CONDITIONS FOR MECHANICAL SELF-ANNEALING IN MOTIONS OF ELASTIC-PLASTIC OSCILLATORS

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Dedicated to Bernard D. Coleman
on the occasion of his sixtieth birthday

Abstract

In this study of the behavior for large times of a class of elastic-plastic oscillators under conservative loading, conditions on the relative stiffness of the oscillator and the loading agent are given that guarantee that permanent displacements arising during a motion automatically are removed in the limit of large times. When these conditions are not met, a sharp bound independent of initial data is obtained on the magnitude of the ultimate permanent displacement.

Contents

1. Introduction

The research described in this article is part of an ongoing study in mechanics that focuses on damping mechanisms that act intermittently, in particular, on the damping that occurs when elastic-plastic materials undergo permanent deformation. In references [1979, 1], [1984, 1], [1985, 1], [1987, 1], [1988, 1], and [1988, 2], various types of elastic-plastic oscillators have been analyzed. These simple mechanical systems are useful model systems for the ongoing study, because they are governed by ordinary differential equations that mathematically are of independent interest and they have solutions displaying a rich variety of physically interesting characteristics. Knowledge of these systems provides a point of reference for studying the partial differential equations that arise in more detailed models of elastic-plastic behavior: the properties of solutions that one deduces from the ordinary differential equations governing elastic-plastic oscillators become conjectured properties of solutions of the partial differential equations of plasticity. Moreover, solutions of the ordinary differential equations describing coupled systems of elastic-plastic oscillators are candidates for lumped approximations to solutions of the partial differential equations. (In the lecture notes [1985,1], MIYOSHI develops the idea of approximation in considerable detail for one- and two-dimensional problems in plasticity.) In a different setting, elastic-plastic oscillators also are useful tools in the design of structures that must withstand large ground motions due to earthquakes, because they are relatively simple to study numerically, and they display some of the observed features of structures that are severely deformed. (See the references in the article [1988,1] for further details.)

The focus of this paper is the description of the behavior for large times of an elastic-perfectly plastic oscillator subject to a linear restoring force. This mechanical system also has been studied, particularly in the engineering literature, in an equivalent formulation as an unforced bilinear elastic-plastic oscillator. From either point of view, the principal issues that surround the long-range character of motions of this system are, first of all, those associated with solutions of every autonomous system of differential equations: simple stability, asymptotic stability, and the limiting character of solution trajectories; second of all, there are issues peculiar to elastic-plastic systems: the extent to which the system can be permanently deformed, and the amount of damping that can occur through permanent deformations. In this paper, we establish for the unforced, bilinear elastic-plastic oscillator the simple stability of equilibrium points, as well as their lack of asymptotic stability, we characterize the limiting behavior of the oscillator for all initial states, and we obtain detailed information about the permanent deformation that can occur. The main results presented have generalizations to cases when non-linear forces replace both the linearly elastic component of the response of the oscillator and the linear restoring force. We have attempted wherever possible to give proofs of results for the bilinear oscillators that carry over without substantial modification to the more general ones.

The most striking results that we obtain center on the discovery of a critical parameter that separates the motions in which oscillations are underdamped from those that are overdamped. Underdamped oscillations occur when that parameter, the stiffness ratio of the oscillator, is one-half or greater. The permanent deformations that arise during underdamped oscillations have the curious property of being automatically removed in the limit of large times, and we say that the oscillator is *self-annealing* in such motions. Overdamped oscillation occur when the stiffness ratio is less than one- half. In overdamped oscillations, permanent deformations generally do not disappear for large times, but we obtain a sharp bound, independent of the initial state, on the ultimate permanent deformation.

This paper contains, to a large extent, refinements and extensions of the results in the Ph.D. Thesis [1984] of THOMAS. Some of these results also are contained in the paper [1988, 1], where different, mathematically less detailed methods were used to describe the phenomenon of self-annealing in the context of earthquake engineering. In particular, unlike the bound obtained in this article, the bound on permanent deformations obtained in the paper [1988, 1] is not sharp.

While we were preparing this article for publication, we learned of the lecture notes of MIYOSHI [1985, 1], in which both ordinary differential equations and partial differential equations in the theory of plasticity are formulated and analyzed. The discussion of Chapter 2 in [1985, 1] covers the elastic-plastic oscillators discussed here in Sections 2-6 when the external force b in MIYOSHI's model with kinematical hardening is put equal to zero, and MIYOSHI's Theorems 2.3 and 2.9 are counterparts of our Theorems 3.1 and 5.1. MIYOSHI does not give results corresponding to our principal results, Theorems 6.1, 6.2 and the bound (6.9).

In Section 2, we formulate the autonomous systems of ordinary differential equations in $\mathbb{R}^3$ that governs the motions of the unforced, bilinear elastic-plastic oscillator, and in Section 3 we show that the corresponding initial value problem is well posed. The main feature of solution trajectories that emerges in Sections 2 and 3 is their piecewise planar character, corresponding to the alternating elastic and plastic character of deformations of the oscillator. In constrast with the oscillators studied in [1978, 1] and [1987, 1], transition times that separate elastic from plastic segments of a trajectory for the oscillators studied here can accumulate only in the limit of large-times. In Sections 4, 5, and 6, the large-time behavior of unforced, bilinear elastic-plastic oscillators is discussed in detail, culminating in Section 6 with the principal results on self-annealing. In Section 7, we describe a more general class of elastic-plastic oscillators for which generalizations of the results in Sections 2-6 can be established.

2. Formulation of the Problem

The mechanical system that we study here consists of a rigid mass supported by a thin rod and acted upon by a conservative restoring force. We assume that the rod undergoes homogeneous, longitudinal motions and that the restoring force is applied in the direction of motion of the rod, so that the mass moves parallel to the longitudinal axis of the rod. In Sections 2 through 6 we consider the case where the rod is comprised of a linearly elastic-perfectly plastic material and where the restoring force is linear. In Section 7 we discuss briefly more general cases that allow for a non-linearly elastic-perfectly plastic material in the rod and a non-linear restoring force.

Let X denote the position of a point in the rod in a reference configuration, and let X_u and X_ℓ denote the positions of the upper and lower ends of the rod in that configuration. The lower endpoint is attached to the rigid mass m, and the length $X_\ell - X_u$ of the rod in the reference configuration is taken to be one. The restriction to homogeneous deformations is obtained by considering only motions of the rod of the form

$$x(X,t) = X_u + \lambda(t)(X - X_u), \tag{2.1}$$

where $x(X,t)$ is the position at time t of the point in the rod at X in the reference configuration and $\lambda(t)$ is the deformation gradient in the rod at time t. The velocity of the mass at time t is given by

$$v(t) := \frac{\partial x}{\partial t}(X_\ell, t) = \dot{\lambda}(t)(X_\ell - X_u) =$$

$$= \dot{X}(t) \quad . \qquad (2.2)$$

where the superposed dot denotes differentiation with respect to time. We assume that the conservative restoring force $f(t)$ acting on the mass is proportional to the displacement of the mass

$$u(t) := ?x(X_r t) - X_\ell = X(t) - 1. \qquad (2.3)$$

so that there is a positive constant k such that

$$f(t) = -ku(t). \qquad (2.4)$$

The force $cr(t)$ that the elastie-perfectly plastic rod exerts on the mass cannot exceed a yield *value* a_y,

$$k(t)l \; io_y. \qquad ' \; (2.5)$$

and is related to the motion of the lower endpoint of the bar through the constitutive equation:

$$\dot{\sigma}(t) = \begin{cases} 0, \text{ if } |\sigma(t)| = \sigma_y \text{ and} \\ \qquad \sigma(t)\dot{\lambda}(t) \geq 0 \\ \mu\dot{\lambda}(t), \text{ otherwise} \end{cases} \tag{2.6}$$

in which μ is a positive number called the *elastic modulus* of the bar. The equation of motion of the mass

$$m\dot{v}(t) = f(t) - \sigma(t)$$

and the constitutive equation (2.6) now can be expressed in terms of the displacement $u(t)$ and the velocity $v(t) = \dot{u}(t)$ of the mass:

$$m\dot{v}(t) = -ku(t) - \sigma(t) \tag{2.7}$$

$$\dot{\sigma}(t) = \begin{cases} 0, \text{ if } |\sigma(t)| = \sigma_y \text{ and} \\ \qquad \sigma(t)v(t) \geq 0 \\ \mu v(t), \text{ otherwise} \end{cases} \tag{2.8}$$

It is useful to introduce the *elastic displacement*

$$u_e(t) := \frac{\sigma(t)}{\mu} \tag{2.9}$$

and the *plastic displacement*

$$u_p(t) := u(t) - u_e(t) \tag{2.10}$$

into the relations (2.5), (2.7) and (2.8) to obtain:

$$|u_e(t)| \leq \alpha, \tag{2.11}$$

$$m\dot{v}(t) = -k(u_e(t) + u_p(t)) - \mu u_e(t), \tag{2.12}$$

$$\dot{u}_e(t) = \begin{cases} 0 \ , & \text{if } |u_e(t)| = \alpha \text{ and} \\ & \quad u_e(t)v(t) \geq 0 \\ v(t) \ , & \text{otherwise} \end{cases} , \tag{2.13}$$

$$\dot{u}_p(t) = \begin{cases} v(t), & \text{if } |u_e(t)| = \alpha \text{ and} \\ & \quad u_e(t)v(t) \geq 0 \\ 0 \ , & \text{otherwise} \end{cases} . \tag{2.14}$$

The positive number $\alpha := \sigma_y/\mu$ is called the *yield displacement*. Relations (2.12)-(2.14) are the basic governing equations of the linearly elastic-perfectly plastic oscillator subject to a linear restoring force. Together with the constraint (2.11), the governing equations determine the evolution of the triple (v, u_e, u_p).

Because the velocity v is the derivative of the displacement u of the mass, specification of the displacement on a time interval in the relations (2.13) and (2.14) along with the bound (2.11) leads to a system of relations that should determine u_e and u_p as functionals of u. When the net force $F := (k + \mu)u_e + ku_p$ is plotted in a $u - F$ plane, with u_e and u_p the functionals of u determined by (2.11), (2.13) and

(2.14), one obtains the characteristic force-displacement curves of a bilinear hysteretic element. Therefore, a solution (v, u_e, u_p) of (2.11)-(2.14) determines the displacement $u = u_e + u_p$ of a freely vibrating mass supported by a bilinear hysteretic element. It is in this formulation that one finds studies of the system (2.11)-(2.14) in the engineering literature. (See the article [1988, 1] for an analysis of some aspects of this system from the point of view of earthquake engineering.) The term "bilinear" follows from the fact that the slope $\frac{dF}{du}$ of the force-displacement curve is given by:

$$\frac{dF}{du} = \begin{cases} k \quad , & \text{if } |u_e| = \alpha \\ & \text{and } u_e \dot{u} \geq 0 \\ k + \mu \quad , & \text{otherwise,} \end{cases} \tag{2.15}$$

so that the force-displacement curve is comprised of line-segments of slopes

$$k_1 := k + \mu \quad \text{and} \quad k_2 := k. \tag{2.16}$$

The *stiffnesses* k_1 and k_2 and the yield strain α characterize the bilinear, hysteretic element. As our discussion in Section 6 will reveal, the *stiffness ratio*

$$\kappa := \frac{k_2}{k_1} = \frac{k}{k + \mu} \tag{2.17}$$

plays an important role in describing the long-term behavior of solutions (v, u_e, u_p) of (2.11)-(2.15).

