ON MATCHINGS AND HAMILTON CYCLES IN RANDOM GRAPHS

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§1. Introduction

The aim of this paper is to review what is currently known about large matchings and cycles in random graphs, in particular perfect matchings and Hamilton cycles. It may seen odd to treat these two subjects together, but we will see in §2 that proofs of theorems on the two topics can be similar.

We will first discuss the two basic results concerning $G_{n,m}$. This graph has vertex-set $[n] = \{1,2,\ldots,n\}$ and and edge-set $E_{n,m}$ which is a random m-subset of the $N = \binom{n}{2}$ possbilities.

We start with perfect matchings and a theorem of Erdös and Rényi [1966], the "founding fathers" of the subject of random graphs.

Theorem 1.1

Let $m = \frac{1}{2}n(\log n + c_n)$. then

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \Pr(G_{n,m} \text{ has a perfect matching})$$

$$= \lim_{\substack{n\to\infty \\ n \text{ even}}} \Pr(\delta(G_{n,m}) \geq 1)$$

$$= \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to +\infty \end{cases}$$

□

(When n is odd $G_{n,m}$ has a matching of size $\lfloor n/2 \rfloor$ with the same limiting probability that there is at most one isolated vertex).

Their proof is complicated and based on Tutte's theorem [Tutte, 1947] for the existence of perfect matchings. We give an outline of a relatively simple proof in §2.

This theorem sets the scene nicely. A simple necessary condition ($\delta \geq 1$) is nearly always sufficient. One can guess that $\delta \geq 2$ is nearly always sufficient for Hamilton cycles. In fact we have the following theorem of Komlós and Szemerédi [1983].

Theorem 1.2

Let $m = \frac{1}{2} n (\log n + \log\log n + c_n)$. then

$$\lim_{n\to\infty} \Pr(G_{n,m} \text{ is Hamilton})$$

$$= \lim_{n\to\infty} \Pr(\delta(G_{n,m}) \geq 2)$$

$$= \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to +\infty \end{cases} \qquad \square$$

Theorem 1.2 took somehwat longer to prove than Theorem 1.1. There were several interim results showing that if m grows large enough then almost every $G_{n,m}$ is hamiltonian. Amongst these the most important is probably that of Posa [1976] showing that $m = Kn\log n$ for sufficiently large K suffices. The paper contains a result (see Lemma 2.2) which is the foundation for many proofs of Hamiltonicity. Also Korsunov [1976] claimed a proof for $c_n \to \infty$ in an extended abstract.

Erdös and Rényi [1961] envisaged a *graph process* in which a random graph grows one edge at a time, the additional edge being chosen randomly from the edges remaining. Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_m, \ldots, \Gamma_N$ denote the random sequence of graphs produced. Let $m_k^* = \min\{m : \delta(\Gamma_m) \geq k\}$. Now clearly, $\Gamma_{m_2^*-1}$ is not Hamiltonian and it is rather nice that the following strengthening of Theorem 1.2 is possible:

Theorem 1.3

$$\lim_{n\to\infty} \Pr(\Gamma_{m_2^*} \text{ is hamiltonian}) = 1$$

□

This theorem was claimed in [Komlós and Szemeredi, 1983] without proof as a reformulation of Theorem 1.2. This is not quite true and Bollobás gave a complete proof in [Bollobás, 1984]. Subsequently, Ajtai, Komlós and Szemerédi [1985] gave a different proof. If Theorem 1.3 is true one should not be surprised with

Theorem 1.4

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \Pr(\Gamma_{m_1^*} \text{ has a perfect matching}) = 1$$

□

For a proof see Bollobás [1985] (Theorem VII.22).

In §2 we will give outline proofs of Theorem 1.1 and 1.2. They demonstrate the general approach to this topic. In §3 we give generalizations of these theorems. In §4–6 we discuss other models of random graphs, in §7 we discuss random digraphs and in §8 we end with some open problems.

§2."Proofs" of Theorems 1.1 and 1.2

We first consider Theorem 1.1. Suppose first that n is even, $m = \frac{1}{2}n(\log n + c)$, and $G_{n,m}$ has no perfect matching. Let $X = \{x: x$ is left exposed by some maximum matching$\}$. For $x \in X$ let $Y(x) = \{y: \exists$ a maximum matching which leaves x and y exposed$\}$.

For $S \subseteq [n]$ let $N(S) = \{t \notin S: \exists\ s \in S$ such that $st \in E_{n,m}\}$.

Lemma 2.1

$|N(Y(x))| < |Y(x)|$ for $x \in X$.

Proof

Suppose $x \in X$. Fix a maximum matching M leaving v exposed. Let S be the remaining set of vertices left exposed by M. Then $y \in Y = Y(x)$ iff there exists an even length alternating path P_y from $s \in S$ to y.

Suppose now that $zy \in E_{n,m}$ with $y \in Y$, $z \notin Y$. The lemma follows from the claim that there exists $y' \in Y$ such that $y'z \in M$. To see this note that either $z \in P_y$ or $P_y + yz + zy'$ is an even length alternating path, where zy' is the edge of M covering z. □

Now clearly

$$(2.1) \qquad x \in X,\ y \in Y(x) \text{ implies } xy \notin E_{n,m}.$$

Now let a graph with vertex set $[n]$ be in EX_k if

$$S \subseteq [n], \quad |S| \leq \frac{n}{k+1} \quad \text{implies} \quad |N(S)| \geq k|S|$$

Now it is not difficult to show that a.e. $G_{n,m}$ with $\delta \geq 1$ satisfies

(2.2) $G_{n,m} - A \in EX_1$ for all matchings A which avoid vertices of degree 1.

