta^\circ \tag{2.22}$$

(with K_t the derivative of K with respect to t holding θ fixed).

Proposition. For a convex evolving curve with curvature, normal velocity, and arc velocity expressed as functions of (θ,t),

$$K_t = K^2(V_{\theta\theta} + V), \qquad v_\theta = -V. \tag{2.23}$$

In addition, if $\mathbf{r}$ has endpoints, and if the angle θ at an endpoint $\mathbf{R}(t)$ has the constant value θ_0, then

$$\mathbf{R}^\circ(t) = V(\theta_0,t)\mathbf{N}(\theta_0) - V_\theta(\theta_0,t)\mathbf{T}(\theta_0). \tag{2.24}$$

Proof. Clearly,

$$V_s = V_\theta K, \qquad V_{ss} = V_{\theta\theta}K^2 + V_\theta K_\theta K,$$

and these relations, (2.22), and $(2.18)_{2,3}$ yield $(2.23)_1$. On the other hand, $v_s = v_\theta K$ and $(2.23)_2$ follows from $(2.18)_1$. Finally, (2.20) with $\Theta^\circ = 0$ and $V_s = V_\theta K$ imply (2.24). ■

The next definition will be useful in discussing evolving curves that represent interfaces between phases. An **interfacial motion** is an evolving curve $\mathbf{r}$ with $\mathbf{r}(\cdot,t)$ a boundary curve at each t. The trace $\mathfrak{z}(t)$ of $\mathbf{r}$ then divides $\mathbb{R}^2$ into two regions. The region $\Omega(t)$ with $\mathbf{N}(s,t)$ as outward normal is called the

reference region, and, without loss in generality, $\Omega(t)$ is taken to be the bounded region interior to $\mathfrak{z}(t)$ when **r** is closed (Figure 2B).

By a **steady motion** we mean an interfacial motion **r** with the following property: there is a vector $\mathbf{U} \neq \mathbf{0}$ such that, for some choice of arc-length map,

$$\mathbf{r}(s,t) = \mathbf{r}_0(s) + t\mathbf{U}; \tag{2.25}$$

U is then the **steady velocity**, the curve $\mathbf{r}_0$ the **portrait**.

Proposition. Given a boundary curve $\mathbf{r}_0$ and a vector $\mathbf{U} \neq 0$, there is a unique[16] steady motion **r** with $\mathbf{r}_0$ as portrait and **U** as steady velocity.

Proof. Assume that $\mathbf{r}_0(s)$ is parametrized by arc length and let $(s,t) \mapsto \mathbf{k}(s,t)$ $(\mathbb{R}\times[0,\infty) \to \mathbb{R}^2)$ be the time-dependent curve defined by (2.25). Then $\mathfrak{v}(s,t)$, as defined in $(B3)_2$ of Appendix B, is independent of t and given by

$$\mathfrak{v}(s) = -\mathbf{U} \cdot \mathbf{T}(s), \tag{2.26}$$

with $\mathbf{T}(s)$ the tangent for $\mathbf{r}_0(s)$. Thus **k** obeys hypothesis (iii) of Theorem 1 (Appendix B), and there is a reparametrization $\boldsymbol{\phi}$ of **k** such that $\mathbf{r} = \mathbf{k}\circ\boldsymbol{\phi}$ is an evolving curve. Writing $\boldsymbol{\phi}(p,t) = (\phi(p,t),t)$, and differentiating $\mathbf{r}(p,t) = \mathbf{k}(\phi(p,t),t)$ with respect to p yields the conclusion that $\phi(p,t)$ is an arc-length map for **r**. Thus $\mathbf{k}(s,t)$ is an arc-length description of $\mathbf{r}(p,t)$, which is the desired conclusion. If **g** is a second steady motion with **k** as an arc-length description, then, trivially, **g** must be a reparametrization of **r**. ■

As is clear from the proof of Theorem 1 (Appendix B), a

[16]The term "unique", when used relative to evolving curves, will always signify "unique up to a reparametrization".

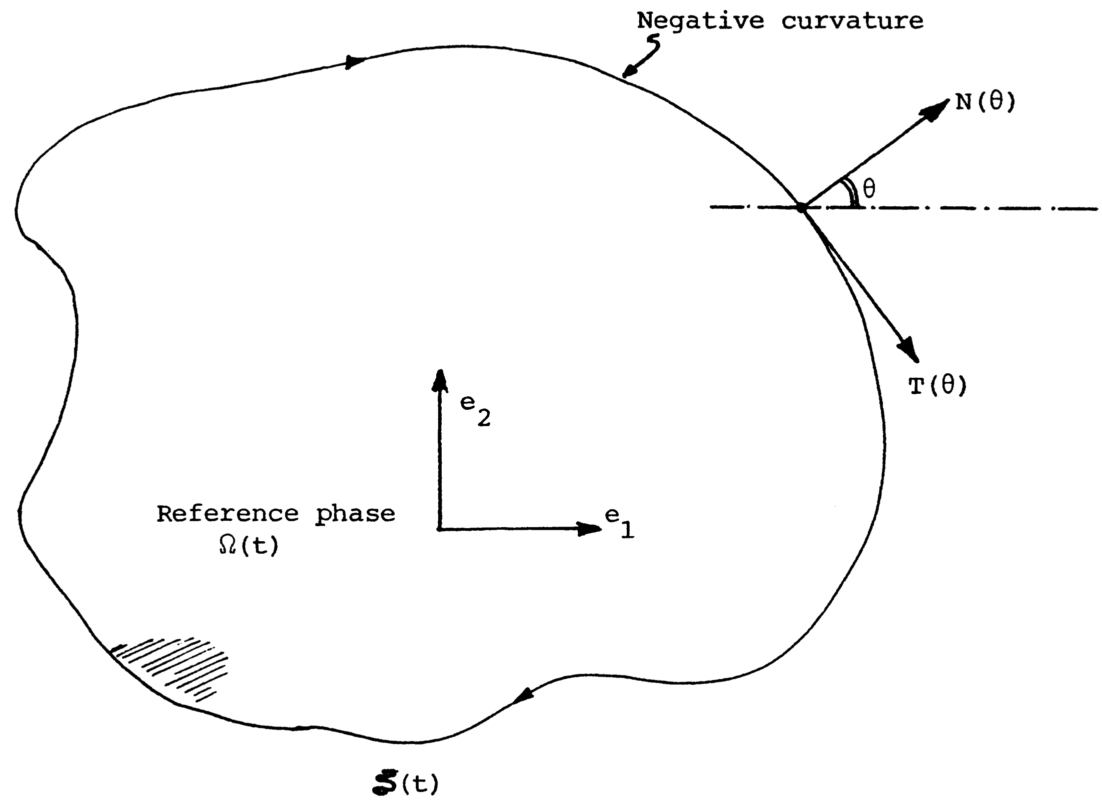

Figure 2B. Sign conventions for interfacial motions. $\Omega(t)$, the region occupied by the reference phase, has N as outward unit normal and is the region interior to the trace $\mathcal{S}(t)$.

parameter change $\boldsymbol{\phi}(p,t) = (s(p,t),t)$ that converts the arc-length description (2.25) to a description $\mathbf{r}(p,t)$ as an evolving curve has $s(p,t)$ a solution of the initial-value problem

$$s^{\circ}(p,t) = v(s(p,t)), \qquad s(p,0) = p; \tag{2.27}$$

in this case $v(s)$, given by (2.26), is the arc velocity and the parameter p is the initial arc-length.

Proposition. <u>For a steady motion the normal velocity</u> $V(s)$, <u>the curvature</u> $K(s)$, <u>the normal</u> $\mathbf{N}(s)$, <u>the tangent</u> $T(s)$, <u>and the angle</u> $\theta(s)$ <u>are independent of time, and</u>

$$V(s) = \mathbf{U} \cdot \mathbf{N}(s). \tag{2.28}$$

<u>Proof</u>. By (2.25), $\mathbf{T}(s)$ and (hence) $\mathbf{N}(s)$ and $\theta(s)$ are independent of t. Further, since $V\mathbf{N} = \mathbf{r}^{\circ} = \mathbf{r}_t + v\mathbf{r}_s$ and $T = \mathbf{r}_s$, $V(s)$ is independent of t and given by (2.28). Finally, since $K = \mathbf{T}_s \cdot \mathbf{N}$ (cf. (2.6)), $K(s)$ is also independent of t. ■

For a convex, steady motion, $V(\theta)$ and $K(\theta)$ are independent of time, and

$$V(\theta) = \mathbf{U} \cdot \mathbf{N}(\theta). \tag{2.29}$$

By a **steadily evolving bump** we mean a convex, steady motion such that

$$K(\theta)\mathbf{U} \cdot \mathbf{N}(\theta) \text{ never vanishes.} \tag{2.30}$$

A steadily evolving bump is **advancing** or **receding** according as

$$K(\theta)\mathbf{U} \cdot \mathbf{N}(\theta) < 0 \quad \text{or} \quad K(\theta)\mathbf{U} \cdot \mathbf{N}(\theta) > 0 \tag{2.31}$$

(Figure 2C). Note that steadily evolving bumps are necessarily

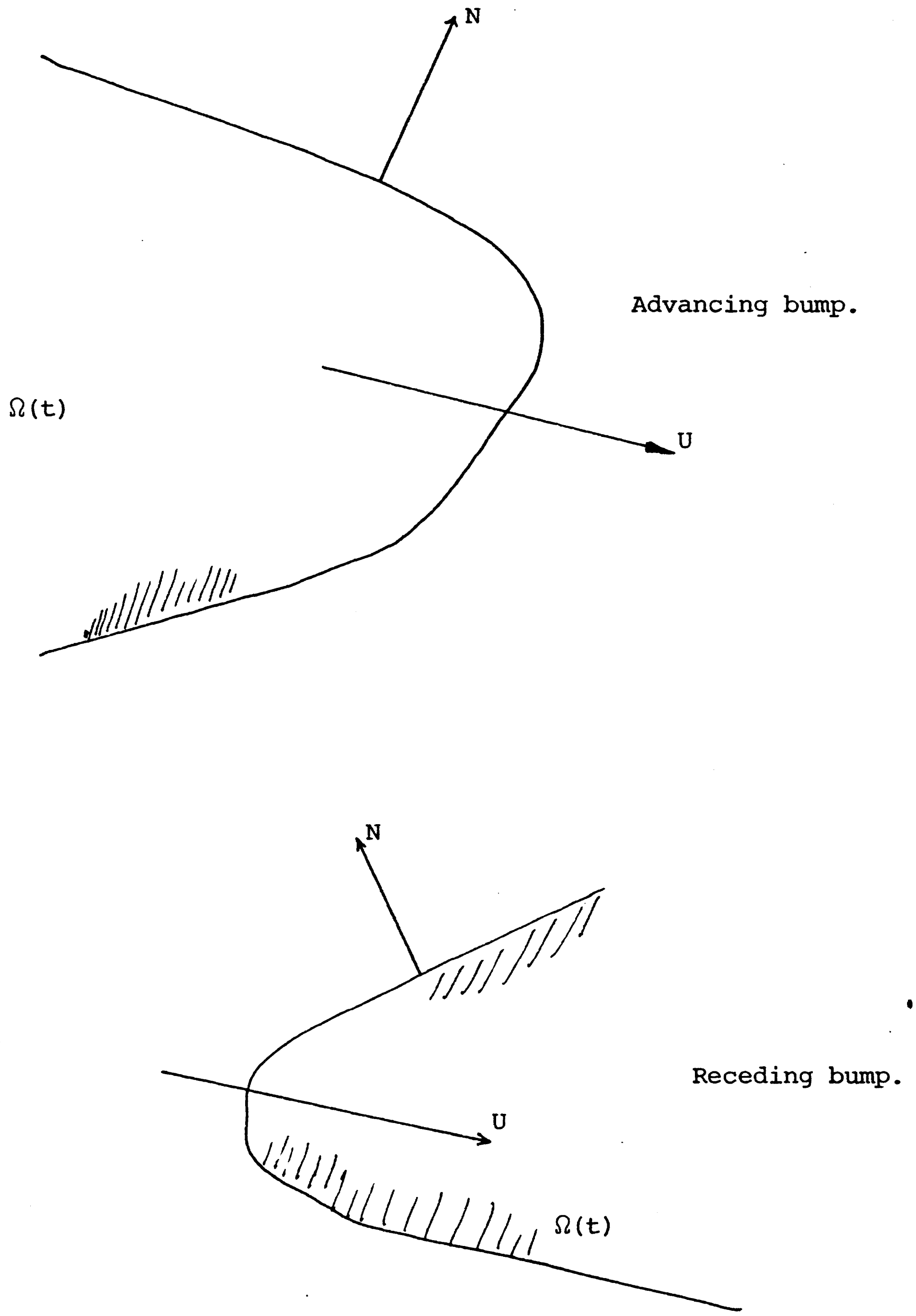

Figure 2C. Steadily evolving bumps.
The steady velocity is given by U.

unbounded.

Finally, we note that a trivial example of a steady motion is a **steadily evolving facet**; such motions are completely determined by **U** and the corresponding angle θ ($\equiv$ constant).

2.3. Integral identities.

Let **r** be an evolving curve with arc length lying in an interval $[S_1(t), S_2(t)]$. We use the notation

$$\int_{\varsigma(t)} \phi\, ds = \int_{S_1(t)}^{S_2(t)} \phi(s,t)\, ds, \qquad \int_{\partial\varsigma(t)} \phi = \phi(S_2(t),t) - \phi(S_1(t),t);$$

trivially,

$$\int_{\partial\varsigma(t)} \phi = \int_{\varsigma(t)} \phi_s\, ds.$$

The intrinsic measure of length on the curve is arc length. In general, the endpoints of **r** will not be normal trajectories; hence $\varsigma(t)$ will loose arc length across its boundary at a rate given by $\{v(S_2(t),t) - S_2^{\circ}(t)\} - \{v(S_1(t),t) - S_1^{\circ}(t)\}$. Thus, for $\phi(s,t)$ a smooth function,

$$\begin{aligned} \mathrm{outflow}(\phi,\partial\varsigma)(t) := {} & \phi(S_2(t),t)\{v(S_2(t),t) - S_2^{\circ}(t)\} \\ & - \phi(S_1(t),t)\{v(S_1(t),t) - S_1^{\circ}(t)\} \end{aligned} \tag{2.32}$$

represents the rate at which ϕ is <u>carried out</u> of $\varsigma(t)$ across $\partial\varsigma(t)$ with this loss in arc length; note that, by (2.21),

$$\mathrm{outflow}(\phi,\partial\varsigma)(t) = 0 \text{ when the endpoints are normal trajectories.} \tag{2.33}$$

Transport theorem for integrals.

$$(d/dt)\int_{\mathfrak{s}(t)}\phi\,ds + \mathrm{outflow}(\phi,\partial\mathfrak{s})(t) = \int_{\mathfrak{s}(t)}(\phi^\circ - \phi KV)\,ds. \qquad (2.34)$$

<u>Proof</u>. By (2.16) (suppressing the argument t where convenient),

$$(d/dt)\int_{S_1(t)}^{S_2(t)}\phi(s,t)\,ds = S_2{}^\circ\phi(S_2) - S_1{}^\circ\phi(S_1) + \int_{S_1}^{S_2}(\phi^\circ - v\phi_s)\,ds.$$

On the other hand, $(2.18)_1$ implies

$$\int_{S_1}^{S_2} v\phi_s\,ds = \phi(S_2)v(S_2) - \phi(S_1)v(S_1) + \int_{S_1}^{S_2}\phi KV\,ds.$$

The last two results and (2.32) yield (2.34). ■

If we take $\phi = 1$ in (2.34) and appeal to (2.20) and (2.32), we arrive at an important identity involving the **length**

$$L(t) := \mathrm{length}(\mathfrak{s}(t)).$$

Transport theorem for length.

$$L^\circ(t) = \mathbf{R}_2{}^\circ(t)\cdot\mathbf{T}_2(t) - \mathbf{R}_1{}^\circ(t)\cdot\mathbf{T}_1(t) - \int_{\mathfrak{s}(t)}KV\,ds \qquad (2.35)$$

<u>if</u> **r** <u>has initial and terminal points</u> $\mathbf{R}_1(t)$ <u>and</u> $\mathbf{R}_2(t)$ <u>with corresponding tangents</u> $\mathbf{T}_1(t)$ <u>and</u> $\mathbf{T}_2(t)$;

$$L^\circ(t) = -\int_{\mathfrak{s}(t)}KV\,ds; \qquad (2.36)$$

<u>if</u> **r** <u>is closed, or if its endpoints are normal trajectories</u>.

2.4. Piecewise-smooth evolving curves.

We now extend some of the previous definitions and results to curves that are continuous but not smooth. For convenience, we write "PS" as an abbreviation for "**piecewise smooth**".

Let $\mathbf{r} = \{\mathbf{r}_1, \mathbf{r}_2, \ldots\}$ denote a finite or countably infinite list of evolving curves $\mathbf{r}_i$, called **arcs** of $\mathbf{r}$, of equal duration $[0,T)$, with $[P_i(t),Q_i(t)]$ the parameter interval for $\mathbf{r}_i$ at time t. Then $\mathbf{r}$ is a **PS evolving curve** if:

(i) at each **juncture** i

(i.e., each i with $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ arcs of $\mathbf{r}$),

$$\mathbf{r}_i(Q_i(t),t) = \mathbf{r}_{i+1}(P_{i+1}(t),t) =: \mathbf{R}_i(t); \tag{2.37}$$

(ii) there is an integer $N > 0$ such that either:

(a) $\mathbf{r}$ consists of N arcs, in which case $\mathbf{r}_1$ and $\mathbf{r}_N$ are **terminal arcs**, and $\mathbf{r}$ has **endpoints** $\mathbf{r}_{init}(t) = \mathbf{r}_1(P_1(t),t)$ and $\mathbf{r}_{term}(t) = \mathbf{r}_N(Q_N(t),t)$; or

(b) $\mathbf{r}$ consists of an infinite number of arcs, but $\mathbf{r}_i = \mathbf{r}_{i+N}$ for all i; in this case $\mathbf{r}$ is **closed**, and the smallest such integer N is the **essential number of arcs** of $\mathbf{r}$.

In addition: $\mathbf{r}_i$ is an **internal arc** if both i and $i-1$ are junctures; $\mathbf{R}_i(t)$ and $\mathbf{R}_i^\circ(t)$ are the **position** and **total velocity of the juncture** i; $\varsigma(t)$, the **trace** of $\mathbf{r}$, is the union of the individual traces

$$\varsigma_i(t) = \{\, \mathbf{r}_i(p,t) : \ p \in [P_i(t),Q_i(t)] \,\}.$$

An **evolving subcurve** $\mathbf{r}_0$ of $\mathbf{r}$ is defined in the obvious manner, as is the phrase "$\mathbf{r}$ is **simple**".

Let $\mathbf{r}$ be a PS evolving curve. An **arc-length map** for $\mathbf{r}$ is a list $\{s_1,s_2,\ldots\}$ with $s_i(p,t)$ an arc-length map for $\mathbf{r}_i$ and

$$s_i(Q_i(t),t) = s_{i+1}(P_{i+1}(t),t) =: S_i(t) \tag{2.38}$$

at each juncture i. It is not difficult to construct an arc-length map for $\mathbf{r}$; granted one is prescribed, we can define arc length s at time t by $s = s_i(p,t)$ for any i and $p \in [P_i(t), Q_i(t)]$. This allows us to consider the tangent, normal, orientation, curvature, normal velocity, and arc velocity as functions $\mathbf{T}(s,t)$, $\mathbf{N}(s,t)$, $\theta(s,t)$, $K(s,t)$, $V(s,t)$, and $v(s,t)$ of arc length and time. These functions will generally suffer jump discontinuities across $s = S_i(t)$; with this in mind, given any function $\phi(s,t)$, we write

$$\phi_i^{\pm}(t) := \phi(S_i(t) \pm 0, t). \tag{2.39}$$

We associate with each juncture i three functions of time:

$$\begin{aligned} \kappa_i &= \mathbf{T}_i^+ \cdot \mathbf{N}_i^- = -\mathbf{T}_i^- \cdot \mathbf{N}_i^+, \\ k_i &= 2\kappa_i/(1 + \mathbf{T}^+ \cdot \mathbf{T}^-), \\ \upsilon_i &= (V_i^+ + V_i^-)/2; \end{aligned} \tag{2.40}$$

k_i is the **transition curvature** and υ_i the **average velocity** of the juncture. Note that

$$k_i = 2\sin\vartheta_i/(1 + \cos\vartheta_i) = 2\tan(\vartheta_i/2), \qquad \vartheta_i = \theta_i^+ - \theta_i^-, \tag{2.41}$$

so that k_i, as a function of ϑ_i, is strictly increasing on $[0,\pi)$, is asymptotic to ϑ_i for ϑ_i small, and tends to ∞ as $\vartheta_i \to \pi$. (In the literature, it is more common to refer to ϑ_i as the curvature of the corner.) Note also that for $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ convex, the PS evolving curve $\{\mathbf{r}_i, \mathbf{r}_{i+1}\}$ is convex (in the usual sense for continuous curves) if and only if k_i and the curvatures of $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ have the same sign.

Corner conditions. <u>At each juncture</u> i

$$V_i^+ \mathbf{N}_i^+ + (S_i^\circ - v_i^+)\mathbf{T}_i^+ = V_i^- \mathbf{N}_i^- + (S_i^\circ - v_i^-)\mathbf{T}_i^- = \mathbf{R}_i^\circ, \qquad (2.42)$$
$$\mathbf{R}_i^\circ \cdot (\mathbf{T}_i^+ - \mathbf{T}_i^-) = k_i \upsilon_i.$$

<u>Proof</u>. The identities $(2.42)_1$ are a direct consequence of $(2.20)_1$ and (2.37). The inner product of

$$V_i^+ \mathbf{N}_i^+ + (\mathbf{R}_i^\circ \cdot \mathbf{T}_i^+)\mathbf{T}_i^+ = V_i^- \mathbf{N}_i^- + (\mathbf{R}_i^\circ \cdot \mathbf{T}_i^-)\mathbf{T}_i^-$$

with $\mathbf{T}_i^+$ and $\mathbf{T}_i^-$ gives two equations, whose difference yields $(2.42)_2$. ■

The next proposition gives an evolution equation for the **length**

$$L(t) = \text{length}(\mathfrak{s}(t))$$

of $\mathbf{r}$. As this result shows (and as is clear from $(2.39)_2$), $k_i \upsilon_i$ measures the rate at which the juncture i generates arc length.

Transport of length.

(i) <u>Let</u> $\mathbf{r}$ <u>have</u> N <u>arcs</u>. <u>Further, let</u> $\mathbf{R}_{init}(t)$ <u>and</u> $\mathbf{R}_{term}(t)$ <u>denote the endpoints of</u> $\mathbf{r}$ <u>with</u> $\mathbf{T}_{init}(t)$ <u>and</u> $\mathbf{T}_{term}(t)$ <u>the corresponding tangents</u>. <u>Then</u>

$$L^\circ(t) = \mathbf{R}_{term}^\circ(t) \cdot \mathbf{T}_{term}(t) - \mathbf{R}_{init}^\circ(t) \cdot \mathbf{T}_{init}(t) - \int_{\mathfrak{s}(t)} KV\,ds - \sum_{i=1}^{N-1} k_i \upsilon_i. \qquad (2.43)$$

(ii) <u>Let</u> $\mathbf{r}$ <u>be closed with</u> N <u>the essential number of arcs</u>. <u>Then</u>

$$L^\circ(t) = -\int_{\mathfrak{s}(t)} KV\,ds - \sum_{i=1}^{N} k_i \upsilon_i. \qquad (2.44)$$

Proof. We apply (2.35) to each arc of $\mathbf{r}$, add the resulting equations, and appeal to $(2.42)_2$ and (2.42). ■

A **PS interfacial motion** is a closed[17] PS evolving curve $\mathbf{r}$ with $\mathbf{r}(\cdot,t)$ a boundary curve at each t. As before, the **reference region** $\Omega(t)$, with outward normal $\mathbf{N}$, is the bounded region interior to the trace $\partial\Omega(t)$ of $\mathbf{r}$. A standard result is the following equation for the evolution of $A(t)=\text{area}(\Omega(t))$:

$$A^{\circ}(t) = \int_{\partial\Omega(t)} V\,ds. \tag{2.45}$$

[17] With the exception of Section 9.6, the underlying PS curves will be bounded.

3. Basic laws.[18]

We consider a body which occupies all of $\mathbb{R}^2$ and consists of two phases separated, at each time t, by an interface. We assume that the interface, as a function of t, is an <u>interfacial motion</u> $\mathbf{r}$. Let $\mathcal{S}$ denote the trace of $\mathbf{r}$. The curve $\mathcal{S}(t)$ represents the **interface** at time t; by definition, $\mathcal{S}(t)$ divides $\mathbb{R}^2$ into two sets, the regions occupied by the two phases. The reference region, $\Omega(t)$, has $\mathbf{N}$ as its outward unit normal; we will refer to the phase occupying $\Omega(t)$ as the **reference phase**.

3.1. Balance of forces.[19]

Consider an interfacial motion $\mathbf{r}$. The micromechanics of the interface is described by two functions of s and t: $\mathbf{C}(s,t)$, the force exerted across the interface at s; and $\mathbf{b}(s,t)$, the force exerted on $\mathcal{S}(t)$ per unit length. $\mathbf{C}(s,t)$ is the **capillary force**; if we write

$$\mathbf{C} = \sigma\mathbf{T} + \xi\mathbf{N}, \tag{3.1}$$

then $\sigma(s,t)$ is the **surface tension**, $\xi(s,t)$ is the **surface shear**.

Balance of internal forces is, to us, the requirement that, if c is the trace of an arbitrary evolving subcurve $\mathbf{r}_0$ of $\mathbf{r}$, then

$$\int_{\partial c(t)} \mathbf{C} + \int_{c(t)} \mathbf{b}\,ds = \mathbf{0} \qquad (3.2)^{20}$$

for the duration of $\mathbf{r}_0$. This law has the local form

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[18]The underlying physics is discussed at greater length in [1988g] (see also [1986g,1988gg]).

[19][1988g].

[20](3.2) should be viewed as a conservation law over and above the usual (gross) balance laws for forces and moments (cf. the discussion of [1988g]).

$$\mathbf{c}_s + \mathbf{b} = \mathbf{0}, \tag{3.3}$$

or equivalently, by (3.1) and (2.6),

$$\xi_s + \sigma K + b = 0, \qquad \sigma_s - \xi K + b_{tan} = 0, \tag{3.4}$$

where

$$b = \mathbf{N} \cdot \mathbf{b}, \qquad b_{tan} = \mathbf{T} \cdot \mathbf{b}$$

are the normal and tangential components of $\mathbf{b}$.

Motion tangential to the interface depends on the choice of parameterization and is hence irrelevant to the underlying physics; the intrinsic evolution of the interface is normal to itself, through the velocity $\mathbf{r}^\circ$. As is consistent with a "constraint" of this type, we leave b_{tan} as indeterminate and consider only the normal component $(3.4)_1$ of the force balance.

We assume that the normal force b consists of two terms:

$$b = b_{ext} + \lambda;$$

b_{ext}, the **external force**, represents the normal force exerted on the interface by the external world;[21] λ, the **interactive force**, gives the normal force exerted on the interface by the bulk material. With this decomposition, the **normal force-balance** takes the form

$$\xi_s + \sigma K + \lambda + b_{ext} = 0. \tag{3.5}$$

3.2. Energy. The second law.

[21] This force is essential to the thermodynamical development of Section 4 (cf. [1988g], Footnote 13). In later sections we will restrict attention to interfacial motions with $b_{ext} = 0$.

We associate with each interfacial motion an **interfacial energy** $f(s,t)$ per unit length. In addition, the individual phases possess bulk energies; in accord with our tacit assumption of isothermal conditions, we assume that the energy of each phase is constant, and we write F for the **energy of the reference phase** minus that of the other phase.