We shall use either the phrase *linearly elastic-perfectly plastic oscillator under a linear restoring force* or the shorter phrase *unforced bilinear elastic-plastic oscillator* to describe the mechanical system embodied in equations (2.11)-(2.15). It is natural to try to decompose locally in time each motion of that system into finitely many *elastic segments*, i.e., segments on which u_p is constant, and finitely many *plastic segments*, i.e., segments on which u_e is constant. On an elastic segment the motion is governed by (2.11), (2.12), $(2.13)_2$, and $(2.14)_2$, while on a plastic segment the motion is governed by (2.11), (2.12), $(2.13)_1$ and $(2.14)_1$. As natural as this decomposition appears, there are examples of elastic-plastic systems where such a decomposition does not exist. For example, the linearly elastic-perfectly plastic oscillators under general forces studied in the article [1978, 1] can undergo motions during which elastic segments and plastic segments alternate and accumulate backwards in time to an initial time. The existence of such an initial time precludes a local decomposition of the motion into finitely many elastic and plastic segments; the mathematical consequence of this fact in the analysis of general forces is the necessity of seeking solutions of the governing equations that are not necessarily locally piecewise smooth. Nevertheless, for the linear restoring force $-k(u_e(t) + u_p(t))$ studied here, as well as for the more general conservative forces discussed in Section 7, it will emerge that every motion can be decomposed locally in time into finitely many elastic and plastic segments, and this fact underlies the choice of smoothness requirements that we make in Section 3 on solutions of (2.11)-(2.14). These smoothness requirements translate into the geometrical restriction that the solution trajectories of (2.11)-(2.14) in $v - u_e - u_p$ space locally be piecewise planar, i.e., each trajectory can be partitioned locally in time into finitely many segments in planes u_p = constant or planes u_e = constant. The possibility that these planar segments can accumulate as the time t tends to $+\infty$ forms the central theme of our discussion in Sections 5 and 6.

3. Qualitative Results for Initial Value Problems

Our discussion in Section 2 leads us to formulate initial value problems (IVP), $(IVP)_{\mathcal{E}}$, $(IVP)_{+}$, and $(IVP)_{-}$ associated with the unforced bilinear elastic-plastic oscillator. In stating these problems below, we let $\alpha, k, \mu \in \mathbb{R}^{++}$ be given, and we use the following additional notation and terminology:

$$\mathcal{S} := \{(v, u_e, u_p) \in \mathbb{R}^3 \mid |u_e| \leq \alpha\}$$

··· state space

$$\mathcal{Y}^{+} := \{(v, u_e, u_p) \in \mathbb{R}^3 \mid u_e = \alpha,\ v \geq 0\}$$

··· upper yield surface

$$\mathcal{Y}^{-} := \{(v, u_e, u_p) \in \mathbb{R}^3 \mid u_e = -\alpha,\ v \leq 0\}$$

··· lower yield surface

$\mathcal{Y} := \mathcal{Y}^{+} \cup \mathcal{Y}^{-}$ *··· space of plastic states*

$\mathcal{E} := \mathcal{S} \setminus \mathcal{Y}$ *··· space of elastic states*

(IVP): *For each* $x^{\circ} \in \mathcal{S}$, *find a continuous, locally piecewise continuously differentiable function* $x = (v, u_e, u_p) : [0, \infty) \to \mathcal{S}$ *such that, for every* t *at which* x *is differentiable,* x *satisfies* (2.11)-(2.14), *and* x *satisfies the initial condition*

$$x(0) = x^{\circ}. \tag{3.1}$$

(IVP)$_\varepsilon$: For each $x_0 \in V$ *and* $t_0 \in [0,\infty)$, find a *continuously differentiable function* $x = (v,u_e,u_p) : [t_0,\infty) \to \mathbb{R}^3$ that satisfies (2.12), $(2.13)_2$ *and* $(2.14)_2$ at each $t \in [t_0,\infty)$ and that satisfies *the* initial *condition*

$$x(t_0) = x_0. \tag{3.2}$$

(IVP)$_\pm$: *For each* $x_0 = (v_0,u_{e0},u_{p0}) \in \mathcal{S}^\pm$ *with* $v_0 \neq 0$, *and for each* $t_0 \in [0,\infty)$, find $t_1 > t_0$ and a continuously differentiable function $x = (v,u_e,u_p) : [t_0,t_1) \to \mathbb{R}^3$ that satisfies (2.12), $(2.13)_\pm$ *and* $(2.14)_\pm$ at each $t \in [t_0,t_1)$ *and that* satisfies *the* initial *condition* (3.2).

To save space, we have combined the statements of $(IVP)_+$ and $(IVP)_-$ into one statement in which either only the plus signs should be read or only the minus signs should be read. Note that the initial time is 0 for (IVP), but it is an arbitrary non-negative number t_0 for $(IVP)_\varepsilon$, $(IVP)_+$, and $(IVP)_-$. Moreover, solutions of (IVP) are required only to be continuous and locally piecewise continuously differentiable, i.e., continuous and, on every interval $[0,T]$ with $T > 0$, piecewise continuously differentiable, whereas solutions of the other initial value problems are required to be continuously differentiable. Solutions of (IVP) and of $(IVP)_\varepsilon$ have domains an interval of the form $[t_0,+\infty)$, with $t_0 \geq 0$, whereas solutions of $(IVP)_+$ and of $(IVP)_-$ can have domains that are bounded intervals. Finally, solutions of (IVP) take their values in the state space V, solutions of $(IVP)_+$ and $(IVP)_-$ take their values in proper subsets of $\mathcal{Y}^+$ and $\mathcal{Y}^-$, respectively, and solutions of $(\text{IVP})_\varepsilon$ can take their values anywhere in $\mathbb{R}^3$.

Our goal in this section is to show that (IVP) is well-posed, and that all solutions of (IVP) can be constructed by patching together solutions of the problems $(\text{IVP})_\varepsilon$, $(\text{IVP})_+$, and $(\text{IVP})_-$. We use explicit properties of solutions of the latter problems to show that the patching procedure never breaks down, i.e., always yields a solution of (2.11)-(2.14) that is continuous and locally piecewise continuously differentiable on the entire half-line $[0,\infty)$. We then use uniqueness of solutions of (IVP), established by means of a simple energy inequality, to show that the patching procedure yields all solutions of (IVP).

Before we discuss the details of the method of patching, we record the energy inequality for solutions $x = (v,u_e,u_p)$ and $\bar{x} = (\bar{v},\bar{u}_e,\bar{u}_p)$ of (2.11)-(2.14) that implies uniqueness and continuous dependence of solutions of (IVP) on initial data:

$$\left[\frac{1}{2}m(v-\bar{v})^2 + \frac{1}{2}k(u_e+u_p-\bar{u}_e-\bar{u}_p)^2 + \frac{1}{2}\mu(u_e-\bar{u}_e)^2\right]^{\bullet} \le$$

$$\left[\frac{\mu}{m}+1\right]\left[\frac{1}{2}m(v-\bar{v})^2 + \frac{1}{2}k(u_e+u_p-\bar{u}_e-\bar{u}_p)^2 + \frac{1}{2}\mu(u_e-\bar{u}_e)^2\right]. \quad (3.3)$$

This inequality is obtained by using the relations (2.11)-(2.14) at times when both x and $\bar{x}$ are differentiable. Similarly, one obtains for a single solution $x = (v,u_e,u_p)$ of (IVP):

$$\left[\frac{1}{2}mv^2 + \frac{1}{2}k(u_e+u_p)^2 + \frac{1}{2}\mu u_e^2\right]^{\bullet} \le 0, \quad (3.4)$$

and this inequality shows that the trajectory of each solution x lies in the compact subset of $\mathbb{R}^3$ consisting of all points (v,u_e,u_p) whose total energy

$$\Phi(v,u_e,u_p) := \frac{1}{2}mv^2 + \frac{1}{2}k(u_e + u_p)^2 + \frac{1}{2}\mu u_e^2 \tag{3.5}$$

does not exceed the total energy $\Phi(v^o,u_e^o,u_p^o)$ associated with the initial point $(v^o,u_e^o,u_p^o) = x(0)$.

The initial value problem $(IVP)_\varepsilon$ has a unique solution $x = (v,u_e,u_p)$ satisfying the linear system for (v,u_e)

$$m\dot{v} = -(k + \mu)u_e - ku_p^o$$

$$\dot{u}_e = v \tag{3.6}$$

so that x is periodic of period $\left[\frac{k + \mu}{m}\right]^{1/2}$ and has trajectory the ellipse

$$\frac{1}{2}mv^2 + \frac{1}{2(k + \mu)}[(k + \mu)u_e + ku_p^o]^2 =$$

$$\frac{1}{2}m(v^o)^2 + \frac{1}{2(k + \mu)}[(k + \mu)u_e^o + ku_p^o]^2 \tag{3.7}$$

centered at the point $(0, -\frac{k}{k + \mu}u_p^o, u_p^o)$ and lying in the plane $u_p = u_p^o$. Figure 1 depicts such an oriented trajectory in relation to the sets $\mathcal{S}$,

$\mathscr{Y}^+$; and $\mathscr{Y}^-$ when u_p^o is positive. We refer to such trajectories as *elastic trajectories*, and to the portions of such trajectories in $\mathscr{S}$ as *elastic segments*. The initial value problem $(IVP)_+$ has a unique, graph-maximal solution $x = (v, u_e, u_p)$ satisfying the linear system for (v, u_p)

$$m\dot{v} = -(k + \mu)\alpha - ku_p$$
$$\dot{u}_p = v, \tag{3.8}$$

so that x has trajectory a portion of the half-ellipse

$$\frac{1}{2} mv^2 + \frac{1}{2k}[ku_p + (k + \mu)\alpha]^2 =$$
$$\frac{1}{2}m(v^o)^2 + \frac{1}{2k}[ku_p^o + (k + \mu)\alpha]^2, \tag{3.9}$$
$$v \geq 0,$$

centered at $(0, \alpha, -\frac{(k + \mu)}{k}\alpha)$ and lying in the half-plane $\mathscr{Y}^+$. Figure 2 depicts such a trajectory. Similarly, $(IVP)_-$ has a unique graph maximal solution $x = (v, u_e, u_p)$ satisfying

$$m\dot{v} = +(k + \mu)\alpha - ku_p$$
$$\dot{u}_p = v, \tag{3.10}$$

so that x has trajectory in the half-ellipse

$$\frac{1}{2}mv^2 + \frac{1}{2k}[ku_p - (k+\mu)\alpha]^2 =$$

$$\frac{1}{2}m(v^\circ)^2 + \frac{1}{2k}[ku_p^\circ - (k+\mu)\alpha]^2, \qquad (3.11)$$

$$v \le 0,$$

centered at $(0,-\alpha,\frac{k+\mu}{k}\alpha)$ and lying in the half-plane $\mathcal{Y}^-$, as depicted in Figure 3. We refer to trajectories for solutions of $(IVP)_+$ or of $(IVP)_-$ as *plastic segments*.