(The proof of this is through a somewhat tedious calculation where it is useful to consider [n] partitioned into small vertices of degree at most $\frac{1}{10}$ logn and large vertices - for details see similar calculations in [Bollobás, Fenner and Frieze, 1987]). We exploit the existence of this "hole" in $G_{n,m}$ implied by (2.1) and (2.2).

Observe also that a.e. $G_{n,m}$ has fewer than logn vertices of degree 1 and $\Delta \leq 3$ logn. We can now finish the proof fairly quickly using an argument based on that in Fenner and Frieze [1983]. Let $\mathcal{G}(n,m)$ denote the set of all n edge graphs on [n] and $\mathcal{G}_1(n,m)$ those which have $\delta \geq 1$, no perfect matching, fewer than logn vertices of degree 1, $\Delta \leq 3$ logn and satisfy (2.2). Let $\omega = \lceil \log n \rceil$ and A be a random ω-subset of $E_{n,m}$. Let $\mathcal{A}$ be the event {(i) A avoids at least one maximum matching of $G_{n,m}$, (ii) A avoids vertices of degree 1 and (iii) A is a matching}. Then

$$\Pr(\mathcal{A} \mid G_{n,m} \in \mathcal{G}_1(n,m)) \geq \frac{1}{3}.$$

(A simple calculation once one has fixed $G \in \mathcal{G}_1(n,m)$. Hence

$$\Pr(G_{n,m} \in \mathcal{G}_1(n,m)) \leq 3 \Pr(\mathcal{A}).$$

But

$$\Pr(\mathscr{A}) = \sum_{H} \Pr(\mathscr{A} \mid G_{n,m} - A = H)\Pr(G_{n,m} - A = H)$$

$$\leq \sum_{H\in EX_1} \Pr(A \cap F_H = \phi \mid G_{n,m} - A = H)\Pr(G_{n,m} - A = H)$$

where $F_H = \{xy\colon x \in X,\ y \in Y(x)\}$ for some sets $X, Y(x)$, $x \in X$ defined in terms of H only. $A \cap F_H = \phi$ follows from $\mathscr{A}(i)$ and $H \in EX_1$ follows from $\mathscr{A}(ii)$, $\mathscr{A}(iii)$ and (2.2). But if H is fixed then A is a random ω-subset of $\overline{E(H)}$. Hence

$$\Pr(\mathscr{A}) \leq \sum_{H\in EX_1} \left(\frac{9}{10}\right)^{\omega} \Pr(G_{n,m} - A = H)$$

$$\leq \left(\frac{9}{10}\right)^{\omega}$$

and the theorem follows after tidying up.

Now let us turn to Theorem 1.2. We will see that in some sense we can prove this theorem by replacing 1 by 2 in the relevant places. Suppose $G = G_{n,m}$ is not hamiltonian. Let $X = \{x\colon x$ is an endpoint of some longest path of $G\}$. For each $x \in X$ choose some longest path $P_x = (x = x_0, x_1, x_2, \ldots, x_p)$ with x as one endpoint. Suppose that the edge $x_p x_i$, $i \leq p-2$, exists. A *rotation* with x as fixed endpoint creates the new longest path $(x_0, x_1, \ldots, x_i, x_p, x_{p-1}, \ldots, x_{i+1})$, also having x as one endpoint. Let now $Y(x)$ be the set of vertices which are the endpoints other than x of those longest paths which can be obtained from P_x by a sequence of rotations with x as fixed endpoint.

In place of Lemma 2.1 we have

Lemma 2.2 (Posa [1976])

$$|N(Y(x))| < 2|Y(x)| \quad \text{for} \quad x \in X.$$

Proof

If $z \in N(Y(x))$ then z is the neighbour, on P_x, of some vertex in $Y(x)$. □

Clearly $|X| > |Y(x)|$ for all $x \in X$ and furthermore (2.1) holds if G is connected (using the current definition for X,Y).

We replace (2.2) by a.e. $G_{n,m}$ with $\delta \geq 2$ satisfies

(2.3) $G_{n,m} - A \in EX_2$ for all matchings A which avoid vertices of degree 2.

Observe also that a.e. $G_{n,m}$ is connected, has fewer than logn vertices of degree ≤ 2 and $\Delta \leq 3$ logn. We finish the proof as before. Now define $\mathcal{G}_2(n,m)$ as those graphs in $\mathcal{G}(n,m)$ which are connected, non-hamiltonian and have $\delta \geq 2$, $\Delta \leq 3$ logn and fewer than logn vertices of degree ≤ 2. Now define $\mathcal{A}$ as the event

{(i) A avoids at least one longest path of G, (ii) A avoids vertices of degree 2, (iii) A is a matching.}

Then

$$\Pr(\mathcal{A}|\ G_{n,m} \in \mathcal{G}_2(n,m)) \geq \frac{1}{3} \quad \text{and} \quad \Pr(\mathcal{A}) \leq (\tfrac{32}{33})^{\omega}$$

by similar arguments to those for the previous theorem.

