COMPOSITIONS FOR BALANCED HYPERGRAPHS

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Abstract

This paper presents two hypergraph compositions which preserve balancedness. Polynomial decomposition algorithms are also presented. The hope is that such compositions will lead to a polynomial algorithm for recognizing balanced hypergraphs.

The relationships between totally unimodular, balanced and perfect matrices are also briefly discussed.

Introduction

Consider the integer linear program given below.

$$\text{(IP)} \qquad \max \quad cx \quad \text{subject to} \quad Ax \leq b, \quad x \text{ integer}$$

where A is an $m \times n$ zero-one matrix, c is an n-vector of integers, and b is an m-vector of integers. The linear programming relaxation of IP is:

$$\text{(LP)} \qquad \max \quad cx \quad \text{subject to} \quad Ax \leq b, \quad x \geq 0.$$

Currently, there is no polynomial algorithm for solving IP, but there is a good algorithm for solving LP. The value of the objective function for an optimal solution to LP is always greater than or equal to the value of the objective function for an optimal solution to IP, so a solution to LP provides an upper bound on solutions to IP. We are interested in under what conditions on A and b this bound is tight.

Definition: A 0,1,-1 matrix A is totally unimodular if the determinant of every square submatrix is 0 or ± 1.

In 1956, Hoffman and Kruskal showed that a basic optimal solution to LP is an optimal solution to IP for all b if and only if A is totally unimodular. By restricting b to $b = e$, a vector of ones, we get a more general class of matrices for which a basic optimal solution to LP is also an optimal solution to IP.

Define IP' to be the integer linear program below:

$$
\text{(IP')} \qquad \max \quad c'x' \qquad \text{subject to} \quad A'x' \leq e', \quad x' \ \text{integer}
$$

where A' is a submatrix of A and e', c' and x' are the compatibly dimensioned vectors. Let LP' be the linear programming relaxation of IP'.

<u>Definition</u> (Berge 1972): Let A be any zero-one matrix. A is <u>balanced</u> if a basic optimal solution to LP' is also an optimal solution IP' for all submatrices A' of A.

Since A being totally unimodular implies a basic optimal solution to LP is an optimal solution to IP for any integer vector b, a totally unimodular zero-one matrix is balanced.

There is a nice characterization of balanced matrices due to Berge [1], and Fulkerson, Hoffman, and Oppenheim.

<u>Theorem 1</u>: Let A be any $m \times n$ matrix of zeroes and ones. Then A is balanced if and only if A does not contain any odd submatrices with row sums and column sums equal to two.

Although there is a nice characterization of balanced matrices, there is no polynomial for determining whether an arbitrary zero-one matrix is balanced. The remainder of this paper is devoted to observing some properties of balanced matrices which may help with this problem. Before examining balanced matrices more closely, there is one more type of matrix the reader should be familiar with. Again we require $b = e$ and define IP and LP as before.

Definition: Let A be any zero-one matrix. A is perfect if a basic optimal solution to LP is also an optimal solution to IP.

Since A being balanced implies a basic optimal solution to LP' is also an optimal solution to IP' for all submatrices A' of A including, of course, $A' = A$, a balanced matrix is perfect.

If the reader is familiar with perfect graphs, he may be wondering about the connection between perfect matrices and perfect graphs. Any graph G can be described by a zero-one matrix $A = [a_{ij}]$ with the rows of A corresponding to the cliques of G and the columns of A corresponding to the vertices of G. $a_{ij} = 1$ if clique i includes vertex j. An augmented clique matrix is the clique matrix described above, possibly augmented with some redundant rows corresponding to non-maximal cliques.

Theorem 2: A is perfect if and only if A is an augmented clique matrix of its derived graph and its derived graph is perfect.

In a similar way, balanced matrices can be related to balanced hypergraphs.

Background

The definitions and results of this section are due to Berge.

Definition: Given a finite set $X = \{x_{1\#}...,x_m\}$ and a family £ = $(E_i|i$ € I) of subsets of X, if £ satisfies:

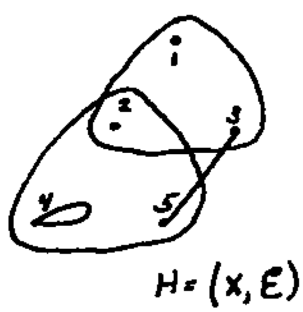
(i) $E_\pm$ * θ, i £ I,

(ii) $\bigcup_{i \in I} E_i = X$,

then H = (X,£) is called a hypergraph. The elements $x_1, x_2, ..., x_m$ are called the vertices of H and the sets $E_1, E_2, ..., E_n$ are called the edges of H. Two vertices x_i, x_j are adjacent if $[x_i, x_j\}$ 5 E_k for some k € I.