We describe now a procedure for patching together elastic and plastic segments to obtain solutions of (IVP). Consider first initial data x° for (IVP) lying in the set $\mathcal{E}$ of elastic states. Because $\mathcal{E}$ does not contain points of $\mathcal{Y}^+$ or of $\mathcal{Y}^-$ and because of the orientation of elastic trajectories, we may follow the solution $x^{(0)} = (v^{(0)},u_e^{(0)},u_p^{(0)})$ of $(IVP)_{\mathcal{E}}$ satisfying $x^{(0)}(0) = x^\circ$ on a non-trivial time interval $[0,\tau_1)$ until the trajectory of $x^{(0)}$ leaves the state-space $\mathcal{S}$ by passing through $\mathcal{Y}^+$ or $\mathcal{Y}^-$ at time $\tau_1 > 0$. Suppose for definiteness that this occurs through $\mathcal{Y}^+$, so that $v^{(0)}(\tau_1) > 0$ and $u_e^{(0)}(\tau_1) = \alpha$. In order to continue along a solution of (IVP), i.e., in order that a continuation of $x^{(0)}$ remain in $\mathcal{S}$, we put $x^\circ := (v^{(0)}(\tau_1),\alpha,u_p^{(0)}(\tau_1))$ and $t_0 := \tau_1$ in $(IVP)_+$ and follow the graph maximal solution $x^{(1)} = (v^{(1)},u_e^{(1)},u_p^{(1)})$ of $(IVP)_+$ until it reaches the line $v = 0$ in $\mathcal{Y}^+$ at a time $\tau_2 > \tau_1$. In doing so, we have followed the elastic segment determined by $x^{(0)}$ with the plastic segment determined by $x^{(1)}$. The form of the plastic segment tells us that $v^{(1)}(\tau_2) = 0$ and

$$-(k + \mu)u_e^{(1)}(\tau_2) - ku_p^{(1)}(\tau_2) < 0. \tag{3.12}$$

If we put $t_0 := \tau_2$ and $x^\circ := (v^{(1)}(\tau_2), u_e^{(1)}(\tau_2), u_p^{(1)}(\tau_2)) = (0, \alpha, u_p^{(1)}(\tau_2))$ in $(IVP)_{\mathcal{E}}$, then we may follow the solution $x^{(2)} = (v^{(2)}, u_e^{(2)}, u_p^{(2)})$ of $(IVP)_{\mathcal{E}}$ and find that its trajectory remains in the state-space $\mathcal{S}$ during a non-trivial time interval $[\tau_2, \tau_3]$ with $\tau_3 > \tau_2$. In fact, we have from (2.12) and (3.12),

$$v^{(2)\cdot}(\tau_2^+) = \lim_{\tau \downarrow \tau_2} \frac{1}{m}(-(k + \mu)u_e^{(2)}(\tau) - ku_p^{(2)}(\tau))$$

$$= \frac{1}{m}(-(k + \mu)u_e^{(1)}(\tau_2) - ku_p^{(1)}(\tau_2)) < 0,$$

and $v^{(2)}(\tau_2) = v^{(1)}(\tau_2) = 0$, so that $u_e^{(2)\cdot} = v^{(2)}$ is negative on an interval of the form $[\tau_2, \tau_3]$ with $\tau_3 > \tau_2$. Therefore, because $u_e^{(2)}(\tau_2) = u_e^{(1)}(\tau_2) = \alpha$, we find that $|u_e^{(2)}| \leq \alpha$ on $[\tau_2, \tau_3]$, i.e., the trajectory of $x^{(2)}$ remains in $\mathcal{S}$ during $[\tau_2, \tau_3]$.

This discussion shows how initial data x° in the set $\mathcal{E}$ of elastic states leads to a path consisting of an elastic, a plastic, and, again an elastic segment; the path starts at x° and lies in $\mathcal{S}$. The assumption that the first elastic segment ends in $\mathcal{Y}^+$ was not crucial for the construction of this path. Consequently, we have shown that every elastic trajectory that leaves $\mathcal{S}$ can be cut off at $\mathcal{Y}^+$ or at $\mathcal{Y}^-$ to form an elastic segment that can be followed by a plastic segment and then another elastic segment. This fact forms the basis of a recursive definition of a mapping $x : [0, t_*) \to \mathcal{S}$, with $t_* > 0$, that is continuous, locally piecewise continuously differentiable, satisfies (2.11) and, at its points of differentiability, (2.12)-(2.14), and obeys the initial condition $x(0) = x^o \in \mathcal{E}$. We remark without proof that such a function x also can be constructed when the initial data x^o is in $\mathcal{S} \setminus \mathcal{E} = \mathcal{Y}$.

We wish to indicate here why the function x constructed above is a solution of (IVP). It is clear that we need only show that $t_* = +\infty$, or, in other words, that the times separating successive elastic and plastic segments in the trajectory of x cannot accumulate at a finite time. We do so by considering only intermediate elastic segments of x, i.e., only those elastic segments that occur between plastic segments. (If there are no intermediate segments, then it is easy to show that the trajectory of x ultimately is an elastic trajectory lying entirely within $\mathcal{S}$, so that the recursive construction of x terminates at a finite state.) Each intermediate elastic segment has associated with it a traversal time, and we give now a geometrical argument that indicates why there is a positive number δ, depending only upon α, k, μ, m and the initial data x^o for x, such that every traversal time is no less than δ. In fact, the inequality (3.4) tells us that the trajectory of x and, in particular, u_p is bounded. Therefore, all the intermediate elastic segments and all the centers of intermediate elastic segments lie in a bounded set. It is clear that an intermediate elastic segment must start in the plane $v = 0$ at $\mathcal{V}^+$ or at $\mathcal{V}^-$ and must end at $\mathcal{V}^-$ or at $\mathcal{V}^+$, respectively, so that each intermediate elastic segment must span the thickness $2a$ of the state-space $\mathcal{V}$ and must have it and its center lying in a predetermined bounded set. It follows that each angle subtended at its center by an intermediate elastic segment is bounded below by a positive number depending only upon a, k, μ, m and the initial data x^0. Because every elastic trajectory is periodic with period $((k + \mu)/m)^{1/2}$, it follows that the traversal times of the elastic segments are also bounded below by a positive number depending only upon the same quantities.

The arguments given in this section can be used to produce a formal proof of the following result.

Theorem 3.1. The initial value problem (IVP) for the unforced, bilinear elastic-plastic oscillator is well posed, and every solution of (IVP) can be obtained by patching together elastic and plastic segments.

In closing this section, we emphasize the fact that the trajectories of solutions of (IVP) are piecewise planar; each planar segment forms a portion of an ellipse in a plane $u_p = \text{const.}$, $u_e = a$ or $u_e = -a$. The center of the ellipses in a plane $u_p = u_p^0$ is the point $v = 0$, $u_e = -\frac{k}{k+\mu} u_p^0$. If $\left|-\frac{k}{k+\mu} u_p^0\right| \le a$, i.e., if $|u_p^0| \le \frac{k+\mu}{k} a$, then the point $(0, -\frac{k}{k+\mu} u_p^0, u_p^0)$ is an equilibrium state for the system (2.11)-(2.15), i.e., the constant function with value $(0, -\frac{k}{k+\mu} u_p^0, u_p^0)$ is a solution of (IVP) with $x^o = (0, -\frac{k}{k+\mu} u_p^o, u_p^o)$. In fact, *all the equilibrium states of* (2.11)-(2.15) *are of the form*

$$(0, -\frac{k}{k+\mu} u_p^c, u_p^c), \text{ with } |u_p^c| \le \frac{k+\mu}{k}\alpha . \tag{3.13}$$

Thus, the equilibrium states form a line segment in the plane $v = 0$. The endpoint $(0, \alpha, -\frac{k+\mu}{k}\alpha) = (0, \alpha, -\frac{\alpha}{\kappa})$ of this line segment is the center of the plastic segments in the half-plane $\mathcal{Y}^+$, and the endpoint $(0, -\alpha, \frac{k+\mu}{k}\alpha) = (0, -\alpha, \frac{\alpha}{\kappa})$ is the center of the plastic segments in the half-plane $\mathcal{Y}^-$. Here, κ is the stiffness ratio defined in (2.17).

Another feature associated with the piecewise planar character of solutions (IVP) was already employed tacitly in the arguments leading to Theorem 3.1 and is recorded here for future use.

Remark 3.1: *Each intermediate elastic segment of a solution of* (IVP) *starts and ends in different yield planes. In other words, successive plastic segments alternate between* $\mathcal{Y}^+$ *and* $\mathcal{Y}^-$. •

4. Stability of Equilibrium States

The equilibrium states for the mechanical system under consideration here are of interest because they represent solutions of the initial value problem (IVP) which do not evolve in time. Knowledge of the location of the equilibrium states also helps to characterize the behavior of the solutions which do evolve in time. As mentioned at the end of Section 3, the equilibrium state in each plane u_p = const. is the center of the ellipses that contain the trajectories of the solutions of $(IVP)_{\varepsilon}$. Similarly, the equilibrium state $(0,\alpha, -\frac{\alpha}{\kappa})$ (resp. $(0,-\alpha,\frac{\alpha}{\kappa})$) is the center of the ellipses that contain the trajectories of the solutions of $(IVP)_+$ (resp. of (IVP_-)). (Here, as above, κ is the stiffness ratio $\frac{k}{k+\mu}$.)

One issue of fundamental significance regarding the equilibrium states of the unforced bilinear elastic-plastic oscillator is the question of stability. We can ensure stability of the equilibrium states if we can construct an appropriate Liapunov function for each equilibrium state. Letting $(0,u_e^c,u_p^c)$ be an equilibrium state of the system (2.11)-(2.14), we consider

$$\hat{\Phi}(v,u_e,u_p):=\frac{1}{2}mv^2+\frac{1}{2}\mu(u_e-u_e^c)^2+\frac{1}{2}k[(u_e+u_p)-(u_e^c+u_p^c)]^2 \quad (4.1)$$

as a possible Liapunov function for this equilibrium state. This function has a strict minimum at $(0,u_e^c,u_p^c)$ and has the required smoothness. It remains to ensure that $\hat{\Phi}$ is non-increasing on solutions. To confirm this, we first differentiate $\hat{\Phi}$ along a solution $x=(v,u_e,u_p)$ to obtain

$$\hat{\Phi}^{\bullet} = m\dot{v}v + \mu(u_e - u_e^c)\dot{u}_e + k[(u_e + u_p) - (u_e^c + u_p^c)](\dot{u}_e + \dot{u}_p). \quad (4.2)$$

Since we are considering the rate of change of $\hat{\Phi}$ along solutions, we may use equation (2.12) to substitute for $m\dot{v}$ in equation (4.2):

$$\hat{\Phi}^{\bullet} = v[-k(u_e + u_p) - \mu u_e] + \mu(u_e - u_e^c)\dot{u}_e$$

$$+ k[(u_e + u_p) - (u_e^c + u_p^c)](\dot{u}_e + \dot{u}_p). \quad (4.3)$$

Furthermore, since $v = \dot{u} = (u_e + u_p)^{\bullet}$, equation (4.3) becomes

$$\hat{\Phi}^{\bullet} = v[-k(u_e + u_p) - \mu u_e] + \mu(u_e - u_e^c)\dot{u}_e$$

$$+ k[(u_e + u_p) - (u_e^c + u_p^c)]v, \quad (4.4)$$

or, equivalently,

$$\hat{\Phi}^{\bullet} = -\mu v u_e + \mu(u_e - u_e^c)\dot{u}_e - k(u_e^c + u_p^c)v. \quad (4.5)$$

Since $\kappa = \frac{k}{k + \mu}$, we may eliminate u_p^c from equation (4.5) using (3.13) to obtain

$$\hat{\Phi}^{\bullet} = -\mu v u_e + \mu(u_e - u_e^c)\dot{u}_e - k[u_e^c - \frac{(k + \mu)}{k} u_e^c]v, \quad (4.6)$$

which becomes after rearrangement

$$\dot{\hat{\Phi}} = \mu(\dot{u}_e - v)(u_e - u_e^c). \tag{4.7}$$

When the solution is undergoing an elastic segment, we have from equation (2.13) $\dot{u}_e = v$, so that the right-hand side of equation (4.7) is identically zero on elastic segments. When the solution is undergoing a plastic segment in $\mathcal{Y}^+$, we have from equation (2.13), $\dot{u}_e = 0$, so that equation (4.7) becomes

$$\dot{\hat{\Phi}} = -\mu v(u_e - u_e^c). \tag{4.8}$$

Since the solution lies in $\mathcal{Y}^+$, $u_e = +\alpha$, so that equation (4.8) reduces to

$$\dot{\hat{\Phi}} = -\mu v(\alpha - u_e^c). \tag{4.9}$$

Since the solution is undergoing a plastic segment in $\mathcal{Y}^+$, we have $v \geq 0$. Furthermore, because we are considering an equilibrium state $(v, u_e, u_p) = (0, u_e^c, u_p^c)$ of the initial value problem with governing equations (2.12)-(2.14) and constraint (2.11), we have $|u_e^c| \leq \alpha$. Thus, we may conclude from (4.9) that $\dot{\hat{\Phi}} \leq 0$ on solutions undergoing a plastic segment in $\mathcal{Y}^+$. The proof that $\dot{\hat{\Phi}} \leq 0$ on solutions undergoing a plastic segment in $\mathcal{Y}^-$ is similar. Therefore, we may conclude that $\hat{\Phi}$ is non-increasing on solutions. Thus, $\hat{\Phi}$ is a suitable Liapunov function for the equilibrium state, and we have

Theorem 4.1. *The equilibrium states of an unforced bilinear elastic-plastic oscillator are stable.*

Note that since $\hat{\Phi}$ is constant on elastic segments of solutions, we cannot claim asymptotic stability. In fact, recalling from Section 3 that the elastic trajectories of $(IVP)_\varepsilon$ in the plane $u_p = u_p^c$ are ellipses centers at $(0, u_e^c, u_p^c)$, we may conclude that when $|u_e^c| < \alpha$, the equilibrium state is a center for $(IVP)_\varepsilon$.