§3. Generalisations

Let a graph G have property $\mathcal{A}_k$ if it contains $\lfloor k/2 \rfloor$ edge disjoint Hamilton cycles and, if k is odd, a further edge disjoint matching of size $\lfloor n/2 \rfloor$, $n = |V(G)|$. Theorems 1.3, 1.4 can be generalised to

Theorem 3.1 (Bollobás and Frieze [1985])

$$\lim_{n\to\infty} \Pr(\Gamma_{m_k^*} \in \mathcal{A}_k) = 1$$

for fixed k. □

This is not too difficult to prove. One shows that in a.e. $\Gamma_{m_k^*}$, is such that $\Gamma_{m_k^*} - A \in EX_k$ for all matchings A which avoid vertices of degree k. If then for example we can only find $t < \frac{1}{2} k$ Hamilton cycles then removing these cycles still leaves $|N(S)| \geq (k - 2t)|S| \geq 2|S|$ for $|S| \leq n/(k+2)$. Posa's Theorem plus the colouring argument finishes the proof.

It seems then that if we find that a random graph does not have property $\mathcal{A}_k$ then the most likely cause is a vertex of degree $k-1$ or less. But what if we exclude this possibility by only considering graphs with minimum degree at least k? Now let $\mathcal{G}^{(k)}_{n,m} = \{G \in \mathcal{G}_{n,m} : \delta(G) \geq k\}$ and let $G^{(k)}_{n,m}$ be sampled uniformly from this set. Bollobás and Frieze [1985] ($k = 1$) and Bollobás, Fenner and Frieze [1988] questions to be answered are (i) how large should m be so that the probability tends to a positive constant and (ii) what is the most likely obstruction to the occurrence of $\mathcal{A}_k$? The answer to (ii) is the existence of $k+1$ vertices of degree k having a common neighbour – a *k-spider*. As for (i):

Theorem 3.2

Let $m = \frac{n}{2}\left(\frac{\log n}{k+1} + k \log n + c_n\right)$. Then

$$\lim_{n\to\infty} \Pr(G^{(k)}_{n,m} \in \mathcal{A}_k)$$

$$= \lim_{n\to\infty} \Pr(G^{(k)}_{n,m} \text{ has no k-spider}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-\theta(c,k)} & c_n \to c \\ 1 & c_n \to +\infty \end{cases}$$

where $\theta(c,k) = \dfrac{e^{-(k+1)c}}{(k+1)!((k-1)!)^{k+1}(k+1)^{k(k+1)}}$.

□

[There is a caveat here for the case $c_n \to -\infty$. We should not allow it to go there too fast. For example if $k = 1$ and $m = \frac{1}{2}n$ exactly then $G^{(1)}_{n,m}$ is always a matching.]

The approach once again is to show that (in the key case, $c_n \to c$) a.e. $G^{(k)}_{n,m}$ is such that $G^{(k)}_{n,m} - A \in EX_k$ for matchings A that avoid vertices of degree k and to then apply the colouring argument. The first problem is quite difficult. It is hard to prove properties of $G^{(k)}_{n,m}$ since $\mathcal{G}^{(k)}_{n,m}$ is a very small subset of $\mathcal{G}_{n,m}$.

The k-core $\sigma_k(G)$ of a graph G is the largest vertex induced subgraph of G which has minimum degree at least k. Now it is not too difficult to show that if $\sigma_k(G_{n,m})$ has n' vertices and m' then it has the same distribution as $G^{(k)}_{n',m'}$. Now when m is as in Theorem 3.2, we find that in a.e. $G_{n,m}$ we have $n - n' = o(n)$, $m - m' = o(n)$ and hence we obtain the result of Luczak [1987a] as a corollary.

Corollary 3.3

Let m, $\theta(c,k)$ be as in Theorem 3.2. Then

$$\lim_{n\to\infty} \Pr(\sigma_k(G_{n,m}) \in \mathcal{A}_k) + \begin{cases} 0 & c_n \to -\infty \\ e^{-\theta(c,k)} & c_n \to c \\ 1 & c_n \to +\infty \end{cases}$$

□

Note that the number of edges required in Theorem 3.2 for property $\mathcal{A}_k$ is much less than that required in Theorem 3.1.

The next thing to look at would be $\mathcal{G}^{(k+1)}_{n,m}$. Here we find that only a linear number of edges is needed, although the result is not yet as precise as that in Theorem 3.2.

Theorem 3.4 (Bollobás, Cooper, Fenner and Frieze [1988])

There exist constants c_k, $k \geq 2$ such that if $m \geq c_k n$ then

$$\lim_{n\to\infty} \Pr(G^{(k+1)}_{n,m} \in \mathcal{A}_k) = 1.$$

□

Let us now for the moment turn to subgraphs of $G_{n,m}$. In particular let

$m = \frac{1}{2}cn$ for some constant $c > 1$. Erdös conjectured that there exists a function $\alpha(c) > 0$, $\alpha(c) \to 0$ as $c \to \infty$ such that a.e. $G_{n,m}$ contains a path of length $\geq (1 - \alpha(c))n$. Ajtai, Komlós and Szemerédi [1981] proved this for all $c > 1$ and independently de la Vega [1979] proved that $\alpha(c) \leq \frac{c_0}{c}$ for some small constant $c_0 > 1$. Ajtai, Komlós and Szemerédi showed that there was a long line of descendants in a certain branching process and de la Vega considered a simple depth-first-search for a long path. Bollobás [1982] was able to show that $\alpha(c)$ was much smaller, for large c, by showing that a.e. $G_{n,m}$ contained a large hamiltonian subgraph of size $n(1 - c^{24}e^{-c/2})$. The proof that the subgraph is Hamiltonian is based on Posa's Theorem plus the colouring argument. Looking for Hamiltonian subgraphs, at least for large c, seems to be the correct approach. Bollobás, Fenner and Frieze [1984] showed $\alpha(c) \leq c^6e^{-c}$ and in Frieze [1986a] we proved the following: let now k be fixed and

$$\alpha(k,c) = \inf\{\alpha \in \mathbb{R}: \text{ a.e. } G_{n,\frac{1}{2}cn} \text{ contains a subgraph } H \in \mathcal{A}_k \text{ with at least } (1-\alpha)n \text{ vertices}\}.$$

Theorem 3.5

$$\alpha(k,c) \leq (1 + \epsilon(k,c)) \sum_{t=1}^{k-1} \frac{c^t e^{-c}}{t!}$$

where $\lim_{c\to\infty} \epsilon(k,c) = 0$.