Let R denote a fixed bounded region of space, and let

$$\Omega_R(t) = \Omega(t)\cap R, \qquad \varsigma_R(t) = \varsigma(t)\cap R, \tag{3.6}$$

so that $\Omega_R(t)$ is the portion of $\Omega(t)$ in R, while $\varsigma_R(t)$ (assumed nonempty) is the portion of $\varsigma(t)$ in R. We assume that the boundary of R is sufficiently smooth that ς_R is the trace of an evolving subcurve $\mathbf{r}_R$ of $\mathbf{r}$, at least on a sufficiently small time interval. Then (modulo an inconsequential constant) the total energy of R is given by

$$F\,\mathrm{area}(\Omega_R(t)) + \int_{\varsigma_R(t)} f\,ds.$$

Interfacial energy is carried out of R whenever the normal trajectories of the interface cross ∂R; in view of (2.32), this <u>outflow</u> is given by the quantity $\mathrm{outflow}(f,\partial\varsigma_R)(t)$.

The terms

$$\int_{\partial\varsigma_R} \mathbf{C}\cdot\mathbf{r}^\circ = \int_{\partial\varsigma_R} \xi V, \qquad \int_{\varsigma_R} b_{ext}V\,ds$$

represent power supplied to R by the portion of the interface outside of R and by the external world. (For convenience, we write ς_R for $\varsigma_R(t)$.) The surface tension σ and the interaction λ do not supply power: σ because it acts in a direction orthogonal to the velocity $\mathbf{r}^\circ$; λ because it represents interactions <u>within</u> R.

The second law for R is the assertion that the rate at which the energy increases plus the energy outflow cannot be greater than the power supplied to R:

$$(d/dt)\Big\{F\,\mathrm{area}(\Omega_R) + \int_{\mathcal{S}_R} f\,ds\Big\} + \mathrm{outflow}(f,\partial\mathcal{S}_R) \le \int_{\partial\mathcal{S}_R} \xi V + \int_{\mathcal{S}_R} b_{ext} V\,ds \quad (3.7)$$

during the duration of $\mathbf{r}_R$. This **global energy-inequality** is assumed to hold for every such region R.

The transport theorem (2.34), the identity (2.45), and the fact that R is arbitrary imply that

$$FV + f^\circ - fKV - (\xi V)_s - b_{ext}V \le 0; \quad (3.8)$$

hence $(2.18)_2$ and (3.5) yield the **local energy-inequality**

$$f^\circ - \xi\theta^\circ + (\sigma - f)KV + (\lambda + F)V \le 0. \quad (3.9)$$

Remark. One could also consider, as an additional postulate, an energy inequality for the <u>interface</u> <u>itself</u> of the form

$$(d/dt)\int_{c} f\,ds + \mathrm{outflow}(\phi,\partial c) \le \int_{\partial c} \xi V + \int_{c} bV\,ds \quad (3.10)$$

with c the trace of an arbitrary evolving subcurve. (Note that we use $b = b_{ext} + \lambda$ to account for the power supplied to the interface by the bulk material.) As we shall see, (3.10) follows as a <u>consequence</u> of our constitutive assumptions. Note that, by $(3.4)_1$, (3.10) has the local form

$$f^\circ - \xi\theta^\circ + (\sigma - f)KV \le 0. \quad (3.11)$$

4. Constitutive equations. Consequences of thermodynamics. Stability.

4.1. Constitutive equations. The compatibility theorem.

As **constitutive equations** we allow the energy, surface tension, capillary shear, and interactive force to depend on the orientation of the interface through a dependence on θ, and on the kinetics of the interface through a dependence on V:

$$f = f\hat{}(\theta,V), \qquad \xi = \xi\hat{}(\theta,V),$$
$$\sigma = \sigma\hat{}(\theta,V), \qquad \lambda = \lambda\hat{}(\theta,V). \tag{4.1}$$
[22]

The first three relations characterize the interface, the last models the interaction between the interface and the bulk material.

Given an interfacial motion $\mathbf{r}$, the constitutive equations may be used to compute a corresponding **process** (f,ξ,σ,λ). The normal force-balance (3.5) then determines the external force b_{ext} needed to support the process. Granted this, the second law (3.7) will hold in every region R if and only if the local-energy inequality (3.9) is satisfied. This should motivate the following definition: the **constitutive equations are compatible with thermodynamics** if given any interfacial motion, the corresponding process satisfies (3.9).

[22] Each of these functions is assumed to be 2π-periodic with respect to θ.

Compatibility theorem.[23] The constitutive equations are compatible with thermodynamics if and only if:

(i) the energy, surface tension, and surface shear are independent of V and satisfy

$$\sigma\hat{\ }(\theta) = f\hat{\ }(\theta), \qquad \xi\hat{\ }(\theta) = f\hat{\ }_{\theta}(\theta); \tag{4.2}$$

(ii) the interactive force has the form

$$\begin{aligned} &\lambda\hat{\ }(\theta,V) = -F - \beta(\theta,V)V, \\ &\beta(\theta,V) \geq 0. \end{aligned} \tag{4.3}$$

Proof. The following simple result will be useful:

> Let $\phi(x)$ be smooth and satisfy $\phi(x)x \geq 0$ for all $x \in \mathbb{R}$. Then there is a smooth function $\mu(x) \geq 0$ such that $\phi(x) = \mu(x)x$. (4.4)

The proof is simple: $\phi(x)x$ has a minimum at $x=0$; thus $\phi(0)=0$, so that $\mu(x) = x^{-1}\phi(x) \geq 0$ is well defined and smooth at $x=0$.

To prove the theorem, we note that, in view of the constitutive equations, (3.9) is equivalent to the inequality

$$\begin{aligned} &[f\hat{\ }_{\theta}(\theta,V) - \xi\hat{\ }(\theta,V)]\theta^{\circ} + f\hat{\ }_{V}(\theta,V)V^{\circ} + \\ &[\sigma\hat{\ }(\theta,V) - f\hat{\ }(\theta,V)]KV + [\lambda\hat{\ }(\theta,V) + F]V \leq 0. \end{aligned} \tag{4.5}$$

Assume that (4.5) holds for all motions of the interface. It is a simple matter to construct an interfacial motion for which, at some point and time, the fields θ, V, K, θ°, and V° have arbitrary values (cf. the Variation Lemma of Gurtin [1988]); this implies (i) and the inequality

[23] This is a special case of a more general theorem [1988g] in which the bulk material is allowed to conduct heat.

$$\{F + \lambda\hat{}(\theta,V)\}V \leq 0. \tag{4.6}$$

Assertion (ii) follows from (4.6) and (4.4) with $x = V$ and $\phi(x) = -F - \lambda\hat{}(\theta,x)$.

Conversely, the assertions (i) and (ii) trivially yield (4.5) in all processes. ■

We will refer to $\beta(\theta,V)$ as the **kinetic coefficient**.

By definition, λ and V are components with respect to the same direction, so that, for V positive, λ may be regarded as a force <u>in the direction of motion</u> exerted on the interface by the bulk material. Equation (4.3) gives this force as the sum of two terms. The first term is a force $-F$ which is positive if the phase into which the interface is moving has higher energy (and is thus less stable) than the other phase. The second term $-\beta V$ is, by $(4.3)_1$, negative, and represents a <u>drag force</u> opposing interfacial motion. Note that, for small values of the velocity V, (4.3) has the approximation

$$\begin{aligned} &\lambda = - F - \beta_0(\theta)V, \\ &\beta_0(\theta) = \beta(\theta,0) \geq 0. \end{aligned} \tag{4.7}$$

Some important consequences of the compatibility theorem are expressed in the following remarks.

Remark 1. The relations (4.2) imply (3.11) with "$\leq$" replaced by "=", and this yields the interfacial energy-inequality (3.10) as an <u>equality</u>. Thus <u>the interface does not dissipate energy</u>; energy is dissipated at most in the interaction between the interface and the bulk material. The right side of (3.7) minus the left side gives this dissipation, which a simple calculation shows to be

$$\int_{\mathcal{S}_R} \beta(\theta,V)V^2 ds . \tag{4.8}$$

Thus $\beta(\theta,V)V^2$ represents the *energy dissipated by the interaction* per unit length.

Remark 2. By (2.4) and (4.2), the *capillary force* (3.1) *may be regarded as a function of orientation*:

$$\mathbf{C} \doteq \mathbf{C}(\theta) = \hat{f}(\theta)\mathbf{T}(\theta) + \hat{f}'(\theta)\mathbf{N}(\theta). \tag{4.9}$$

The relations (4.2) also imply that $\sigma_s = \xi K$ in every process. Therefore, by $(3.4)_2$, *tangential forces are balanced with* $b_{tan} = 0$, which obviates the need for constraint forces. Thus, granted $b_{tan} = b_{ext} = 0$, we may use (4.3) to write the capillary balance law (3.2) in the equivalent integral form

$$\int_{\partial c(t)} \mathbf{C}(\theta) \quad = \quad \int_{c(t)} (F + \beta(\theta,V)V)\mathbf{N}\,ds \tag{4.10}$$

for every c, or, by (3.5), (4.2), and (4.3), in the *equivalent* local form

$$\beta(\theta,V)V = [f(\theta) + f''(\theta)]K - F. \tag{4.11}$$

A final consequence of the compatibility theorem is the following result, which is essentially a statement of the second law for evolving curves.

Corollary. *Consider a curve* $\mathbf{r}$ *that evolves according to the normal force-balance* (3.5) *and the thermodynamic relations* (4.2) *and* (4.3). *Let* ς *denote the trace of* $\mathbf{r}$, *and let* $S_1(t) < S_2(t)$ *denote the arc lengths corresponding to the endpoints of* $\varsigma(t)$. *Then*

$$(d/dt)\int_{\varsigma(t)} f\,ds + F\int_{\varsigma(t)} V\,ds = -\int_{\varsigma(t)} \beta(\theta,V)V^2\,ds + \int_{\varsigma(t)} b_{ext}V\,ds + W_2(t) - W_1(t),$$
$$W_i(t) = \mathbf{C}(S_i(t),t)\cdot(d/dt)\mathbf{r}(S_i(t),t). \tag{4.12}$$

<u>Proof</u>. By (4.2) and $(2.18)_2$, $f^\circ - fKV = \xi V_s - \sigma KV$. If we substitute this relation into (2.34) with $\phi = f$, integrate the term ξV_s by parts, and appeal to (2.32), (3.5), $(2.20)_1$, and (3.1), we arrive at (4.12). ■

4.2. General assumptions. Admissibility for evolving curves.

Since there is no danger of confusion, we will use the shorthand:

$$f(\theta) = \hat{f}(\theta).$$

Further, to avoid repeated hypotheses, we will assume, for the remainder of the paper, that the following hypotheses are satisfied:

Assumptions.

(i) <u>The constitutive equations are compatible with thermodynamics</u>.

(ii) <u>The interfacial energy and kinetic coefficient satisfy</u>:

$$f(\theta) > 0, \quad \beta(\theta,V) > 0,$$
$$\beta(\theta,V) \ \underline{\text{is independent of}} \ V. \tag{4.13}$$

(iii) <u>We henceforth restrict attention to evolving curves that correspond to vanishing external forces</u>:

$$b_{ext} = 0. \tag{4.14}$$

We will refer to an evolving curve as **admissible** if it is consistent with (4.11). By Remark 2 of the Section 4.1, admissibility for an evolving curve is equivalent to the requirement that the curve be consistent with balance of capillary forces.

4.3. Stability of the interfacial energy.

The following calculation leads to a condition on f which ensures that straight line segments locally minimize interfacial energy. Let ℓ denote an oriented, straight line segment with initial point $\mathbf{r}_0$ and terminal point $\mathbf{r}_1$, and let $\mathbf{r}$ denote a (not necessarily admissible) evolving curve whose initial and terminal points are <u>fixed</u> at $\mathbf{r}_0$ and $\mathbf{r}_1$, respectively, and whose trace satisfies $\mathcal{S}(0)=\ell$. Let $\mathcal{F}(t)$ denote the energy of $\mathcal{S}(t)$:

$$\mathcal{F}(t) = \int_{\mathcal{S}(t)} f(\theta)ds.$$

Then

$$\mathcal{F}^{\circ}(0) = 0, \qquad \mathcal{F}^{\circ\circ}(0) = [f(\theta_0) + f''(\theta_0)]\int_{\ell}(V_s)^2 ds, \qquad (4.15)$$

with θ_0 the angle of ℓ. We will establish (4.15) at the end of the section. Thus[24] <u>a necessary and sufficient condition that</u> $\mathcal{F}(t)$ <u>have a strict local minimum at</u> $t=0$ <u>is that</u>

$$f(\theta_0) + f''(\theta_0) > 0.$$

This proposition should motivate the following definition. The interfacial energy f is **strictly stable at** θ, **stable at** θ, or **unstable at** θ according as

$$f(\theta) + f''(\theta) > 0, \qquad f(\theta) + f''(\theta) \geq 0, \qquad f(\theta) + f''(\theta) < 0; \qquad (4.16)$$

f is: **strictly stable** if it is strictly stable for all $\theta\in\mathbb{R}$; **stable** if it is stable for all $\theta\in\mathbb{R}$. As we shall see in Section 6, the partial differential equation describing the evolution of the interface will be <u>parabolic</u> where the interfacial energy is strictly stable and <u>backward</u> <u>parabolic</u> where f is unstable. Since $f(\theta)>0$,

[24]Cf. Herring [1951b], Frank [1963], Gjostein [1963], Gruber (Gjostein [1963]), Taylor [1978], Fonseca [1988].

the interfacial energy cannot be unstable for all θ.

By (2.4) and (4.9),

$$\mathbf{C}'(\theta) = [f(\theta) + f''(\theta)]\mathbf{N}(\theta), \tag{4.17}$$

Thus, if the interfacial energy is strictly stable, then given any angle α, $\mathbf{N}(\alpha)\cdot\mathbf{C}(\theta)$ increases strictly with θ for $\alpha - \pi/2 \leq \theta \leq \alpha + \pi/2$ and decreases strictly with θ for $\alpha + \pi/2 \leq \theta \leq \alpha + 3\pi/2$.

We now prove (4.15). By (2.33) and (2.34)

$$\mathcal{F}^\circ(t) = \int_{\mathcal{S}(t)} [f'(\theta)\theta^\circ - f(\theta)KV]ds,$$

and, since $V = 0$ at the endpoints, while $K = 0$ and $\theta = \theta_0$ at $t = 0$, we may use $(2.18)_2$ to conclude that $(4.15)_1$ is satisfied. Similarly,

$$\mathcal{F}^{\circ\circ}(t) = \int_{\mathcal{S}(t)} [f'(\theta)\theta^{\circ\circ} + f''(\theta)(\theta^\circ)^2 - f(\theta)K^\circ V]ds \quad \text{at} \quad t = 0.$$

But $\theta^\circ = V_s$ and, by (2.17) and (2.18), $\theta^{\circ\circ} = KVV_s + (V^\circ)_s$, $K^\circ V = V_{ss}V + K^2V^2$; thus

$$\mathcal{F}^{\circ\circ}(t) = \int_{\mathcal{S}(t)} [f''(\theta_0)(V_s)^2 - f(\theta_0)V_{ss}V]ds \quad \text{at} \quad t = 0,$$

and integrating the last term by parts we arrive at $(4.15)_2$.

II. Smooth interfacial motions.

5. Evolution equations for the interface.

We now discuss the equations that describe admissible evolving curves; that is, evolving curves whose evolution is governed by the constitutive equations and the capillary balance law.

5.1. Isotropic interface.

For an isotropic interface f and β are constants. Without loss in generality, we set $f=\beta=1$; (4.11) then reduces to[25]

$$V = K - F. \tag{5.1}$$

A complete set of partial differential equations for an admissible evolving curve consists of (5.1) suplemented by the kinematical conditions (2.18) (cf. (2.16)) satisfied by all evolving curves:

$$\begin{aligned} &V = K - F, && \theta_t + v\theta_s = V_s, \\ &K_t + vK_s = V_{ss} + K^2 V, && v_s = -KV \end{aligned} \tag{5.2}$$

(where the subscript t denotes the time derivative holding s fixed). The domains of the underlying fields in the arc-length description are not known a-priori, since s varies in the interval $[0,L(t)]$ with $L(t)=\text{length}(\mathfrak{z}(t))$. However, (2.36) relates $L^{\circ}(t)$ to KV, and we can introduce the rescaled variable $s^* = s/L(t)$.[26]

When the curve is convex, the system (5.2) takes a particularly

[25] Allen and Cahn [1979] and Rubinstein, Sternberg, and Keller [1987] deduce the equation $V=K$ as a formal approximation to the Landau-Ginzburg equation. Evolution according to $V=K$ is discussed by many authors; cf. Brakke [1978], Sethian [1985], Abresch and Langer [1986], Gage [1984,1986], Gage and Hamilton [1986], Grayson [1987], Huisken [1987], Osher and Sethian [1987], and the references therein.

[26] Abresch and Langer [1986].

simple form; indeed, (5.1) and (2.23) yield

$$K_t = K^2[K_{\theta\theta} + K - F]. \tag{5.3}$$

(with K_t the time derivative holding θ fixed).

5.2. Anisotropic interface.

5.2.1. Basic equations.

For convenience, we define

$$\Phi = \beta^{-1}(f + f''), \qquad \Psi = \beta^{-1}F, \tag{5.4}$$

then (4.11) becomes

$$V = \Phi(\theta)K - \Psi(\theta). \qquad (5.5)^{27}$$

A complete system of equations for an admissible curve consists of (5.5) in conjunction with $(5.2)_{2-4}$. When the curve is convex, this system reduces to

$$K_t = K^2[\Phi K - \Psi]_{\theta\theta} + K^2[\Phi K - \Psi] \tag{5.6}$$

(with K_t the derivative holding θ fixed). This equation is also valid for nonconvex motions, at least locally where $K \neq 0$.

Remark. The term of highest order on the right side of (5.6) is $K^2\Phi K_{\theta\theta}$; thus (5.6) is parabolic for $\Phi(\theta) > 0$, backward parabolic for $\Phi(\theta) < 0$. (Note that (5.6) degenerates at $K = 0$.) By (4.13), (4.16), and (5.4), parabolicity is equivalent to the strict stability of the interfacial energy, while backward parabolicity is equivalent to instability (cf. (4.16)). There is no compelling physical reason to suppose that the interfacial energy is strictly stable; in fact, material scientists often consider energies which are unstable[28] for

[27] The special case $V = -\Psi(\theta)$ was introduced by Frank [1958].

particular ranges of the orientation θ. Since $f(\theta)>0$ and periodic, at worst we can have an equation which is backward parabolic for some but not all values of θ.

Note that, by (5.5), the general equation (5.6), when expressed in terms of the normal velocity $V(\theta,t)$, has the form

$$\Phi(\theta)V_t = [V + \Psi(\theta)]^2[V_{\theta\theta} + V]. \tag{5.7}$$

5.2.2. Equations when the curve is the graph of a function.

Locally, an evolving curve may be represented as the graph of a function $y = h(x,t)$, provided the x and y axes are chosen appropriately. Consider the choice indicated in Figure 5A (with orientation such that arc length increases with increasing x) and let

$$p = h_x, \tag{5.8}$$

Then

$$p\tan\theta = -1, \qquad h_t = (\sin\theta)^{-1}V, \qquad K = h_{xx}(1+p^2)^{-3/2}, \tag{5.9}$$

and the evolution equation (5.5) takes the form

$$\begin{aligned} h_t &= Q(p)h_{xx} - B(p), \\ Q(p) &= \Phi(\theta)\sin^2\theta, \quad B(p) = \Psi(\theta)/\sin\theta, \end{aligned} \tag{5.10}$$

or, differentiating with respect to x,

$$p_t = [Q(p)p_x - B(p)]_x, \tag{5.11}$$

which is in conservation form.

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[28]Cf., e.g., Gjostein [1963], Cahn and Hoffman [1974].

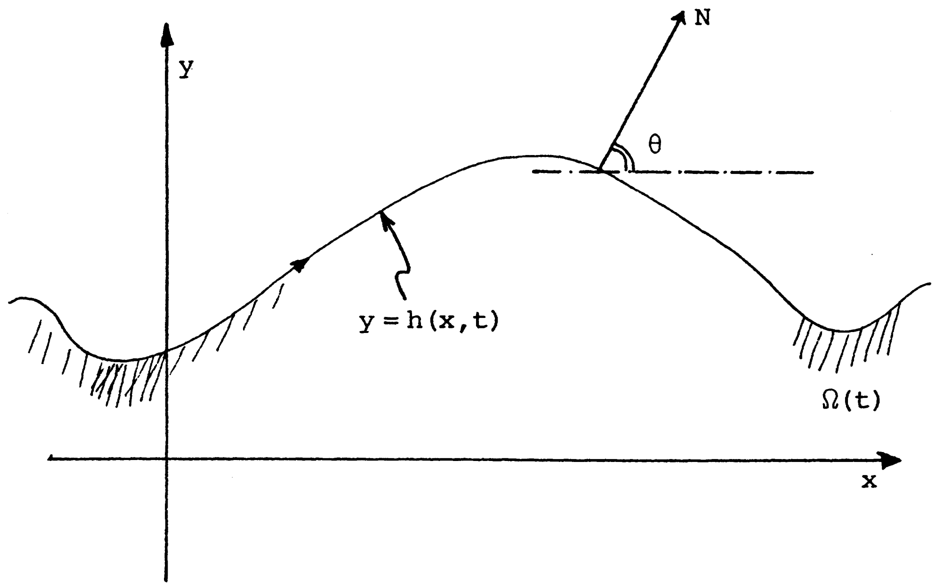

Figure 5A. Sign conventions when the curve is a graph $y = h(x,t)$.

6. Stationary interfaces. Steady interfacial motions.

We assume throughout this section that the interface is strictly stable. Then

$$f(\theta) + f''(\theta) > 0$$

for all θ, and we can write

$$w(\theta) = [f(\theta) + f''(\theta)]^{-1}. \tag{6.1}$$

Note that, by (2.11),

$$\int_0^{2\pi} w(\theta)^{-1} e^{i\theta} d\theta = 0. \tag{6.2}$$

This section is restricted to evolving curves and interfacial motions that are admissible; to avoid repetition, *we shall omit the term "admissible" in most of the ensuing discussion*.

6.1. Stationary interfaces.

By a **stationary interface** we mean an interfacial motion which is independent of time. A trivial consequence of (5.5) is that for $F = 0$ the unbounded time-independent facets form the complete collection of stationary interfaces. The next theorem establishes the existence of stationary interfaces for $F \neq 0$.

Wulff's theorem.[29] *Assume that* $F \neq 0$. *Then*

$$\mathbf{r}(\theta) = F^{-1}[f'(\theta)\mathbf{T}(\theta) - f(\theta)\mathbf{N}(\theta)]. \tag{6.3}$$

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[29]Wulff [1901]. See also Dinghas [1944], Taylor [1978]. The bounded region Ω with boundary defined by (6.3) is actually a Wulff region in the sense of (7.12): Ω has least *interfacial* energy among all regions Γ with $\text{area}(\Gamma) = \text{area}(\Omega)$.

defines a stationary interface which is closed, convex, and parametrized by angle, and any other stationary interface differs from (6.3) by at most a translation. The curvature and support function corresponding to (6.3) are given by

$$K(\theta) = Fw(\theta), \qquad p(\theta) = -f(\theta)/F. \tag{6.4}$$

Proof. In view of (4.11) (with $V=0$), a motion is stationary if and only if it is convex with curvature given by $(6.4)_1$. Moreover, by (6.2), the function $K(\theta)$ defined by $(6.4)_1$ satisfies (2.10). The theorem therefore follows from Lemma 3 (Section 2.1). ■

Let $\mathbf{Q}$ denote the orthogonal transformation that rotates vectors clockwise by $\pi/2$. An interesting consequence of (4.9) is that the capillary force $\mathbf{C}(\theta)$ corresponding to (6.3) is given by

$$\mathbf{C}(\theta) = F\mathbf{Q}\mathbf{r}(\theta).$$

6.2. Steadily evolving facets.

By (2.29), (5.5), and (6.1), steady interfacial motions $\mathbf{r}$ evolve according to

$$K(\theta) = [F + \beta(\theta)\mathbf{U}\cdot\mathbf{N}(\theta)]w(\theta), \tag{6.5}$$

where $\mathbf{U}\,(\neq \mathbf{0})$ is the steady velocity. It is convenient to introduce the vector potential

$$\boldsymbol{\beta}(\theta) = \beta(\theta)\mathbf{N}(\theta), \tag{6.6}$$

whose locus forms the polar diagram $\mathrm{Polar}(\beta)$ of β, and to write (6.5) in the form

$$K(\theta) = G(\theta)w(\theta), \qquad G(\theta) = F + \mathbf{U}\cdot\boldsymbol{\beta}(\theta). \tag{6.7}$$

Appealing to the proposition containing (2.29), we see that a steady motion is a steadily evolving facet if and only if the corresponding angle θ is identically constant and a solution of

$$G(\theta) = 0. \tag{6.8}$$

The equation (6.8) has a simple geometric solution. To state this solution concisely, we introduce the following terminology. For $\mathbf{d} \neq \mathbf{0}$, let $\ell(\mathbf{d})$ denote the straight line

$$\ell(\mathbf{d}) = \{\, \mathbf{x} : \mathbf{d} \cdot \mathbf{x} = |\mathbf{d}|^2 \,\}; \tag{6.9}$$

(6.9) defines a one-to-one correspondence between nonzero vectors and lines that do not pass through the origin; we will refer to $\mathbf{d}$ as the **support vector** for $\ell = \ell(\mathbf{d})$. In the same spirit, for $\mathbf{d} \neq \mathbf{0}$, we write $\ell = \ell(0\mathbf{d})$ for the line through the origin perpendicular to $\mathbf{d}$ and refer to ℓ as the line with **support vector** $0\mathbf{d}$. Then

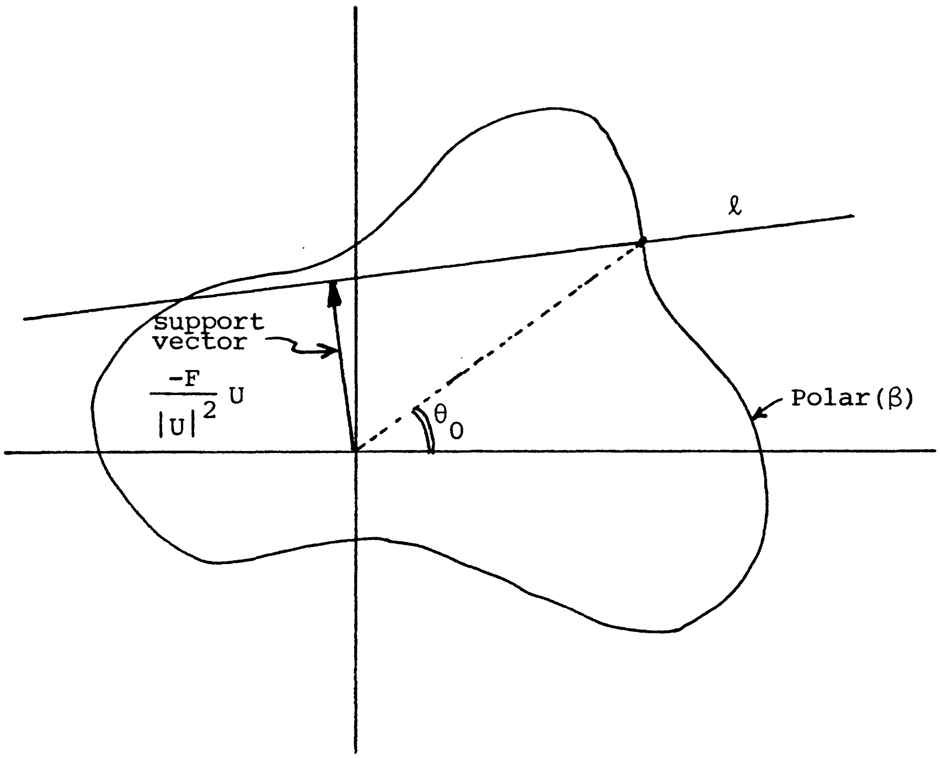
$$\left.\begin{array}{l} G(\theta_0)=0 \text{ if and only if the line with} \\ \text{support vector } (-F/|\mathbf{U}|^2)\mathbf{U} \text{ intersects} \\ \text{the polar diagram Polar}(\beta) \text{ at } \boldsymbol{\beta}(\theta_0); \end{array}\right\} \tag{6.10}$$

hence we have the following result (Figure 6A).