To make one additional observation, let us consider a plane $u_p = u_p^c$ with $|u_p^c| < \frac{\alpha}{\kappa}$. In each such plane of constant plastic displacement, there are ellipses of the form (3.7) centered at the equilibrium state $(0, -\kappa u_p^c, u_p^c)$ that do not intersect the planes $|u_e| = \alpha$, so that the solutions whose trajectories are contained in these ellipses remain elastic for all time. These equilibrium states therefore are isolated from one another in their respective planes of constant plastic displacement.

5. General Behavior of Solutions for Large Times

In this section we obtain a result which gives a characterization of the asymptotic behavior of solutions of the unforced bilinear elastic-plastic oscillator. Of particular interest is the behavior of solutions which undergo an infinite number of plastic segments. The general framework provided in this section will be needed in the discussion of self-annealing in Section 6. We begin with some standard terminology.

Definition 5.1: *Let* $x_0 \in \mathbb{R}^3$ be *given, and let* $x = (v, u_e, u_p)$ be *the corresponding solution of* (IVP). *The curve* $t \mapsto x(t)$ is *called the positive* **half-trajectory *through*** x_0 and is *denoted by* $C^+(x_0)$. *The* **ω-limit set** $L(C^+(x_0))$ *of* $C^+(x_0)$ is the set *of* all points $x^* \in \mathbb{R}^3$ such *that there is a sequence* $\{t_n\}$ *with* limit $+\infty$ *for which* $x(t_n) \to x^*$ as $n \to \infty$.

We need some preliminary results to obtain our characterization of ω-limit sets.

Remark 5.1. **A** necessary *and sufficient condition that a solution which has undergone a plastic segment in* V^+ (resp. V^-) will have a *subsequent* plastic segment in V^- *(resp.* V^+) *after an intervening elastic segment* is $u_{p_1} > 0$ *(resp.* $u_{p_1} < 0$), *where* u_{p_1} is *the* value *of the plastic displacement* as the *solution departs* V^+ (resp. V^-).

Proof: Let a solution which has undergone a plastic segment in V^+ be given, and let u_{p_1} denote the value of the plastic displacement as the solution departs V^+. Then the solution departs V^+ from the point $(0,\alpha,u_{p_1})$. Since, by (3.7), during the elastic segment the velocity v and elastic displacement u_e satisfy

$$\frac{1}{2}mv^2 = \frac{1}{2(k+\mu)}\left[(k+\mu)^2(\alpha^2 - u_e^2) + 2k(k+\mu)(\alpha - u_e)u_{p_1}\right] \qquad (5.1)$$

the elastic segment reaches the line $u_e = -\alpha$ to the left of the point $(0,-\alpha,u_{p_1})$ and thereby undergoes a segment in V^- if and only if

$$\frac{1}{2}mv^2 = 2k\alpha u_{p_1} > 0. \qquad (5.2)$$

(The possibility that the solution might cross the line $v = 0$ above the point $(0,-\alpha,u_{p_1})$ and subsequently enter V^+ with $v > 0$ is ruled out by the fact that the elastic unloading segment is contained in an ellipse symmetric about the line $v = 0$.) The proof of the analogous statement for a solution which has departed V^- is similar. □

The next result shows that the magnitude of the resultant plastic displacements are decreasing:

Remark 5.2. *Let a solution undergo a* plastic segment *in* V^+ (resp. V^-) *and, after an intervening elastic segment, a* plastic segment *in* V^- (resp. V^+), subsequently departing *that* plastic segment as well. *Let* u_{p_1} denote *the* plastic displacement as *the solution departs* V^+ *(resp.* V^-*), and let* u_{p_2} *denote the plastic displacement as the solution departs* V^- *(resp.* V^+*). There then holds* $|u_{p_2}| < |u_{p_1}|$.

Proof: When the solution departs V^+, the total energy is $\frac{1}{2} pa^2 + \frac{1}{2} k(a + u_{p_1})^2$. When the solution departs V^-, the total energy is $\frac{1}{2} pa^2 + \frac{1}{2} k(-a + u_{p_2})^2$. Since the total energy is constant on the intermediate elastic segment and since the change in total energy for a nontrivial plastic segment is negative (Remark 5.3), we have

$$\frac{1}{2} pa^2 + \frac{1}{2} k(a + u_{p_1})^2 > \frac{1}{2} pa^2 + \frac{1}{2} k(-a + u_{p_2})^2. \qquad (5.3)$$

By Remark 5.1, $u_{p_1} > 0$. Since $\dot{u}_p = v < 0$ while the solution lies in V^- with $v \neq 0$, $u_{p_1} > u_{p_2}$. If $u_{p_2} \geq 0$, then the conclusion is immediate. Otherwise, $u_{p_2} < 0$, in which case inequality (5.3) gives $a + u_{p_1} > +a - u_{p_2}$. Since $|u_{p_2}| = -u_{p_2}$ and $|u_{p_1}| = u_{p_1}$, the desired result follows. The proof with V^+ and V^- switched in the statement is analogous. □

The following estimate on the energy of solutions also will be useful:

Remark 5.3: *If a solution of* (IVP) *enters a plastic segment in* $\mathcal{Y}^+$ (*resp. in* $\mathcal{Y}^-$) *at time* t_0, *and subsequently departs the plastic segment in* $\mathcal{Y}^+$ (*resp. in* $\mathcal{Y}^-$) *at time* t_1, *with* $t_1 > t_0$, *then the net decrease in total energy* Φ *experienced during the segment is*

$$\Phi(x(t_1)) - \Phi(x(t_0)) = -\mu\alpha(u_p(t_1) - u_p(t_0)) < 0 \tag{5.4}$$

$$(\text{resp. } \Phi(x(t_1)) - \Phi(x(t_0)) = \mu\alpha(u_p(t_1) - u_p(t_0)) < 0). \tag{5.5}$$

Proof: Recall from equation (3.5) that $\Phi(v,u_e,u_p) = \frac{1}{2} mv^2 + \frac{1}{2} \mu u_e^2 + \frac{1}{2} k(u_e + u_p)^2$. Thus, during a plastic segment in $\mathcal{Y}^+$ (resp. $\mathcal{Y}^-$), we have

$$\dot{\Phi} = mv\dot{v} + \mu u_e \dot{u}_e + k(u_e + u_p)(u_e + u_p)^{\bullet}$$

$$= mv\dot{v} + k(\alpha + u_p)v \tag{5.6}$$

$$(\text{resp. } \dot{\Phi} = mv\dot{v} + k(-\alpha + u_p)v.) \tag{5.7}$$

Substitution of equation (2.12) into equation (5.6) with $u_e = +\alpha$ (resp. into equation (5.7) with $u_e = -\alpha$) yields

$$\dot{\Phi} = -\mu\alpha v \tag{5.8}$$

$$(\text{resp. } \dot{\Phi} = +\mu\alpha v.) \tag{5.9}$$

Putting $v = \dot{u}_p$ into equations (5.8) and (5.9) and integrating yields

$$\Phi(x(t_1)) - \Phi(x(t_0)) = -\eta a(u_p(t_1) - u_p(t_0)) \qquad (5.10)$$

$$(\text{resp. } \Phi(x(t_1)) - \Phi(x(t_0)) = \eta a(u_p(t_1) - u_p(t_0)). \qquad (5.11)$$

Since $v > 0$ (resp. $v < 0$) while the solution has $v \neq 0$ and remains in Φ^+ (resp. Φ^-), the function $t \mapsto u_p(t)$ is strictly increasing (resp. decreasing) during such a plastic segment and the energy difference $\Phi(x(t_1)) - \Phi(x(t_0))$ is negative. D

Note that equations (5.10) and (5.11) may be consolidated so that the net decrease W in energy experienced during a plastic segment can be written as

$$W = \eta a|\Delta u_p|, \qquad (5.12)$$

where Δu_p is the net change in plastic displacement experienced during the segment.

We can now prove the main result of this section:

Theorea 5-1: (Characterization *of* wlimit sets). Let $x_0 \in V$ be giuen. *For the* initial *value problem corresponding to the unforced bilinear* elastic-plastic oscillator, the delimit set $L(C^+(x_0))$ *of* $C^+(x_0)$ consists *of a single point in* $\mathcal{S}$ *(an equilibrium state) or an ellipse which lies in a plane* u_p = const. *Furthermore, if* $x^o \in \mathcal{S}$ *is such that the resulting solution undergoes an infinite number of plastic segments, then the* ω*-limit set of* $C^+(x_0)$ *is the ellipse in the plane* $u_p = 0$ *given by*

$$\frac{1}{2} mv^2 + \frac{1}{2}(\mu + k)u_e^2 = \frac{1}{2}(\mu + k)\alpha^2. \tag{5.13}$$

Proof: Suppose first that $x^o \in \mathcal{S}$ is such that the resulting solution undergoes no nontrivial plastic segments. Then $\Phi(x(t)) = \Phi(x^o)$ and $u_p(t) = u_p^o$, for all $t \in [0,\infty)$. The curves $\Phi(x)$ = const. in the plane $u_p = u_p^o$ consist of a singleton (an equilibrium state) and ellipses. Hence the solution either is constant, in which case the singleton x^o is the ω-limit set, or the solution is elastic and periodic, and the corresponding ellipse

$$\frac{1}{2} mv^2 + \frac{1}{2} \mu u_e^2 + \frac{1}{2} k(u_e + u_e^o)^2$$

$$= \frac{1}{2} mv^{o2} + \frac{1}{2} \mu(u_e^o)^2 + \frac{1}{2} k(u_e^o + u_p^o)^2 \tag{5.14}$$

is the ω-limit set.