□

This gives the correct order of magnitude for $\alpha(k,c)$ because a.e. $G_{n,\frac{1}{2}cn}$

contains approximately $n \sum_{t=1}^{k-1} \frac{c^t e^{-c}}{t!}$ vertices of degree $k-1$ or less. In the proof we start with the k-core and remove a few extra vertices and show that what is left has property $\mathcal{A}_k$ with high probability. Theorem 3.5 is true if we allow c to grow with n and we can deduce Theorem 3.1 as a corollary. Finally, for the case c close to 1, Suen [1985] managed to improve the estimate of Ajtai, Komlós and Szemerédi for the length of the longest path.

Let us now return to the threshold for being Hamiltonian. A graph is said to be *pancyclic* if it has cycles of all lengths between 3 and n. Cooper and Frieze [1987] and Luczak [1987b] independently showed that the threshold for being pancyclic was also that for minimum degree 2. In fact Luczak showed that for large c, a.e. $G_{n,\frac{1}{2}cn}$ contains cycles of all lengths between $\omega(n)$ and the upper bound implied by Theorem 3.5. Here $\omega(n)$ is any function tending "slowly" to infinity with n. In a twist on pancyclicity Cooper [1988] has shown that when $G_{n,m}$ is Hamiltonian it is almost always possible to find a Hamilton cycle from which one can construct cycles of all lengths by using 2 chords for each cycle.

In another variation we showed, Frieze [1988a], that for any fixed k, the threshold for being Hamiltonian was also that for the existence of k vertex disjoint cycles of sizes $\lfloor \frac{n}{k} \rfloor$ and $\lceil \frac{n}{k} \rceil$ which covered $[n]$.

Another variation on Hamiltonicity is that of being *Hamilton connected*. A graph is Hamilton connected if there is a Hamilton path joining each pair of vertices. To be Hamilton connected a graph has to have minimum degree at least 3 since the neighbours of a vertex of degree 2 cannot be connected by a Hamilton path. One can guess that the threshold for being Hamilton connected is the same as that for $\delta \geq 3$. This was proved in Bollobás, Fenner and

Frieze [1987] and independently in Luczak [1988].

We ought to mention bipartite graphs. We showed in Frieze [1985], not surprisingly, that if $m = n(\log n + \log\log n + c_n)$, then

$$\lim_{n\to\infty} \Pr(B_{n,m} \text{ is Hamiltonian}) = \lim_{n\to\infty} \Pr(\delta(B_{n,m}) \geq 2)$$

$$= \begin{cases} 0 & c_n \to -\infty \\ e^{-2e^{-c}} & c_n \to c \\ 1 & c_n \to +\infty \end{cases}$$

where $B_{n,m}$ denotes a random bipartite graph with $2n$ vertices and m edges. The proof of this result was actually a little trickier than that of Theorem 1.2. The problem is that (2.1) is of no help if the longest path is of odd length. The trick used to deal with this was helpful in proving the next result.

We end this section with a recent result of Cooper and Frieze [1988] on the number of Hamilton cycles in a random graph. Theorem 1.3 states that a.e. $\Gamma_{m_2^*}$ has a Hamilton cycle. This raises the question of how many? The expected number in $\Gamma_{m_2^*}$ is certainly at most $\frac{(n-1)!}{2}(\log n + O(\log\log n))^n = (\log n)^{n-o(n)}$. What we show is that a.e. $\Gamma_{m_2^*}$ has $(\log n)^{n-o(n)}$ distinct hamilton cycles. (The $o(n)$ terms in the result and the expectation are quite different.) The idea behind the proof here is roughly to find for (most) $v \leq \frac{1}{2}n$, a collection of $k = \lceil \frac{c(\log n)^2}{r} \rceil$ (c constant, $r = (\log\log n)^2$) sets $W_1^{(v)}, W_2^{(v)}, \ldots, W_k^{(v)}$ of r edges satisfying

$|W_i^{(v)} \cap W_j^{(v)}| \le 1$, $i \ne j$. For each $f\colon [\frac{1}{2}n - o(n)] \to k$ we obtain a graph H_f by deleting all edges incident with v other than $W_{f(v)}^{(v)}$. We show that almost all H_f are Hamiltonian for a.e. $\Gamma_{m_2^*}$. If $f \ne f'$ then a Hamilton cycle in H_f is distinct from one in $H_{f'}$, since $|W_{f(v)}^{(v)} \cap W_{f'(v)}^{(v)}| \le 1$ if $f(v) \ne f'(v)$. The number of different f is $k^{\frac{1}{2}n - o(n)} = (\log n)^{n-o(n)}$.