Write x_i adj x_j if x_i is adjacent to x_j, and x_i *p&i* x_j if x_i is not adjacent to x_j.

Hypergraphs can be represented by drawing a point for each vertex and a curve enclosing the vertices in E_i for each edge with $JE_i|$ > *2.* If $|E_i|$ = *2,* E_i is drawn as a line segment connecting the two vertices as in a graph. An edge with $|E_i$.I =1 is drawn as a loop. See Figure 1. A hypergraph with $|E_i|$ £ 2 for all i 6 I is a multigraph.



x = {1,2,3,4,5}

e = £U,2,3),£2,4#5),{3,5},UJJ

Figure 1. A hypergraph.

Definition: In a hypergraph $H = (X,\mathcal{E})$, a chain of length q is defined to be sequence $(x_1,E_1,x_2,\ldots,E_q,x_{q+1})$ such that

(i) $x_1,x_2,\ldots,x_q$ are all distinct vertices of H,

(ii) $E_1,E_2,\ldots,E_q$ are all distinct edges of H,

(iii) $x_k,x_{k+1} \in E_k$ for $k = 1,\ldots,q$.

If $q > 1$ and $x_{q+1} = x_1$, then this chain is a cycle of length q.

Definition: A hypergraph H is balanced if every odd cycle $(x_1,E_1,x_2,\ldots,E_{2p+1},x_1)$ has an edge E_i that contains at least three vertices x_j of the cycle.

The reader (hopefully) is now wondering about the connection between balanced matrices and balanced hypergraphs. Well, we can consider any zero-one matrix to be the edge-node incidence matrix of a hypergraph. A matrix A is balanced if and only if it is the incidence matrix of a balanced hypergraph.

The next section of this paper describes a method of decomposing a balanced hypergraph into two smaller balanced hypergraphs. This decomposition principle has been used extensively in efforts to characterize perfect graphs. The decomposition discussed here has been shown to preserve perfection when applied to perfect graphs. Before going on, we need a few more definitions.

We can associate a graph G with each hypergraph $H = (X,\mathcal{E})$ by letting X be the vertices of G and two vertices x_i,x_j be joined by an edge if there is an $E_k \in \mathcal{E}$ with $\{x_i,x_j\} \subseteq E_k$.

<u>Definition</u>: A hypergraph $H = (X, \mathcal{E})$ is <u>conformal</u> if the set of maximal cliques in G is equal to the set of maximal edges of H.

Berge has shown that balanced hypergraphs are conformal.

<u>Definition</u>: A <u>partial hypergraph</u> of $H = (X, \mathcal{E})$ generated by a family $\mathcal{F} \subset \mathcal{E}$ is defined to be the hypergraph $(X_{\mathcal{F}}, \mathcal{F})$ where $X_{\mathcal{F}} = \cup_{E_i \in \mathcal{F}} E_i$. This corresponds to an edge-induced subgraph. The <u>subhypergraph</u> of $H = (X, \mathcal{E})$ generated by the set $A \subseteq X$ is defined to be the hypergraph $H_A = (A, \mathcal{E}_A)$, where

$$\mathcal{E}_A = \{E_i \cap A \mid E_i \in \mathcal{E},\ E_i \cap A \neq \emptyset\}.$$

This corresponds to a node-induced subgraph.

If H is a balanced hypergraph, then every partial hypergraph and every subhypergraph are also balanced.

<u>Definition</u>: Given a hypergraph $H = (X, \mathcal{E})$, a <u>q-coloring</u> of H is defined to be a partition of X into q sets $S_1, S_2, \ldots, S_q$, each corresponding to a color, with no edge $E_i \in \mathcal{E}$ contained in S_j, $j = 1, \ldots, q$.

<u>Theorem 3</u> (Berge): A hypergraph $H = (X, \mathcal{E})$ is balanced if and only if for every $S \subseteq X$, the subhypergraph H_S is bicolorable.

<u>Proof</u>:

($\Leftarrow$) If every subhypergraph of H is bicolorable, then H is balanced because otherwise there exists an odd cycle $(a_1, E_1, a_2, \ldots, E_p, a_1)$ with no E_i containing three of the a_j and $S = \{a_1, a_2, \ldots, a_p\}$ generates a subhypergraph H_s which is not bicolorable.