Theorem. Given any vector $\mathbf{U} \neq \mathbf{0}$, there is a steadily evolving facet with steady velocity $\mathbf{U}$ and angle θ_0 if and only if the line with support vector $(-F/|\mathbf{U}|^2)\mathbf{U}$ intersects Polar(β) at $\beta(\theta_0)$.

6.3. Steady motions that are not flat.

Let F and $\mathbf{U} \neq \mathbf{0}$ be given. Let $\mathbf{r}$ be a steady motion corresponding to F with $\mathbf{U}$ as steady velocity, and assume that $\mathbf{r}$ is **not flat** in the sense that its curvature is not identically zero.



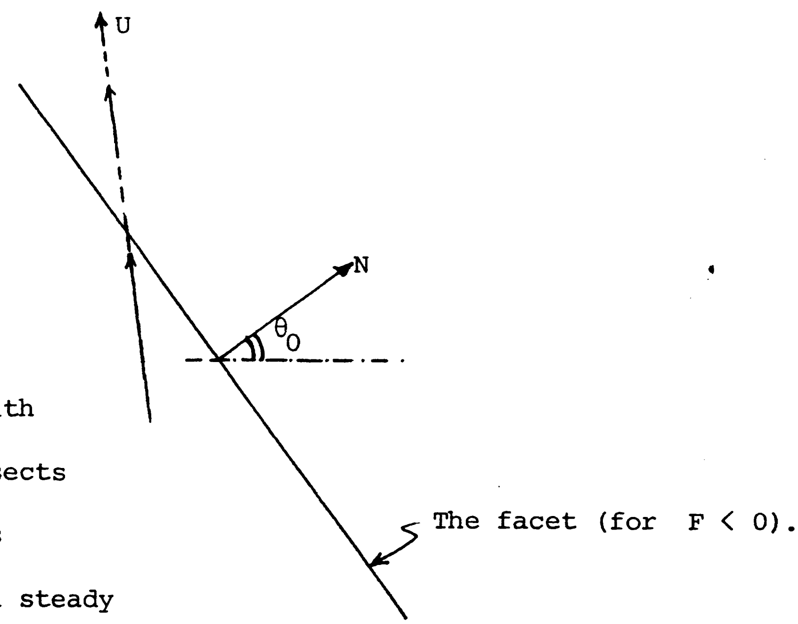


Figure 6A. When the line ℓ with support vector $\frac{-F}{|U|^2} U$ intersects Polar(β) at θ_0, then there is a steadily evolving facet with steady velocity U and direction θ_0.

Then Lemma 2 (Section 2.1) implies that $\mathbf{r}$ is <u>convex</u>. By definition, $\mathbf{r}$ is a boundary curve at each t, and hence is simple and either closed or unbounded. Assume that $\mathbf{r}$ is closed. Let $U = |\mathbf{U}|$, $\mathbf{e} = \mathbf{U}/U$. Then (2.10) and (6.7) yield

$$\int_0^{2\pi} [G(\theta)w(\theta)]^{-1}\mathbf{e}\cdot\mathbf{N}(\theta)d\theta = 0. \tag{6.11}$$

Let $M(\theta,\mathbf{e},U)$ denote the left side of (6.11). By (6.2) and $(6.7)_2$, $M(\theta,\mathbf{e},0) = 0$. Further, differentiating $M(\theta,\mathbf{e},U)$ with respect to U yields the conclusion that $M(\theta,\mathbf{e},U)$ is strictly monotone in U. Thus (6.11) is possible only if $U = 0$, which violates the definition of a steady motion; hence $\mathbf{r}$ cannot be closed. Thus $\mathbf{r}$ is <u>unbounded</u>.

Therefore, appealing to Lemma 3 of Section 2.1, the angle range of $\mathbf{r}$ is a bounded interval (θ_1,θ_2), and this interval and the accompanying curvature $K(\theta)$ must be a solution of the following problem:

$$\left.\begin{array}{l}\text{find an angle interval } (\theta_1,\theta_2),\ \theta_2-\theta_1 \le \pi,\\ \text{such that } K(\theta), \text{ defined by (6.5), is nonvanishing}\\ \text{on } (\theta_1,\theta_2) \text{ and has } K(\theta_1)=K(\theta_2)=0.\end{array}\right\} \tag{6.12}$$

Conversely, if (θ_1,θ_2) is consistent with (6.12), then Lemma 3 of Section 2.1 implies that $K(\theta)$ restricted to (θ_1,θ_2) is the curvature of a convex, bounded curve $\mathbf{r}_0$. By virtue of the proposition following (2.25), $\mathbf{r}_0$ is the portrait and $\mathbf{U}$ the steady velocity of a (unique) steady motion $\mathbf{r}$; trivially, $\mathbf{r}$ has $K(\theta)$ as its curvature.

Thus we are reduced to solving (6.12); since $w(\theta) > 0$, (θ_1,θ_2) is a solution of (6.12) if and only if

$$\theta_1,\theta_2,\ 0 < \theta_1 - \theta_2 \le \pi, \text{ are } \underline{\text{consecutive}} \text{ zeros of } G(\theta). \tag{6.13}$$

To facilitate the discussion of such zeros, let us agree to call a line ℓ a **chord for** Polar(β) **between** θ_1 **and** θ_2 if $0<\theta_1-\theta_2\leq\pi$ and ℓ intersects Polar(β) at $\boldsymbol{\beta}(\theta_1)$ and $\boldsymbol{\beta}(\theta_2)$, but not at any other point $\boldsymbol{\beta}(\theta)$ with $\theta\in(\theta_1,\theta_2)$. In view of (6.10), (6.13) is then equivalent to the requirement that

> the line ℓ with support vector $(-F/|\mathbf{U}|^2)\mathbf{U}$
> be a chord for Polar(β) between θ_1 and θ_2.

Given a line ℓ with ℓ a chord for Polar(β) between θ_1 and θ_2, we write

$$\ell|_{(\theta_1,\theta_2)} := \{\ \mathbf{x}:\ \ \mathbf{x}=\boldsymbol{\beta}(\theta_1)+a[\boldsymbol{\beta}(\theta_2)-\boldsymbol{\beta}(\theta_1)],\ \ a\in(0,1)\ \}$$

for the segment of ℓ between $\boldsymbol{\beta}(\theta_1)$ and $\boldsymbol{\beta}(\theta_2)$.

Continuing as before, let F and $\mathbf{U}\neq\mathbf{0}$ be given, and let $\mathbf{r}$ be a steady motion corresponding to F and $\mathbf{U}$. Let (θ_1,θ_2) be the angle range for $\mathbf{r}$, so that the line ℓ with support vector $(-F/|\mathbf{U}|^2)\mathbf{U}$ is a chord for Polar(β) between θ_1 and θ_2. Choose $\theta\in(\theta_1,\theta_2)$. Then there is a unique point $\mathbf{x}\in\ell|_{(\theta_1,\theta_2)}$ such that $\mathbf{x}=\alpha\mathbf{N}(\theta)$ for some $\alpha\geq 0$; in addition,

$$\begin{aligned}&\beta(\theta)>\alpha \ \text{ or } \ \beta(\theta)<\alpha \ \text{ according as } \ \ell|_{(\theta_1,\theta_2)}\\ &\text{is interior or exterior to Polar}(\beta).\end{aligned} \tag{6.14}$$

Since $\mathbf{x}\in\ell$, $\mathbf{x}\cdot\mathbf{U}=-F$; thus (6.6) and (6.7) yield

$$G(\theta) = [\beta(\theta) - \alpha]\mathbf{U}\cdot\mathbf{N}(\theta).$$

Further $K=wG$ with $w>0$ and $K(\theta)\neq 0$; hence

$$K(\theta)\mathbf{U}\cdot\mathbf{N}(\theta) = C[\beta(\theta) - \alpha], \qquad C > 0,$$

and we conclude, with the aid of (2.30), (2.31), and (6.14), that $\mathbf{r}$ is a <u>steadily evolving bump</u>, which recedes or advances according as $\ell|_{(\theta_1,\theta_2)}$ is interior or exterior to Polar(β).

The results established above are summarized in the next theorem, in which F and $\mathbf{U} \neq \mathbf{0}$ are assumed prescribed.

Theorem on steady motions. *Let ℓ denote the line with support vector $(-F/|\mathbf{U}|^2)\mathbf{U}$. Let $\mathbf{r}$ be a nonflat steady motion which corresponds to F and has $\mathbf{U}$ as steady velocity. Then $\mathbf{r}$ is a steadily evolving bump. If (θ_1,θ_2) is the angle range of $\mathbf{r}$, then ℓ is a chord for Polar(β) between θ_1 and θ_2, and $\mathbf{r}$ is receding or advancing according as $\ell|_{(\theta_1,\theta_2)}$ is interior or exterior to Polar(β).*

Conversely, if ℓ is a chord for Polar(β) between θ_1 and θ_2, then there is a unique steady motion $\mathbf{r}$ corresponding to F with (θ_1,θ_2) as angle range and $\mathbf{U}$ as steady velocity.

Corollary. *There are no advancing bumps if Polar(β) is convex, and none when $F=0$.*

Remark. For an isotropic material $(f=\beta=1)$ and $F=0$, (6.12) reduces to $K(\theta)=\mathbf{U}\cdot\mathbf{N}(\theta)$; letting $\mathbf{U}=U(1,0)$, $U>0$, there are two steady motions, one with angle range $(-\pi/2,\pi/2)$, the other with angle range $(\pi/2,3\pi/2)$, and both motions are receding bumps.[30]

Remark. It is generally believed that dendritic growth requires diffusion in the bulk material. It is interesting that a steadily advancing bump is possible even without diffusion. In the present theory such growth is a consequence of anisotropy in the kinetic coefficient and results when certain orientations suffer drag forces sufficiently lower than neighboring orientations.

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[30]This solution is known; it is referred to as the "grim reaper" by geometers (cf. Grayson [1987], p. 298).

7. Global behavior for an interface with stable energy.

In this section we analyze the global behavior of the interface under the assumption of a *strictly stable* interfacial energy; in particular, we consider the general anisotropic equation

$$\beta(\theta)V = [f(\theta) + f''(\theta)]K - F \tag{7.1}$$

with

$$f(\theta) + f''(\theta) > 0, \qquad \beta(\theta) > 0 \tag{7.2}$$

for all $\theta \in \mathbb{R}$ (cf. (4.13), (4.16)).

7.1. Existence of interfacial motions from a prescribed initial curve.

The existence of an interface evolving from a given initial configuration is ensured by the

Existence theorem. *Let* f *and* β *be* C^{∞}. *Let* Ω_0 *be a given initial domain, which we assume to be bounded with boundary a Lipschitz-continuous simple closed curve. Then there is a unique, maximal family of domains* $\Omega(t)$ $(0 \le t < T_{max})$ *such that:*
(i) $\partial\Omega(t)$ *is a* C^2 *simple closed curve, continuous for* $0 < t < T_{max}$;
(ii) the evolution of $\partial\Omega(t)$ *is governed by* (7.1);
(iii) $\Omega(0) = \Omega_0$.
In fact, this solution is C^{∞} *for* $0 < t < T_{max}$.

A proof of this theorem is given by Angenent [1988], who shows that, for $T_{max} < \infty$, one of the following must be true:
(E1) $\sup_{s \in \mathbb{R}} |K(s,t)| \to \infty$ as $t \to T_{max}$;

(E2) K and its derivatives remain bounded as $t \to T_{max}$, so that

$\partial\Omega(t)$ converges to a C^{∞} curve Γ; however Γ is not simple.

(E1) will occur whenever the interface shrinks to a point, or whenever the interface develops a kink; (E2) indicates the formation of self-intersections or self-tangencies (Figure 7A).

When Ω_0 is smooth, $\partial\Omega(t)$ admits a parametrization $(p,t) \mapsto \mathbf{r}(p,t)$ as an admissible interfacial motion of duration $[0,T_{max})$. In the next section we will study the behavior of such motions as measured by their **perimeter** $L(t)$ and **enclosed area** $A(t)$,

$$L(t) = \text{length}(\partial\Omega(t)), \qquad A(t) = \text{area}(\Omega(t)). \tag{7.3}$$

We will, however, restrict our attention to motions, termed regularly maximal, whose singularity at $t = T_{max}$ (for $T_{max} < \infty$) is not too pathological.

Precisely, an admissible interfacial motion with duration $[0,T)$ is **regularly maximal** if either $T = \infty$ or

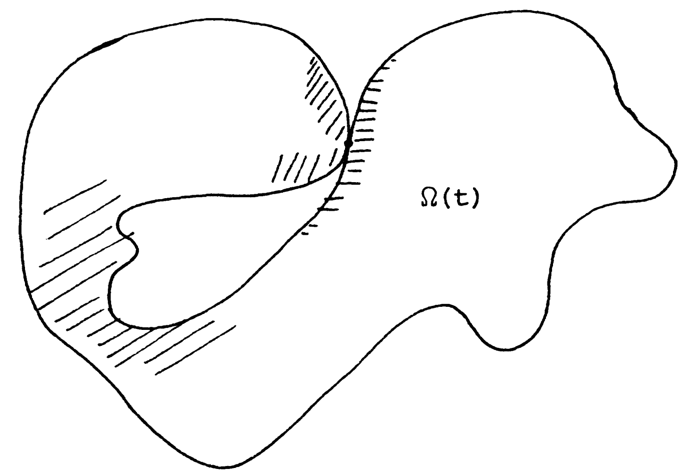
$$T < \infty \quad \text{and} \quad A(t) \to 0 \quad \text{as} \quad t \to T. \tag{7.4}$$

Regularly maximal motions cannot be extended beyond $t = T$, but for T finite exhibit fairly regular behavior as $t \to T$: they either explode or disappear. This class of motions does not include motions that develop self-tangencies, self-intersections, or kinks at $t = T$.

7.2. Growth and decay of the interface.

Let $\mathbf{r}$ denote an admissible interfacial motion. For convenience, we write

$$\mathcal{F}(t) := \int_{\partial\Omega(t)} f(\theta)\,ds \tag{7.5}$$



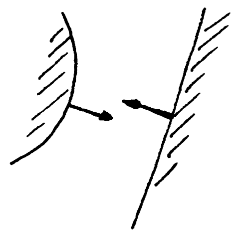


Figure 7A. A simply connected region may evolve to a multiply connected region.

for the **total interfacial energy**. The next result, essentially the second law, is an immediate consequence of (3.7) and the discussion of the paragraph containing (4.8).

Growth theorem.[31]

$$\mathcal{F}^\circ(t) + FA^\circ(t) = -\int_{\partial\Omega(t)} \beta(\theta)V^2 ds \le 0. \tag{7.6}$$

It is convenient to rewrite (7.1) in the form

$$\begin{gathered} V = \Phi(\theta)K - \Psi(\theta), \\ \Phi(\theta) = [f(\theta) + f''(\theta)]/\beta(\theta), \qquad \Psi(\theta) = F/\beta(\theta). \end{gathered} \tag{7.7}$$

Also, for any 2π-periodic function $g(\theta)$ on $\mathbb{R}$, we write

$$g_{av} = (2\pi)^{-1}\int_0^{2\pi} g(\theta)d\theta, \qquad g_{max} = \sup_{\theta\in\mathbb{R}} g(\theta), \qquad g_{min} = \inf_{\theta\in\mathbb{R}} g(\theta) \tag{7.8}$$

If we use (2.5) to change variable in $(7.8)_1$ from θ to s, using the fact that as s increases from 0 to $L(t)$, θ goes from 0 to -2π, we arrive at

$$\int_{\partial\Omega(t)} g(\theta)K ds = -2\pi g_{av}. \tag{7.9}$$

Note that, by (7.2),

$$\Phi_{av} > 0, \qquad \Phi_{min} > 0.$$

Given any bounded region Γ, we will refer to the number

$$\mathrm{isoper}(\Gamma) := \mathrm{length}(\partial\Gamma)^2/4\pi\,\mathrm{area}(\Gamma) \tag{7.10}$$

as the **isoperimetric ratio** for Γ;

---

[31] Gurtin [1988g].

$$\text{isoper}(\Gamma) \geq 1 \quad \text{(isoperimetric inequality)} \tag{7.11}$$

with equality holding if and only if $\partial\Gamma$ is a circle.[32]

The following generalization of the isoperimetric ratio is useful. Let $e(\theta)$ be a continuous, piecewise smooth, strictly positive, 2π-periodic function on $\mathbb{R}$. Then the **Wulff ratio for** $e(\theta)$ is the number

$$W(e) = (4\pi)^{-1} \inf \Big\{ \int_{\partial\Gamma} e(\theta) ds \Big\}^2 / \text{area}(\Gamma) \tag{7.12}$$

with the infimum taken over all bounded regions Γ with $\partial\Gamma$ piecewise smooth. This infimum is actually attained:[33] minima Γ are called **Wulff regions for** $e(\theta)$, and are convex regions, unique modulo translation and scaling. When $e=1$, the minimum is a circle and $W(e)=1$. More generally, taking Γ to be a circle yields $W(e) \leq (e_{av})^2$, so that

$$0 < (e_{min})^2 \leq W(e) \leq (e_{av})^2. \tag{7.13}$$

The inequality $F>0$ occurs when the reference phase has higher bulk energy than the other phase; in this instance (7.6) indicates a tendency for the less stable reference phase to shrink. On the other hand, $F<0$ when the reference phase has lower bulk energy; here $FA(t)$ is negative and of the wrong sign for a Lyapunov function, indicating a tendency for the more stable reference phase to grow, at least in situations for which area dominates length. The next theorem shows that this is indeed the case. In fact, we show that for $F \geq 0$ the reference phase shrinks to zero; for $F<0$ the reference phase shrinks to zero when initially

[32]The condition $\text{isoper}(\Omega(t)) \to \infty$ might indicate the formation of a dendritic structure. In this connection Gurtin [1986] (cf. eq. (7.6)) discusses $A(t) \to 0$, $L(t) \to L_0 > 0$.

[33]Cf. Taylor [1978], who uses the term "Wulff crystal" rather than "Wulff region".

small, but grows unboundedly when initially large.

Theorem on the growth of the reference phase. Consider a regularly maximal, admissible interfacial motion with duration $[0,T)$.

(i) If $F \geq 0$, then $T < \infty$ and $A(t) \to 0$ as $t \to T$.

(ii) If $F < 0$, then:

(a) if $L(0)$ is sufficiently small, then $T < \infty$ and $A(t) \to 0$ as $t \to T$.

(b) if $A(0)$ is sufficiently large, then $T = \infty$ and $A(t) \to \infty$ as $t \to \infty$. In this case isoper$(\Omega(t)$ remains bounded

$$\limsup_{t \to \infty} \operatorname{isoper}(\Omega(t)) \leq [(\beta^{-1})_{av}]^2 / W(\beta^{-1}); \qquad (7.14)$$

thus, for $\beta = $ constant, $\operatorname{isoper}(\Omega(t)) \to 1$.

Proof. We begin with the identities:

$$A^\circ(t) = -2\pi\Phi_{av} - F\int_{\partial\Omega(t)} \beta(\theta)^{-1} ds,$$
$$L^\circ(t) = -2\pi\Psi_{av} - \int_{\partial\Omega(t)} \Phi(\theta) K^2 ds. \qquad (7.15)$$

To derive $(7.15)_1$ we integrate $(7.7)_1$ over $\partial\Omega(t)$ and use (2.45) and (7.9); to derive $(7.15)_2$ we multiply $(7.7)_1$ by K, integrate over $\partial\Omega(t)$, and use (2.36) and (7.9).

If we can show that $L(t) \to 0$ in finite time provided the solution persists that long, then we can conclude from (7.4) that $T < \infty$ and $A(t) \to 0$ as $t \to T$; we cannot conclude (from this alone) that $L(t) \to 0$ as $t \to T$.

Assume that $F \geq 0$. Then, by $(7.15)_2$ and the remark made in the previous paragraph, (i) follows.

Assume that $F < 0$. By (7.5), $f_{min} L \leq \mathcal{F} \leq f_{max} L$; thus (7.6) and $(7.15)_1$ yield

$$\mathcal{F}^\circ \leq -FA^\circ \leq -2\pi|F|\Phi_{av} + (F^2/f_{min}\beta_{min})\mathcal{F},$$

which implies (iia).

Next, $(7.15)_2$ yields

$$L(t) \leq L(0) + 2\pi |F| (\beta^{-1})_{av} t. \tag{7.16}$$

On the other hand, (7.12) yields

$$\int_{\partial\Omega(t)} \beta(\theta)^{-1} ds \geq 2[\pi W(\beta^{-1}) A(t)]^{\frac{1}{2}};$$

therefore, by $(7.15)_1$,

$$\begin{aligned} &A^{\bullet}(t) \geq DA(t)^{\frac{1}{2}} - C, \\ &C = 2\pi\Phi_{av} > 0, \qquad D = 2|F|[\pi W(\beta^{-1})]^{\frac{1}{2}} > 0. \end{aligned} \tag{7.17}$$

By (7.4), (7.16), and (7.17), if $A(0)$ is sufficiently large, then $T = \infty$ and $A(t) \to \infty$ as $t \to \infty$. In fact, $(7.17)_1$ is easily integrated to give

$$A(t)^{\frac{1}{2}} + \kappa \ell n[A(t)^{\frac{1}{2}} - \kappa] \geq A(0)^{\frac{1}{2}} + \kappa \ell n[A(0)^{\frac{1}{2}} - \kappa] + Dt/2, \tag{7.18}$$

where $\kappa = C/D$. Since $A(t) \to \infty$ as $t \to \infty$, (7.10), (7.16), and (7.18) imply (7.14). ■

Conjecture. Consider case (iib) of the last theorem, in which $F < 0$ and $A(0)$ is sufficiently large that the interface grows without bound. We conjecture that, as $t \to \infty$,

$$\partial\Omega(t) \ \underline{\text{is asymptotic to a Wulff region for}} \ \beta(\theta)^{-1}. \tag{7.19}$$

Our argument in support of (7.19) is as follows. As the interface grows the curvature term in (5.5) should ultimately be negligible.

Granted this, $V \simeq -F\beta(\theta)^{-1}$ and the total energy dissipated

$$\int_{\partial\Omega(t)} \beta(\theta)V^2 ds$$

(cf. (4.8)) should be asymptotic to

$$\mathfrak{D} = F^2 \int_{\partial\Omega(t)} \beta(\theta)^{-1} ds.$$

On the other hand, as $\Omega(t)$ grows, bulk energy should dominate interfacial energy; hence the total energy should be asymptotic to

$$\mathcal{E} = F\,\mathrm{area}(\Omega(t)).$$

It seems reasonable to expect that the interface should ultimately minimize both $\mathfrak{D}$ and $\mathcal{E}$. Thus, since $F<0$, and since $\mathcal{E}$ scales as $\mathfrak{D}^2$, one might expect the ultimate shape of the interface to minimize $\mathfrak{D}^2/|\mathcal{E}|$, which is exactly what a Wulff region for $\beta(\theta)^{-1}$ does.

Remarks.

(1) Assume that (7.19) is valid. Then for β constant $\Omega(t)$ is asymptotic to a circle as $t \to \infty$. More generally, it follows from properties of Wulff regions that $\partial\Omega(t)$ will have a smooth asymptotic shape if and only if $\beta(\theta)$ has a strictly convex polar diagram, $\mathrm{Polar}(\beta)$; if not the asymptotic shape will have corners in which the angle jumps across the Maxwell lines (Appendix A) of $\mathrm{Polar}(\beta)$.

(2) When $\mathrm{Polar}(e^{-1})$ is strictly convex, Wulff regions for $e(\theta)$ have the common isoperimetric ratio

$$\mathrm{isoper}_{\mathrm{Wulff}}(e) := (e_{av})^2/W(e). \tag{7.20}$$

Thus, for $\mathrm{Polar}(\beta)$ strictly convex, (7.14) yields

$$\limsup_{t\to\infty} \text{isoper}(\Omega(t)) \le \text{isoper}_{\text{Wulff}}(\beta^{-1}).$$

(3) Assume that $F<0$. One can conclude from the proof of the last theorem that, for the <u>interface initially a circle</u>,

$$A(t)\to 0 \quad \text{if} \quad L(0)<\delta_0\ell, \qquad A(t)\to\infty \quad \text{if} \quad L(0)>\ell,$$
$$\ell = 2\pi\beta_{max}\Phi_{av}/|F|, \qquad \delta_0 = \beta_{min}f_{min}/\beta_{max}f_{av}. \tag{7.21}$$

ℓ represents a <u>critical circumference</u> for a circular interface, while $\delta_0\in(0,1]$ is a measure of the underlying <u>anisotropy</u>; for initial circumferences between $\delta_0\ell$ and ℓ, (7.21) furnishes no information. For an <u>isotropic material</u>, $\delta_0=1$, and the reference phase grows or shrinks according as the initial circumference is greater than or less than $\ell=2\pi f/|F|$.

7.3. Evolution of curvature. Fingers.

Let $\mathbf{r}$ denote a bounded, admissible interfacial motion. The next theorem shows that the total curvature between inflection points cannot increase.

Theorem. <u>Let c be the trace of an evolving subcurve of</u> $\mathbf{r}$. <u>Assume that the curvature does not change sign on c and vanishes at the end points of c. Then</u>[34]

$$(d/dt)\int_{c(t)}|K|ds \le 0. \tag{7.22}$$

<u>In fact, if $\Theta(t)$ denotes the interval of angles $\theta(s,t)$ for arc lengths s comprising $c(t)$, then $\Theta(t)$ nests as t increases.</u>[35]

[34]Cf. Brakke [1978], Prop. 2, p. 230 and Albresch and Langer [1986] for the case $V=K$.

[35]Cf Grayson [1987], Lemma 1.9(iii), for the case $V=K$.

<u>Proof</u> Let $[S_1(t),S_2(t)]$ denote the arc-length interval corresponding to $\mathcal{c}(t)$. We will give a proof for $K(s,t) \geq 0$ on $[S_1(t),S_2(t)]$. (The proof for $K \leq 0$ is analogous.) For any function $\phi(s,t)$, let $\phi_i(t) = \phi(S_i(t),t)$. By hypothesis,

$$K_i(t) = 0, \tag{7.23}$$

so that

$$(K_s)_1 \geq 0, \qquad (K_s)_2 \leq 0. \tag{7.24}$$

By (5.5), $V = U - \Psi$ with $U = \Phi(\theta)K$, and, since $\Phi > 0$, (7.23) and (7.24) yield the conclusion that (7.24) holds with K replaced by U. On the other hand, since $\Psi = \Psi(\theta)$, $\Psi(\theta)_s = \Psi'(\theta)K = 0$ at $S_1(t)$ and $S_2(t)$. Thus (7.24) holds with $(K_s)_i$ replaced by $(V_s)_i$, and, in view of (7.23) and $(2.20)_2$, with $(K_s)_i$ replaced by $(\theta_i)^\circ$. The desired conclusions follow from these assertions and (since $K = |K|$) from the identity

$$(d/dt)\int_{\mathcal{c}(t)} K ds = \theta_2{}^\circ(t) - \theta_1{}^\circ(t). \quad \blacksquare \tag{7.25}$$

For a convex interface,

$$(d/dt)\int_{\mathcal{c}(t)} |K| ds = 0$$

(cf (7.8)). But one can prove more. Intuitively, dividing the motion into subcurves on which K does not change sign, and then appealing to (7.22) on each subcurve, makes the following result plausible. We will give a careful proof of this theorem as well as of the remaining results of this section in [1989ag].