Now suppose that $x^o \in \mathcal{S}$ is such that the resulting solution undergoes one or more plastic segments, the number of such segments being finite. During its final plastic segment, the solution trajectory is contained in a half-ellipse of the type given by equation (3.7). One can explicitly solve the equations governing the evolution of the solution during the plastic segments, and using the point at which the solution entered $\mathcal{Y}^+$ as initial data, an explicit formula for the time required to traverse the plastic segment can be obtained. This formula confirms that the final plastic segment is departed in finite time, and the plastic displacement thereafter remains constant. Let u_{p_f} denote the ultimate plastic displacement, and suppose that the final plastic segment occurred in $\mathcal{Y}^+$ (resp. $\mathcal{Y}^-$). Then by Remark 5.1, $u_{p_f} \leq 0$ (resp. $u_{p_f} \geq 0$), and for all subsequent time the solution remains in the ellipse given by

$$\frac{1}{2} mv^2 + \frac{1}{2} \mu u_e^2 + \frac{1}{2} k(u_e + u_{p_f})^2$$

$$= \frac{1}{2} \mu\alpha^2 + \frac{1}{2} k(\alpha + u_{p_f})^2 \qquad (5.15)$$

(resp. the ellipse given by

$$\frac{1}{2} mv^2 + \frac{1}{2} \mu u_e^2 + \frac{1}{2} k((u_e + u_{p_f})^2$$

$$= \frac{1}{2} \mu\alpha^2 + \frac{1}{2} k(-\alpha + u_{p_f})^2.) \qquad (5.16)$$

Since there are no additional plastic segments, this ellipse touches $\mathcal{Y}^+$ (resp. $\mathcal{Y}^-$) at the single point $(0,\alpha,u_{p_f})$ (resp. $(0,-\alpha,u_{p_f})$), and does not intersect the line $u_e = -\alpha$ (resp. the line $u_e = +\alpha$) in the plane $u_p = u_{p_f}$. Hence, subsequent to departing its final plastic segment the solution is elastic and periodic, and the elliptical orbit in which it lies for all times thereafter is the ω-limit set of $C^+(x^o)$.

Finally, let $x^o \in \mathcal{S}$ be such that the resulting solution undergoes an infinite number of plastic segments. Let W_i be the net decrease in Φ associated with the i^{th} plastic segment. Then Remark 5.3 and equation (5.12) imply that

$$W_i = \mu\alpha|(\Delta u_p)_i|, \tag{5.17}$$

where $(\Delta u_p)_i$ is the net change in plastic displacement for the i^{th} plastic segment. Now let $(\Phi)_n$ be the value of Φ as the solution departs the n^{th} plastic segment. Then $(\Phi)_n$ is given by

$$(\Phi)_n = \Phi(x^o) - \sum_{i=1}^{n} \mu\alpha|(\Delta u_p)_i|. \tag{5.18}$$

Since $\Phi(x) \geq 0$ for all $x \in \mathcal{S}$, equation (5.18) implies that the series $\sum_{i=1}^{\infty} \mu\alpha|(\Delta u_p)_i|$ consists of positive terms and its partial sums are bounded above by $\Phi(x^o)$, so that the series is convergent. We have $(u_p)_n = u_p^o + \sum_{i=1}^{n} (\Delta u_p)_i$. By Remark 3.1, the plastic segments alternate between the yield surfaces $\mathcal{Y}^+$ and $\mathcal{Y}^-$. By Remark 5.1, the $(u_p)_n$'s must then alternate in sign, and therefore, $(u_p)_n \to 0$ as $n \to \infty$.

Suppose for definiteness that the first plastic segment occurs in $\mathcal{Y}^+$ (the proof if the first plastic segment occurs in $\mathcal{Y}^-$ is analogous to what follows). Then the subsequence $\{(u_p)_{2n-1}\}$, $n = 1,2,\ldots$ corresponds to the resultant plastic displacements as the solution departs plastic segments in $\mathcal{Y}^+$, and the subsequence $\{(u_p)_{2n}\}$, $n = 1,2,\ldots$ corresponds to the resultant plastic displacements as the solution departs plastic segments in $\mathcal{Y}^-$.

When the solution has departed $(0,\alpha,(u_p)_{2n-1})$ after a plastic segment in $\mathcal{Y}^+$, then during the subsequent elastic segment the solution lies in the ellipse given by

$$\frac{1}{2}mv^2 + \frac{1}{2}\mu u_e^2 + \frac{1}{2}k(u_e + (u_p)_{2n-1})^2$$
$$= \frac{1}{2}\mu\alpha^2 + \frac{1}{2}k(\alpha + (u_p)_{2n-1})^2. \tag{5.19}$$

Similarly, when a solution has departed $(0,-\alpha,(u_p)_{2n})$ after a plastic segment in $\mathcal{Y}^-$, then during the subsequent elastic segment the solution lies in the ellipse given by

$$\frac{1}{2}mv^2 + \frac{1}{2}\mu u_e^2 + \frac{1}{2}k(u_e + (u_p)_{2n})^2$$
$$= \frac{1}{2}\mu\alpha^2 + \frac{1}{2}k(-\alpha + (u_p)_{2n})^2. \tag{5.20}$$

As a result of Remarks 5.1 and 5.2, the planes u_p = const. on which these elastic unloading segments are located approach the plane $u_p = 0$ in the following manner: the planes $u_p = (u_p)_{2n-1}$ approach monotonically from the $u_p > 0$ direction, and the planes $u_p = (u_p)_{2n}$ approach monotonically from the $u_p < 0$ direction. Finally, because $(\Delta u_p)_n \to 0$ as $n \to \infty$ the curves containing the plastic segments in the half-planes $\mathcal{Y}^+$ and $\mathcal{Y}^-$ shrink down respectively to the points $(0,\alpha,0)$ and $(0,-\alpha,0)$.

Our final task is to confirm that the ellipse given by equation (5.13) is the ω-limit set of $C^+(x^\circ)$ for each $x^\circ \in \mathcal{S}$ whose solution undergoes an infinite number of plastic segments. Let t_{2n-1} be the time at which the solution occupies the point $(0,\alpha,(u_p)_{2n-1})$ (i.e., the time when the solution departs the $(2n-1)^{st}$ plastic segment). Note that the sequence $\{(0,\alpha,(u_p)_{2n-1}\}$ of points monotonically approaches the point $(0,\alpha,0)$ as $n \to \infty$. Moreover, from the discussion preceding Theorem 3.1, $t_{2n-1} \to +\infty$ as $n \to \infty$. Thus, $(0,\alpha,0) \in L(C^+(x^\circ))$ by the definition of ω-limit set. Similarly, it follows that $(0,-\alpha,0) \in L(C^+(x^\circ))$. Thus, the ellipse E in (5.13) has two of its points in $L(C^+(x^\circ))$ and is itself the trajectory of the solution of (IVP) starting at $(0,\alpha,0)$. Well known properties of ω-limit sets (see, [1980,1], Theorem 8.1, p. 47) then tell us that $L(C^+(x^\circ))$ includes the ellipse E in (5.13). To show that $L(C^+(x^\circ))$ equals E, we first note from earlier arguments that $L(C^+(x^\circ))$ is included in the plane $u_p = 0$. Let $(v,u_e,0)$ be given in the exterior of E (with respect to the plane $u_p = 0$). The trajectory of the solution of (IVP) with $x^\circ = (v,u_e,0)$ must contain a non-trivial plastic segment and, hence, points with $u_p \neq 0$. Because $L(C^+(x^\circ))$ is an invariant set for (2.12)-(2.15) and $u_p = 0$ at each point of $L(C^+(x^\circ))$, it follows that $(v,u_e,0)$ is not in $L(C^+(x^\circ))$. Let a point $(v,u_e,0)$ be given that is in the interior of E (with respect to the plane $u_p = 0$). The form of the elastic segments of $C^{+}(x^{o})$ through the points $(0, \alpha, (u_p)_{2n-1})$ and the form of the elastic segments of $C^{+}(x^{o})$ through the points $(0, -\alpha, (u_p)_{2n})$, for $n = 1,2,3,\cdots$, tell us that these segments, when projected on the plane $u_p = 0$, lie in the exterior of E. It follows that no point $(v, u_e, 0)$ interior to E can be in $L(C^{+}(x^{o}))$. Thus, $L(C^{+}(x^{o})) = E$. □

6. Conditions for Self-Annealing

In this section we find non-trivial subsets of the state space ^ with the property that all initial data in a given subset lead to solution trajectories of (IVP) with the same character for large times. We already know that when the initial data is close to an equilibrium state other than the two endpoints of the line of equilibrium states, the solution remains elastic and periodic for all time. Here, we shall delimit sets of initial data for which the resulting solutions undergo infinitely many plastic segments. From Theorem 5.1, we know that the plastic displacement associated with each such solution satisfies $u_p(t) \to 0$ as $t \to \infty$, and that the solution has the ellipse

$$\tfrac{1}{2} mv^2 + \tfrac{1}{2}(\mu + k)u_e^2 = \tfrac{1}{2}(\mu + k)a^2 \tag{5.13}$$

as its ω-limit set. Specifically, we delimit here the data in the plane $u_p = 0$ that produce such solutions. In doing so, we find that the stiffness ratio $\kappa = \frac{k}{\mu + k}$ is a critical parameter: when $\frac{1}{2} \le \kappa < 1$, solutions of the (IVP) with an infinite number of plastic segments do indeed exist, and the choices of initial data whose solutions have this property can be completely characterized; on the other hand, when $0 < \kappa < \frac{1}{2}$, solutions of (IVP) can have at most finitely many plastic segments. (See the thesis [1984, 1] for a detailed partition of initial data with $u_p \neq 0$ according to the long-term behavior of u_p.)

Let $x_0 \in \mathcal{Y}$ in the plane $u_p = 0$ lie outside of the ellipse given by equation (5.13). As we show below, when $\frac{1}{2} \le \kappa < 1$, the solution of (IVP) starting at x^o must undergo an infinite number of plastic segments. Furthermore, the plastic displacement of the solution satisfies $\lim_{t\to\infty} u_p(t) = 0$. Although the initial point x^o in the plane $u_p = 0$ can be chosen so that the plastic displacement at the end of the first plastic segment has a magnitude that is arbitrarily large, the system nevertheless *completely removes* that plastic displacement from the solution in the limit of large times. We call this behavior *self-annealing*, or *mechanical self-annealing* to emphasize the absence of thermal devices to effect the removal of plastic displacements. The behavior of the unforced bilinear elastic-plastic oscillator when $\frac{1}{2} \leq \kappa \leq 1$ also can be considered to be a type of *underdamped oscillation*. Indeed, the solution of (IVP) for initial $x^o \in \mathcal{S}$ in the plane $u_p = 0$ and outside the ellipse (5.13) has plastic displacements at the end of the plastic segments that alternate in sign and decrease monotonically in magnitude to zero. This behavior is analogous to the behavior of the displacements of a linear spring with linear viscosity in the underdamped case.

As mentioned above, we shall show that when $0 < \kappa < \frac{1}{2}$, all solutions of (IVP) have only a finite number of plastic segments. Moreover, if x^o is in the plane $u_p = 0$ and lies outside the ellipse (5.13), then the final plastic displacement is not necessarily zero, and that displacement is achieved in finite time. We also shall give a sharp bound, independent of the choice of initial data, on the magnitude of the final plastic displacement. The portion of Theorem 5.1 applicable to solutions with a finite number of plastic segments governs this case, so that we may conclude that after its final plastic segment, the solution remains in an elastic orbit contained in an ellipse which touches the yield surface in which the solution had its final plastic segment. Here again, until the final plastic segment is attained, the plastic displacements at the end of the plastic segments change sign and decrease monotonically in magnitude. Thus, for μ and k such that $0 < \kappa < \frac{1}{2}$, the unforced bilinear elastic-plastic oscillator acts as if it were *overdamped*, because it allows only a finite number of plastic segments and, hence, only a finite number of oscillations in the sign of the resultant plastic displacements. Moreover, the value $\kappa = \frac{1}{2}$ is a *critical stiffness ratio*, because it separates the underdamped and overdamped cases.