($\Rightarrow$) Let H be a balanced hypergraph that is not bicolorable, with minimum order $n = |X|$. We shall contradict the minimality of n.

1) Show that each vertex x_0 belongs to at least two distinct edges of H with exactly two elements. The subhypergraph H_0 generated by $X - \{x_0\}$ is balanced (since H is balanced) and has order $n - 1$. Since H is of minimum order, H_0 has a bicoloring (S_1^0, S_2^0). If x_0 did not belong to an edge with exactly two elements, then $(S_1^0 \cup \{x_0\}, S_2^0)$ would be a bicoloring of H. If x_0 belongs to only one edge with two elements, and if the edge's other endpoint lies in S_1^0, then $(S_1^0, S_2^0 \cup \{x_0\})$ would be a bicoloring of H. Hence, x_0 belongs to at least two edges with two elements, say (x_0, y) and (x_0, z) with $y \neq z$.

2) Denote by $\mathcal{G}$ the family of edges with exactly two elements. Consider the partial hypergraph $G = (X, \mathcal{G})$. G is balanced and since $|G_i| = 2$ for all $G_i \in \mathcal{G}$, G is a multigraph. Since G is balanced, G is bipartite. (No edge of G contains three vertices, so G cannot have an odd cycle.) Consider a connected component C of G. Since G has at least three vertices (from 1), there exists at least one vertex x_1 in C which is not an articulation point.

3) Consider the subhypergraph H_1 generated by $X - \{x_1\}$. It is balanced and has order $n - 1$. Therefore H_1 admits a bicoloring (S_1, S_2), and all vertices adjacent to x_1 in G have the same color. Let S_1 be the set of vertices with this

color. Then, $(S_1, S_2 \cup \{x_1\})$ is a bicoloring of H because each edge of H with two elements is bicolored (since it is an edge of G), and each edge of H with more than two elements is also bicolored (because its intersection with $X - \{x_1\}$ is bicolored). ∎

Decompositions

Given two graphs G_1 and G_2, define the i-join composition as follows:

For $j = 1,2$, consider a clique of size i in G_j with vertices $\{v_1^j,\dots,v_i^j\}$ and let U_j be the remaining vertices in G_j. Assume no vertex of U_j is adjacent to more than one vertex v_h^j. The composed graph $G_1 * G_2$ is obtained by deleting v_h^1 and v_h^2 for each $h = 1,\dots,i$ and joining every neighbor of v_h^1 to every neighbor of v_h^2.

It is known that when $i = 1$ or 2, the i-join composition preserves perfection. (See, for example, Bixby [3], Cornuejols and Cunningham [4].)

We can generalize this composition to hypergraphs as follows:

Let H_1 and H_2 be hypergraphs. For $j = 1,2$, consider an edge E_j in H_j with $|e_j| = i$ and $e_j = \{v_1^j,\dots,v_i^j\}$ and let U_j be the remaining set of vertices in H_j. Assume no vertex of U_j is adjacent to more than one vertex v_h^j. The composed hypergraph $H_1 * H_2$ is obtained by deleting v_h^1 and v_h^2 and for each pair of edges E_h^1 containing v_h^1 and E_h^2 containing v_h^2 replacing them with the single edge $E = E_h^1 \cup E_h^2 \backslash \{v_h^1, v_h^2\}$.

<u>Lemma 1</u>: Let H be obtained from the i-join of H_1 and H_2. Let K be a subhypergraph of H, but not a subhypergraph of H_1 or H_2. Then $K = K_1 * K_2$ for some subhypergraph K_1 of H_1 and K_2 of H_2.

Proof; Define

$$V(K_1) = \{V(K)\ Hvd\text{^})\}\ U\ \{v_1^*,...,v_i^*3$$

$$V(K_2) = [v(K)\ nv(H_2)\}\ u\ W^*_1, ...,v_i^*3$$

where [v^,...,v-?3 are the vertices joined in the i-join of $H_1 * H_2$. Let K_j denote the subhypergraph of H_j generated by $V(K_j)$, j = 1,2.

Claim: K = 1^ * K_2«

Clearly, V(K) = V(K_1 * K_2>.