Theorem.[36]

$$(d/dt)\int_{\mathcal{S}(t)} |K|ds \le 0. \tag{7.26}$$

The next result shows that an initially convex interface will remain convex for all time. To state the theorem concisely, we use the term **inflection point** for an interfacial point[37] at which the curvature changes sign.

Theorem. The number of inflection points cannot increase with time.

Remark. Roughly speaking, a finger may be defined as a section of the interface between inflection points. The last theorem and (7.22) then have the following corollary:

the total number of fingers as well as the total curvature of each finger cannot increase with time. (7.27)

The final result, essentially a consequence of (5.10) and the comparison theorem[38] for parabolic equations, shows that nested interfaces remain nested.

Theorem. Let $\Omega(t)$ and $\Omega^*(t)$, $0 \le t < T$, be reference regions for two admissible interfacial motions (corresponding to the same β, f, and F). Assume that $\Omega(0) \subset \Omega^*(0)$. Then $\Omega(t) \subset \Omega^*(t)$ for $0 \le t < T$.

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[36]Cf. Albresch and Langer [1986] for the case $V = K$.
[37]This definition makes sense: from the parabolicity of (5.10), straight line segments in the interfacial curve disappear immediately.
[38]Cf. Protter and Weinberger [1967].

III. Interfacial motions with corners.

8. Corners. The Frank diagram.

When the interfacial energy is not stable, an admissible interfacial motion must, at each time, exhibit orientations for which the evolution equations are backward parabolic. Two ways of overcoming this difficulty are: (i) to regularize the evolution equations; (ii) to allow corners which correspond to jumps in θ across the unstable portions of $f(\theta)$. Here we restrict our attention to (ii).[39]

8.1. Corners.

In this section we will make extensive use of the relation (4.9) expressing the capillary force as a function of the angle θ:

$$\mathbf{C} = \mathbf{C}(\theta) = f(\theta)\mathbf{T}(\theta) + f'(\theta)\mathbf{N}(\theta). \tag{8.1}$$

Corners are defined by a jump discontinuity in the dependence of θ on s. An immediate consequence of balance of forces (3.2) is the continuity of $\mathbf{C}(\theta(s,t))$ with respect to s. Thus at a corner defined by a jump in orientation from θ^- to θ^+ we must have $\mathbf{C}(\theta^-) = \mathbf{C}(\theta^+)$.[40] This discussion should motivate the following definition. Let θ^-, θ^+ be distinct angles with $\mathbf{C}(\theta^-) = \mathbf{C}(\theta^+)$. Then the ordered pair $\{\theta^-, \theta^+\}$ is: a **corner** if $|\theta^+ - \theta^-| < \pi$; a **cusp** if $\theta^+ - \theta^- = \pi$. One should visualize $\{\theta^-, \theta^+\}$ as representing a jump in angle from θ^- to θ^+ as arc length increases. If $\{\theta^-, \theta^+\}$ is a corner, then $\{\theta^+, \theta^-\}$ is a corner.

[39] A suitable regularized theory will be the subject of a future paper.

[40] Herring [1951a], eq. (19).

Cusp and corner theorem. Cusps are not possible. Corners are not possible when the interfacial energy is strictly stable. In fact, if $\{\theta^-,\theta^+\}$ is a corner (labelled so that $0 < \theta^+ - \theta^- < \pi$), then either f is unstable somewhere in (θ^-,θ^+) or $f(\theta) + f''(\theta) \equiv 0$ on (θ^-,θ^+).

Proof. Let $\{\theta, \theta + \pi\}$ be a cusp. Since $\mathbf{N}(\theta) = -\mathbf{N}(\theta + \pi)$ and $\mathbf{T}(\theta) = -\mathbf{T}(\theta + \pi)$, if we take the inner product of $\mathbf{T}(\theta)$ with $\mathbf{C}(\theta) = \mathbf{C}(\theta + \pi)$, we conclude, with the aid of (8.1), that $f(\theta) = -f(\theta + \pi)$, which contradicts the assumption $f > 0$.

Next, since $\mathbf{N}(\theta) = (\cos\theta, \sin\theta)$, we may use (4.17) to conclude that $\mathbf{C}(\theta^-) = \mathbf{C}(\theta^+)$ if and only if

$$\int_{\theta^-}^{\theta^+} [f(\theta) + f''(\theta)]\cos(\theta + \alpha)\,d\theta = 0$$

for all $\alpha \in \mathbb{R}$, and this implies the remaining assertions of the theorem. ∎

8.2. The Frank diagram.

In this section we will use the notation and terminology of Appendix A on polar diagrams. Here, because the interfacial energy is smooth, the polar diagrams we will encounter will not have sharp spots; for that reason we will give a direct proof of certain assertions, even though these assertions actually follow from the more general results of Appendix A.

The **Frank diagram**[41] is the polar diagram of $f(\theta)^{-1}$, and is hence the locus of the **Frank potential**

$$\boldsymbol{\sigma}(\theta) = f(\theta)^{-1}\mathbf{N}(\theta). \tag{8.2}$$

The Frank potential and the capillary force (8.1) have an interesting relationship. First of all, by (2.4),

---

[41] Frank [1963].

$$C(\theta) = -f(\theta)^2\sigma'(\theta), \tag{8.3}$$

so that the capillary force is tangent to the Frank diagram and points in the direction of decreasing θ. Further, (A1) with $g(\theta)=f(\theta)^{-1}$ and (T2) of Appendix A yield the following (cf. Figure 8A)

Theorem. The capillary force is the negative of the supporting tangent of the Frank diagram. Thus

$$C(\theta) = -\sigma^*(\theta), \tag{8.4}$$

and $|C(\theta)|^{-1}$ is the support function of the Frank diagram.

In view of of this result, $\{\theta^-,\theta^+\}$ is a corner if and only if θ_1 and θ_2 have a common supporting tangent (relative to the Frank diagram).

The next theorem[42] establishes the existence of corners for unstable interfacial energies and shows how this energy may be decomposed into stable sections separated by corners.

Convexity-stability theorem.

(i) The Frank diagram is convex if and only if f is stable.

(ii) More generally, f is stable on the globally-convex sections of the Frank diagram. If (θ^-,θ^+) is an open interval separating two adjacent globally-convex sections, then $\{\theta^-,\theta^+\}$ is a corner, and f is unstable somewhere in (θ^-,θ^+).

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[42]The ideas underlying this theorem are due to Wulff [1901], Herring [1951b], Frank [1963], Gjostein [1963], and Gruber (cf. Gjostein [1963]).

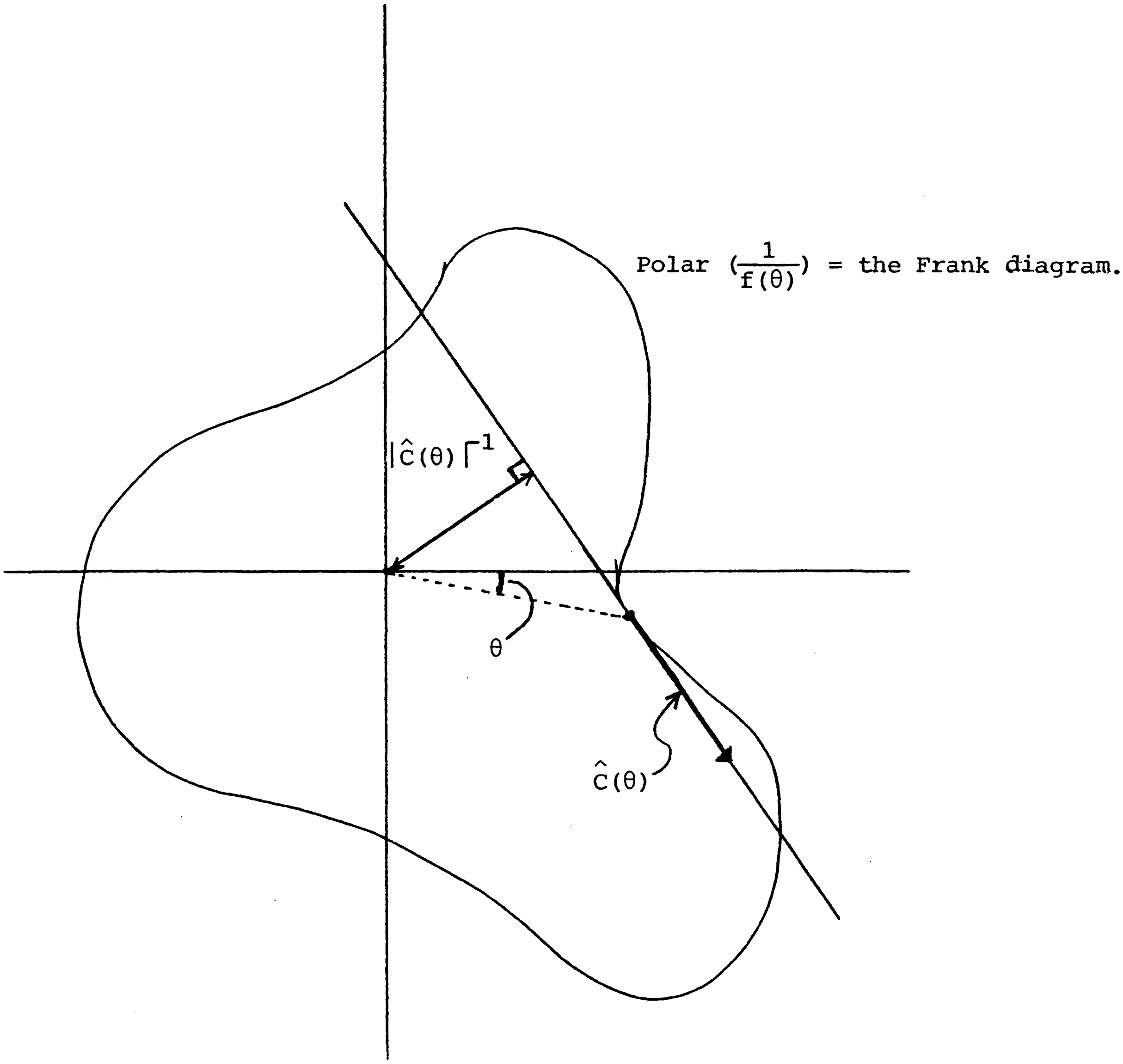


Figure 8A. The Frank diagram = Polar (f^{-1}).

The capillary force $\hat{C}(\theta)$ is the negative of the supporting tangent of the Frank diagram.

Proof. Let $k_F(\theta)$ denote the curvature of the Frank diagram, so that $k_F(\theta)$ is given by (A5) with $g(\theta) = f(\theta)^{-1}$. Then

$$k_F(\theta) \geq 0 \quad \text{if and only if} \quad f(\theta) + f''(\theta) \geq 0, \tag{8.5}$$

and (i) and the first assertion of (ii) follow.

Suppose that (θ^-,θ^+) is an open interval separating two adjacent globally-convex sections. Then (θ_1,θ_2) is an angle interval for a Maxwell line, and the final assertion of (ii) follows from (ii) of the Maxwell theorem (Appendix A). ■

This theorem should motivate the following terminology in which we use **"GS"** as shorthand for the term **"globally-stable"**. The globally-convex sections of the Frank diagram will be referred to as **GS sections of the energy**; angles θ that belong to GS sections will be referred to as **GS angles**; an open interval (θ^-,θ^+) that separates two adjacent GS sections will be referred to as a **globally-unstable section**; the corresponding corners $\{\theta^-,\theta^+\}$ and $\{\theta^+,\theta^-\}$ will be referred to as **GS corners** and the angles θ^- and θ^+ as **GS corner angles**.

We will refer to an energy f as **regular** if:

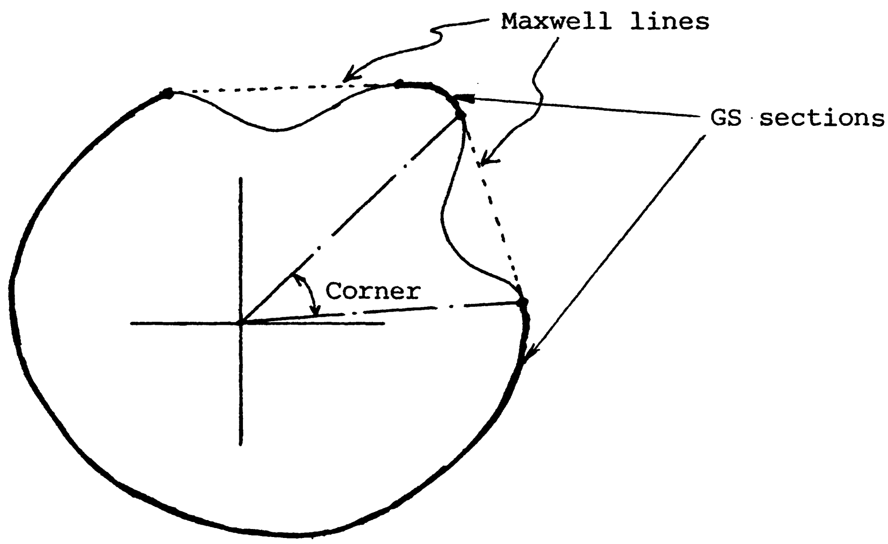
(R1) the GS sections of f are finite in number;

(R2) each GS angle θ is strictly stable,

(R3) the Maxwell lines (Appendix A) of the Frank diagram are mutually disjoint.

(Cf. Figure 8B). A consequence of the proposition containing (8.3) and the properties of the convex hull is the following alternative characterization of regular energies.

Regular Frank diagram



Irregular Frank diagram

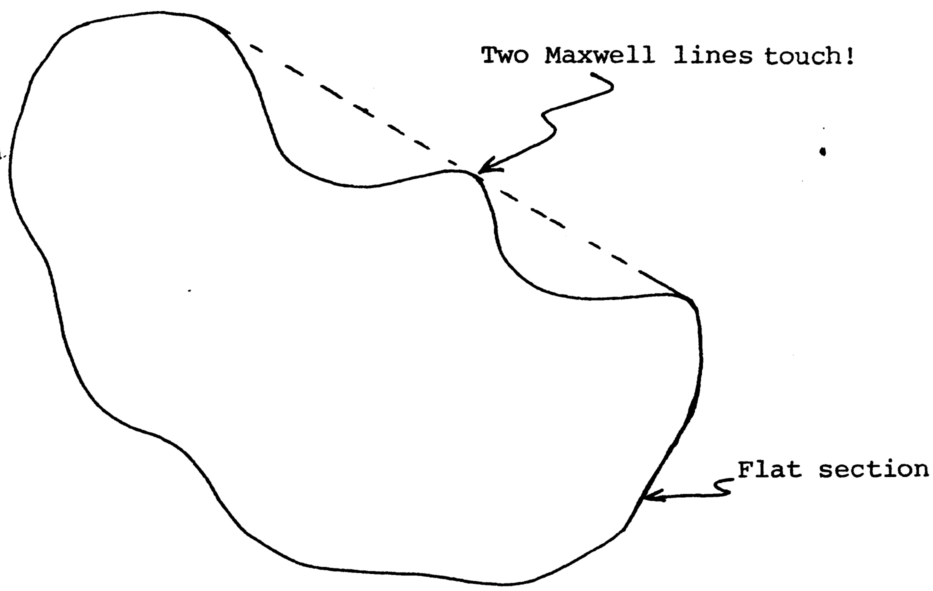


Figure 8B. Examples of Frank diagrams.

Proposition. *Regular interfacial energies have the following properties*:

(R4) GS *sections are not singletons*.

(R5) *Let* $\{\theta^-,\theta^+\}$ *be a corner with* θ^- *and* θ^+ GS *angles*. *Then* $\{\theta^-,\theta^+\}$ *is a* GS *corner, and* (*aside from* $\{\theta^+,\theta^-\}$) *there is no other corner involving* θ^- *or* θ^+.

In fact, granted (R1) *and* (R2), (R3) *is equivalent to either* (R4) *or* (R5).

9. Unstable interfacial energies. Motions with corners.

Consider now an energy f such that

$$f \text{ \underline{is not stable, but regular}.} \tag{9.1}$$

It seems reasonable to consider "motions" in which the interface at each time is a piecewise-smooth closed curve whose regular arcs and "corners", respectively, correspond to GS sections and GS corners of f. For such an "interfacial motion" the evolution equations are parabolic, since the nonparabolic portions are removed by corners, but the positions of the corners as functions of time are not known a-priori and hence constitute free boundaries. The next section begins our discussion of such motions.

9.1. Corner conditions for piecewise-smooth evolving curves that are admissible.

Let $\mathbf{r}=\{\mathbf{r}_1,\mathbf{r}_2,\dots\}$ be a PS (piecewise-smooth) evolving curve. We will use the notation and terminology of Section 2.4; thus $\mathbf{R}_i(t)$ and $\mathbf{R}_i^\circ(t)$ are the position and total velocity of the juncture i; s is the arc length; $\mathbf{T}(s,t)$, $\mathbf{N}(s,t)$, $\theta(s,t)$, $K(s,t)$, $V(s,t)$, and $v(s,t)$ are the tangent, normal, angle, curvature, normal velocity, and arc velocity.

Our interest is in PS evolving curves that are consistent with balance of capillary forces and have arcs which correspond to GS sections of the energy. Such curves are the subject of the next definition.

Let $\mathbf{r}$ be a PS evolving curve. Then $\mathbf{r}$ is **admissible** if:[43]
(A1) $\theta(s,t)$ is always GS;

[43]There is a slight ambiguity in our use of the word "admissible": in Section 4.2, which concerned smooth evolving curves, admissibility meant consistency with balance of capillary forces and the underlying constitutive equations. Here and in what follows we require, in addition, that admissible evolving curves have angles $\theta(s,t)$ that are GS.

(A2) the capillary force $\mathbf{C}(\theta)$ defined by (8.1) is consistent with capillary balance in the form (cf.(4.10))

$$\int_{\partial c(t)} \mathbf{C}(\theta) = \int_{c(t)} (F + \beta(\theta)V)\mathbf{N}\,ds; \tag{9.2}$$

whenever c is the trace of an evolving subcurve of $\mathbf{r}$.

Given an admissible PS evolving curve, smoothness dictates that on each evolving arc the angle $\theta(s,t)$ belong to a single GS section. Further, we may, without loss in generality, assume that on adjacent arcs $\theta(s,t)$ belongs to different GS sections; were this not the case $\theta(s,t)$ would be continuous across the juncture of the arcs, and the arcs may be combined to form a single (smooth) evolving curve.

Alternative characterization of admissibility. <u>Let</u> $\mathbf{r}$ <u>be</u> <u>a</u> PS <u>evolving</u> <u>curve</u> <u>consistent</u> <u>with</u> (A1). <u>Then</u> $\mathbf{r}$ <u>is</u> <u>admissible</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u>:

(A3) <u>each</u> <u>of</u> <u>the</u> <u>arcs</u> $\mathbf{r}_i$ <u>evolves</u> <u>according</u> <u>to</u>

$$V = \Phi(\theta)K - \Psi(\theta), \qquad \Phi(\theta) = [f(\theta) + f''(\theta)]/\beta(\theta), \qquad \Psi(\theta) = F/\beta(\theta); \tag{9.3}$$

(A4) <u>each</u> <u>of</u> <u>the</u> <u>pairs</u> $\{\theta(S_i(t)-0,t), \theta(S_i(t)+0,t)\}$ (i <u>a</u> <u>juncture</u>) is <u>independent</u> <u>of</u> <u>time</u> <u>and</u> <u>is</u> <u>a</u> GS <u>corner</u> <u>of</u> <u>the</u> <u>interfacial</u> <u>energy</u>.

<u>Proof</u>. Note that, by Remark (ii) of Section 4.1,

$$\text{(A2) is equivalent to (A3) and the continuity in } s \text{ of } \mathbf{C}(\theta(s,t)). \tag{9.4}$$

Thus, in view of the definition of a corner, (A3) and (A4) yield admissibility. Conversely, suppose $\mathbf{r}$ is admissible. Then, by (9.4), (A3) holds and each $\{\theta(S_i(t)-0,t), \theta(S_i(t)+0,t)\}$ is a corner.

Thus, in view of (9.1) and the last proposition of Section 8.2, (A4) follows. ■

In view of (A4), the corner-angles

$$\theta_i^{\pm} := \theta(S_i(t) \pm 0, t) \tag{9.5}$$

are <u>constants</u>, as are the quantities (cf. (2.40))

$$\kappa_i = \mathbf{T}(\theta_i^+) \cdot \mathbf{N}(\theta_i^-) = -\mathbf{T}(\theta_i^-) \cdot \mathbf{N}(\theta_i^+). \tag{9.6}$$

It is often convenient to use (θ,t) as independent variables on a given convex arc, irrespective of the sign the curvature takes on the other arcs.

Corner conditions for admissible PS evolving curves. <u>At each juncture</u> i:

(CC1) <u>the capillary force</u> $\mathbf{C}(s,t)$ <u>is continuous across</u> $s = S_i(t)$;

(CC2) $K(S_i(t) \pm 0,t)\kappa_i \geq 0$, $\quad K(S_i(t)-0,t)K(S_i(t)+0,t) \geq 0$;

(CC3) $V_s(S_i(t) \pm 0,t) = [v(S_i(t) \pm 0,t) - S_i^{\circ}(t)]K(S_i(t) \pm 0,t)$;

(CC4) <u>if the evolving arcs</u> $\mathbf{r}_i$ <u>and</u> $\mathbf{r}_{i+1}$ <u>are convex, then</u> $V(\theta,t)\mathbf{N}(\theta) - V_\theta(\theta,t)\mathbf{T}(\theta,t)$ <u>is continuous across</u> $\{\theta_i^-, \theta_i^+\}$ <u>and its value at</u> $\theta = \theta_i^{\pm}$ <u>is the total velocity</u> $\mathbf{R}_i^{\circ}(t)$.

<u>Proof</u>. (9.4) yields (CC1), (CC3) follows from $(2.20)_2$ and (A4), and (CC4) follows from $(2.42)_1$, (2.24), and (2.18). We have only to prove (CC2). Since $\{\theta_i^-, \theta_i^+\}$ is a GS corner of f, one of the intervals (θ_i^-, θ_i^+), (θ_i^+, θ_i^-) is a globally unstable section of f. Assume that (θ_i^-, θ_i^+) is such a section. Then, by property (ii) of admissible PS evolving curves, $\theta \leq \theta_i^-$ on ς_i, while $\theta \geq \theta_i^+$ on ς_{i+1}, so that $K(S_i(t) \pm 0,t) \geq 0$. On the other hand, since $0 < \theta_i^+ - \theta_i^- < \pi$, (9.6) implies that $\kappa_i \geq 0$, so that (CC2) is satisfied. A similar argument applies when (θ_i^+, θ_i^-) is a globally

unstable section of f. ■

Remark. It should be emphasized that the corner inequalities (CC2) are based on the hypothesis that the underlying arcs correspond to adjacent GS sections of the interfacial energy. These inequalities imply that corners preserve local convexity. In particular (cf. (2.40)),

$$\left.\begin{array}{l}\text{if two adjacent arcs are convex, then their}\\ \text{curvatures as well as the transition curvature}\\ \text{of the corner between them are of the same sign.}\end{array}\right\} \quad (9.7)$$

9.2. Facetings. Evolving curves with wrinkles.

Let $\mathbf{r}=\{\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N\}$ be an admissible PS evolving curve. Then $\mathbf{r}$ is a **faceting** if each of its arcs is a facet, and if no two adjacent arcs combine to form a single facet. On each facet, $\theta(s,t)$ is independent of s; if $\theta(s,t)\equiv\text{constant}$ on each facet, then $\mathbf{r}$ has **fixed orientations**. An example of a faceting with fixed orientations is a **wrinkling** (Figure 9A); here there are fixed angles θ_{odd} and θ_{even} such that, for all t,

$$\begin{aligned}\theta(s,t) &= \theta_{odd} \quad \text{on } \mathcal{S}_i(t) \text{ for i odd,}\\ \theta(s,t) &= \theta_{even} \quad \text{on } \mathcal{S}_i(t) \text{ for i even;}\end{aligned} \qquad (9.8)$$

in this case $\mathbf{r}$ is a **wrinkling between angles** θ_{odd} and θ_{even}. Note that, by $(9.3)_1$, for a wrinkling the normal velocity V of each facet is constant (in space and time) with

$$\begin{aligned}V &= -\Psi(\theta_{odd}) \quad \text{on } \mathcal{S}_i \text{ for i odd,}\\ V &= -\Psi(\theta_{even}) \quad \text{on } \mathcal{S}_i \text{ for i even.}\end{aligned} \qquad (9.9)$$

By definition, on a facet, $\theta(t)=\theta(s,t)$ is independent of s. Therefore its normal velocity is given by $V=-\Psi(\theta(t))$ and $V_s=0$. Thus, by $(2.18)_2$, $\theta^\circ=0$; since $\theta_t=\theta^\circ-v\theta_s$, $\theta_t=0$. Thus the angle θ

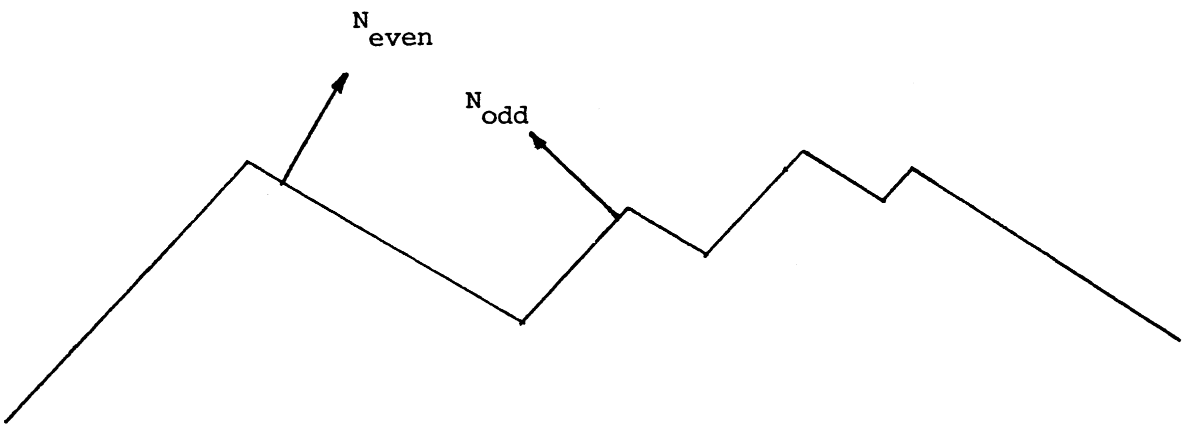


Figure 9A. Wrinkling between the angles θ_{odd} and θ_{even}. Here $N_{odd} = N(\theta_{odd})$ and $N_{even} = N(\theta_{even})$.

is identically constant on a facet. Consider now a faceting. Each internal facet meets two corners, $\{\alpha^-,\alpha^+\}$ and $\{\theta^-,\theta^+\}$, say. Since the orientation of the facet is constant, $\alpha^+=\theta^-$. Our assumption that the energy be regular then implies that $\alpha^-=\theta^+$ (cf. (R5)). Thus and by (A1) we have the following

Proposition.[44] The only possible facetings are wrinklings between the fixed angles of a GS corner.

Let $\mathbf{r}_A$ and $\mathbf{r}_B$ be admissible evolving curves, and let $\mathbf{r}=\{\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N\}$ be a faceting. Then $\mathbf{r}$ **connects** $\mathbf{r}_A$ and $\mathbf{r}_B$ if $\{\mathbf{r}_A,\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N,\mathbf{r}_B\}$ is an admissible PS evolving curve. The next theorem, the main result of this section, shows that wrinkles decay with time.

Wrinkle-decay theorem. Let $\mathbf{r}$ be a wrinkling that connects admissible evolving curves $\mathbf{r}_A$ and $\mathbf{r}_B$. Then the total length of the wrinkling decreases with time. In fact, the lengths of the initial and terminal facets decrease with time, while the lengths of the internal facets remain constant.

Our proof is based on two subsidiary results.

Proposition. The length of each internal facet of a wrinkling is independent of time. Thus (since the orientation of each facet is fixed) the system of internal facets behaves as a rigid body undergoing translational motion.