§4. Regular Graphs, k-out and Planar Maps

There are two other graph models which have received some attention from the point of view fo Hamiltonicity. We first consider random regular graphs. These are usually studied via the configuration model of Bollobás [1980]. Suppose r is constant and we wish to generate a random regular graph with vertex set $[n]$. We let $W = W_1 \cup W_2 \cup \ldots \cup W_n$ where the W_i's are a disjoint collection of r-sets. A configuration is a partition of W into $m = \frac{1}{2}rn$ pairs. Associate with F the multigraph $\mu(F)$ with vertex set $[n]$ and an edge xy whenever F contains a pair $\{\xi,\eta\}$ with $\xi \in W_x$, $\eta \in W_y$. If Φ is the set of configurations and F is chosen randomly from Φ then (i) each simple regular graph has the same probability of occurrence as $\phi(F)$ and (ii) $\Pr(\phi(F)$ is a simple graph $\sim e^{-(r^2-1)/4}$. Hence if r is constant it is enough to show that a.e. $\phi(F)$ is Hamiltonian in order to show that a.e. r-regular graph is. This idea has been used to prove

Theorem 4.1

There exists a constant $r_0 \ge 3$ such that if $r \ge r_0$ is constant then a.e. r-regular graph is Hamiltonian. □

Bollobás [1983] showed $r_0 \le 10^{10}$ and independently Fenner and Frieze [1984] showed $r_0 \le 776$. We gave an algorithmic proof that $r_0 \le 85$ in [1988b]. Robinson and Wormald have claimed a proof that $r_0 = 3$, but as yet there is no paper. They have already proved that a.e. cubic bipartite graph is Hamiltonian and that 98% of cubic graphs are too [Robinson and Wormald, 1984]. They use the Chebycheff inequality to prove both results; the second result requires a clever argument relating cubic graphs and triangle free graphs. It would appear that the proof of $r_0 = 3$ is an extension of this argument.

One curious point about random regular graphs is that knowing a.e. r-regular graph is Hamiltonian does not imply a.e. (r+1)-regular graph is. Thus Theorem 4.1 does not say anything about r(n)-regular graphs when $r(n) \to \infty$. However by comparing rates at which various probabilities go to zero, one can show [Frieze, 1988c], that the result continues to hold for $r = O(n^{1/3-\epsilon})$. The upper bound seems unnecessary but is there for the moment.

There is not much to say about perfect matchings in this case since the existence question is covered by Hamiltonicity or r-connectivity.

Another model which has received some attention is $G_{k\text{-out}}$. Here the vertex set is $[n]$ and each $v \in [n]$ independently chooses k neighbours. (Equivalently, sample uniformly from the space of digraphs with vertex set $[n]$ and regular out degree k. Then ignore orientation). Observe that $G_{1\text{-out}}$ is the graph induced by a random function and is well studied (see e.g. Kolchin [1986]). We proved the following in [Frieze, 1986b]:

Theorem 4.2

a.e. $G_{2\text{-out}}$ has a matching of size $\lfloor n/2 \rfloor$. □

This implies the earlier result of Shamir and Upfal [1982] the a.e. $G_{6\text{-out}}$ has such a matching.

Now it is easy to see that a.e. G_{1-out} has no matching of this size since it will contain a large number of degree one vertices with a common neighbour. So how big is the largest matching in G_{1-out}. Let $\nu(n)$ denote its expected size.

Theorem 4.3 (Meir and Moon [1974])

$$\frac{\nu(n)}{n} \to 1 - \rho$$

where $\rho = .4328....$ is the unique solution to $x = e^x$. □

Actually, Meir and Moon proved this result for random labelled trees. J.W. Moon pointed out to me that this implies the result for random amppings. This becomes obvious once one considers Joyal's proof of Cayley's formula for the number of spanning trees of K_n and the fact that a.e. G_{1-out} has $O(\sqrt{n})$ vertices on cycles. In the case of Hamilton cycles we have the following

Theorem 4.4

There exists a constant $k_0 \geq 3$ such that if $k \geq k_0$ then a.e. G_{k-out} is Hamiltonian. □

It was shown in Fenner and Frieze [1983] that $k_0 \leq 23$ and it was here that we first used the colouring argument of §2. We gave an algorithmic proof in [Frieze, 1988b] that $k_0 \leq 10$ but recently, in Luczak and Frieze [1988], we have shown $k_0 \leq 5$.

The idea of the last paper is simple enough to explain in a few paragraphs. G_{5-out} is (or contains) the union of 2 independent G_{2-out}'s plus a G_{1-out}. Take the 2 independent matchings from Theorem 4.2 and make up 2-factor which almost always has at most 2 logn cycles.

Now repeat the following procedure until a Hamilton cycle is constructed. Since a.e. G_{4-out} is connected there must be an edge joining 2 cycles. Delete 2 edges to make a path through all the vertices of these cycles. Now using only the G_{4-out} either extend the path to drag in another cycle and make the path longer, or do rotations and use the independent G_{1-out} to join 2 endpoints of one of the paths produced. Since G_{4-out} "expands" we can use Posa's Theorem to show there is always a high probability of being able to do this.

As a final model we consider planar maps and the result of Richmond, Robinson and Wormald [1985]. Consider the set of unlabelled 3-connected cubic planar maps with n vertices. They show that a.e. map contains a large number of vertex induced copies of any fixed size map M. By choosing M to be any 3-connected map which has no Hamilton path, they show that a.e. such map in non-Hamiltonian. Their proof is based on a clever use of generating functions.