Let E be an edge of K. If E 5 V(K_j) for some j = 1,2, then E 6 £(K_1 * K_2). Otherwise, partition E into two sets $E_\pm$ = {E n Vd^) j c: *V(E_X)* and E_2 = {E H V(K_2> } c V(H_2> . E € e(K) so E_c E^1 € £(H) since K is a subhypergraph of H. An edge of H contains vertices of H_1 and H_2 only if there exist edges EJ = [E^1 flV(H_1)] U [v^j and *E'*$_2$ = [E' flV(H_2)] U {v^j for some h = l,...,i. Hence, E_j U {v^} € £(K_j), j = 1,2 and E € £(1^ * K_2) .

Now let E be an edge of *$_1$ * K_2. If E c v(K_j) for some j = 1,2, then E € G(K). Otherwise partition E into two sets E_x = [EH V(K_1) 3 c: V(H_1) and E_2 = {E fl V(K_2) 3 cV(H_2> • Then there exist edges EJ e £(1^) containing E_1 U [v^3 and E'_2 € fc(H_2) containing E_2 U {v_h} for some h = l,...,i which implies E € £(H); E € £(K) since K is a subhypergraph of H. ■

Theorem 4; Let two balanced hypergraphs H_1 and H_2 be given. For j = 1,2, consider a subset e_j of an edge E_j in H_j with |e_j| = i and e_j = {v^,____,v_i?} and let U_j be the remaining set of vertices in H_j. Assume no vertex of U_j is adjacent to more than one vertex v_h^j. Let $c(v_h^j)$ denote the color of vertex v_h^j in a bicoloring of H_j. If H_1 and H_2 admit bicolorings such that $c(v_h^1) \neq c(v_h^2)$, then the hypergraph $H = H_1 * H_2$ formed by this i-join composition is balanced.

Before proving the theorem, let's think about the hypotheses. At first they seem terribly restrictive, especially since there is no easy way to find bicolorings of hypergraphs, but for $i = 1$ or 2, the hypotheses always hold and for $i = 3$ there is always a bicoloring for which they will hold. In other words, this theorem says that for $i = 1,2$ the i-join preserves balancedness and there is always a way to apply the 3-join so it preserves balancedness. An interesting open question is whether there is always a way to apply the 3-join to perfect graphs so it preserves perfection.

There is also a class of hypergraphs, called unimodular hypergraphs (their edge-node incidence matrices are unimodular), which admit equitable bicolorings, i.e., the edges can be colored with two colors in such a way that an even cardinality edge has exactly the same number of vertices of each color and an odd cardinality edge has one more vertex of one color than the other. So this theorem also states that as long as there is an edge E_j in H_j of cardinality i for $j = 1,2$, there is a way to apply the i-join to two unimodular hypergraphs so that the resulting hypergraph will be balanced.

<u>Proof of Theorem 4</u>: Consider the bicoloring of H_1 and H_2 and the family of edges $\mathfrak{F}_h^j = \{E_h^j : v_h^j \in E_h^j, E_h^j \neq \{v_h^j\}\}$. For all $E_h^j \in \mathfrak{F}_h^j$, $E_h^j \backslash \{v_h^j\}$ contains a vertex whose color is not the color of v_h^j, $h = 1,\ldots,i$, $j = 1,2$. All new edges E formed by the i-join contain $E_h^1 \backslash \{v_h^1\}$ and $E_h^2 \backslash \{v_h^2\}$ for some $h = 1,\ldots,i$. Since $c(v_h^1) \neq c(v_h^2)$, E contains vertices of both colors, i.e. H admits a bicoloring.

To show H is balanced, we must also show that any subhypergraph K of H admits a bicoloring. If K is a subhypergraph of H_j for some $j = 1$ or 2, K is balanced and so K admits a bicoloring. Otherwise, by Lemma 1, $K = K_1 * K_2$ for some subhypergraphs K_1 of H_1 and K_2 of H_2. K_1 and K_2 are balanced so we can apply the above argument to $K = K_1 * K_2$ to show K admits a bicoloring. ∎

Such a decomposition is useful in helping to identify balanced hypergraphs only if there is a polynomial algorithm for recognizing how to decompose a hypergraph into its smaller components. The algorithms for decomposing graphs are given in Cunningham [5] and Cornuejols and Cunningham [4]. The algorithms given below and the proofs that they work are essentially those given for graphs.