Proof. Let $\mathbf{r}_i$ be an internal facet (Figure 9B). The two adjacent facets $\mathbf{r}_{i\pm1}$ have the same normal velocity $V_{i\pm1}=-\Psi(\theta_{i\pm1})$, so that the distance d between them does not change with time. The middle facet $\mathbf{r}_i$ will move relative to its two neighbors, but it

[44]Here the smoothness of $f(\theta)$ is crucial. In Section 10.3 we will show that certain nonsmooth energies exhibit more general types of facetings.

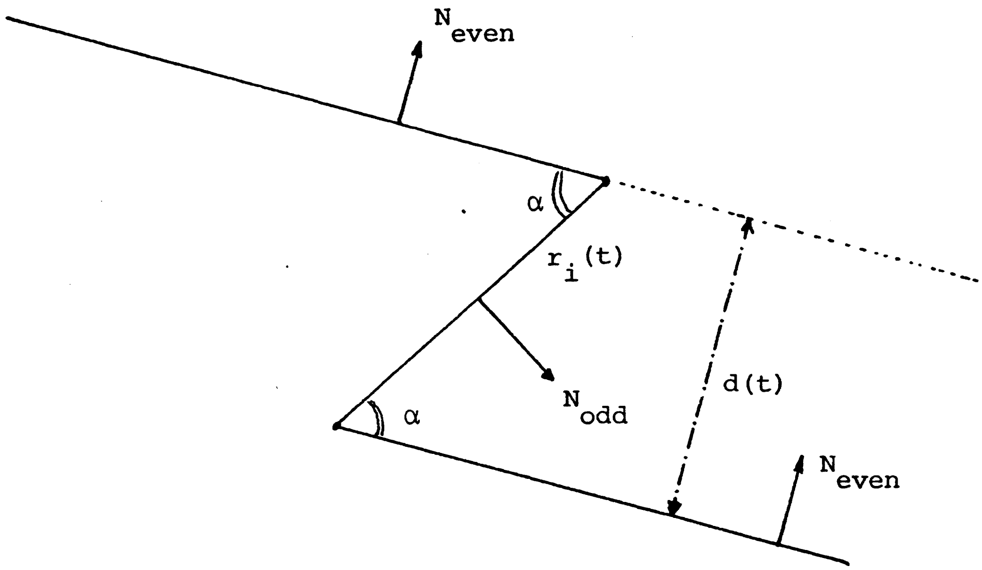


Figure 9B. The length of an internal facet $r_i(t)$ does not change with time.

will always meet them at the fixed angle $|\theta_i - \theta_{i-1}| = \alpha$. Therefore its length $L(t) = d/\sin\alpha$ is also independent of time. ■

Let $\mathbf{r} = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$ be a wrinkling that connects admissible evolving curves $\mathbf{r}_A$ and $\mathbf{r}_B$. It is convenient to use the following notation: $\mathbf{R}_A^\circ$, θ_A, $\mathbf{T}_A$, $\mathbf{N}_A$, K_A, and V_A, respectively, denote the total velocity, orientation, tangent, normal, curvature, and normal velocity of the **terminal** point of $\mathbf{r}_A$; an analogous notation applies to the corresponding quantities associated with the **initial** point of $\mathbf{r}_B$; κ_A denotes the corner curvature between $\mathbf{r}_A$ and the initial facet; κ_B denotes the corner curvature between the terminal facet and $\mathbf{r}_B$; L_A and L_B, respectively, denote the lengths of the initial and terminal facets; given any function $g(\theta)$, we write $g_{odd} = g(\theta_{odd})$, $g_{even} = g(\theta_{even})$, with θ_{odd} and θ_{even} as in (9.8).

Properties of wrinklings that connect evolving curves. <u>Let</u> $\mathbf{r} = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$ <u>be a wrinkling that connects admissible evolving curves</u> $\mathbf{r}_A$ <u>and</u> $\mathbf{r}_B$. <u>Then</u>:

<u>for</u> N <u>even</u>:

$$\begin{aligned} \theta_A &= \theta_{even}, & \mathbf{R}_A^\circ \cdot \mathbf{N}_{odd} &= -\Psi_{odd}, \\ \theta_B &= \theta_{odd}, & \mathbf{R}_B^\circ \cdot \mathbf{N}_{even} &= -\Psi_{even}; \end{aligned} \tag{9.10}$$

<u>for</u> N <u>odd</u>:

$$\theta_A = \theta_B = \theta_{even}, \quad \mathbf{R}_A^\circ \cdot \mathbf{N}_{odd} = \mathbf{R}_B^\circ \cdot \mathbf{N}_{odd} = -\Psi_{odd}; \tag{9.11}$$

<u>in either case</u>:

$$\begin{aligned} &\Phi(\theta_A)K_A = -\kappa_A L_A^\circ, \quad \Phi(\theta_B)K_B = -\kappa_B L_B^\circ \quad \text{for} \quad N \geq 2, \\ &\Phi(\theta_A)[K_A - K_B] = -\kappa_A L_A^\circ \qquad\qquad\qquad\; \text{for} \quad N = 1. \end{aligned} \tag{9.12}$$

Proof. We will establish only those results which concern A. The conclusions concerning B are verified analogously. Assume that $N \geq 2$. Then the corner condition $(2.42)_1$, applied at $s = S_A(t)$ to the corner between $\mathbf{r}_A$ and $\mathbf{r}_1$ and at $s = S_1(t)$ to the corner between $\mathbf{r}_1$ and $\mathbf{r}_2$, yields the relations

$$\mathbf{R}_A{}^\circ = -\Psi_{odd}\mathbf{N}_{odd} + (S_A{}^\circ - v_1)\mathbf{T}_{odd},$$
$$-\Psi_{odd}\mathbf{N}_{odd} + (S_1{}^\circ - v_1)\mathbf{T}_{odd} = -\Psi_{even}\mathbf{N}_{even} + (S_1{}^\circ - v_2)\mathbf{T}_{even}.$$

Thus $\mathbf{R}_A{}^\circ \cdot \mathbf{N}_{odd} = -\Psi_{odd}$. Further, since $V_A = \mathbf{R}_A{}^\circ \cdot \mathbf{N}_{even}$ and $\kappa_A = \mathbf{T}_{odd} \cdot \mathbf{N}_{even}$ (cf. (9.6)), the above relations yield

$$V_A = -\Psi_{odd}(\mathbf{N}_{odd} \cdot \mathbf{N}_{even}) + (S_A{}^\circ - v_1)\kappa_A,$$
$$-\Psi_{odd}(\mathbf{N}_{odd} \cdot \mathbf{N}_{even}) + (S_1{}^\circ - v_1)\kappa_A = -\Psi_{even}.$$

Thus $V_A = -\Psi_{even} + (S_A - S_1)^\circ \kappa_A$. But $L_A = S_1 - S_A$ and, by (9.3), $V_A + \Psi_{even} = \Phi(\theta_A)K_A$; thus $\Phi(\theta_A)K_A = -\kappa_A L_A{}^\circ$.

We have established all of the results for A except $(9.12)_2$. Thus let $N = 1$. Applying $(2.42)_1$ to the corners at $s = S_A(t)$ and $s = S_B(t)$, taking the inner product of the resulting relations with $\mathbf{N}_{even}$, and then subtracting the two relations yields $V_A - V_B = (S_A - S_B)^\circ \kappa_A$. But $\theta_A = \theta_B$ and $L_A = L_B = S_B - S_A$; in view of (9.3), $(9.12)_2$ follows. ■

Proof of the wrinkle-decay theorem. For $N \geq 2$ the proof follows from (CC2), $(9.12)_1$, and the proposition following the wrinkle-decay theorem. For $N = 1$, (9.6) yields $\kappa_B = -\kappa_A$, so that $(K_A - K_B)/\kappa_A = (K_A/\kappa_A) + (K_B/\kappa_B)$; (CC2) and $(9.12)_2$ then yield $L_A{}^\circ \leq 0$. ■

9.3. Curves that are convex except for wrinkles.

A **PS admissible evolving curve is convex** if each of its arcs is convex. The remark (9.7) renders this definition meaningful. In particular, at each t the corresponding PS curve $\mathfrak{s}(t)$ is convex in the usual sense for continuous curves. For convenience, we write

"CPS" as an abbreviation for "convex piecewise-smooth".

An admissible PS evolving curve **r** is **convex except for wrinkles** if **r** consists of CPS evolving curves, called **convex sections**, connected by wrinklings. Each convex section is the union of convex arcs (which are smooth). A convex section $\mathbf{r}_C$ is **internal** if **r** is closed, or if the initial and terminal arcs of $\mathbf{r}_C$ are internal arcs of **r**; in this case the initial and terminal points of $\mathbf{r}_C$ connect to wrinklings. It is generally most convenient to take (θ,t) as independent variables on each convex section and to express the underlying system of evolution equations in terms of the normal velocity $V(\theta,t)$. For an internal convex section, (5.7), (CC4), (9.10), and (9.12), yield an interesting system of equations for $V(\theta,t)$.

Evolution equations for convex sections. Let **r** be a PS evolving curve that is closed, admissible, and convex except for wrinkles. Consider a convex section of **r**, let θ_A and θ_B, respectively, denote the angles corresponding to the initial and terminal points of the section, and let $\{\gamma_A,\theta_A\}$ and $\{\theta_B,\gamma_B\}$ denote the corners at these terminal points. Then the evolution of $V(\theta,t)$ for this section is governed by the following conditions:

(E1) on each convex arc:

$$\Phi(\theta)V_t = [V + \Psi(\theta)]^2[V_{\theta\theta} + V];$$

(E2) $V\mathbf{N} - V_\theta\mathbf{T}$ is continuous across corners separating convex arcs;

(E3) at the initial and terminal points:

$$[V(\theta_A,t)\mathbf{N}(\theta_A) - V_\theta(\theta_A,t)\mathbf{T}(\theta_A)]\cdot\mathbf{N}(\gamma_A) = -\Psi(\gamma_A),$$

$$[V(\theta_B,t)\mathbf{N}(\theta_B) - V_\theta(\theta_B,t)\mathbf{T}(\theta_B)]\cdot\mathbf{N}(\gamma_B) = -\Psi(\gamma_B).$$

Remark. It seems reasonable to expect that (E1)-(E3), with compatible initial data, yield a well-posed problem for $V(\theta,t)$ on each convex section. Thus wrinklings between convex sections essentially decouple these sections from each other, at least until the wrinkles decay. If $V(\theta,t)$ and (hence) $K(\theta,t)$ are known on each section, then the evolution of the wrinkles (from prescribed initial positions) is easily determined using (9.12) and properties (i) and (ii) of wrinklings. Granted this is done, the initial and terminal positions of the convex sections are known as functions of time, and this data and a knowledge of the corresponding curvatures yields the complete evolutionary behavior of the convex sections, and hence of the complete evolving curve.

9.4. Equations near a corner when the curve is a graph.

Consider the situation shown in Figure 9C, in which an evolving PS curve is represented, in a neighborhood of a corner $\{\theta^-,\theta^+\}$, as the graph of a function $y=h(x,t)$. Here x ranges in an interval (x_0,x_1); $t\in[0,T)$; $x=\zeta(t)$ is the position at time t of the corner; and the curve is oriented so that arc length increases with increasing x. Then $h(x,t)$ is continuous and piecewise smooth, with a jump discontinuity in $p=h_x$ at the **free boundary** $x=\zeta(t)$.

The function p satisfies (5.11) away from the free boundary and is consistent with two free-boundary conditions. The first of these, a direct consequence of (5.9), is given by

$$p(\zeta(t)\pm 0,t) = P^{\pm} \tag{9.13}$$

with

$$P^{\pm} = -\cot\theta^{\pm}.$$

The second condition is more complicated. For any function $\phi(x,t)$, let $\phi^{\pm}(t)=\phi(\zeta(t)\pm 0,t)$. By $(2.20)_1$,

$$V^{\pm} = \mathbf{R}^{\circ}\cdot\mathbf{N}^{\pm}; \tag{9.14}$$

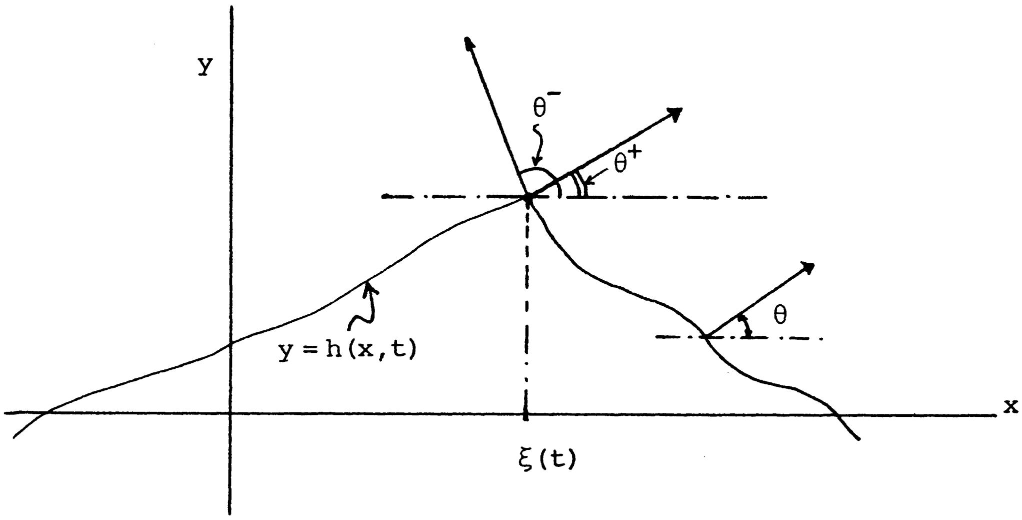


Figure 9C. A corner when the evolving curve is a graph $y = h(x,t)$; $x = \xi(t)$ marks the corner.

thus, since $\mathbf{N} = (\cos\theta, \sin\theta)$, if we eliminate the y-component of $\mathbf{R}^\circ$ between the two equations (9.14), and use the fact that ζ° is the x-component of $\mathbf{R}^\circ$, we arrive at an expression for ζ° as a function of $V^\pm$ and $\theta^\pm$; the expressions $V = \Phi K - \Psi$ and $K = p_x(1 + p^2)^{-3/2}$ then lead to the free-boundary condition:

$$\zeta^\circ = A^+(p_x)^+ - A^-(p_x)^- - C \tag{9.15}$$

with $A^\pm$ and C the constants defined by

$$A^\pm = \Phi(\theta^\pm)D^\pm/W^\pm, \quad C = \Psi(\theta^+)D^+ - \Psi(\theta^-)D^-.$$
$$W^\pm = (1 + (P^\pm)^2)^{3/2}, \quad D^+ = \sin\theta^-/a, \quad D^- = \sin\theta^+/a,$$
$$a = \sin(\theta^- - \theta^+).$$

The basic system of equations then consists of (5.11) away from $x = \zeta(t)$ supplemented by (9.13) and (9.15) at $x = \zeta(t)$. A change in dependent variable renders this system more transparent. Thus let

$$u(x,t) = \begin{cases} A^-(p(x,t) - P^-) & \text{for } x < \zeta(t) \\ A^+(p(x,t) - P^+) & \text{for } x > \zeta(t), \end{cases}$$

so that $u(x,t)$ is continuous across $x = \zeta(t)$. Further, let Q and B be as specified in (5.10), and define

$$Q^\pm(u) = Q((u/A^\pm) + P^\pm), \qquad B^\pm(u) = A^\pm B((u/A^\pm) + P^\pm).$$

Then the system under consideration reduces to the <u>partial differential equations</u>

$$\begin{aligned} u_t &= [Q^-(u)u_x - B^-(u)]_x \quad \text{for } x < \zeta(t), \\ u_t &= [Q^+(u)u_x - B^+(u)]_x \quad \text{for } x > \zeta(t), \end{aligned} \tag{9.16}$$

in conjunction with the free-boundary conditions

$$u(\zeta(t)\pm 0,t) = 0,$$
$$u_x(\zeta(t)+0,t) - u_x(\zeta(t)-0,t) = \zeta^{\circ}(t) + C. \qquad (9.17)$$

Apart from the constant C, which may be transferred from (9.17) to (9.16) by the coordinate change $x^* = x + Ct$, (9.17) are exactly the free-boundary conditions of the classical Stefan problem.

9.5. Stationary interfaces and steady interfacial motions with corners.

Stationary interfaces and steady interfacial motions are defined as for smooth interfaces,[45] except that we now add the requirement that all angles be GS angles and all corners GS corners.

An argument analogous to that given in Section 6.1 then implies that $\mathbf{r}(\theta)$ defined on the set of GS angles θ by

$$\mathbf{r}(\theta) = F^{-1}[f(\theta)\mathbf{N}(\theta) - f'(\theta)\mathbf{T}(\theta)]$$

yields a closed, convex stationary interface that is PS. (The remark given in the last paragraph of Section 6.1 and the fact that θ jumps only at GS corners implies that $\mathbf{r}(\theta)$ is continuous across such jumps).

We also have the possibility of steady motions with corners. Let $F \neq 0$ be given. Let $\{\theta_1,\theta_2\}$, $0 < \theta_2 - \theta_1 < \pi$, be a GS corner, let ℓ be a line with ℓ a chord for Polar(β) between θ_1 and θ_2 (if β is strictly convex, then exactly one such line exists), and compute $\mathbf{U}$ by the requirement that $-(F/|\mathbf{U}|^2)\mathbf{U}$ be the support vector for ℓ (cf. Section 6.2). Then any (stationary) infinite wrinkling in which θ jumps back and forth between θ_1 and θ_2 is a portrait of a steady interfacial motion with steady velocity $\mathbf{U}$.

[45]To begin with, our definition of a PS evolving curve is restricted to bounded curves, but the extension to unbounded curves is straightforward.

One might refer fo this motion as a <u>steady wrinkling</u> (Figure 9D).

Other solutions are possible. For example, when the Frank diagram and the polar diagram of β are of the form shown in Figure 9E, there is a steadily receding bump (with a corner) as shown. Similarly, one can construct advancing bumps with corners for nonconvex Frank and β-diagrams of certain prescribed shapes.

9.6. Existence.

Let Ω_0 be a bounded, simply connected domain in $\mathbb{R}^2$. Then $\partial\Omega_0$ is a **GS boundary** if it is piecewise $C^{2+\alpha}$ for some $\alpha\in(0,1)$, if its outward normal always points in a GS direction, and if all of its corners are GS corners (cf. Section 8.2).

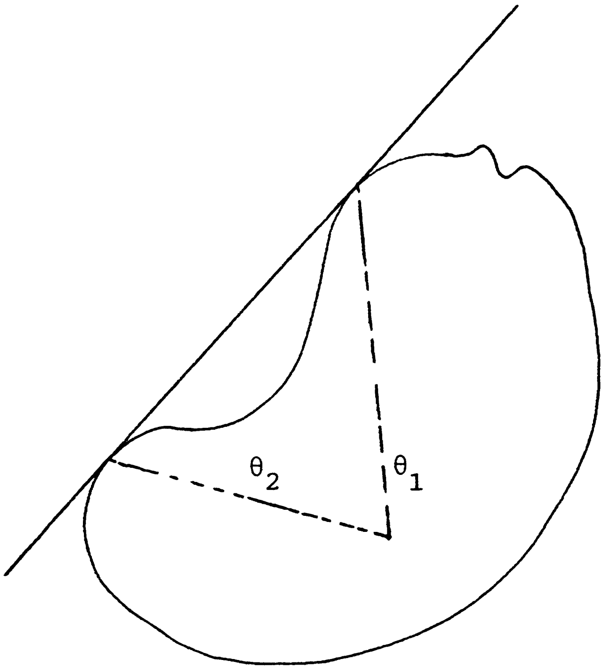
Existence theorem.[46] <u>Let</u> f <u>and</u> β <u>be</u> C^∞. <u>Let</u> Ω_0 <u>be a given initial domain, assumed to be admissible</u>. <u>Then there is a unique, maximal family of domains</u> $\Omega(t)$ $(0 \le t < T_{max})$ <u>such that</u>:
(i) $\partial\Omega(t)$ <u>is an admissible PS evolving curve</u>;
(ii) $\Omega(0) = \Omega_0$.
<u>In fact, this evolving curve is piecewise</u> C^∞ <u>for</u> $0 < t < T_{max}$.

Further, for $T_{max} < \infty$, as $t \to T_{max}$ either (E1) or (E2) (of Section 7.1) must hold, or
(E3) an arc of $\partial\Omega(t)$ shrinks to zero.
The condition (E3) is possible with the facets of a wrinkling and with arcs whose initial and terminal points correspond to the same corner angle.

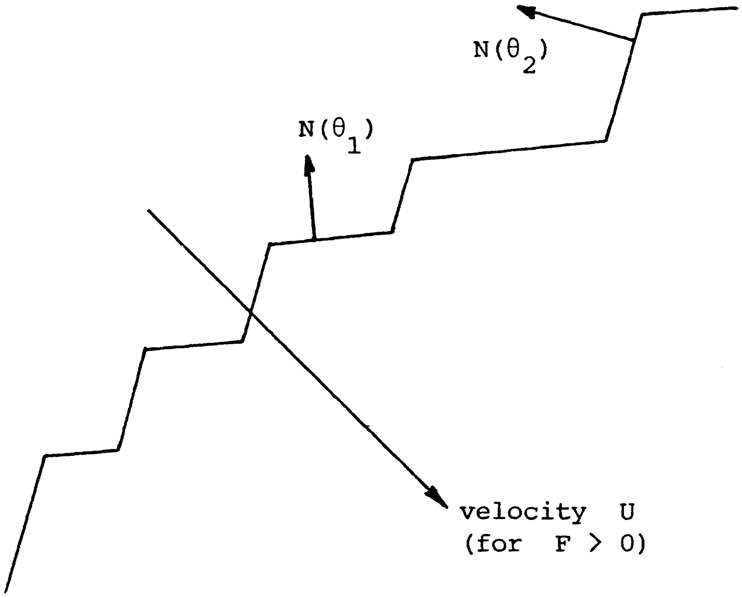
9.7. A note on regularized equations.

Another method of treating situations in which the evolution equations are backward parabolic is to develop a suitable regularization of these equations. Such a regularization will be discussed elsewhere; under certain simplifying assumptions (among them that $\beta = 1$) this regularization reduces to the following fourth-

[46] Angenent and Gurtin [1989].



Frank diagram



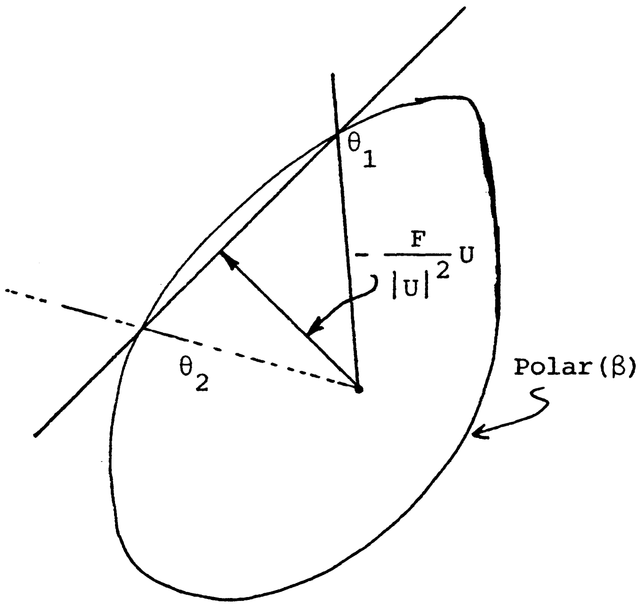


Figure 9D. Construction of a steady wrinkling.

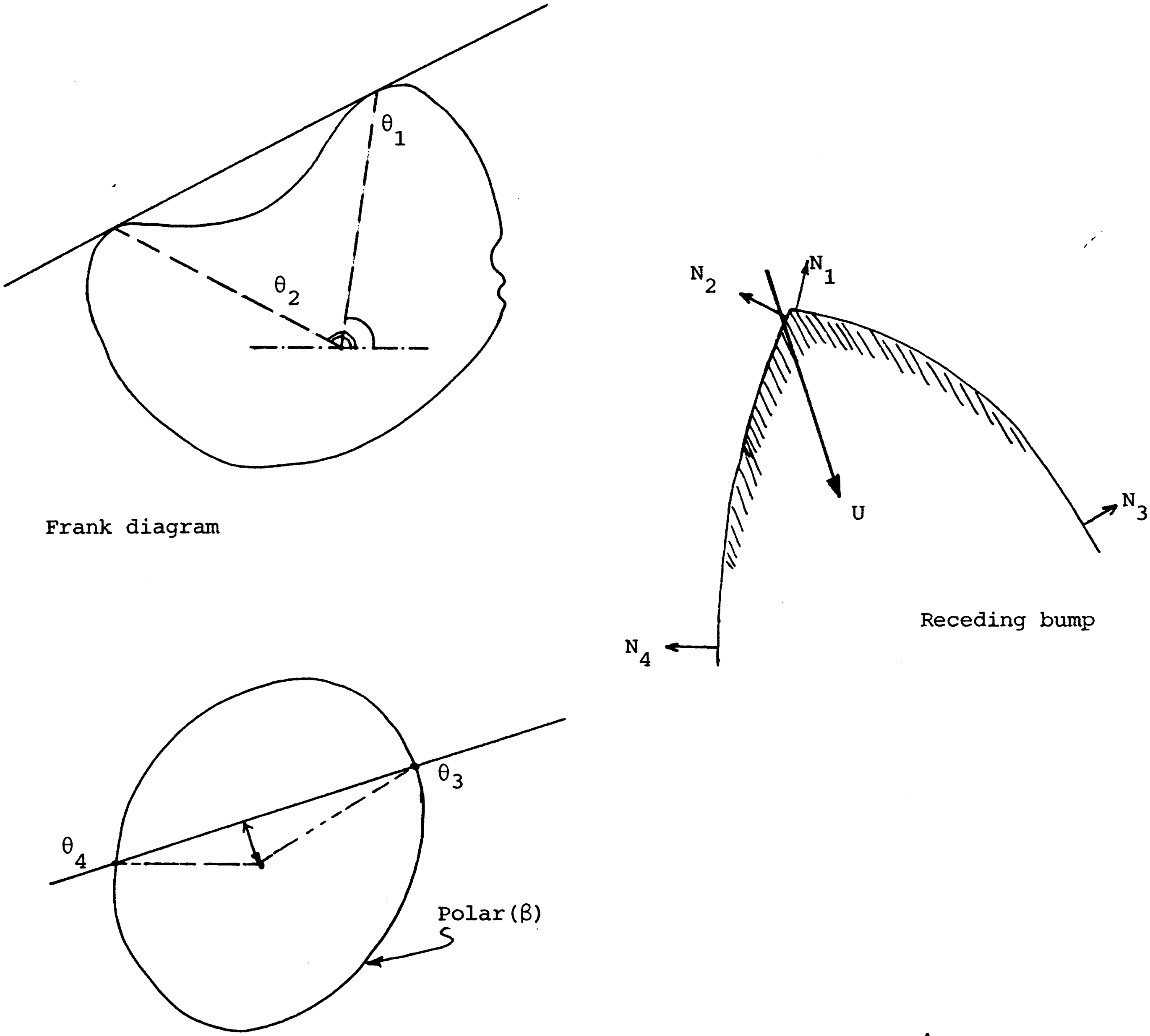


Figure 9E. Construction of a receding bump.

order parabolic system for a convex section:

$$K_t = K^2(V_{\theta\theta} + V),$$
$$V = \Phi(\theta)K - \varepsilon K[(K^2)_{\theta\theta} + K^2] - F,$$

with $\varepsilon > 0$ a small constant.

IV. Nonsmooth interfacial energies.

10. Interfacial energies with sharp spots.

Material scientists often consider interfacial energies that are continuous but have derivatives which suffer jump discontinuities.[47] We now discuss energies of this type.