§5. Algorithmic Aspects

The study of random graphs is by and large a study of the likely existence of objects. Random graphs can also be used to analyse the expected performance of algorithms that search for these objects. The Hamilton cycle problem is interesting from this point of view in that while the problem of finding a Hamilton cycle in a graph is NP-hard (see e.g. Garey and Johnson [1978]) there exist polynomial time algorithms which succeeds in almost every case. More precisely Bollobás, Fenner and Frieze [1987] devised a (deterministic) algorithm HAM which runs in $O(n^3 \log n)$ time on an n vertex graph and satisfies

Theorem 5.1

(i) Let $m = \frac{n}{2}(\log n + \log\log n + c_n)$ assuming $c_n \to \pm\infty$ or a constant we have

$$\lim_{n\to\infty} \Pr(\text{HAM finds a Hamilton cycle in } G_{n,m}) = \lim_{n\to\infty} \Pr(G_{n,m} \text{ is Hamiltonian})$$

(ii)

$$\Pr(\text{HAM fails to find a Hamilton cycle in } G_{n,.5} \mid \delta(G_{n,.5}) \geq 2) = o(2^{-n}). \quad \square$$

The main point of (ii) is that if the input to HAM is equally likely to be any graph with vertex set [n] then HAM fails so infrequently that these cases can be handled by dynamic programming (Held and Karp [1962]) and we will have a deterministic algorithm which correctly determines whether any graph is Hamiltonian and runs in polynomial expected time.

The algorithm uses the rotations discussed in §2. It runs in stages. At the start of stage k there is a path P of length k. If either endpoint of P is adjacent to a vertex not in P we extend the path and the stage ends. Otherwise we construct all paths obtainable by a single rotation and see if any of these can be extended. We continue this "breadth-first" construction of paths to depth approximately $T = \frac{2 \log n}{\log\log n}$, unless we find a path that we can extend. We show that in a.e. $G_{n,m}$, $m = \frac{n}{2}(\log n + \log\log n + c)$, the number of paths grows by a factor of at least $\alpha \log n$, $\alpha > 0$ constant. Thus at depth T we have approximately βn^2 paths and we use the colouring argument to show that with probability $1 - o(1)$ one of these paths has adjacent endpoints. Since a.e. $G_{n,m}$ will be connected we find that either we have a Hamilton cycle we can find a longer path.

If we allow randomized algorithms then we have

Theorem 5.2 (Gurevich and Shelah [1987], Thomason [1987])

There exist linear expected time randomized algorithms for deciding graph Hamiltonicity for input distribution $G_{n,p}$, p constant. (Actually Thomason's paper treats $p \geq n^{-1/3}$). □

Neither of the 2 algorithms mentioned in this theorem have good expected performance at the threshold. On the other hand they can be modified to solve the Hamilton cycle problem on digraphs (see §6).

Earlier results for this problem were obtained by Angluin and Valiant [1979] who gave an $O(n(\log n)^2)$ randomized algorithm that finds a Hamilton cycle in a.e. $G_{n,Kn\log n}$ for large K. Shamir [1983] gave an algorithm for m slightly above the threshold.

Before moving on to sparse graphs we mention the result of Gimbel, Katz, Lesniak, Scheinermann and Weirman [1987]. It is well known that the graph $G = xy$ is Hamiltonian if and only if G is Hamiltonian whenever, $d_G(x) + d_G(y) \geq |V(G)|$ (Bondy and Chvatal [1976]). By adding all such edges, we obtain $c(G)$. We can repeat this and compute $c^2(G), c^3(G), \ldots, c^*(G)$, the closure of G. Gimbel et al proved

Theorem 5.3

In the following p is a constant, $0 \leq p \leq 1$ and the statements hold with probability $1 - o(1)$:

(a) $p < \frac{1}{2}$ implies $c(G_{n,p}) = G_{n,p}$

(b) $p = \frac{1}{2}$ implies $c^2(G_{n,\frac{1}{2}}) \neq c^3(G_{n,\frac{1}{2}}) = c^*(G_{n,\frac{1}{2}}) = K_n$

(c) $p > \frac{1}{2}$ implies $c(G) = K_n$.

□

Let us now consider $G_{n,m}$, $m = \frac{1}{2}cn$ where $c > 1$ is constant. Karp and Sipser [1981] discuss a simple algorithm for finding a large matching in such a graph. Their algorithm is as follows: Suppose the input graph is G;

begin

H: = G - {isolated vertices}; M := ∅

repeat

<u>if</u> $\delta(H) = 1$ **then** randomly choose an edge u incident with a vertex of degree 1 **else** randomly choose any edge u;

M := M + u; remove u and all edges indicdent with u; remove all isolated vertices

until H has no vertices

output M

end

They prove

Theorem 5.4

With probability $1 - o(1)$

(i) The matching produced by M is within $o(n)$ in size of a largest matching.

(ii) If $c < e$ (= basis of natural logarithms) then for a.e. $G_{n,m}$ we find $\delta(H) = 1$ throughout the algorithm, otherwise there are phases in which $\delta(H) = 2$. □

For the problem of finding a long path or cycle we essentially have de la Vega's algorithm which for large c finds a path with in $O(\frac{n}{c})$ of the maximum and that in Frieze (1988b) which finds one within $O(n\,\epsilon(c)e^{-c})$ of the maximum, where $\lim_{c\to\infty} \epsilon(c) = 0$. Both algorithms are polynomial, the latter being a modification of HAM of Bollobás, Fenner and Frieze [1987]. This latter algorithm can also be used on random regular graphs.