Consider the hypergraph $H = (X,\mathcal{E})$ with $X = \{x_1,\ldots,x_n\}$ and $\mathcal{E} = \{E_1,\ldots,E_m\}$. Assume for convenience that H is connected. Given a partition of X into two sets (A_1,A_2), let $B_1 = \{u \in A_1 : u \text{ adj } v \text{ for some } v \in A_2\}$ and similarly for B_2. We say (A_1,A_2) is a (<u>join</u>) <u>split</u> of X if:

(i) $|A_1| \geq 2 \leq |A_2|$,

(ii) u adj v whenever $u \in B_1$, $v \in B_2$.

H has a join decomposition if and only if it admits a split (A_1,A_2).

To find such a split, fix an adjacent pair of nodes x,y and fix $z \neq x,y$. We will now give an algorithm to find a split (A_1,A_2) with $x,z \in A_1$ and $y \in A_2$, or show no such split exists. To find a join decomposition or show none exists, it is necessary to apply this algorithm twice, once with the roles of x and y interchanged, for each pair of adjacent nodes x,y. Thus, Algorithm 1 is applied 2K times where K is the number of pairs of adjacent nodes. Note, $K \leq \binom{n}{2} < n^2$. Algorithm 1 starts with and maintains a partition $(S,X\backslash S)$ with $S \subseteq A_1$, $A_2 \subseteq X\backslash S$.

<u>Algorithm 1</u>:

Initialization: $S = \{x,z\}$, $T = S$.

While $T \neq \emptyset$ do

Select $u \in T$

$T = T\backslash\{u\}$

For $v \in X\backslash S$

(1) If u adj y, x adj v, and u ~~adj~~ v, then $S = S \cup \{v\}$, $T = T \cup \{v\}$.

(2) if u adj v, x ~~adj~~ v, then $S = S \cup \{v\}$, $T = T \cup \{v\}$.

(3) If u adj v, u ~~adj~~ y,

then $S = S \cup \{v\}$, $T = T \cup \{v\}$.

End for

End while

<u>Justification for Algorithm 1</u>:

(1) Since u adj y and $u \in S$, $u \in B_1$. If $v \in A_2$ then since x adj v, $v \in B_2$ and so u adj v by (ii); contradiction, so $v \in A_1$.

(2) Since x adj y, $x \in B_1$ and if $v \in A_2$ then since u adj v, $v \in B_2$. If $v \in A_2$ then x adj v by (ii); contradiction. so $v \in A_1$.

(3) Since u ~~adj~~ y, $y \in B_2$ and $u \in S$, $u \in A_1 \backslash B_1$. Since u adj v, $v \in A_1$.

<u>Proposition 1</u>: Suppose Algorithm 1 is applied to H with S initialized to $\{x,z\}$. If $|X \backslash S| \geq 2$, then $A_1 = S$, $A_2 = X \backslash S$ is a split with $\{x,z\} \subseteq A_1$, $y \in A_2$. Otherwise, no such split exists.

<u>Proof</u>: The second part of the proposition, that $|X \backslash S| < 2$ implies no such split exists follows immediately from the fact that $A_2 \subseteq X \backslash S$ and $|A_2| \geq 2$ for any split. Now suppose that $|X \backslash S| \geq 2$. We must show $A_1 = S$, $A_2 = X \backslash S$ satisfies (i) and (ii). Of course (i) is satisfied. Suppose $u \in B_1$, $v \in B_2$. By the definition of B_1, B_2 there exist $p \in A_1$, $q \in A_2$ with u adj q and p adj v. Since (2) left $v \in X \backslash S$, x adj v and since (3) left $v \in X \backslash S$, u adj y. Then since (1) left

$v \in X \backslash S$, u adj v. Thus (ii) holds, so $(S, X \backslash S)$ is a split.

Algorithm 1 can be implemented in $O(n^2)$ time (see Cornuejols and Cunningham [4]). Thus finding a join decomposition or showing none exists can be done in $O(n^2K)$ time.

To find a 2-join decomposition, we need to find a (<u>2-join</u>) <u>split</u> satisfying:

(i') $|A_1| \geq 3 \leq |A_2|$

(ii') There exists a partition $\{B_{i1}, B_{i2}\}$ of B_i, $i = 1,2$, such that if $u \in B_{1j}$, $v \in B_{2k}$, then u adj v if and only if $j = k$.