For the analysis that supports these results, two pairs of curves will be particularly useful: (1) the plastic segment in $\mathcal{Y}^+$ that passes through $(0,\alpha,0)$ and its counterpart in $\mathcal{Y}^-$ which passes through $(0,-\alpha,0)$, and (2) the locus of points (v,α,u_p) with $u_p < 0$ (resp. $(-v,-\alpha,u_p)$ with $u_p > 0$) where the elastic segment through $(0,-\alpha,u_p)$ (resp. $(0,\alpha,u_p)$) enters $\mathcal{Y}^+$ (resp. $\mathcal{Y}^-$).

The single curve, obtained from the pair of curves in (1) by joining at the origin (0,0) the projection on $u_e = 0$ of the curve in $\mathcal{Y}^+$ to the projection on $u_e = 0$ of its counterpart in $\mathcal{Y}^-$, will be denoted by $\mathcal{P}_0$. The portion of $\mathcal{P}_0$ with $u_p \geq 0$ is the half-ellipse

$$\frac{1}{2}mv^2 + \frac{1}{2}k(-\alpha + u_p)^2 - \mu\alpha u_p = \frac{1}{2}k\alpha^2 \quad , \quad v \leq 0, \qquad (6.1)$$

while the portion of $\mathcal{P}_0$ with $u_p \leq 0$ is the half-ellipse

$$\frac{1}{2}\,mv^2 + \frac{1}{2}k\,(\alpha + u_p)^2 + \mu\alpha u_p = \frac{1}{2}\,k\alpha^2 \quad , \quad v \geq 0. \tag{6.2}$$

Thus, on $\mathscr{S}_0$ with $u_p \geq 0$, the kinetic energy is given in terms of u_p by rewriting (6.1) in the form

$$\frac{1}{2}\,mv^2 = \frac{1}{2}\,k\alpha^2 + \mu\alpha u_p - \frac{1}{2}k\,(-\alpha + u_p)^2, \tag{6.3}$$

with a similar equation on $\mathscr{S}_0$ holding when $u_p \leq 0$. We note that the curve $\mathscr{S}_0$ is symmetric with respect to the origin in the plane $u_e = 0$.

The curve $\mathscr{S}_{entry}$ in the plane $u_e = 0$ is defined to be the union of the projections on $u_e = 0$ of the pair of curves in (2). For $u_p \geq 0$, it is the half-parabola

$$\frac{1}{2}\,mv^2 + \frac{1}{2}k\,(-\alpha + u_p)^2 + \frac{1}{2}\,\mu\alpha^2$$
$$= \frac{1}{2}k\,(\alpha + u_p)^2 + \frac{1}{2}\,\mu\alpha^2, \quad v \leq 0, \tag{6.4}$$

or

$$\frac{1}{2}\,mv^2 = 2k\alpha u_p \quad , \quad v \leq 0, \tag{6.5}$$

and for $u_p \leq 0$, $\mathscr{S}_{entry}$ is the half-parabola

$$\frac{1}{2}\,mv^2 = -2k\alpha u_p \quad , \quad v \geq 0. \tag{6.6}$$

The curve $\mathcal{F}_{entry}$ is also symmetric with respect to the origin in the plane $u_e = 0$. Representative curves $\mathcal{F}_0$ and $\mathcal{F}_{entry}$ are shown in Figures 4 and 5. Subtracting the right hand side of the equation in (6.3) from the right hand side of the equation in (6.5) gives after some simplification

$$\frac{1}{2}m(v_{entry}^2 - v_0^2) = (k - \mu)\alpha u_p + \frac{1}{2}ku_p^2. \tag{6.7}$$

In (6.7) we have written v_{entry} and v_0 for the values of v on $\mathcal{F}_{entry}$ and on $\mathcal{F}_0$ corresponding to a given u_p.

Suppose a solution has just departed a plastic segment in $\mathcal{Y}^+$ (resp. $\mathcal{Y}^-$) with resultant plastic strain $u_{p_1} \in \mathbb{R}$. If $u_{p_1} \leq 0$ (resp. $u_{p_1} \geq 0$), then Remark 5.1 ensures that no further plastic segments occur. If $u_{p_1} > 0$, then Remark 5.1 guarantees that a subsequent plastic segment occurs in $\mathcal{Y}^-$ (resp. $\mathcal{Y}^+$) after an intermediate elastic segment. Knowledge of the curves $\mathcal{F}_0$ and $\mathcal{F}_{entry}$ enables one to determine whether there occurs yet another plastic segment in $\mathcal{Y}^+$ (resp. $\mathcal{Y}^-$). For example, a subsequent plastic segment **will not** take place if the point (v, u_{p_1}) on $\mathcal{F}_{entry}$ lies on $\mathcal{F}_0$ or in the interior of the region bounded by the curve $\mathcal{F}_0$ and the line $v = 0$; otherwise, a subsequent plastic segment **will** take place.

To illustrate more fully how the curves $\mathcal{F}_0$ and $\mathcal{F}_{entry}$ can be used to determine whether subsequent plastic segments occur, let $y^o = (v^o, \alpha, u_p^o) \in \mathcal{Y}^+$ be given such that $v^o > 0$ and $u_p^o > 0$. The solution of (IVP) for this initial data undergoes an initial plastic segment. The fact that $t \mapsto u_p(t)$ is strictly increasing on solutions in $\mathcal{Y}^+$ ensures that the solution departs $\mathcal{Y}^+$ with positive resultant plastic displacement. The solution then undergoes an intermediate elastic segment. Since the resultant plastic displacement at the end of the first plastic segment is positive, by Remark 5.1 the solution will undergo a subsequent plastic segment. If we let $u_p = u_p^1$ denote the resultant plastic strain at the end of the first plastic segment, then the intersection of the line $u_p = u_p^1$ with the curve $\mathcal{F}_{entry}$ in Figure 4 indicates where the solution enters the yield surface $\mathcal{Y}^-$. For the case indicated in Figure 4, the point of intersection lies between the curve $\mathcal{F}_0$ and the line $v = 0$. This ensures that the plastic strain at the end of the resulting plastic segment in $\mathcal{Y}^-$ is positive. We may then use Remark 5.1 to conclude that no further plastic segments occur.

Now consider Figure 5, where the same initial data are used, but the curves $\mathcal{F}_0$ and $\mathcal{F}_{entry}$ are different. Once again, the solution undergoes an initial plastic segment with positive resultant plastic displacement $u_p^1 > 0$. After an intermediate elastic segment, the solution enters $\mathcal{Y}^-$ at the intersection of the line $u_p = u_p^1$ with the curve $\mathcal{F}_{entry}$. For the case indicated in Figure 5, the intersection point lies outside the region between $\mathcal{F}_0$ and $v = 0$. This ensures that the plastic strain at the end of the subsequent plastic segment in $\mathcal{Y}^-$ is negative. We may then conclude by Remark 5.1 that the solution will undergo a subsequent plastic segment in $\mathcal{Y}^+$.

It is now possible to use these observations to draw some conclusions about the asymptotic behavior of solutions for particular choices of μ

and k. Consider first an unforced bilinear elastic-plastic oscillator for which $k \geq \mu$, or, equivalently, $\frac{1}{2} \leq \frac{\mu}{\mu + k} = \kappa < 1$. Notice that this condition on k and μ ensures that the right hand side of equation (6.7) is strictly positive for all $u_p > 0$. Hence, we may conclude that for all u_p not equal to zero, the curve $\mathcal{F}_0$ lies in the interior of the region bounded by the curve $\mathcal{F}_{entry}$ and the line $v = 0$. With this in mind, consider $x^o = (v^o, u_e^o, 0) \in \mathcal{S}$, i.e., consider an initial condition in $\mathcal{S}$ with zero initial plastic displacement. For a choice of x^o on or in the interior of the ellipse given by equation (5.13), the solution remains elastic and periodic for all time. Consider instead an $x^o = (v^o, u_e^o, 0) \in \mathcal{S}$ with $v^o > 0$ which lies in the plane $u_p \equiv 0$ but in the exterior of the above ellipse. Since the initial elastic trajectory for x^o eventually reaches $\mathcal{Y}^+$, the trajectory undergoes an initial plastic segment in $\mathcal{Y}^+$. Since the plastic displacement is strictly increasing along plastic segments in $\mathcal{Y}^+$, the trajectory for x^o emerges from $\mathcal{Y}^+$ with positive resultant plastic displacement. By Remark 5.1, there will be a subsequent plastic segment in $\mathcal{Y}^-$. The above remark in the case $\frac{1}{2} \leq \kappa < 1$ concerning the curves $\mathcal{F}_0$ and $\mathcal{F}_{entry}$ ensures that the resultant plastic displacement at the end of this plastic segment in $\mathcal{Y}^-$ is negative. Again by Remark 5.1, the trajectory has another plastic segment in $\mathcal{Y}^+$, and the same remark about $\mathcal{F}_0$ and $\mathcal{F}_{entry}$ implies that the resultant plastic displacement at the end of the segment is positive. Let us define a *plastic cycle* to be a process in which a solution with elastic and plastic segments departs $\mathcal{Y}^+$, undergoes an elastic segment and then a subsequent plastic segment in $\mathcal{Y}^-$, followed by another elastic segment and then a subsequent plastic segment in $\mathcal{S}^+$. Then for $x^0 \in \mathcal{S}$ as given, the following statement has been demonstrated:

(1) The solution undergoes a plastic cycle, and the resultant plastic displacement upon completion of this cycle is positive.

Label additional statements as follows:

(n) The solution undergoes n plastic cycles, and the resultant plastic displacement upon completion of cycle n is positive.

(n + 1) The solution undergoes n + 1 plastic cycles, and the resultant plastic displacement upon completion of cycle n + 1 is positive.

The arguments used to prove that the solution corresponding to x^0 as given satisfies statement (1) may also be used to show that if statement (n) is assumed, then statement (n + 1) follows. Thus, it follows by induction that each $x^0 = (v_0, u_0, 0) \in \mathcal{Y}_e$ with $v_0 > 0$ which lies in the exterior of the ellipse given by equation (5.13) has a solution which undergoes an infinite number of plastic segments. The proof for $x^0 = (v_0, u_0, 0) \in \mathcal{Y}_e$ with $v_0 < 0$ lying in the exterior of the above ellipse is completely analogous. Thus, invoking Theorem 5.1, we have

Theorem 6.1. *Let* $h_\ell \leq K < 1$, *and* let $x^0 = (v_0, u_0, 0) \in \mathcal{Y}_e$ be given such that x^0 lies *in* the *exterior of the* ellipse *given* by

$$\frac{1}{2}\,mv^2 + \frac{1}{2}(\mu + k)u_e{}^2 = \frac{1}{2}(\mu + k)\alpha^2. \qquad (5.13)$$

The solution $t \mapsto (v(t), u_e(t), u_p(t))$ *then undergoes an infinite number of plastic segments,* $u_p(t) \to 0$ *as* $t \to \infty$, *and the* ω*-limit set of* $C^+(x^o)$ *is the ellipse given by the equation* (5.13).

One may analyze similarly the asymptotic behavior of solutions for initial data lying in planes u_p = const., but $u_p \neq 0$. It turns out that in addition to the regions which correspond to solutions that remain elastic for all time, and the regions which correspond to solutions that undergo an infinite number of plastic segments, there is a third type of region when $\frac{1}{2} \leq \kappa < 1$. For the third region, the solution undergoes exactly one plastic segment, after which it is elastic and periodic for all time. The region lying in the half-planes $\mathcal{Y}^+$ and $\mathcal{Y}^-$ bounded by the curve $\mathcal{S}_0$ and the line $v = 0$ is an example of such a region.