10.1. Sharp spots. The capillary set. Stability.

By an **interfacial energy with sharp spots**[48] we mean a 2π-periodic, strictly-positive function $f(\theta)$ on $\mathbb{R}$ that is smooth except for sharp spots (Appendix A). The **sharp spots** are then the angles across which $f'(\theta)$ jumps, the remaining angles are **smooth spots**. Tacit in this definition is the requirement that the number of sharp spots be nonzero.

Let θ_0 be a sharp spot. By (8.1), the capillary force[49] $\mathbf{C}\hat{}(\theta)$ is discontinuous across θ_0: the tangential component $\sigma = f(\theta)$ is continuous, but the normal component $\xi = f'(\theta)$ is not:

$$\mathbf{C}\hat{}(\theta_0 + 0) - \mathbf{C}\hat{}(\theta_0 - 0) = [f'(\theta_0 + 0) - f'(\theta_0 - 0)]\mathbf{N}(\theta_0).$$

If we think of the energy at a sharp spot as the limit of a sequence of smooth, locally-convex energies, then it seems reasonable to allow the capillary shear ξ at θ_0 to have values between the two extremes $f'(\theta_0 \pm 0)$. With this in mind, we define the **capillary set** $\{\mathbf{C}\hat{}(\theta_0)\}$ at θ_0 to be the vector fan between $\mathbf{C}\hat{}(\theta_0 - 0)$ and

[47] Cf., e.g., Herring [1951ab], Cahn and Hoffman [1974].

[48] We use this terminology to avoid confusion: a sharp spot marks a loss in smoothness for the interfacial energy; corners denote jumps in orientation that are consistent with capillary balance, and hence denote possible discontinuous tangencies for an evolving curve.

[49] It is convenient to write $\mathbf{C}\hat{}(\theta)$, rather than $\mathbf{C}(\theta)$, for the capillary force defined by (8.1) away from sharp spots, and to reserve $\mathbf{C}(s,t)$ for the capillary force on an evolving curve.

$C^\wedge(\theta_0+0)$; $\{\mathbf{C}^\wedge(\theta_0)\}$ is thus the set of all vectors of the form

$$\mathbf{C} = f(\theta_0)\mathbf{T}(\theta_0) + \xi\mathbf{N}(\theta_0) \tag{10.1}$$

with ξ in the closed interval bounded by the numbers $f'(\theta_0-0)$ and $f'(\theta_0+0)$. It is convenient to also use this terminology at smooth spots, in which case the capillary set is the singleton $\{C^\wedge(\theta)\}$. The proposition following (8.4) then generalizes:

Proposition. The capillary set is the negative of the supporting tangent-fan of the Frank diagram:

$$\{\mathbf{C}^\wedge(\theta)\} = -\{\boldsymbol{\sigma}^*(\theta)\}. \tag{10.2}$$

There is a natural extension of the notions of **corners** and **cusps**: we simply replace the condition $\mathbf{C}^\wedge(\theta^-)=\mathbf{C}^\wedge(\theta^+)$ by

$$\{\mathbf{C}^\wedge(\theta^-)\}\cap\{\mathbf{C}^\wedge(\theta^+)\}\neq\emptyset;$$

the theorem on common tangents (Appendix A) then implies that θ^- and θ^+ have a common supporting tangent:

Corner-force theorem. Cusps are not possible. If $\{\theta^-,\theta^+\}$ with $0<\theta^+-\theta^-<\pi$ is a corner, then $\{\mathbf{C}^\wedge(\theta^-)\}\cap\{\mathbf{C}^\wedge(\theta^+)\}$ consists of exactly one vector $\mathbf{a}$, and $\mathbf{a}$ points in the same direction as $\boldsymbol{\sigma}(\theta^-)-\boldsymbol{\sigma}(\theta^+)$. We write $\mathbf{a}=\mathbf{C}^\wedge(\theta^-,\theta^+)$ and refer to $\mathbf{C}^\wedge(\theta^-,\theta^+)=\mathbf{C}^\wedge(\theta^+,\theta^-)$ as the **corner force** corresponding to $\{\theta^-,\theta^+\}$.

We can also extend the terminology concerning stability to sharp spots. Indeed, the paragraph containing (4.16) with $f''(\theta)$ considered as a distribution yields the following definitions: f is **strictly-stable** or **unstable at a sharp spot** θ_0 according as

$$f'(\theta_0^+) > f'(\theta_0^-) \quad \text{or} \quad f'(\theta_0^+) < f'(\theta_0^-). \tag{10.3}$$

With these definitions <u>the convexity-stability theorem of Section 8 holds without change</u>.

GS sections and **GS corners** of the interfacial energy are defined as for smooth f. **GS angles** (whether sharp spots or smooth spots) are, as before, angles that belong to GS sections; GS sharp spots are then, necessarily, <u>strictly stable</u>.

The <u>convexification</u> of the Frank diagram is the polar diagram of a function $\Sigma(\theta)$ and is hence the locus of a vector potential

$$\mathbf{\Sigma}(\theta) = \Sigma(\theta)\mathbf{N}(\theta),$$

the **convexified Frank potential**. On GS sections $\mathbf{\Sigma}(\theta)$ coincides with $\boldsymbol{\sigma}(\theta)$; between such sections $\mathbf{\Sigma}(\theta)$ coincides with the Maxwell lines of the Frank diagram. The function

$$\mathbf{\Sigma}^*(\theta) = \Sigma(\theta)^{-2}\mathbf{\Sigma}'(\theta) = -\Sigma(\theta)^{-1}\mathbf{T}(\theta) + \Sigma(\theta)^{-2}\Sigma'(\theta)\mathbf{N}(\theta) \qquad (10.4)$$

is the supporting tangent of the the convexified Frank diagram. By (i) of the Maxwell theorem (Appendix A), we have the following analog of the "thermodynamic relation" (8.4) (cf. Figure 10A).

Proposition. <u>Let</u> $\{\theta^-,\theta^+\}$ <u>with</u> $0<\theta^+-\theta^-<\pi$ <u>be a</u> GS <u>corner</u>. <u>Then</u>, <u>for all</u> $\theta\in(\theta^-,\theta^+)$,

$$\mathbf{C}\hat{\ }(\theta^-,\theta^+) = -\mathbf{\Sigma}^*(\theta). \qquad (10.5)$$

Remark. Let $\{\theta_0,\theta_1\}$ be a GS corner with θ_0 a sharp spot. Let $\mathcal{N}$ be a fixed neighborhood of θ_0 containing no sharp spots other than θ_0. Consider $f(\theta)$ as the limit, as $\varepsilon\to 0$, of energies $f_\varepsilon(\theta)$ which are C^1 on $\mathcal{N}$, which are <u>strictly stable</u> on $\mathcal{I}_\varepsilon := (\theta_0-\varepsilon,\theta_0+\varepsilon)$, and which satisfy $f(\theta)=f_\varepsilon(\theta)$ outside $\mathcal{I}_\varepsilon$. Let $\mathbf{C}_\varepsilon(\theta)$ denote the capillary force for $f_\varepsilon(\theta)$, and consider the set

$$\mathcal{C}_\varepsilon = \{\,\mathbf{C}_\varepsilon(\theta):\ \theta\in\mathcal{I}_\varepsilon \text{ is a GS angle of } f_\varepsilon(\theta)\,\}.$$

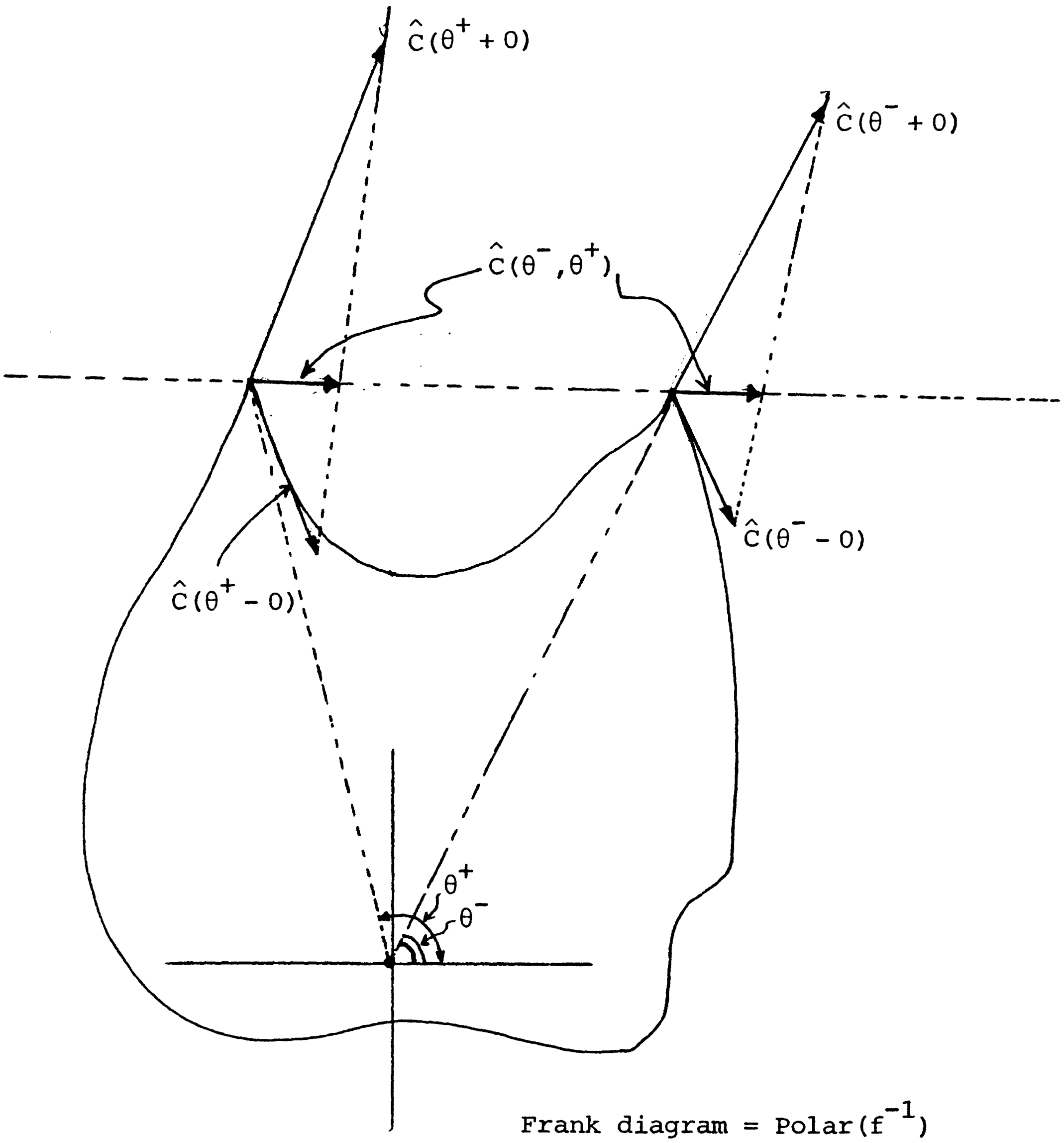


Figure 10A. The corner force $\hat{C}(\theta^-, \theta^+)$.

The limit $\mathcal{C}$ of the sets $\mathcal{C}_\varepsilon$ (that is, the intersection of the sets $\mathcal{C}_\varepsilon$ over all sufficiently small ε) in some sense represents the globally-stable set of capillary forces at θ_0. It is not difficult to show that $\mathcal{C}$ is independent of the choice of $f_\varepsilon(\theta)$; in fact, $\mathcal{C}$ is the negative of the supporting tangent-fan $\{\Sigma^*(\theta_0)\}$ of the convexified Frank diagram. Since $\{\Sigma^*(\theta_0)\}\subset\{\sigma^*(\theta_0)\}$, $\mathcal{C}$ is a subset of the capillary set $\{\mathbf{C}\hat{}(\theta_0)\}$. For these reasons, we refer to $\{-\Sigma^*(\theta_0)\}$ as the **GS capillary set** at θ_0.

Regularity for an interfacial energy with sharp spots is defined exactly as in Section 8.2.

Remark. A major difference between smooth energies and energies with sharp spots is that for the latter regularity does not rule out singleton GS sections, which are possible at sharp spots. Further, granted regularity, if $\{\theta^-,\theta^+\}$ $(0<\theta^+-\theta^-<\pi)$ is a corner with θ^- and θ^+ GS angles, then $\{\theta^-,\theta^+\}$ is a GS corner. Moreover, there is another corner involving θ^- (other than $\{\theta^+,\theta^-\}$) if and only if $\{\theta^-\}$ is a singleton GS section, in which case the corner is of the form $\{\alpha,\theta^-\}$ $(0<\theta^--\alpha<\pi)$; an analogous assertion applies to θ^+.

10.2. Admissibility in the presence of sharp spots.

We consider now an energy f with the following properties:

f is an interfacial energy with sharp spots; f is regular.

The following notation is useful:

$\mathcal{F}$:= the convexified Frank diagram of f,
$\mathcal{D}$:= the set of GS angles of f,
$\mathcal{D}_{sm}$:= the interior of the set of GS angles which are smooth spots[50] of $\mathcal{F}$,

[50]By the smooth and sharp spots of $\mathcal{F}$ we mean the smooth and

$\mathcal{D}_{smco}$:= the set of all smooth spots of $\mathcal{T}$ which are GS corner angles,

$\mathcal{D}_{sh}$:= the set of GS angles which are sharp spots of $\mathcal{T}$.

Note that $\mathcal{D}_{sm}$, $\mathcal{D}_{smco}$, and $\mathcal{D}_{sh}$ are mutually disjoint and have $\mathcal{D}$ as their union. We use the term **smooth GS part** to designate the closure of a connected component of $\mathcal{D}_{sm}$.

Let $\mathbf{r}$ be a PS evolving curve. Fix t and the angle θ_0, and let $\mathcal{R}$ be the set of all s such that $\theta(s,t) = \theta_0$. We will refer to the closures of the connected components of $\mathcal{R}$ as the **θ_0-sections** at time t.

We use the following terminology for arcs $\mathbf{r}_i$ of $\mathbf{r}$: (i) $\mathbf{r}_i$ is **curved** if, at each t, $\theta_s(s,t)=0$ at most at a discrete set of points of $\mathcal{S}_i(t)$; (ii) if $\mathbf{r}_i$ is a facet with $\theta_i(t)$ the corresponding angle, then $\mathbf{r}_i$ is **maximal** if $\mathcal{S}_i(t)$ is a $\theta_i(t)$-section at each t.

At sharp spots θ_0 there is no uniquely defined capillary force; instead there is a vector fan $\{\mathbf{C}^\wedge(\theta_0)\}$ of possible forces. It thus seems reasonable to allow the capillary force $\mathbf{C}(s,t)$ associated with a given evolving curve to vary within $\{\mathbf{C}^\wedge(\theta_0)\}$ whenever $\theta(s,t)=\theta_0$. This should motivate the following notion of admissibility.

A PS evolving curve $\mathbf{r}$ is **admissible** if:[51]

(AP1) $\theta(s,t)$ is always GS;

(AP2) there is an **associated capillary force** $\mathbf{C}(s,t)$ such that $\mathbf{C}(s,t)\in\{\mathbf{C}^\wedge(\theta(s,t))\}$ for all (s,t) and

$$\int_{\partial c(t)} \mathbf{C} = \int_{c(t)} (F + \beta(\theta)V)\mathbf{N}\, ds \tag{10.6}$$

whenever c is the trace of an evolving subcurve of $\mathbf{r}$;

---

sharp spots of $\Sigma(\theta)$. Sharp spots of $\mathcal{T}$ are GS sharp spots of $f(\theta)$, but the converse is not always true.

[51]We will also need one technical assumption: for $\theta_0\in\mathcal{D}_{sh}$, the set of all s such that $\theta(s,t) = \theta_0$ has a finite number of connected components.

(AP3) each arc of $\mathbf{r}$ is either a maximal facet or a curved arc, and adjacent curved arcs correspond to different smooth GS parts.

The associated capillary force is, in fact, unique, a result we shall prove by showing that $\mathbf{C}(s,t)$ is essentially characterized by the <u>convexified</u> <u>Frank</u> <u>diagram</u>.

Let $\mathbf{r}$ be a <u>closed</u> PS evolving curve. Fix t and the angle θ_0 with $\theta_0 \in \mathcal{D}_{sh}$, and consider a given θ_0-section $\mathcal{S} = [H,S]$. Then $\mathcal{S}$ is **trivial** or **nontrivial** according as $H = S$ or $S > H$. Further:

(T1) $\theta(s,t)$ **increases across** $\mathcal{S}$ if, for all sufficiently small $\varepsilon > 0$,

$$\theta(H-\varepsilon,t) < \theta_0, \qquad \theta(S+\varepsilon,t) > \theta_0;$$

the statement "$\theta(s,t)$ **decreases across** $\mathcal{S}$" has an analogous meaning; in either case we refer to $\mathcal{S}$ as **transitional**;

(T2) $\mathcal{S}$ is a **local maximum** for $\theta(s,t)$ if, for all sufficiently small $\varepsilon > 0$,

$$\theta(H-\varepsilon,t_0) \le \theta_0, \quad \theta(S+\varepsilon,t_0) \le \theta_0;$$

the statement "$\mathcal{S}$ a **local minimum** for $\theta(s,t)$" has an analogous meaning; in either case we refer to $\mathcal{S}$ as **nontransitional**.

Finally, a function $\mathbf{g}(s)$ **goes from a to b** if $\mathbf{g}(H+0) = \mathbf{a}$ and $\mathbf{g}(S-0) = \mathbf{b}$.

The degree of smoothness assumed for PS evolving curves ensures that each $\mathcal{S}$ is either transitional or nontransitional. Indeed, if $\mathcal{S} = [H,S]$, then there is an $\varepsilon > 0$ such that θ is different from θ_0 everywhere in $(H-\varepsilon,H) \cup (S,S+\varepsilon)$; the continuity of θ then yields the following four possibilities: $\pm(\theta-\theta_0) > 0$ on $(H-\varepsilon,H)$, $\pm(\theta-\theta_0) > 0$ on $(S,S+\varepsilon)$.

Characterization of the capillary force. *Let* $\mathbf{r}$ *be a closed*,[52] *admissible* PS *evolving curve*. *Then there is exactly one capillary force* $\mathbf{C}(s,t)$ *associated with* $\mathbf{r}$, *and* $\mathbf{C}(s,t)$ *has the following properties*:

(i) $\mathbf{C}(s,t)$ *belongs to the* GS *capillary set* $\{-\Sigma^*(\theta(s,t))\}$, *so that* $\mathbf{C}(s,t)=-\Sigma^*(\theta(s,t))$ *whenever* $\theta(s,t)$ *is a smooth spot of* $\mathcal{C}$.

(ii) *Let* θ_0 *a sharp spot of* $\mathcal{C}$, *and let* $\mathcal{S}=[H,S]$ *be a* θ_0-*section at some fixed time* $t\in(0,T)$.

(F1) *If* $\mathcal{S}$ *is transitional, then* $\mathcal{S}$ *is nontrivial, and* $\mathbf{C}(s,t)$ *varies linearly with* s *on* $\mathcal{S}$, *going from* $-\Sigma^*(\theta_0-0)$ *to* $-\Sigma^*(\theta_0+0)$ *or from* $-\Sigma^*(\theta_0+0)$ *to* $-\Sigma^*(\theta_0-0)$ *according as* $\theta(s,t)$ *increases or decreases across* $\mathcal{S}$.

(F2) *If* $\mathcal{S}$ *is nontransitional, then* $\mathbf{C}(s,t)$ *is constant on* $\mathcal{S}$ *with value* $-\Sigma^*(\theta_0-0)$ *or* $-\Sigma^*(\theta_0+0)$ *according as* $\mathcal{S}$ *is a local maximum or a local minimum for* $\theta(s,t)$.

Proof. By (AP1) and (10.2), $\mathbf{C}(s,t)=-\Sigma^*(\theta(s,t))$ whenever $\theta(s,t)$ is a smooth spot. Further, this result, (F1), and (F2), imply that $\mathbf{C}(s,t)$ is uniquely determined with $\mathbf{C}(s,t)\in\{-\Sigma^*(\theta(s,t))\}$ for all (s,t). Thus we have only to establish (F1) and (F2). We will prove only (F1), and only when $\theta(s,t)$ increases across $\mathcal{S}$; the remaining assertions are proved analogously. Since the time t is fixed, we shall suppress it as an argument. Suppose that

$$\theta_0+\delta \text{ is GS for all sufficiently small } \delta>0. \tag{10.7}$$

Then $\theta(S+0)=\theta_0$ and

$$\mathbf{C}(S-0)=\mathbf{C}(S+0)=\mathbf{C}^\wedge(\theta_0+0)=-\boldsymbol{\sigma}^*(\theta_0+0)=-\Sigma^*(\theta_0+0). \tag{10.8}$$

[52]For $\mathbf{r}$ not closed $\mathbf{C}(s,t)$ is determined uniquely (and (F1), (F2) hold) on the internal arcs of $\mathbf{r}$, and also on the terminal arcs provided they are not sharp-spot facets (facets whose angles are sharp spots). Boundary conditions are needed to determine $\mathbf{C}(s,t)$ uniquely on terminal sharp-spot facets.

On the other hand, if (10.7) is not satisfied, then there is a corner $\{\theta_0, \theta_1\}$ such that $\theta(S+0) = \theta_1$, and, appealing to (10.5) and the corner-force theorem,

$$\mathbf{C}(S-0) = \mathbf{C}(S+0) = \mathbf{C}\hat{}(\theta_0, \theta_1) = -\mathbf{\Sigma}^*(\theta_0 + 0). \tag{10.9}$$

Similar results hold at H. The results (10.8) and (10.9) and their counterparts for H have the following consequences: (i) $\mathcal{S}$ is nontrivial, for otherwise $\mathbf{\Sigma}^*(\theta_0 - 0) = \mathbf{\Sigma}^*(\theta_0 + 0)$, which is not possible when θ_0 is a sharp spot of the convexified Frank diagram; (ii) $\mathbf{C}(s,t)$ goes from $-\mathbf{\Sigma}^*(\theta_0 - 0)$ to $-\mathbf{\Sigma}^*(\theta_0 + 0)$.

Finally, since $\theta(s) \equiv \theta_0$ on $\mathcal{S}$, capillary balance (10.6) must there have the local form

$$\mathbf{C}_s = [F + \beta(\theta_0)V]\mathbf{N}(\theta_0) \tag{10.10}$$

with $V = V(t)$ independent of s. Thus $\mathbf{C}(s)$ varies <u>linearly</u> on $\mathcal{S}$. ∎

Let

$$\begin{aligned} \Delta(\theta) &:= [\mathbf{\Sigma}^*(\theta - 0) - \mathbf{\Sigma}^*(\theta + 0)] \cdot \mathbf{N}(\theta) \\ &= \Sigma(\theta)^{-2}[\Sigma'(\theta - 0) - \Sigma'(\theta + 0)] \geq 0 \end{aligned} \tag{10.11}$$

(cf. (A11) of Appendix A). Because of (F1), evolving facets on which $\theta \equiv \theta_0$, with θ_0 a sharp spot, are to be expected. By (F1) and (10.10), the length $L(t)$ and the normal velocity $V(t)$ of a "transitional" facet are related by

$$\pm\Delta(\theta_0) = L(t)[F + \beta(\theta_0)V(t)] \tag{10.12}$$

with the plus or minus sign chosen according as $\theta(s,t)$ increases or decreases across the corresponding θ_0-section. For a "nontransitional" facet, (10.12) remains valid, but with $\Delta(\theta) = 0$.

Remark 1. Let $S_i(t)$ be the arc length at a juncture i of a closed, admissible PS evolving curve. Then $\theta_i^- := \theta(S_i(t)-0,t)$ and $\theta_i^+ := \theta(S_i(t)+0,t)$ are independent of time, and either:

(i) $\{\theta_i^-,\theta_i^+\}$ is a GS corner, in which case the juncture is **nontrivial**; or

(ii) $\theta_i^- = \theta_i^+ \in \mathfrak{D}_{sh}$ in which case the juncture is **trivial**.

Remark 2. There are exactly three possibilities for an arc $\mathbf{r}_i$ of an admissible PS evolving curve.

(1) $\mathbf{r}_i$ is a curved arc. Since each boundary angle of a smooth GS part must belong to $\mathfrak{D}_{smco} \cup \mathfrak{D}_{sh}$, it is a clear from (F1) that (on $\mathbf{r}_i$) $\theta(s,t)$ must belong to a <u>single</u> smooth GS part $\mathfrak{D}_i$. In this case $\mathbf{r}_i$ evolves according to

$$V = \Phi(\theta)K - \beta(\theta)^{-1}F,$$
$$\Phi(\theta) = \beta(\theta)^{-1}[f(\theta) + f''(\theta)] > 0. \tag{10.13}$$

with $f''(\theta)$ computed by restricting $f(\theta)$ to $\mathfrak{D}_i$. Further, the initial and terminal orientations, θ_{init} and θ_{term}, are constants belonging to $\mathfrak{D}_{smco} \cup \mathfrak{D}_{sh}$ (cf. the paragraph containing (9.4)). We assign a **transition number** to $\mathbf{r}_i$ as follows: the transition number is +1, -1, or 0 according as $\theta_{term} > \theta_{init}$, $\theta_{term} < \theta_{init}$, or $\theta_{term} = \theta_{init}$.

(2) $\mathbf{r}_i$ is a facet with orientation $\theta_i \in \mathfrak{D}_{smco}$. Then $\mathbf{r}_i$ evolves according to

$$\beta(\theta_i)V = -F.$$

(3) $\mathbf{r}_i$ is a facet with orientation $\theta_i \in \mathfrak{D}_{sh}$. Then either $\theta(s,t)$ increases across $\varsigma_i(t)$ for all t, or decreases across $\varsigma_i(t)$ for all t, or $\varsigma_i(t)$ is nontransitional for all t; we define the **transition number** χ_i for $\mathbf{r}_i$ to be +1, -1, or 0,

respectively, for these three possibilities. Then $\mathbf{r}_i$ evolves according to (cf. (10.12))

$$\chi_i \Delta(\theta_i) = L_i(t)[F + \beta(\theta_i)V(t)] \tag{10.14}$$

with $L_i(t)$ the length of $\mathbf{r}_i$.

10.3. Crystalline energies.

Let $f(\theta)$ be an interfacial energy with sharp spots. Then $f(\theta)$ is **crystalline**[53] if its convexified Frank diagram is a polygon, and if the vertices of this polygon form the complete set of GS angles (cf. Figure 10B). Such energies are clearly regular.

Let $f(\theta)$ be crystalline, and consider an admissible PS evolving curve $\mathbf{r}$. Each arc of $\mathbf{r}$ must have $\theta(s,t)$ equal to one of the GS angles, and hence must have $\theta(s,t) \equiv$ constant. Thus we have the following[54]

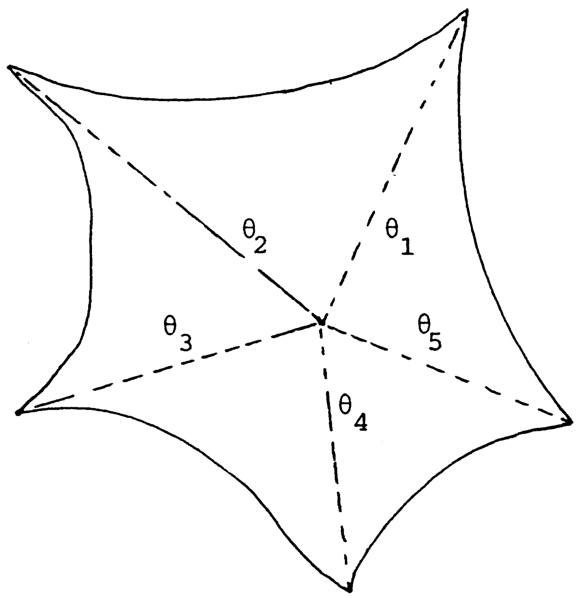
Proposition. Facetings with fixed orientations are the only admissible PS evolving curves for a crystalline energy.