The algorithm for finding a 2-join split is an extension of Algorithm 1. Start with 4 distinct vertices x_1, x_2, y_1, y_2 with x_1 adj y_1, x_2 adj y_2, x_1 ~~adj~~ y_2, and x_2 ~~adj~~ y_2. We look for a split (A_1, A_2) for which $x_j \in B_{1j}$, $y_j \in B_{2j}$, $j = 1,2$. Choose $z \neq x_1, x_2, y_1, y_2$. Algorithm 2 below finds a partition with $x_1, x_2, z \in A_1$ and $y_1, y_2 \in A_2$. Again we need to apply the algorithm with the roles of x_i and y_i interchanged, $i = 1,2$ and for all possible choices of x_1, x_2, y_1, y_2. Algorithm 2 also starts with and maintains $S \subseteq A_1$, $A_2 \subseteq X \backslash S$.

<u>Algorithm 2</u>:

Initialization: $S = \{x_1, x_2, z\}$, $T = S$.

While $T \neq \emptyset$ do

Select $u \in T$

$T = T \backslash \{u\}$

For $v \in X \backslash S$

(1') If $u \not\text{adj}\ v$ and for some i u adj y_i, x_i adj v,
then $S = S \cup \{v\}$, $T = T \cup \{v\}$.

(2') If u adj v and $x_i \not\text{adj}\ v$, $i = 1,2$,
then $S = S \cup \{v\}$, $T = T \cup \{v\}$.

(3') If u adj v and $u \not\text{adj}\ y_i$, $i = 1,2$,
then $S = S \cup \{v\}$, $T = T \cup \{v\}$.

(4') If x_1 adj v and x_2 adj v,
then $S = S \cup \{v\}$, $T = T \cup \{v\}$.

(5') If u adj y_1 and u adj y_2,
then stop.

End for

End while

Justification for Algorithm 2:

(1')-(3') are justified similarly to (1)-(3) in Algorithm 1.

(4') Since x_1 adj v, $v \in B_{21}$ and since x_2 adj v, $v \in B_{22}$, but $B_{i1} \cap B_{i2} = \emptyset$, so $v \in A_1$.

(5') Since u adj y_1, $u \in B_{11}$ and since u adj y_2, $u \in B_{12}$, but $B_{i1} \cap B_{i2} = \emptyset$, so there is no 2-join split.

Proposition 2: Suppose Algorithm 2 is applied to H with S initialized to $\{x_1,x_2,z\}$. If $|X \setminus S| \geq 3$, then $A_1 = S$, $A_2 = X \setminus S$ is a 2-join split with $x_1,x_2,z \in A_1$ and $y_1,y_2 \in A_2$. Otherwise, no such split exists.

The proof of Proposition 2 is similar to the proof of Proposition 1. Algorithm 2 can also be implemented in $O(n^2)$

time. There are K^2 possibilities for x_1, x_2, y_1, y_2 so we have an $O(n^2K^2)$ algorithm for the recognition of the 2-join decomposition.

Note* the time bounds given here are given only to show that the algorithms are polynomial. They are not necessarily optimal bounds.

Applying the 1 or 2-join decomposition to an arbitrary hypergraph reduces it to two smaller hypergraphs. If these hypergraphs are balanced, the original hypergraph is also balanced. Otherwise, it is not. Presumably, it is easier to determine whether or not these smaller hypergraphs are balanced than to determine whether the original hypergraph is balanced. Ideally, a class of irreducible hypergraphs will be identified for which it is easy to determine whether or not they are balanced. Then to determine whether an arbitrary hypergraph is balanced, one would apply the decompositions necessary to reduce the hypergraph to its irreducible components and check whether these irreducible components are balanced.

Conclusion

This paper presented two compositions which preserve balancedness. It is hoped that they will lead to a method of recognizing balanced hypergraphs and so balanced matrices. Both compositions also preserve perfection. There are several other compositions which are known to preserve perfection, but which do not preserve balancedness, e.g., clique identification, complementation, substitution. There is a lot of on-going work in discovering new compositions which preserve perfection, but not a whole lot in discovering new compositions which preserve balancedness. Since balanced matrices are perfect, compositions which preserve balancedness may also preserve perfection.

There are other ways to approach the problem of determining when an arbitrary matrix A is balanced. One way would be to define a minimally unbalanced matrix analogously to a minimally imperfect matrix and attempt to characterize minimally unbalanced matrices. (A matrix is minimally imperfect if A is not perfect, but every column-induced submatrix of A is perfect.) Another approach is to attempt to develop an algorithm directly from the fact that any balanced hypergraph can be bicolored.

One would also hope that a method for identifying balanced matrices could be generalized to recognizing perfect or unimodular matrices. The decomposition approach and the minimally unbalanced approach have potential for being extended if they are successful. The bicoloring approach does not have as much potential.

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