We consider one additional case in this section. Suppose now that the unforced bilinear elastic-plastic oscillator is such that $0 < k < \mu$, which means that $0 < \kappa < \frac{1}{2}$. The right hand side of equation (6.7) then is negative for every positive but sufficiently small plastic displacement. Thus, for u_p positive and sufficiently small, points (v, u_p) on $\mathcal{S}_{entry}$ lie in the interior of the region enclosed by $\mathcal{S}_0$ and the line $v = 0$. However, as u_p is increased, the curves intersect, and for u_p sufficiently large, (v, u_p) in $\mathcal{S}_{entry}$ lies outside that region. The equations of the curves for $\mathcal{S}_0$ and $\mathcal{S}_{entry}$ are for $u_p \geq 0$ respectively

$$\frac{1}{2}mv^2 + \frac{k}{2}(u_p - \frac{\alpha}{\kappa})^2 = \frac{k}{2}(\frac{\alpha}{\kappa})^2 \tag{6.8}$$

and

$$\frac{1}{2}mv^2 = 2k\alpha u_p. \tag{6.5}$$

Notice from equation (6.8) that the portion of $\mathcal{F}_0$ contained in the projection of $\mathcal{Y}^-$ into the plane $u_e = 0$ is a half-ellipse centered at $(0,\frac{\alpha}{\kappa})$, and that this half-ellipse crosses the line $v = 0$ at $u_p = 2\frac{\alpha}{\kappa}$. On the other hand, as mentioned earlier, equation (6.5) describes the half-parabola representing that portion of $\mathcal{F}_{entry}$ contained in the projection of $\mathcal{Y}^-$ into the plane $u_e = 0$. This half-parabola is tangent to the line $v = 0$. The two curves described above intersect (in the projection of $\mathcal{Y}^-$) at the point $(-2\alpha\sqrt{\frac{2(\mu - k)}{m}}, \frac{2(\mu - k)}{k}\alpha)$.

Let us again examine the asymptotic behavior of a solution with initial data $x^o = (v^o,u_e^o,0) \in \mathcal{S}$, i.e., an initial condition in $\mathcal{S}$ with zero initial plastic displacement. Once again, if x^o lies on or in the interior of the ellipse given by equation (5.13), the solution remains elastic and periodic for all time. Now consider $x^o = (v^o,u_e^o,0) \in \mathcal{S}$ that lies in the exterior of this ellipse. Since the initial elastic trajectory for x^o eventually reaches $\mathcal{Y}^+$, it will undergo an initial plastic segment in $\mathcal{Y}^+$. Since the plastic displacement is strictly increasing along plastic segments in $\mathcal{Y}^+$, the trajectory will emerge from $\mathcal{Y}^+$ with positive resultant plastic displacement. By Remark 5.1, the solution will have a subsequent plastic segment in $\mathcal{Y}^-$. There is a special segment of an ellipse $\mathcal{F}_2$ in $\mathcal{Y}^+$ (see Figure 6) which represents a plastic solution segment. It is that segment of an ellipse which crosses the line $v = 0$

at exactly the value of u_p at which $\mathscr{F}_0$ and $\mathscr{F}_{entry}$ intersect. The solution corresponding to this curve will undergo an intermediate elastic segment, enter $\mathscr{Y}^-$, and undergo a plastic segment in $\mathscr{Y}^-$ contained in the $\mathscr{F}_0$ curve. Thus, this solution departs its second plastic segment with zero resultant plastic displacement, and thereupon enters the ellipse given by equation (5.13) and remains elastic and periodic for all subsequent time. For $x^o = (v^o, u_e^o, 0) \in \mathscr{S}$ with $v^o > 0$ in the exterior of the ellipse given by equation (5.13) whose solution enters $\mathscr{Y}^+$ between $(0,\alpha,0)$ and the point at which $\mathscr{F}_2$ crosses the line $u_p = 0$, the corresponding solution undergoes plastic segment in $\mathscr{Y}^+$, an intermediate elastic segment, and then a plastic segment in $\mathscr{Y}^-$. Since this solution enters $\mathscr{Y}^-$ along $\mathscr{F}_{entry}$ at a point in the interior of $\mathscr{F}_0$, it emerges from its plastic segment in $\mathscr{Y}^-$ with positive resultant plastic displacement, and by Remark 5.1 then can undergo no further plastic segments. Thus, when $0 < \kappa < \frac{1}{2}$, there is a region of the portion of $\mathscr{S}$ in the plane $u_p = 0$ with $v \geq 0$ such that if x^o lies in this region, then the corresponding solution undergoes exactly two plastic segments, and the solution is subsequently elastic and periodic in a plane $u_p = \text{const.} \geq 0$.

There is an additional special segment of an ellipse $\mathscr{F}_3$ in $\mathscr{Y}^+$ (see Figure 6) which represents a plastic solution segment. A solution contained in that segment undergoes an initial plastic segment in $\mathscr{Y}^+$, followed by an intermediate elastic segment, followed by a plastic segment in $\mathscr{Y}^-$ such that, after the solution departs that plastic segment and undergoes an additional intermediate elastic segment, the solution reenters

$\mathcal{Y}^+$ at exactly the point where $\mathcal{F}_0$ and $\mathcal{F}_{entry}$ intersect in $\mathcal{Y}^+$. This solution will then be contained in a portion of $\mathcal{F}_0$, and will subsequently emerge from its third and final plastic segment with zero resultant plastic displacement. It will subsequently remain in the ellipse given by equation (5.13) for all time, and the solution is then elastic and periodic. For each $x^o = (v^o, u_e^o, 0) \in \mathcal{S}$ with $v^o > 0$ whose solution enters its initial plastic segment in $\mathcal{Y}^+$ between the points at which $\mathcal{F}_2$ and $\mathcal{F}_3$ cross the line $u_p = 0$, the solution undergoes a plastic segment in $\mathcal{Y}^+$, an intermediate elastic segment, a plastic segment in $\mathcal{Y}^-$, an additional intermediate elastic segment, and then a plastic segment in $\mathcal{Y}^+$ with a resultant plastic displacement that is negative, so that by Remark 5.1, it cannot undergo a further plastic segment in $\mathcal{Y}^-$. Thus, when $0 < \kappa < \frac{1}{2}$, there is a region of the portion of $\mathcal{S}$ in the plane $u_p = 0$ with $v = 0$ such that if x^o lies in this region, then the corresponding solution undergoes exactly three plastic segments, and the solution is subsequently elastic and periodic in a plane $u_p = \text{const.} \leq 0$.

It is clear that one can continue to construct such special elliptical segments forming regions of $\mathcal{Y}^+$ such that solutions corresponding to initial data in one region undergo one more plastic segment than those for the previous region. The portion of $\mathcal{S}$ in the plane $u_p = 0$ lying in $v \leq 0$ can be analyzed in a similar manner. Thus, one has the result:

Theorem 6.2. *Let* $0 < \kappa < \frac{1}{2}$, *and let* $x^o = (v^o, u_e^o, 0) \in \mathcal{S}$ *be given such that* x^o *lies in the exterior of the ellipse given by*

$$\frac{1}{2}mv^2 + \frac{1}{2}(\mu + k)u_e^2 = \frac{1}{2}(\mu + k)\alpha^2. \tag{5.13}$$

For each natural number n, *there exists a region* $\mathcal{D}_n$ *in the exterior of the ellipse such that if* $x° \in \mathcal{D}_n$, *then the solution undergoes exactly* n *plastic segments, and is subsequently elastic and periodic. If* $v° > 0$ *(resp.* $v° < 0$*) and* $x° \in \mathcal{D}_n$, *and if* n *is even, the solution undergoes its final plastic segment in* $\mathcal{Y}^-$ *(resp.* $\mathcal{Y}^+$*) and the final resultant plastic strain is nonnegative (resp. nonpositive); on the other hand, if* n *is odd the solution undergoes its final plastic segment in* $\mathcal{Y}^+$ *(resp. in* $\mathcal{Y}^-$*) and the final resultant plastic strain is nonpositive (resp. nonnegative).*

We note finally, when $0 < \kappa < \frac{1}{2}$, there is an explicit bound for the maximum possible final resultant plastic displacement u_{p_f}. A solution which yields the extremal value is one whose final plastic segment lies in the region formed by $\mathcal{F}_{entry}$ and the line $v = 0$ and is tangent to $\mathcal{F}_{entry}$ at exactly one point. The maximum possible value of u_{p_f} is found by determining the smaller value at which the ellipse containing the final plastic segment intersects the line $v = 0$. The calculations to determine the point of tangency yield the bound

$$|u_{p_f}| \leq \alpha\left[\frac{1}{\kappa} - 2\sqrt{\frac{1}{\kappa} - 1}\,\right]. \tag{6.9}$$

It should be observed that this estimate is sharp, and that it is independent of $x°$.

7. Generalizations of the Main Results

We consider now elastic-plastic oscillators for which the dependence of the force σ upon the elastic displacement u_e and the dependence of the restoring force f upon the total displacement $u = u_e + u_p$ both can be non-linear. Specifically, we replace the dynamical equation (2.12) by the relation

$$m\dot{v}(t) = - \phi'(u_e(t) + u_p(t)) - \Sigma'(u_e(t)), \tag{7.1}$$

where ϕ and Σ are smooth functions with further properties to be specified below, and where $'$ denotes differentiation. Of course, when

$$\phi(u) = \frac{1}{2} ku^2 \quad \text{and} \quad \Sigma(u) = \frac{1}{2} \mu u^2 \tag{7.2}$$

we obtain again (2.12), so that (7.1) together with (2.11), (2.13) and (2.14) does include the system (2.11)-(2.14) studied in Sections 2 through 6.

In this section we give further conditions on the potentials ϕ and Σ that imply conclusions similar to those obtained when ϕ and Σ are quadratic. Specifically, the following conditions on ϕ and Σ yield generalizations of the main results in Sections 3, 5, and 6:

$$\Sigma, \phi \in C^2(\mathbb{R}, \mathbb{R}), \tag{7.3}$$

$$\phi(0) = 0 \quad . \quad 2(0) = 0 \tag{7.4}$$

$$2(-u) - 2(u) = *(-u) - *(u) = 0 \quad \text{for all } u \in R, \tag{7.5}$$

$$*'(u) > 0 \quad \text{for all} \quad u > 0. \tag{7.6}$$

$$\lim_{|U|HO} *(u) = + ». \tag{7.7}$$

$$2''(u) > 0 \quad \text{for all} \quad u \in R\backslash\{0\}, \tag{7.8}$$

$$4>''(u) > 0 \quad \text{for all} \quad u \in R \quad \text{and} \quad <fr'(a) \; I \; 2'(a) \tag{7.9a}$$

or

$$4>''(u) \; £ \; 0 \quad \text{for all} \quad u > a \quad \text{and} \quad \sup_{u \in R} 4>'(u) < 2'(a). \tag{7.9b}$$

When ^ and 2 are the quadratics in (7.2), then conditions (7.3)-(7.8) are satisfied; if $k \geq jx$, then condition (7.9a) also holds. Examples of other functions that satisfy (7.3)-(7.9a) are the functions

$$\phi(u) = au^2 + bu^4$$

$$2(u) = cu^4 \tag{7.10}$$

with a.b.c positive and $2aa + 4(b - c)a^{[illegible]}$ positive. Examples of functions that satisfy (7.3)-(7.8). (7.9b) are the functions

$$\phi(u) = \begin{cases} 1 + \sin \dfrac{\pi(u - \alpha)}{2\alpha} & , \quad 0 \leq u \leq \alpha \\ 1 + \dfrac{\pi}{2\alpha}(u - \alpha) & , \quad \alpha < u \\ \phi(-u) & , \quad u < 0 \end{cases} \tag{7.11}$$

$$\Sigma(u) = cu^4,$$

with $c > \pi/8\alpha^4$.