By an **evolving crystal** we mean a faceting $\mathbf{r}$ that is simple and closed. In view of the proposition, these are the only admissible interfacial motions consistent with a crystalline energy. As before, we let the reference region $\Omega(t)$ denote the bounded region interior to the corresponding curve at each time, so that $\Omega(t)$ has $\mathbf{N}$ as its outward unit normal. We will refer to $\mathbf{r}$ as **essentially convex** if $\Omega(t)$ is a convex region at each t. (Note that $\mathbf{r}$ cannot be convex as an evolving curve, since $K=0$ on each facet.)

Let $\mathbf{r} = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$ be an evolving crystal, and let θ_i, V_i,

[53]Cf. Taylor [1978], who studies the stable equilibria of interfaces coresponding to crystalline energies.

[54]Facetings are as defined in Section 9.2, except that the underlying notion of admissibility is as defined in Section 10.2.



Frank diagram = Polar(f^{-1}).

Figure 10B. The Frank diagram of a crystalline energy f. The GS angles are the angles corresponding to the five sharp spots.

and L_i denote the orientation, normal velocity, and length of $\mathbf{r}_i$. Further, for any function $g(\theta)$ let $g_i = g(\theta_i)$, and define

$$\rho_{i,j} := [\mathbf{T}_j \cdot \mathbf{N}_i]^{-1}, \qquad \alpha_{i,j} := (\mathbf{T}_i \cdot \mathbf{T}_j)\rho_{i,j}. \tag{10.15}$$

Remark. From (10.11) and the definition of a crystalline energy, it is clear that each of the (fixed) angles $\theta_i \in \mathfrak{D}_{sh}$; thus

$$\Delta_i := \Delta(\theta_i) > 0. \tag{10.16}$$

Moreover, our agreement that $\mathbf{N}$ be outward implies that, for an essentially convex crystal, θ decreases across each facet and

$$\rho_{i,i+1} < 0, \qquad \alpha_{i,i+1} < 0. \tag{10.17}$$

Curves governed by smooth energies have, as evolution equations, a fairly complex system of *partial differential equations*; the next theorem shows that, in contrast, crystals evolve according to a finite set of *ordinary differential equations*.

Evolution equations for a crystal. *Crystals corresponding to a crystalline energy evolve according to the equations*

$$\begin{aligned} L_i^{\circ} &= [\alpha_{i,i+1} + \alpha_{i-1,i}]V_i - \rho_{i-1,i}V_{i-1} - \rho_{i,i+1}V_{i+1}, \\ V_i &= (\beta_i)^{-1}[-F + \chi_i(\Delta_i/L_i)] \end{aligned} \tag{10.18}$$

at each juncture i.

Proof. The second of (10.18) is (10.12). To derive the second, let v_i denote the arc velocity and $[S_i, S_{i+1}]$ the arc-length interval for $\mathbf{r}_i$. Then

$$\begin{aligned} V_{i-1}\mathbf{N}_{i-1} + [S_i^{\circ} - v_{i-1}]\mathbf{T}_{i-1} &= V_i\mathbf{N}_i + [S_i^{\circ} - v_i]\mathbf{T}_i, \\ V_i\mathbf{N}_i + [S_{i+1}^{\circ} - v_i]\mathbf{T}_i &= V_{i+1}\mathbf{N}_{i+1} + [S_{i+1}^{\circ} - v_{i+1}]\mathbf{T}_{i+1}. \end{aligned} \tag{10.19}$$

If we take the inner product of these equations with $\mathbf{T}_i$ and then add the resulting equations, we find that

$$L_i^\circ = -(\rho_{i-1,i})^{-1}V_{i-1} - (\rho_{i,i+1})^{-1}V_{i+1} + \\ [S_{i+1}^\circ - v_{i+1}](\mathbf{T}_i \cdot \mathbf{T}_{i+1}) - [S_i^\circ - v_{i-1}](\mathbf{T}_i \cdot \mathbf{T}_{i-1}). \qquad (10.20)$$

Next, we take the inner product of (10.19) with $\mathbf{N}_i$; the result is

$$\begin{aligned} (\rho_{i-1,i})^{-1}[S_i^\circ - v_{i-1}] &= -V_i + V_{i-1}(\mathbf{T}_i \cdot \mathbf{T}_{i-1}), \\ (\rho_{i-1,i})^{-1}[S_{i+1}^\circ - v_{i+1}] &= V_i - V_{i+1}(\mathbf{T}_i \cdot \mathbf{T}_{i+1}). \end{aligned} \qquad (10.21)$$

The relations (10.20) and (10.21) combine to give $(10.18)_1$. ■

A geometric derivation of (10.18) follows upon noting that, for $\vartheta_i = \theta_i - \theta_{i-1}$ (Figure 10C),

$$L_i^\circ = -[\cot\vartheta_i + \cot\vartheta_{i+1}]V_i + (\sin\vartheta_i)^{-1}V_{i-1} + (\sin\vartheta_{i+1})^{-1}V_{i+1},$$

which is easily obtained, at least formally, by incrementing the time by a "small" amount.

10.3.1. Evolution of a rectangular crystal.

A simple example that yields useful information occurs when the convexified Frank diagram is quadrilateral with vertices at $\theta = 0, \pi/2, \pi, 3\pi/2$. A corresponding evolving crystal, if essentially convex, is rectangular at each t with sides having these angles as orientations. Therefore $L_1 = L_3$, $L_2 = L_4$, and, by (10.15), $\rho_{i,i+1} = -1$, $\alpha_{i,i+1} = 0$. Thus, defining

$$\begin{aligned} F_1 &= F(\beta_2 + \beta_4)/\beta_2\beta_4, & F_2 &= F(\beta_1 + \beta_3)/\beta_1\beta_3, \\ \delta_1 &= (\Delta_4\beta_2 + \Delta_2\beta_4)/\beta_2\beta_4, & \delta_2 &= (\Delta_3\beta_1 + \Delta_1\beta_3)/\beta_1\beta_3, \end{aligned} \qquad (10.22)$$

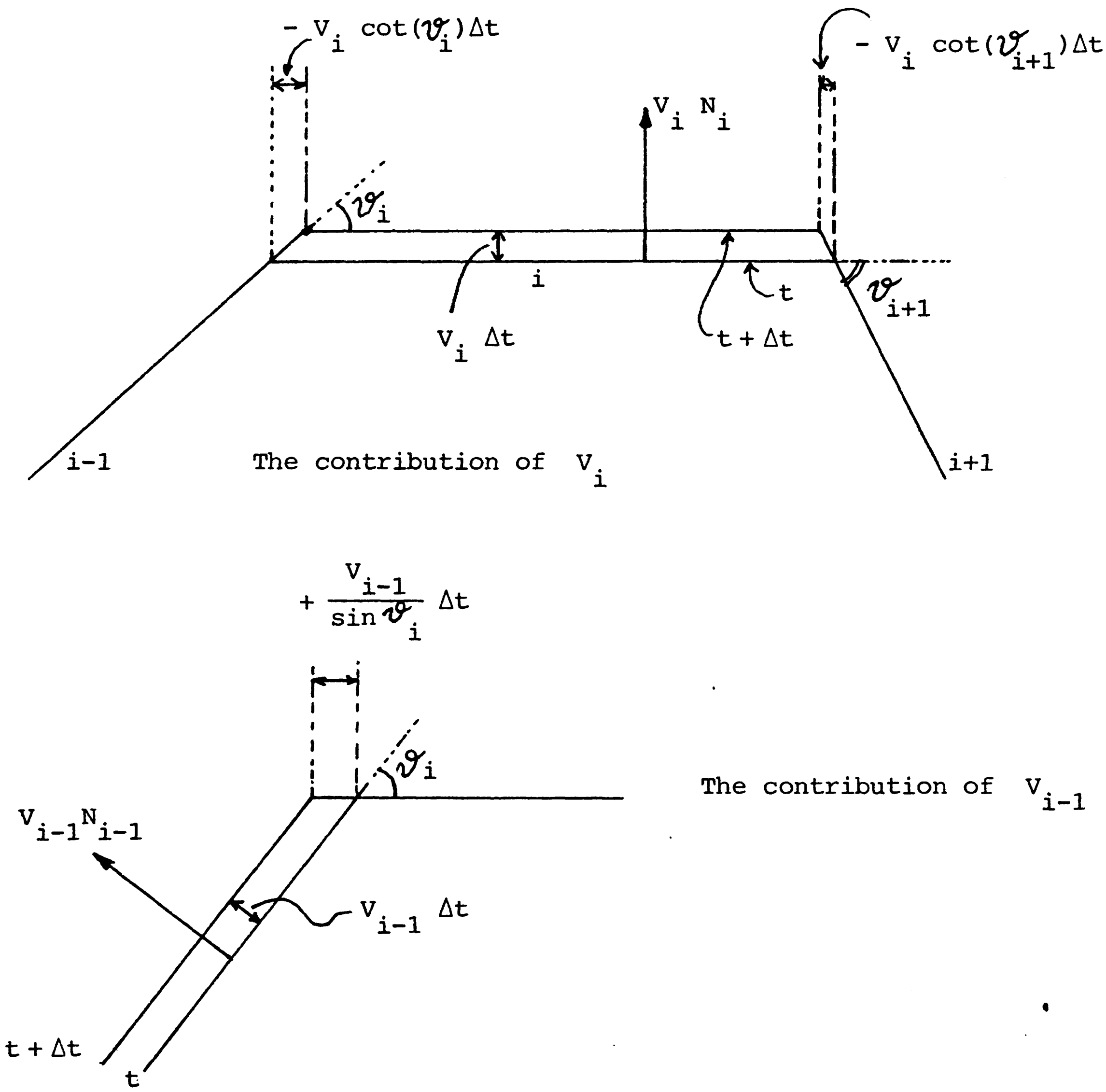


Figure 10C. Derivation of $\dot{L}_i(t)$, expressed as a linear combination of V_{i-1}, V_i and V_{i+1}.

the evolution equations (10.18) reduce to

$$L_1^\circ = -F_1 - \delta_1/L_2,$$
$$L_2^\circ = -F_2 - \delta_2/L_1 \qquad (10.23)$$

with (cf. (10.16))

$$\operatorname{sgn} F_i = \operatorname{sgn} F, \qquad \delta_i > 0. \qquad (10.24)$$

Case 1: $F=0$. Solutions of (10.23) approach zero in finite time T. Defining

$$\delta = \delta_1/\delta_2,$$

there is a constant $C>0$ such that

$$L_1(t) = CL_2(t)^{\delta}. \qquad (10.25)$$

Thus for $\delta=1$ the isoperimetric ratio $\rho(t)= [\mathrm{length}(\partial\Omega)]^2/4\pi\,\mathrm{area}(\Omega)$ is constant, but for $\delta \neq 1$

$$\rho(t) \to \infty \quad \text{as} \quad t \to T. \qquad (10.26)$$

For $\delta>1$, $L_1(t)$ approaches zero faster than $L_2(t)$, so that the crystal shrinks to a point, but is ultimately in the shape of a "needle oriented by θ_2 and θ_4". This is in sharp contrast to an interfacial motion for an isotropic interface (cf. Section 5.1); there the interface shrinks to a round point.[55]

[55]Gage [1984], Gage and Hamilton [1986], Grayson [1987].

<u>Case</u> 2: $F>0$. Solutions still approach zero in finite time T. The result (10.25) holds asymptotically, and the discussion of Case 1 for $\delta \neq 1$ is appropriate.

<u>Case</u> 3: $F<0$. This case corresponds to a crystal evolving in a supercooled liquid. Here (10.23) has an equilibrium at

$$L_2 = |F_1|/\delta_1, \quad L_1 = |F_2|/\delta_2, \tag{10.27}$$

which is a saddle. For any given initial value $L_1(0)$ there is a number $\ell > 0$ such that: (i) if $L_2(0) < \ell$, the sides shrink to zero in finite time, in which case the asymptotic behavior of the crystal is as discussed in Case 1; (ii) if $L_2(0) > \ell$, the sides grow to infinity as $t \to \infty$, asymptotically as

$$L_1(t) \approx |F_1|t, \qquad L_2(t) \approx |F_2|t.$$

The equilibrium (10.27) represents the Wulff shape of the crystal. Interestingly, none of the asymptotic shapes of the crystal are of this form.

11. General global behavior.

Section 7 established results for the growth and shrinking of an interface whose energy is smooth and stable. We now generalize these results. We assume that

> f <u>is a</u> (<u>not necessarily stable</u>) <u>regular interfacial energy with sharp spots</u>.

Let $\mathbf{r}$ be a **PS interfacial motion** that is **admissible** in the sense of (AP1), (AP2), and (AP3),[56] and let $L(t)$, $A(t)$, and $\mathcal{F}(t)$ denote the perimeter, area, and total interfacial energy as defined in (7.3) and (7.5).

Theorem.

$$\mathcal{F}^\circ(t) + FA^\circ(t) = -\int_{\partial\Omega(t)} \beta(\theta)V^2 ds \leq 0. \tag{11.1}$$

<u>Proof</u>. Our first step will be to show that (4.12) (with $b_{ext} = 0$) holds for each arc $\mathbf{r}_i$ of $\mathbf{r}$. It is clear from its proof that (4.12) holds for $\mathbf{r}_i$ in cases (1) and (2) of Remark 2 (Section 10.2). We now show that (4.12) is also satisfied in case (3). For this case $\mathbf{N}$ and $\mathbf{T}$ are constant, $K \equiv 0$, and V is independent of arc length. Let $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$ denote the initial and terminal points of $\mathbf{r}_i$, with $\mathbf{C}_1(t)$ and $\mathbf{C}_2(t)$ the corresponding capillary forces. Since $f(\theta_i) \equiv \text{constant}$, (2.35) yields

$$(d/dt)\int_{\varsigma_i(t)} f(\theta_i)ds = f(\theta_i)[\mathbf{R}_2^\circ - \mathbf{R}_1^\circ] \cdot \mathbf{T},$$

while (10.6) implies

[56] Cf. the paragraphs containing (2.45) and (10.6).

$$F\int_{\xi_i(t)} V\,ds = -\int_{\xi_i(t)} \beta(\theta_i)V^2 ds + V[\mathbf{C}_2 - \mathbf{C}_1]\cdot\mathbf{N}.$$

Further, by (10.1), $f(\theta_i) = \mathbf{C}_1 \cdot \mathbf{T} = \mathbf{C}_2 \cdot \mathbf{T}$; this relation, $(2.20)_1$, and the last two identities imply (4.12).

Thus (4.12) holds on each arc of $\mathbf{r}$. If we apply (4.12) to each such arc, add the resulting equations, and use (2.45), (2.37), and the fact that, by (10.6), $\mathbf{C}(s,t)$ is continuous in s, we arrive at (11.1). ∎

Let $e(\theta)$ be a continuous, piecewise smooth, strictly positive, 2π-periodic function on $\mathbb{R}$. We need a generalization of the Wulff ratio (7.12); a generalization in which the exterior normals of the underlying regions Γ are restricted to point in GS directions.[57] This is easily accomplished by replacing $e(\theta)$ by $+\infty$ whenever θ is not GS. Thus, letting

$$e^*(\theta) = \begin{cases} e(\theta) & \text{for } \theta \text{ a GS angle} \\ +\infty & \text{otherwise,} \end{cases} \tag{11.2}$$

the **GS Wulff ratio for** $e(\theta)$ is the number

$$W_{GS}(e) := (4\pi)^{-1} \inf \Big\{ \int_{\partial\Gamma} e^*(\theta)ds \Big\}^2 / \,\mathrm{area}(\Gamma) \tag{11.3}$$

with the infimum taken over all bounded regions Γ with $\partial\Gamma$ piecewise smooth. Corresponding minima Γ are called **GS Wulff regions for** $e(\theta)$.

Let $\{\Theta_m^-, \Theta_m^+\}$, $m = 1,2,\dots,M$, denote the GS corners in the order encountered as the Frank diagram is traversed in the <u>clockwise</u> direction, so that $\Theta_1^- > \Theta_1^+ \geq \Theta_2^- > \Theta_2^+ \geq \Theta_3^-$, and so forth;[58] for each such corner, let[59]

[57] GS is with respect to the interfacial energy $f(\theta)$, so that $W_{GS}(e)$ depends also on $f(\theta)$.

[58] We use this numbering scheme since for a closed, convex curve the angle decreases with increasing arc length.

$$\mathcal{k}_m := 2\tan(\vartheta_m/2) < 0, \qquad \vartheta_m = \Theta_m^{+}-\Theta_m^{-}. \tag{11.4}$$

We define the **GS average** g_{GSav} of a function $g(\theta)$ by

$$2\pi g_{GSav} := \int_{\theta\in\mathcal{D}} g(\theta)d\theta + \frac{1}{2}\sum_{m=1}^{M} |\mathcal{k}_m| \, [g(\Theta_m^{+})+g(\Theta_m^{-})]. \tag{11.5}$$

Theorem on the growth of the reference phase. Consider a regularly maximal,[60] admissible PS interfacial motion with duration $[0,T)$.

(i) If $F \geq 0$, then $T < \infty$ and $A(t) \to 0$ as $t \to T$.

(ii) If $F < 0$, then:

(a) if $L(0)$ is sufficiently small, then $T < \infty$ and $A(t) \to 0$ as $t \to T$.

(b) if $A(0)$ is sufficiently large, then $T = \infty$ and $A(t) \to \infty$ as $t \to \infty$. In this case isoper$(\Omega(t))$ remains bounded

$$\limsup_{t\to\infty} \text{isoper}(\Omega(t)) \leq [(\beta^{-1})_{GSav}]^2 / W_{GS}(\beta^{-1}). \tag{11.6}$$

Proof. Our first step will be to show that

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[59]The $\mathcal{k}_m$ are M numbers associated with a given interfacial energy; these should be differentiated from the transition curvatures (2.41), which correspond to the actual junctures in an interfacial motion (cf. (11.9)).

[60]In the sense of the sentence containing (7.4). Here it must be kept in mind that we have, as yet, no existence theorem appropriate to an energy with sharp spots, although we do have an existence theorem for a smooth but unstable energy (cf. Section 9.6).

$$A^\circ(t) = -C - F\int_{\partial\Omega(t)} \beta(\theta)^{-1} ds,$$

$$\mathcal{F}^\circ \le FC + [F^2/(f_{min}\beta_{min})]\mathcal{F}, \qquad (1$$ [redacted]

$$L^\circ(t) \le -2\pi F(\beta^{-1})_{GSav},$$

where $C>0$ is a constant.

To prove $(11.7)_1$, let N denote the essential number of a [redacted] $\mathbf{r}$, let

$$\mathcal{I}_{arc} := \{\ i:\ \mathbf{r}_i \text{ is a curved arc, } 1 \le i \le N\ \},$$

$$\mathcal{I}_{sh} := \{\ i:\ \mathbf{r}_i \text{ is a facet with } \theta_i \in \mathcal{D}_{sh},\ 1 \le i \le N\ \},$$ [redacted]

and, for each $i \in \mathcal{I}_{arc}$, let $\mathcal{D}_i$ denote the smooth GS part [redacted] corresponding to $\mathbf{r}_i$. Since $\mathbf{N}$ is the outward normal to $\partial\Omega$, [redacted] interface is negatively (clockwise) oriented as a closed curve. [redacted] Moreover, because of the regularity of f, as s increases, θ [redacted] varies continuously through smooth GS parts, crosses sharp sp [redacted] <u>adjacent</u> smooth GS parts, or jumps across GS corners to <u>adjac</u> [redacted] sharp spots or smooth GS parts. Thus:

(i) The sum of the transition numbers for the totality of $\mathcal{D}_i$, [redacted] $i \in \mathcal{I}_{arc}$, corresponding to a given smooth GS part is -1, a [redacted]

$$\mathcal{D}_{sm} = \text{interior } \Big(\bigcup_{i \in \mathcal{I}_{arc}} \mathcal{D}_i \Big).$$

(ii) For each $\theta_0 \in \mathcal{D}_{sh}$ there is at least one $i \in \mathcal{I}_{sh}$ such that [redacted] and the sum of the transition numbers χ_i of the $\mathbf{r}_i$ wit [redacted] $i \in \mathcal{I}_{sh}$ and $\theta_i = \theta_0$ is -1.

Because of these conclusions, and since $K = \theta_s$, for any contin [redacted] 2π-periodic function $g(\theta)$,

$$\sum_{i\in\mathcal{J}_{arc}} \int_{\mathcal{S}_i} g(\theta)K\,ds = -\int_{\theta\in\mathcal{D}_{sm}} g(\theta)\,d\theta,$$
$$\sum_{i\in\mathcal{J}_{sh}} \chi_i g(\theta_i)\Delta(\theta_i) = -\sum_{\theta\in\mathcal{D}_{sh}} g(\theta)\Delta(\theta). \tag{11.8}$$

Thus, integrating $V(s,t)$ over $\partial\Omega(t)$, we conclude, with the aid of (2.45) and the remark containing (10.14), that $(11.7)_1$ holds with

$$C = \int_{\theta\in\mathcal{D}_{sm}} \Phi(\theta)\,d\theta + \sum_{\theta\in\mathcal{D}_{sh}} \beta(\theta)^{-1}\Delta(\theta) > 0.$$

Next, by (7.5), $f_{min}L \le \mathcal{F}$; (11.1) and $(11.7)_1$ therefore yield $(11.7)_2$. To verify $(11.7)_3$, let

$$\mathcal{J} = \text{the set of nontrivial junctures } j,\ \ 1 \le j \le N.$$

Each $j\in\mathcal{J}$ will have transition curvature k_j defined by (2.41). These transition curvatures are related to the numbers $\mathscr{k}_m$ defined by (11.4): for each j there is a corresponding $m = m(j)$ such that

$$\begin{aligned} &k_j = \mathscr{k}_m \quad \text{and} \quad \{\theta_j^-,\theta_j^+\} = \{\Theta_m^-,\Theta_m^+\}, \quad \text{or} \\ &k_j = -\mathscr{k}_m \quad \text{and} \quad \{\theta_j^-,\theta_j^+\} = \{\Theta_m^+,\Theta_m^-\}. \end{aligned} \tag{11.9}$$

Further, letting $J(m)$ denote the set of all j with $|k_j| = |\mathscr{k}_m|$,

$$\sum_{j\in J(m)} k_j = \mathscr{k}_m. \tag{11.10}$$

If j is a juncture for a facet whose angle is a sharp spot of the Frank diagram, and if χ denotes the transition number of the facet, then

$$\chi k_j \ge 0. \tag{11.11}$$

The verification of $(11.7)_3$ is based on (2.44), which we write in the form

$$L^\circ(t) = -\int_{\partial\Omega(t)} KV ds - \sum_{j\in\mathcal{J}} k_j v_j \tag{11.12}$$

with v_j the average velocity $(2.40)_3$ of the juncture j. By Remark 2 of Section 10.2 and $(11.8)_1$,

$$\int_{\partial\Omega(t)} KV ds = \sum_{i\in\mathcal{I}_{arc}} \int_{\mathcal{S}_i} [\Phi(\theta)K^2 - F\beta(\theta)^{-1}K] ds \geq F\int_{\theta\in\mathcal{D}_{sm}} \beta(\theta)^{-1} d\theta, \tag{11.13}$$

Let $j\in\mathcal{J}$ be the juncture of an arc $\mathbf{r}_i$ of $\mathbf{r}$. Then, on the arc at that juncture,

$$k_j V \geq -k_j \beta(\theta)^{-1} F. \tag{11.14}$$

Indeed, (11.13) holds trivially if $\mathbf{r}_i$ is a facet with angle $\theta\in\mathcal{D}_{smco}$; for $\mathbf{r}_i$ a facet with angle $\theta\in\mathcal{D}_{sh}$, (11.14) is a consequence of (10.15), (10.11), and (11.11); if $\mathbf{r}_i$ is curved, then (11.14) follows from (10.13) and (CC2) of Section (9.1) (which also holds in the present circumstances). By $(2.40)_3$, (11.10), and (11.14),

$$\sum_{j\in\mathcal{J}} k_j v_j \geq -\tfrac{1}{2} F \sum_{m=1}^{M} \mathcal{k}_m [\beta(\Theta_m^{+})^{-1} + \beta(\Theta_m^{-})^{-1}]. \tag{11.15}$$

Since $\mathcal{k}_m < 0$, (11.5), (11.12), (11.13), and (11.15) imply $(11.7)_3$.

Finally, an argument identical to that given in the paragraph containing (7.16) shows that (11.7) yields all of the desired conclusions. ■

Conjecture. Consider case (iib) of the last theorem. We conjecture that, as $t\to\infty$,

$\partial\Omega(t)$ <u>is asymptotic to a</u> GS <u>Wulff region for</u> $\beta(\theta)^{-1}$.

Remark. Suppose that the energy is <u>crystalline</u> with $\theta_1 > \theta_2 > \cdots > \theta_M$ the complete set of GS angles. Then $\{\theta_m, \theta_{m+1}\}$, $m = 1,2,\cdots,M$ $(\theta_{M+1} = \theta_1)$ are the GS corners,

$$\ell_m = 2\sin\vartheta_m/(1+\cos\vartheta_m), \qquad \vartheta_m = \theta_{m+1} - \theta_m,$$

and it is not difficult to verify, using the Wulff construction,[61] that

$$W_{GS}(1) = (2\pi)^{-1}\sum_{m=1}^{M} |\ell_m| = (1)_{GSav}.$$

Further, as is clear from (11.3), $W_{GS}(1)$ represents the minimum value of the isoperimetric ratio isoper(Γ) over all polygons Γ whose outward unit normals are limited to the GS directions $\theta_1, \theta_2, \cdots, \theta_M$. Thus, when $\beta = \text{constant}$, (11.9) yields, for case (iib) of the last theorem,

$$\text{isoper}(\Omega(t)) \to (2\pi)^{-1}\sum_{m=1}^{M} |\ell_m|$$

as $t \to \infty$.

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[61] Cf. Taylor [1978].

Appendix A. Polar diagrams.

Let $g(\theta)$ be a 2π-periodic, *strictly-positive* function on $\mathbb{R}$. Then g is **smooth except for sharp spots** if:

(i) g is continuous;

(ii) g' and g'' are continuous except possibly for a finite number of jump discontinuities.

The angles at which g' suffers jump discontinuities will be referred to as **sharp spots**; all other angles will be referred to as **smooth spots**.

Let $\mathbf{N}(\theta)$ and $\mathbf{T}(\theta)$ be defined by (2.3). The **polar diagram** Polar(g) of g is the *simple closed curve* in $\mathbb{R}^2$ defined by the vector function

$$\mathbf{g}(\theta) := g(\theta)\mathbf{N}(\theta).$$

We orient Polar(g) by θ, so that Polar(g) is positively (counter-clockwise) oriented as a closed curve. Since $\mathbf{N}' = -\mathbf{T}$,

$$\mathbf{g}'(\theta) = -g(\theta)\mathbf{T}(\theta) + g'(\theta)\mathbf{N}(\theta)$$

whenever θ is a smooth spot; thus

$$\mathbf{n}(\theta) := |\mathbf{g}'(\theta)|^{-1}[g(\theta)\mathbf{N}(\theta) + g'(\theta)\mathbf{T}(\theta)]$$

defines an outward unit normal to Polar(g). A tangent vector on Polar(g) more useful than $\mathbf{g}'(\theta)$ is the **supporting tangent**

$$\mathbf{g}^*(\theta) := g(\theta)^{-2}\mathbf{g}'(\theta)$$

$$= -g(\theta)^{-1}\mathbf{T}(\theta) + g(\theta)^{-2}g'(\theta)\mathbf{N}(\theta), \tag{A1}$$

which is well defined for θ a smooth spot. This notion has an extension which is defined for all angles. By the **vector fan** between vectors $\mathbf{a}$ and $\mathbf{b}$ we mean the set of all vectors $\mathbf{c}$ such that

$$\mathbf{c} = \mathbf{a} + \alpha(\mathbf{b} - \mathbf{a}), \qquad 0 \le \alpha \le 1.$$

For each θ, the **supporting tangent-fan** $\{\mathbf{g}^*(\theta)\}$ is the vector fan defined by $\mathbf{g}^*(\theta - 0)$ and $\mathbf{g}^*(\theta + 0)$, and is hence the <u>set</u> of vectors of the form

$$-g(\theta)^{-1}\mathbf{T}(\theta) + \zeta\mathbf{N}(\theta), \tag{A2}$$

where ζ varies in the closed interval of $\mathbb{R}$ bounded by $g(\theta)^{-2}g'(\theta - 0)$ and $g(\theta)^{-2}g'(\theta + 0)$. This definition has some simple consequences for θ a sharp spot: let γ and Γ, respectively, designate the smaller and larger values of $\mathbf{g}^*(\theta \pm 0) \cdot \mathbf{N}(\theta)$, and choose $\mathbf{a}, \mathbf{b} \in \{\mathbf{g}^*(\theta)\}$; then

$$\mathbf{g}^*(\theta + 0) - \mathbf{g}^*(\theta - 0) = g(\theta)^{-2}[g'(\theta + 0) - g'(\theta - 0)]\mathbf{N}(\theta), \tag{A3}$$
$$\gamma \le \mathbf{a} \cdot \mathbf{N}(\theta) \le \Gamma, \qquad \mathbf{a} - \mathbf{b} \text{ is parallel to } \mathbf{N}(\theta).$$

Somewhat less trivial are the following:

Properties of the supporting tangent.