Let us end this section with a discussion of parallel algorithms. Using the algorithm of Thomason [1987] to construct small cycles and then doing some patching we constructed an $O((\log\log n)\ )$ expected time parallel algorithm for deciding graph or digraph Hamiltonicity on a CRCW PRAM (Frieze [1987a]). The model of random input is $G_{n,p}$, p constant.

For sparse random graphs with $Kn \log n$ edges, K sufficiently large, Coppersmith, Rhagavan and Tompa [1987] have constructed an $O((\log n)^2)$ parallel algorithm that constructs a Hamilton cycle with probability $1 - o(1)$. Finally, in Frieze and Tygar [1988] we have been considering the problem of finding a maximum matching in a random graph. The algorithm is deterministic and purely graph theoretic in nature. It is to be contrasted with the *randomized* NC algorithms of Karp, Upfal and Wigderson [1986] or Mulmuley, Vazirani and Yazirani [1987] which work on <u>all</u> inputs and rely on evaluating determinants.

Weighted Problcms

If we assign weights to the edgs of graphs then we can study the problems of finding minimum (total) wieght perfect matchings and minimum weight Hamilton cycles (the *travelling salesman problem).* We first consider weighted perfect matchings in the complete bipartite graph $K_{xi,n}$ (the assignment problem). The most efficient algorithms for this problem run in $0(n^3)$ worst-case time but Karp [1980] describes an algorithm with $0(n^2 \log n)$ expected running time assuming the weights are i.i.d. random variables.

Suppose next that W_n is the (random minimum weight of a perfect matching in $K_{n,n}$ when the edge weights are independent uniform [0,1] random variables. Walkup [1979] proved the surprising result that $E(W_n) \le 3$, always. In the proof Walkup replaces each edge by a pair of edges, one blue and one red say. Each edge is given a weight which is a random variable with distribution function $F(x) = 1 - (1-x)^{1/3}$ so that the minimum of the red weight and the blue weight is uniform. Now consider the random bipartite graph where one half of the vertex partition chooses the two least weight red edges incident with each vertex and the other half uses blue edges. This graph is like a bipartite 2-out and Walkup [1980] shows it has a matching with probability $1 - o(1)$. (compare with Theorem 4.2). The expected length of each edge in this matching is at most $\frac{3}{n}$ and Walkup's result follows, modulo some technical tidying up.

Subsequently Karp [1987], showed $E(W_n) \leq 2$ (see also Dyer, Frieze and McDiarmid [1986]). It is known that $E(W_n) \geq 1 + e^{-1}$ and the exact limiting value of $E(W_n)$ is unknown.

Now consider the Travelling Salesman Problem (TSP). It was shown by Beardwood, Halton and Hammersley [1959] that if $X_1, X_2, \ldots, X_n$ are independetly chosen randomly from within the unit square $[0,1]^2$ then

$$\Pr\left(\frac{L_n}{\sqrt{n}} \to \beta\right) = 1$$

where L_n is the length of the shortest "tour" through the n points and β is constant whose precise value is not known. Karp [1977] in a very influential paper constructed an $O(n \log n)$ algorithm which computes a tour which, with probability $1 - o(1)$, is very close to optimum.

Karp also considered the asymmetric TSP. Let the arcs of the complete digraph be given independent uniform [0,1] lengths. Karp [1979] gave an $O(n^3)$ algorithm which, with probability $1 - o(1)$, computes a tour which is very close to optimum. Later Karp and Steele [1985] and Dyer and Frieze [1988] further improved these results. The basic idea is to compute in $O(n^3)$ time a minimum weight set of vertex disjoint directed cycles which together cover all vertices. [The optimisation problem here is, essentially, the assignment problem mentioned previously]. A tour is then obtained by "patching" together the cycles. Dyer and Frieze show that this can usually be done at an extra cost of $O(\frac{(\log n)^4}{n})$.

The above algorithms for the travelling salesman problem are approximation algorithms. They do not aim to solve the problem exactly. In Frieze [1987b] we consider the symmetric TSP where the costs are independent random integers in the range [0,B]. We described an $O(n^3 \log n)$ *randomized* algorithm TSPSOLVE which satisfies the following: assume $B = B(n) = o(\frac{n}{\log\log n})$. Then

$$(5.1) \qquad \lim_{n\to\infty} \Pr(\text{TSPSOLVE finds an optimum solution}) = 1.$$

The idea is to identify a set X_0 of "troublesome vertices" and construct a set of vertex disjoint paths $\mathcal{P}$ which contain X_0 as interior points. $\mathcal{P}$ is to have minimum total edge weight among all such sets of paths. Having done this we use zero length edges to construct a Hamilton cycle containing the eges of $\mathcal{P}$. The algorithm used for finding such a cycle is related to that in Frieze (1988b).

§6. Digraphs

Many of the theorems on Hamilton cycles in random graphs have natural analogues in digraphs, most of which have not been proved yet. However the directed analogues of Theorems 1.2, 1.3, 2.1 have been proved in Frieze

[1988d]. Thus for example if $D_{n,m}$ is a random digraph with vertex set [n] and m edges chosen randomly from $[n]^2$ then we have

Theorem 6.1

Let $m = n \log n + c_n n$. Then

$$\lim_{n\to\infty} \Pr(D_{n,m} \text{ is Hamiltonian}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-2e^{-c}} & c_n \to c \\ 1 & c_n \to +\infty \end{cases}$$

□

The proof is via the analysis of an $O(n^{1.5})$ time algorithm.

There is not much else to say about random digraphs except that there is an analogous result to (5.1) for the asymmetric TSP with $B = O(\frac{n}{\log n})$.