The distinction between the conditions (7.9a) and (7.9b) lies chiefly in the relative strengths of the restoring forces ϕ' and Σ'. In fact, when ϕ and Σ satisfy (7.2)-(7.8) and (7.9a), one can show that an analogue of Theorem 6.1 holds, so that all initial data in the plane $u_p = 0$ produce positive half-trajectories whose ω-limit sets are elastic tractories in the plane $u_p = 0$. Thus, condition (7.9a), in which the restoring force $f = \phi'$ is stronger than the force $\sigma = \Sigma'$, leads to self-annealing for all initial data in the plane $u_p = 0$. In particular, if a solution of (2.11), (7.1), (2.13) and (2.14) with $u_p^\circ = 0$ undergoes at least one plastic segment, it undergoes infinitely many plastic segments and u_p tends to zero in the limit of large times. On the other hand, when ϕ and Σ satisfy (7.2)-(7.8), and (7.9b), then one can show that every solution of (2.11), (7.1), (2.13), and (2.14) undergoes at most two plastic segments, and, only for exceptional initial data is it true that u_p has limit zero for large times. Thus, condition (7.9b), in which the restoring force $f = \phi'$ is weaker than $\sigma = \Sigma'$, leads to self-annealing only in exceptional circumstances.

The methods used to obtain generalizations of the results of Sections 3, 5, and 6 to the case of non-quadratic potentials follow closely these presented in those sections for the case of quadratic potentials. Details can be found in the special case when the potential Σ is quadratic in the Ph.D. thesis of THOMAS, [1984,1]. It should be noted that, when ϕ is not quadratic, there also are generalizations of Theorem 4.1 on stability of equilibrium states (see [1984, 1], Theorem 4.2), but the proofs of these generalizations are different from our proof of Theorem 4.1, because we have been unable to find Liapunov functions when the potential ϕ is not quadratic.

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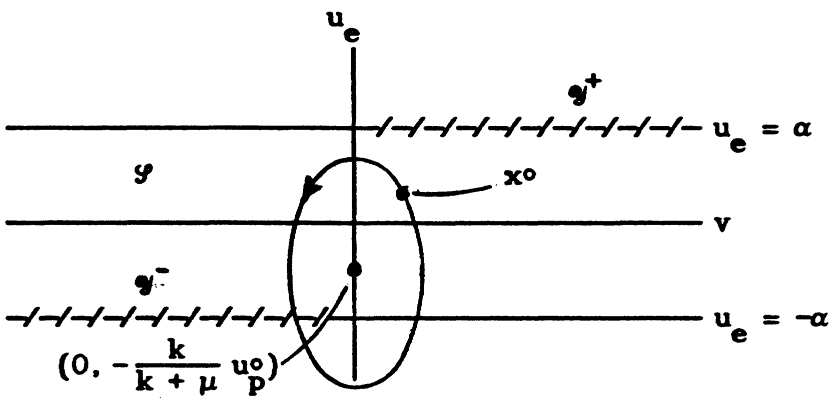

Figure 1: An elastic trajectory

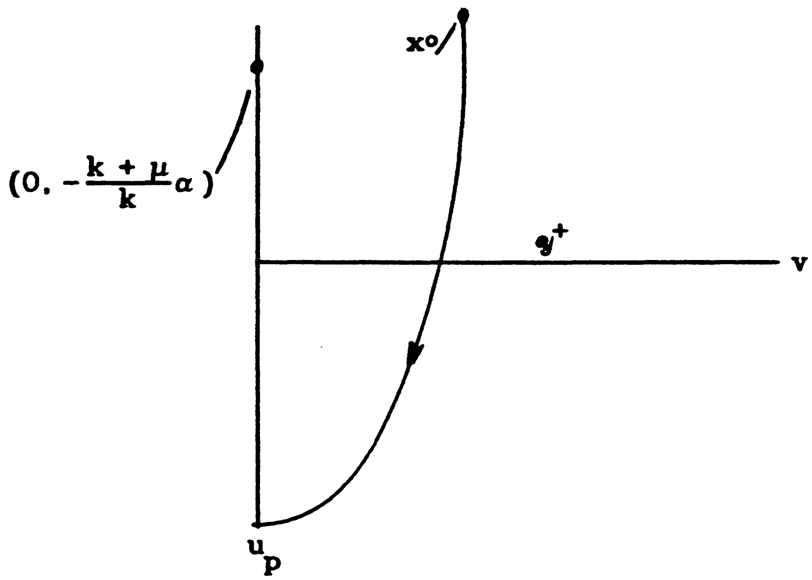

Figure 2: A plastic segment in $\mathscr{Y}^+$

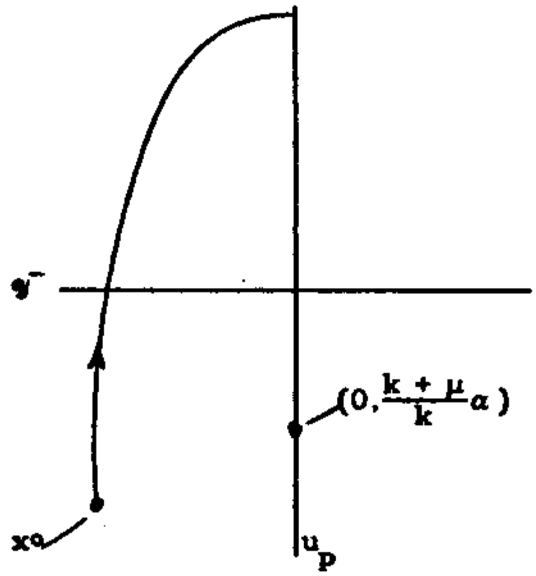

Figure 3: A plastic segment in V''

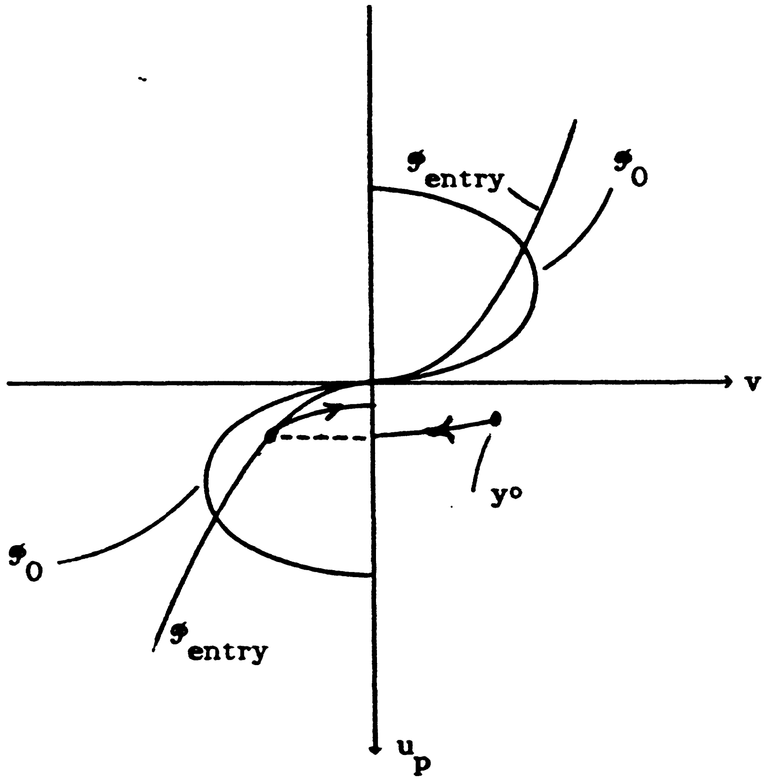

Figure 4: Special curves in $\mathcal{Y}^+ \cup \mathcal{Y}^-$ for overdamped motions

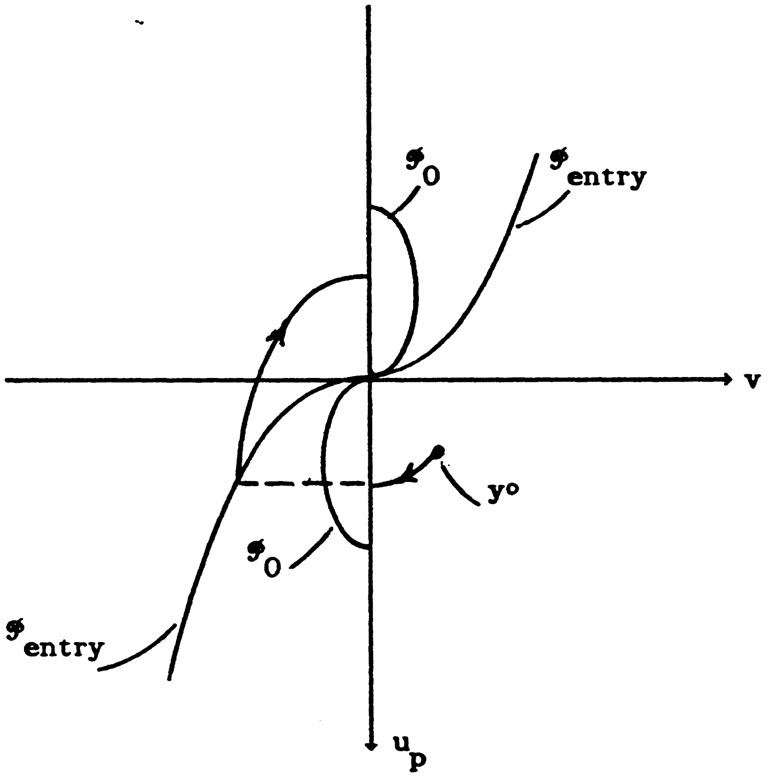

Figure 5: Special curves in $\mathcal{Y}^+ \cup \mathcal{Y}^-$ for underdamped motions

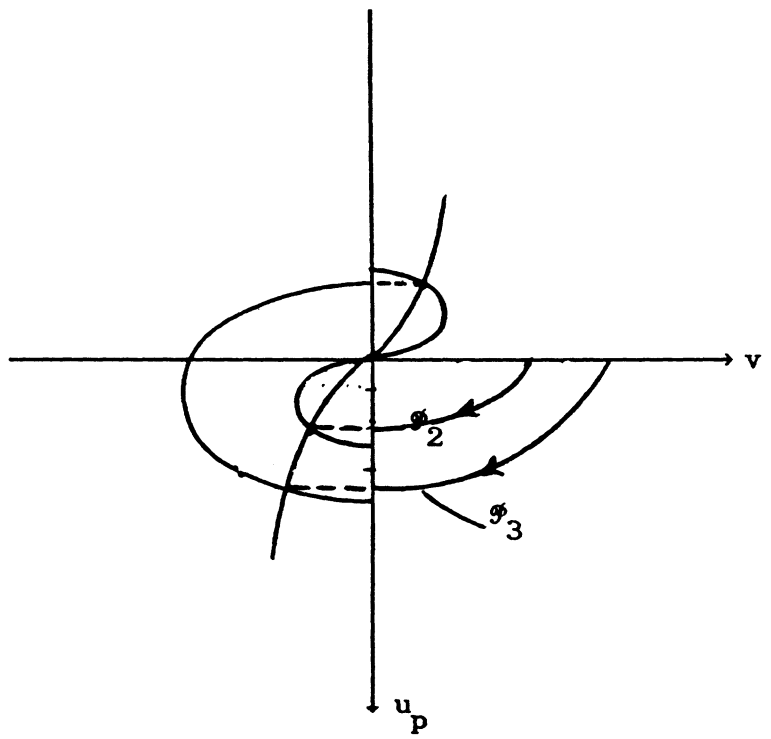

Figure 6: Separating trajectories $\mathscr{S}_2$ and $\mathscr{S}_3$