(T1) <u>Let</u> $\mathbf{a} \in \{\mathbf{g}^*(\theta)\}$, <u>and let</u> ℓ <u>be the line through</u> $\mathbf{g}(\theta)$ <u>in the direction</u> $\mathbf{a}$. <u>Then the perpendicular distance from</u> ℓ <u>to the origin is</u> $|\mathbf{a}|^{-1}$.

(T2) $|\mathbf{g}^*(\theta)|^{-1}$ <u>is the support function of</u> Polar(g):

$$\mathbf{g}(\theta) \cdot \mathbf{n}(\theta) = |\mathbf{g}^*(\theta)|^{-1} \quad \underline{\text{for}}\ \theta\ \underline{\text{a smooth spot}}.$$

(T3) $\mathbf{g}^*(\theta)$ <u>is constant on a connected open subset</u> ℓ <u>of</u> Polar(g) <u>if and only</u> ℓ <u>is a straight line</u>.

<u>Proof</u>. Let $\mathbf{a} \in \{\mathbf{g}^*(\theta)\}$, so that $\mathbf{a} = -g(\theta)^{-1}\mathbf{T}(\theta) + \zeta\mathbf{N}(\theta)$. Omitting the argument θ, the <u>unit</u> vector $\mathbf{u} = |\mathbf{a}|^{-1}[g^{-1}\mathbf{N} + \zeta\mathbf{T}]$ is orthogonal to $\mathbf{a}$, so that $d := |\mathbf{g} \cdot \mathbf{u}|$ is the perpendicular distance from ℓ to the origin. But $d = |\mathbf{a}|^{-1}$. Thus (T1) is valid. (T2) is

an obvious consequence of (T1), and (T3) follows from (T2). ■

Theorem on common tangents. _Let_ $\{\mathbf{g}^*(\theta_1)\}\cap\{\mathbf{g}^*(\theta_2)\} \neq \varnothing$. _Then_:

(CT1) $|\theta_2-\theta_1| \neq \pi$;

(CT2) $\{\mathbf{g}^*(\theta_1)\}\cap\{\mathbf{g}^*(\theta_2)\}$ _consists of a single vector_ $\mathbf{a}$;

(CT3) _letting_ $\theta_2-\theta_1<\pi$, $\mathbf{a}$ _points in the same direction as_ $\mathbf{g}(\theta_2)-\mathbf{g}(\theta_1)$.

In this case we will refer to $\mathbf{a}$ _as the_ **common supporting tangent** _for_ θ_1 _and_ θ_2.

Proof. Assume that $\theta_2-\theta_1 \leq \pi$. Let $\mathbf{a}\in\{\mathbf{g}^*(\theta_1)\}\cap\{\mathbf{g}^*(\theta_2)\}$, and let ℓ_i denote the line through $\mathbf{g}(\theta_i)$ in the direction $\mathbf{a}$. It suffices to show that $\ell_1=\ell_2$. Suppose not. Then ℓ_1 and ℓ_2 are parallel, and, by (T1), the origin lies equidistant between them. Thus $\mathbf{T}(\theta_1)\cdot\mathbf{a}$ and $\mathbf{T}(\theta_2)\cdot\mathbf{a}$ are of opposite sign. But by (A2), $\mathbf{T}(\theta_i)\cdot\mathbf{a}=g(\theta_i)^{-1}>0$, a contradiction. ■

Corollary. _Let_ P(g) _be convex_. _Let_ $\theta_1,\theta_2\in\mathbb{R}$ _with_ $0<\theta_2-\theta_1<\pi$ _and_ $\{\mathbf{g}^*(\theta_1)\}\cap\{\mathbf{g}^*(\theta_2)\} \neq \varnothing$. _Then_ $\mathbf{g}(\theta)$, $\theta_1 \leq \theta \leq \theta_2$, _is a straight line_.

The _curvature_ $k_g(\theta)$ of Polar(g) (positive for Polar(g) strictly convex) is given by

$$k_g = [g^2-gg''+2(g')^2]/[(g')^2+g^2]^{3/2} \qquad \text{(A5)}$$

whenever g' and g'' exist. By (A1),

$$(d/d\theta)\mathbf{g}^*(\theta) = -A(\theta)k_g(\theta)\mathbf{N}(\theta), \qquad \text{(A6)}$$
$$A = g^{-3}[(g')^2+g^2]^{-3/2} > 0.$$

Theorem. <u>Let</u> θ_1 <u>and</u> θ_2 <u>have a common supporting tangent</u>. <u>Then one of the following three conditions must hold</u>:

(C1) $P(g)$ <u>is a straight line between</u> θ_1 <u>and</u> θ_2;

(C2) $g'(\theta+0)>g'(\theta-0)$ <u>at some sharp spot</u> $\theta\in[\theta_1,\theta_2]$;

(C3) $k_g(\theta)<0$ <u>at some smooth spot</u> $\theta\in[\theta_1,\theta_2]$.

<u>Proof</u>. Let $\mathbf{a}$ be the common supporting tangent, and let $\mathbf{b}$ be the unit vector with

$$\mathbf{b}\cdot\mathbf{a}=0 \quad \text{and} \quad \mathbf{b}\cdot\mathbf{N}(\theta)>0 \quad \text{for all} \quad \theta\in[\theta_1,\theta_2]. \tag{A7}$$

Suppose that neither (C2) nor (C3) are satisfied: for $\theta\in[\theta_1,\theta_2]$,

$$g'(\theta+0)\leq g'(\theta-0) \quad \text{for } \theta \text{ sharp}, \quad k_g(\theta)\geq 0 \quad \text{for } \theta \text{ smooth}. \tag{A8}$$

Let Θ denote the set of sharp spots in (θ_1,θ_2). If we integrate $(d/d\theta)\mathbf{g}^*(\theta)$ from θ_1+0 to θ_2-0 using (A6) and $(A3)_1$, and then take the inner product of the resulting relation with $\mathbf{b}$, we find that

$$[\mathbf{g}^*(\theta_2-0)-\mathbf{g}^*(\theta_1+0)]\cdot\mathbf{b} = \sum_{\theta\in\Theta} c(\theta)[g'(\theta+0)-g'(\theta-0)] - \int_{\theta_1}^{\theta_2} C(\theta)k_g(\theta)\,d\theta, \tag{A9}$$
$$c(\theta), C(\theta)>0 \quad \text{on} \quad [\theta_1,\theta_2];$$

thus, by (A8) and (A7),

$$[\mathbf{g}^*(\theta_2-0) - \mathbf{a}]\cdot\mathbf{b} - [\mathbf{g}^*(\theta_1+0) - \mathbf{a}]\cdot\mathbf{b} \leq 0. \tag{A10}$$

Next, in view of $(A3)_{2,3}$ and $(A8)_1$,

$$\mathbf{g}^*(\theta_2-0) - \mathbf{a} = \alpha_2\mathbf{N}(\theta_2), \qquad \alpha_2 \geq 0,$$
$$\mathbf{g}^*(\theta_1+0) - \mathbf{a} = \alpha_1\mathbf{N}(\theta_1), \qquad \alpha_1 \leq 0,$$

and we may use (A7) to conclude that (A10) holds with "$\leq$"

replaced by "="; hence (A9) vanishes, and this yields (A8) with inequalities replaced by equalities. Thus, by $(A3)_1$ and (A6), $g^*(\theta)$ is constant on (θ_1,θ_2); in view of (T3), this implies (C1). ■

The convex hull[62] of Polar(g) is a polar diagram Polar(G) of a function $G(\theta)$. We will refer to Polar(G) as the **convexification** of $g(\theta)$. Let

$$\mathbf{G}(\theta) := G(\theta)\mathbf{N}(\theta).$$

The set Polar(g)$\cap$Polar(G) on which the polar diagram coincides with its convex hull is important. This subset of $\mathbb{R}^2$ is conveniently identified with a set of angles, namely

$$C(g) := \{\theta\in\mathbb{R}: \ \mathbf{g}(\theta)=\mathbf{G}(\theta)\}.$$

We will refer to the <u>connected</u> <u>components</u> of $C(g)$ as the **globally-convex sections** of Polar(g). The portion of Polar(G) that is <u>disjoint</u> from Polar(g) is the union of open line-segments; the closures of these line segments will be referred to as the **Maxwell lines** of Polar(g). Let $\mathscr{m}$ be a Maxwell line with end points $\mathbf{g}(\theta_1)$ and $\mathbf{g}(\theta_2)$, $\theta_1<\theta_2$; we will refer to (θ_1,θ_2) as the **angle interval for** $\mathscr{m}$ and to θ_1 and θ_2 as **Maxwell angles**.

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[62]More precisely, boundary of the convex hull of Polar(g). In our discussion of curves the term "convex" means "strictly convex". Here our terminology is ambiguous, as the boundary of the convex hull will generally not be strictly convex. Thus the globally-convex sections of Polar(g) are allowed to have subsets with vanishing curvature.

Maxwell theorem. Let θ belong to a globally-convex section of Polar(g). Then:

$$\begin{aligned} &\mathbf{g}(\theta) = \mathbf{G}(\theta), \qquad \{\mathbf{G}^*(\theta)\} \subset \{\mathbf{g}^*(\theta)\}, \\ &g'(\theta + 0) \leq G'(\theta + 0) \leq G'(\theta - 0) \leq g'(\theta - 0), \qquad \text{(A11)} \\ &\mathbf{N}(\theta) \cdot [\mathbf{G}^*(\theta + 0) - \mathbf{G}^*(\theta - 0)] \leq 0 \end{aligned}$$

with inequality in $(A11)_4$ if θ is a sharp spot of g.

Let (θ_1, θ_2) be an angle interval for a Maxwell line. Then:

(i) θ_1 and θ_2 have a common supporting tangent $\mathbf{G}_0$ with $\mathbf{G}_0$ the constant value of $\mathbf{G}^*(\theta)$ on (θ_1, θ_2);

(ii) either (C2) or (C3) holds on (θ_1, θ_2).

Proof. $(A11)_1$ is obvious; $(A11)_3$ follows directly from the properties of the convex hull; $(A11)_3$ implies $(A11)_2$ and, by virtue of (A1), also $(A11)_4$. Further $\mathbf{G}^*(\theta) \equiv \mathbf{G}_0$ on (θ_1, θ_2) follows from (T3) applied to G rather than g, while $(A11)_2$ yields $\mathbf{G}_0 \in \{\mathbf{g}^*(\theta_1)\} \cap \{\mathbf{g}^*(\theta_2)\}$; the desired conclusion in (i) then follows from the theorem on common tangents. Finally, between θ_1 and θ_2, P(G) is a straight line disjoint from P(g), thus (C1) is not possible, so that either (C2) or (C3) is satisfied. ■

Appendix B. Invariance under reparametrization.

A suitable discussion of invariance requires a class of time-dependent curves broader than the class of evolving curves. Let T, $0<T\leq\infty$, be fixed. A **time-dependent interval** is a set of the form $\mathbb{R}\times[0,T)$ or a set of the form

$$\{(p,t):\quad p\in[P(t),Q(t)],\quad t\in[0,T)\}\tag{B1}$$

with $P,Q:[0,T)\to\mathbb{R}$ $(P<Q)$ smooth functions. A **time-dependent curve** is a smooth mapping $(p,t)\mapsto\mathbf{r}(p,t)$ such that:
(i) the domain of $\mathbf{r}$ is a time-dependent interval;
(ii) $\mathbf{r}(\cdot,t)$ is a <u>curve</u> for each $t\in[0,T)$.
If $\mathrm{Domain}(\mathbf{r})$ has the form (B1), then $\mathbf{r}$ has **endpoints** $\mathbf{r}(P(t),t)$ and $\mathbf{r}(Q(t),t)$.

We write $\mathcal{C}$ for the set of time-dependent curves. Let $\mathbf{r}\in\mathcal{C}$. Then $\mathbf{r}$ **evolves normally** if

$$\mathbf{r}_t(p,t)\cdot\mathbf{r}_p(p,t)=0$$

for all $(p,t)\in\mathrm{Domain}(\mathbf{r})$; thus the term "**evolving curve**" as used in the main body of the paper is here synonymous with **normally evolving curve**.

Let $\mathbf{r}\in\mathcal{C}$. We consider the tangent $\mathbf{T}(p,t):=\mathbf{r}_p(p,t)/|\mathbf{r}_p(p,t)|$ and normal $\mathbf{N}(p,t)$ to $\mathbf{r}$ as functions of $(p,t)\in\mathrm{Domain}(\mathbf{r})$, and similarly for the **normal velocity**

$$V(p,t)\ :=\ \mathbf{r}_t(p,t)\cdot\mathbf{N}(p,t).\tag{B2}$$

Further, we define

$$\begin{aligned}J(p,t)\ &:=\ |\mathbf{r}_p(p,t)|,\\ \upsilon(p,t)\ &:=\ -J(p,t)^{-1}\mathbf{r}_t(p,t)\cdot\mathbf{T}(p,t).\end{aligned}\tag{B3}$$

Let $Z(t)$ be given with $(Z(t),t)\in\mathrm{Domain}(\mathbf{r})$ for some interval of t.

Then the curve $t \mapsto \mathbf{r}(Z(t),t)$ is a **normal trajectory** provided

$$\mathbf{T}(Z(t),t) \cdot (d/dt)\mathbf{r}(Z(t),t) = 0.$$

But by (B3),

$$\mathbf{T}(Z(t),t) \cdot (d/dt)\mathbf{r}(Z(t),t) = J(Z(t),t)[dZ(t)/dt - \mathfrak{v}(Z(t),t)], \quad \text{(B4)}$$

so that $\mathfrak{v}(p,t)$ gives the rate at which the parameter p changes with time following a normal trajectory. This discussion should motivate the following definition.

Choose $(p_0,t_0) \in \text{Domain}(\mathbf{r})$. The function $t \mapsto Z(t)$, <u>maximally</u> defined as the solution of the problem

$$dZ(t)/dt = \mathfrak{v}(Z(t),t), \qquad Z(t_0) = p_0, \quad \text{(B5)}$$

is the **normal parameter-trajectory** through (p_0,t_0).

Let $\mathbf{r} \in \mathcal{C}$, let Φ be a smooth function on $\text{Domain}(\mathbf{r})$, and choose $(p,t) \in \text{Domain}(\mathbf{r})$. Then the **normal time-derivative** $\Phi^\circ(p,t)$ of Φ at (p,t) is defined as follows:

$$\Phi^\circ(p,t) = (d/d\tau)\Phi(Z(\tau),\tau)\big|_{\tau = t}, \quad \text{(B6)}$$

with $Z(\tau)$ the normal parameter-trajectory through (p,t). Clearly, $\mathbf{r}^\circ \cdot \mathbf{N} = \mathbf{r}_t \cdot \mathbf{N}$, while (B4) and (B5) yield $\mathbf{r}^\circ \cdot \mathbf{T} = 0$; hence (B2) implies

$$\mathbf{r}^\circ(p,t) = V(p,t)\mathbf{N}(p,t). \quad \text{(B7)}$$

The next proposition is easily verified.

Proposition 1. <u>Let</u> $\mathbf{r} \in \mathcal{C}$. <u>Then the following are equivalent</u>:

(i) $\mathbf{r}$ <u>evolves normally</u>;

(ii) <u>the normal parameter-trajectories are of the form</u> $Z = \text{constant}$;

(iii) $\mathbf{r}_t \equiv \mathbf{r}^\circ$;

(iv) $\upsilon \equiv 0$.

<u>Moreover, if</u> $\mathbf{r}$ <u>evolves normally, and if</u> Φ <u>is a smooth function on</u> domain($\mathbf{r}$), <u>then</u>

$$\Phi^\circ = \Phi_t. \tag{B8}$$

Let $\mathbf{r} \in \mathcal{C}$. By a **parameter change** for $\mathbf{r}$ we mean a smooth bijection $\boldsymbol{\phi}$, from a time-dependent interval onto Domain($\mathbf{r}$), of the form

$$(p,t) \mapsto \boldsymbol{\phi}(p,t) = (\phi(p,t),t), \qquad \phi_p > 0;$$

if $\mathbf{r}$ is closed we require, in addition, that there exist a smooth function $\omega > 0$ on $[0,T)$ such that

$$\phi(p + \omega(t),t) = \phi(p,t) + \lambda(t) \tag{B9}$$

for all $(p,t) \in \mathbb{R} \times [0,\infty)$, where $\lambda(t)$ is the minimal period of $\mathbf{r}(\cdot,t)$. Given a parameter change $\boldsymbol{\phi}$ for $\mathbf{r}$, the function $\mathbf{r} \circ \boldsymbol{\phi}$ on Domain($\boldsymbol{\phi}$) defined by

$$(\mathbf{r} \circ \boldsymbol{\phi})(p,t) = \mathbf{r}(\boldsymbol{\phi}(p,t)) = \mathbf{r}(\phi(p,t),t) \tag{B10}$$

is also a member of $\mathcal{C}$ (and is closed if $\mathbf{r}$ is closed); we refer to $\mathbf{r} \circ \boldsymbol{\phi}$ as a **reparametrization** of $\mathbf{r}$. This definition, (B4) with $Z(t) = \phi(p,t)$, and the equivalency of (i) and (iv) in Proposition 1 yield

Proposition 2. *Let* $\mathbf{r} \in \mathcal{C}$. *Then* $\mathbf{r} \circ \boldsymbol{\phi}$ *evolves normally if and only if* $\phi_t(p,t) = \upsilon(\phi(p,t),t)$ *for all* $(p,t) \in \text{Domain}(\boldsymbol{\phi})$, *so that each of the functions* $t \mapsto \phi(p,t)$ *is a normal parameter-trajectory. Thus, when* $\mathbf{r}$ *evolves normally,* $\mathbf{r} \circ \boldsymbol{\phi}$ *evolves normally if and only if* $\phi_t = 0$.

The next result is central; it shows that within a large class of time-dependent curves there is no essential loss of generality in limiting attention to curves that evolve normally.

Theorem 1. *Let* $\mathbf{r} \in \mathcal{C}$ *satisfy one of the following three conditions:*

(i) $\mathbf{r}$ *is closed;*

(ii) $\mathbf{r}$ *has endpoints, and the endpoints are normal trajectories;*

(iii) $\mathbf{r}$ *is unbounded, and there are smooth functions* $a : \mathbb{R} \to \mathbb{R}$ *and* $b : [0,T) \to \mathbb{R}$ *such that, for all* $(p,t) \in \text{Domain}(\mathbf{r})$,

$$|\upsilon(p,t)| \le a(p)b(t). \tag{B11}$$

Then there is a parameter change $\boldsymbol{\phi}$ *for* $\mathbf{r}$ *such that* $\mathbf{r} \circ \boldsymbol{\phi}$ *is a normally evolving curve.*

Proof. For each p in the initial interval $[P(0),Q(0)]$ or $\mathbb{R}$, let $t \mapsto Z(p,t)$ denote the normal parameter-trajectory through $(p,0)$. Consider (ii). In this case $Z(P(0),t) = P(t)$ and $Z(Q(0),t) = Q(t)$ for $0 \le t < T$; thus, since $Z(p,t)$ is, for p fixed, a maximal solution of (B5), and since $P(0) \le p \le Q(0)$, $Z(p,t)$ is also defined for $0 \le t < T$. In fact, the definition of Z as the solution of (B5) renders the mapping $\boldsymbol{\phi}$ defined by $\boldsymbol{\phi}(p,t) = (Z(p,t),t)$ a smooth bijection of $[P(0),Q(0)] \times [0,T)$ onto $\text{Domain}(\mathbf{r})$. Further, differentiating $Z_t(p,t) = \upsilon(Z(p,t),t)$ with respect to p, one easily concludes that $Z_p(p,t) > 0$, since it has this property at $p = 0$. Thus $\boldsymbol{\phi}$ is a parameter change for $\mathbf{r}$. The last proposition then implies that $\mathbf{r} \circ \boldsymbol{\phi}$ is a normally evolving curve.

Consider (iii). Let $\boldsymbol{\phi}_0$ denote the parameter change for $\mathbf{r}$ defined by $\boldsymbol{\phi}_0(p,t) = (\phi_0(p),t)$ with $\phi_0(p)$ any solution of $d\phi_0(p)/dp = a(p)$. Then the reparametrization $\mathbf{r}_0 \circ \boldsymbol{\phi}_0$ obeys (B11) with $a(p) = 1$. Thus it suffices to consider unbounded curves $\mathbf{r} \in \mathcal{C}$ that obey an estimate of the form

$$|\upsilon(p,t)| \leq b(t) \tag{B12}$$

with b continuous on $[0,T)$. This estimate and the definition of Z as the solution of (B5) imply that, for each $p \in \mathbb{R}$, $Z(p,t)$ is defined for $0 \leq t < T$. In fact, arguing as above, the mapping $\boldsymbol{\phi}$ is a parameter change for $\mathbf{r}$, and $\mathbf{r} \circ \boldsymbol{\phi}$ is a normally evolving curve.

Consider (i). Since $\mathbf{r}$ is periodic with period $\lambda(t)$ a smooth function of t, the estimate (B12) again is satisfied. Thus the mapping $\boldsymbol{\phi}$ defined by $\boldsymbol{\phi}(p,t) = (Z(p,t),t)$ is a smooth bijection of $\mathbb{R} \times [0,T)$ onto Domain($\mathbf{r}$), and $Z_p(p,t) > 0$. Since $\mathbf{r}(z,t) = \mathbf{r}(z + \lambda(t),t)$, (B3) yields $\upsilon(z + \lambda(t),t) = \upsilon(z,t) + \lambda_t(t)$, and this, in turn, leads to the conclusion that $z(p,t) := Z(p,t) + \lambda(t)$ satisfies $z_t(p,t) = \upsilon(z(p,t),t)$, $z(p,0) = p + \lambda(0)$. Thus $z(p,t) = Z(p + \lambda(0),t)$, so that $Z(p,t) + \lambda(t) = Z(p + \lambda(0),t)$ for all (p,t), and we have compliance with the condition (B9). Thus $\boldsymbol{\phi}$ is a parameter change for $\mathbf{r}$, and $\mathbf{r} \circ \boldsymbol{\phi}$ is a normally evolving curve. ■

We define the **arc-length derivative** $\Phi_s(p,t)$, the **curvature** $K(p,t)$, and the **angle derivative** $\Phi_\theta(p,t)$ through:

$$\begin{aligned} \Phi_s(p,t) &:= \Phi_p(p,t) J(p,t)^{-1}, \\ K(p,t) &:= \mathbf{N}(p,t) \cdot \mathbf{T}_s(p,t), \\ \Phi_\theta(p,t) &:= K(p,t)^{-1} \Phi_s(p,t) \end{aligned} \tag{B13}$$

(the last definition being appropriate to (p,t) with $K(p,t) \neq 0$).

To discuss invariance under reparametrization, we now write $\mathbf{T}_r(p,t)$, $\mathbf{N}_r(p,t)$, $V_r(p,t)$, and $K_r(p,t)$ to make explicit the dependence of these quantities on the time-dependent curve $\mathbf{r} \in \mathcal{C}$

in question. This allows us to consider, for example, the normal velocity as a mapping V that assigns to each $\mathbf{r} \in \mathcal{C}$ a function $(p,t) \mapsto V_{\mathbf{r}}(p,t)$ on Domain($\mathbf{r}$).

More generally, a **curve descriptor** is a mapping Ψ that assigns to each $\mathbf{r} \in \mathcal{C}$ a function $(p,t) \mapsto \Psi_{\mathbf{r}}(p,t)$ on Domain($\mathbf{r}$). Given a curve descriptor Ψ, we may consider its normal time-derivative, its arc-length derivative, and its angle derivative as curve descriptors; i.e., e.g., $(\Psi_s)_{\mathbf{r}} := (\Psi_{\mathbf{r}})_s$.

A curve descriptor Ψ is intrinsic if it is invariant under reparametrization; that is, if, at each t, its value at p on a reparametrized curve $\mathbf{r} \circ \boldsymbol{\phi}$ is the same as its value at $\phi(p,t)$ on the original curve $\mathbf{r}$. Precisely, Ψ is **intrinsic** if, given any $\mathbf{r} \in \mathcal{C}$ and any parameter change $\boldsymbol{\phi}$ for $\mathbf{r}$,

$$\Psi_{(\mathbf{r} \circ \boldsymbol{\phi})} = (\Psi_{\mathbf{r}}) \circ \boldsymbol{\phi}.$$

The following result is well known.

Invariance theorem. <u>The following curve descriptors are intrinsic: tangent, normal, normal velocity, and curvature. If a curve descriptor is intrinsic, then so also are its normal time-derivative, its arc-length derivative, and its angle derivative.</u>

<u>Proof</u>. Let $\boldsymbol{\phi}$ be a parameter change for $\mathbf{r} \in \mathcal{C}$, let $\mathbf{g} = \mathbf{r} \circ \boldsymbol{\phi}$, and write $\mathbf{g}(p,t) = \mathbf{r}(q,t)$, $q = \phi(p,t)$. Since $\mathbf{g}_p = \mathbf{r}_q \phi_p$, $\mathbf{T}_{\mathbf{g}}(p,t) = \mathbf{g}_p(p,t)/|\mathbf{g}_p(p,t)| = \mathbf{r}_q(q,t)/|\mathbf{r}_q(q,t)| = \mathbf{T}_{\mathbf{r}}(q,t)$, so that $\mathbf{T}$ and (hence) $\mathbf{N}$ are invariant. Let Ψ be an intrinsic curve-descriptor. Then the same argument applied to $(B13)_1$ yields Ψ_s intrinsic. Thus, by $(B13)_{2,3}$, K and Ψ_θ are invariant. Next, let $Z(t)$ be the normal parameter-trajectory for $\mathbf{g}$ through (p_0,t_0). A simple calculation shows that

$$(d/dt)\mathbf{r}(\phi(Z(t),t),t) = (d/dt)\mathbf{g}(Z(t),t) = 0,$$

so that $z(t) := \phi(Z(t),t)$ is the normal parameter-trajectory for $\mathbf{r}$ through $\boldsymbol{\phi}(p_0,t_0)$. On the other hand, since Ψ is intrinsic,

$$(d/d\tau)\Psi_{\mathbf{g}}(Z(\tau),\tau) = (d/d\tau)\Psi_{\mathbf{r}}(z(\tau),\tau). \tag{B14}$$

At $\tau = t_0$, the left side of (B14) is $(\Psi^\circ)_{\mathbf{g}}(p_0,t_0)$, the right side is $(\Psi^\circ)_{\mathbf{r}}(\boldsymbol{\phi}(p_0,t_0))$; thus the normal time-derivative of an intrinsic curve-descriptor is intrinsic. In particular, $\mathbf{r} \mapsto \mathbf{r}^\circ$ is intrinsic; hence, by (B7), V is intrinsic. ■

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