We end this section with a remarkable inequality due to McDiarmid [1981] which gives a result close to Theorem 6.1.

Let $G_{n,p}$, (resp. $D_{n,p}$) denote a random graph (resp. digraph) in which each possible edge is independently included with probability p.

Theorem 6.2

$$\Pr(D_{n,p} \text{ is Hamiltonian}) \geq \Pr(G_{n,p} \text{ is Hamiltonian}).$$

Proof

Let $e_1, e_2, \ldots, e_N$ be any enumeration of the edges of the complete graph K_n. We consider a sequence of random digraphs $H_1, H_2, H_3, \ldots, H_N = D_{n,p}$.

To construct H_i we do the following: if $j < i$ and $e_j = uv$ then we independently add arc uv with probability p and arc vu with probability p. If $j \geq i$ then we either add both uv and vu with probability p or add neither. Thus H_1 is $G_{n,p}$ with each undirected edge replaced by a pair of poositely oriented edges. Thus

$$\Pr(H_1 \text{ is Hamiltonian}) = \Pr(G_{n,p} \text{ is Hamiltonian}).$$

We show

$$\Pr(H_{i+1} \text{ is Hamiltonian}) \geq \Pr(H_i \text{ is Hamiltonian}), \quad 1 \leq i < N$$

and the theorem follows. In fact let ω represent the outcome of our experiment with edges e_j, $j \neq i+1$. We have

$$(6.1) \qquad \Pr(H_{i+1} \text{ is Hamiltonian} \mid \omega) \geq \Pr(H_i \text{ is Hamiltonian} \mid \omega)$$

remembering that edges e_j, $j \neq i+1$ are treated the same in H_i and H_{i+1}. Let D_ω denote the digraph with edges made from the outcomes of experiments with edges e_j, $j \neq i$.

Now let $e_{i+1} = uv$, then there are 5 cases:

(i) D_ω has a Hamilton cycle.

(ii) D_ω has no Hamilton cycle but has Hamilton paths from u to v and from v to u.

(iii) D_ω has no Hamilton cycle and no Hamilton path from u to v but has one from v to u.

(iv) D_ω has no Hamilton cycle and no Hamilton path from v to n but has one from u to v.

(v) D_ω has no Hamilton cycle and no Hamilton path from u to v or from v to u.

Consider the following table in which position $(k, i+\theta)$ gives $\Pr(H_{i+\theta}$ is Hamiltonian $\mid$ (case (k)):

	i	i+1
(i)	1	1
(ii)	p	$2p-p^2$
(iii)	p	p
(iv)	p	p
(v)	0	0

It is clear tht we have proved (6.1) and the theorem. □

Observe that Theorems 1.2 and 6.2 imply the result in Theorem 6.1 for $c_n - \log\log n \to \infty$. (It is straightforward to translate such results for $D_{n,p}$ to $D_{n,m}$ - see Theorem II.2 of Bollobás [1985]).

§7. Open Problems

1. Can Theorem 3.1 be generalized to

$$\lim_{n\to\infty} \Pr(\Gamma_m \in \mathcal{A}_{\delta(\Gamma_m)},\ m = 1,2,\ldots,N) = 1?$$

2. Are the following true (see Theorem 3.4)?

   (a) $c > 1$ implies a.e. $G^{(2)}_{n,cn}$ has a matching of size $\lfloor n/2 \rfloor$.

   (b) $c \ge \frac{3}{2}$ implies a.e. $G^{(3)}_{n,cn}$ is Hamiltonian.

3. Determine $\epsilon(k,c)$ of Theorem 3.5 to within $o(1)$, (as $n \to \infty$).

4. Show that if $m = \frac{n}{2}(\log n + \log\log n + c)$ and $\delta(G_{n,m}) \geq 2$ then with probability tending to 1 either

   (a) $\exists$ a Hamilton cycle H such that cycles of all lengths can be obtained by adding 1 chord

   or

   (b) (a) is false.

   (Cooper [1988] has shown that 2 chords are enough).

5. Determine the threshold for being able to partition the vertices of $G_{n,m}$ into $k = k(n)$ cycles of roughly equal size. (k constant is dealt with in Frieze [1988a]). The problem gets harder as k grows faster.

6. Show that a.e. $r(n)$-regular graph is Hamiltonian where $r(n) \to \infty$. The case $r = O(n^{1/3-\epsilon})$ is treated in Frieze [1988c] (it is not clear at present whether showing a.e. cubic graph is Hamiltonian is open).

7. Show that a.e. $G_{3\text{-out}}$ is Hamiltonian.

8. Find polynomial time algorithms for finding, with probability $1 - o(1)$, Hamilton cycles in (i) random cubic graphs, (ii) $G_{3\text{-out}}$.

9. Find polynomial time algorithms for solving random travelling salesman problems exactly, with probability $1 - o(1)$.

10. (a) Show that there exists a constant $r_Q \geq 2$ such that if $r \geq r_Q$ then a.e. r-regular digraph is Hamiltonian. (By r-regular we mean both the indegree and outdegree of every vertex is r).

(b) Show that there exists a constant $k_Q \geq 2$ such that if $k \geq k_Q$ then a.e. $G_{k\text{-in},k\text{-out}}$ ^ Hamiltonian. (^ ^ ^ ^ is a digraph with vertex set [n] in which each $v \in [n]$ independently chooses k in-neighbours and k out-neighbours. See Fenner and Frieze [1982]).

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