FULLY DISCRETE FINITE ELEMENT SCHEMES FOR THE CAHN-HILLIARD EQUATION

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Introduction

We consider the nonlinear evolution equation

(1.1a) $$u_t + \gamma D^4 u = D^2\varphi(u)\ , \quad x \in I = (0,L),\ t > 0$$

with boundary conditions

(1.1b) $$Du = D^3u = 0 \quad \text{at} \quad x = 0,L$$

and initial condition

(1.1c) $$u(\cdot,0) = u_0$$

where $D \equiv \frac{\partial}{\partial x}$, γ is a prescribed positive constant and $\varphi \in C^1(\mathbb{R})$ is a prescribed function.

If

$$\varphi(u) = u(u^2 - \beta^2)$$

then (1.1a) is the Cahn-Hilliard equation, see Cahn [1968], Novick-Cohen & Segel [1984], Elliott & French [1987], and the references cited therein.

The Cahn-Hilliard generalized diffusion equation has been proposed to model phase separation in a binary mixture where u is a scaled concentration. The solution $u(x,t)$ exhibits pattern formation with interfaces separating regions where u is nearly constant taking the phase values $\pm\ \beta$. Also in the time evolution the solution reaches certain states and remains close to them for a long time before evolving into different states of lower energy; this phenomenon is called metastability.

We assume that (1.1) has a unique solution for $t \in [0,T]$. Further we assume that the solution is sufficiently smooth and

(1.2) $$\| u(\cdot,t)\|_{L^\infty(I)} \leq M \quad \forall\ t \in [0,T]\ .$$

In the case of the Cahn-Hilliard equation global existence and boundedness of u has been proved in Elliott & Zheng [1986].

A weak formulation for this problem is: (P) Find $u(\cdot,t) \in H^2_E(I)$ for $t \in [0,T]$ such that

(1.3) $$(u_t,v) + \gamma(D^2u,D^2v) = (\varphi(u),D^2v) \quad \forall\ v \in H^2_E(I)$$

with $u(\cdot,0) = u_0$;

where

$$(v,w) \equiv \int_I v(x)w(x)dx,$$

$$H^2_E(I) = \left\{ v \in H^2(I) : Dv = 0 \ \text{ at } x = 0 \ \text{ and } \ x = L\right\} .$$

Throughout this note the norms in $L^{\infty}(I)$, $L^2(I)$ and $H^s(I)$ are denoted by $|\cdot|_{0,\infty}$, $|\cdot|_0$ and $\|\cdot\|_s$. The semi-norm $|D^s v|_0$ is denoted by $|v|_s$.

Numerical Method: Let S_h^r be the piecewise polynomial spline space

$$S_h^r = \{\chi \in C^{r-2}(I) : \chi|_{I_j} \in P_{r-1}(I_j) , j = 1 \dots N\}$$

where $r \geq 3$ is an integer;

$$0 = x_0 < x_1 < \dots < x_N = L$$

is a partition of $I = (0,L)$ with $I_j = (x_{j-1}, x_j)$, $h = \max_j (x_j - x_{j-1})$ and $P_{r-1}(I_j)$ denotes the set of all polynomial functions on I_j of degree less than or equal to $r-1$. Define

$$\overset{o}{S}{}_h^r = \{\chi \in S_h^r : D\chi = 0 \text{ at } x = 0 \text{ and } x = L\}$$

and let

$$\alpha \geq h (\min_j (x_j - x_{j-1}))^{-1}$$

where we assume that for a family of partitions α is fixed.

The approximation scheme using a Galerkin method in space and Crank-Nicolson time discretization gives the following problem: $(P_{h,k})$. Find $\{U^n\}_{0 \leq nk \leq T}$ such that $U^n \in \overset{o}{S}{}_h^r$,

(1.4) $$(\partial U^n, \chi) + \gamma(D^2 U^{n+1/2}, D^2\chi) = (\varphi(U^{n+1/2}), D^2\chi)$$

for all $\chi \in \overset{o}{S}{}_h^r$ and $U^0 = u_0^h$;

where u_0^h is an approximation of u_0.

$$\partial V^n = \frac{1}{k} (V^{n+1} - V^n)$$

and

$$U^{n+1/2} = \frac{1}{2} (U^{n+1} + U^n).$$

For $v \in H_E^2(I)$, let $P_h v \in \overset{o}{S}{}_h^r$ be the unique solution of

$$(D^2 P_h v - D^2 v, D^2\chi) = 0 \quad \text{for all } \chi \in \overset{o}{S}{}_h^r$$

where $(\chi,1) = 0$ and $(P_h v,1) = (v,1)$. If $\rho^n = P_h u(\cdot,nk) - u(\cdot,nk)$ then

(1.5) $$\| (\frac{\partial}{\partial t})^j \rho^n \|_0 \leq Ch^{\bar{r}} \quad ; \; j = 0,1 \; ; \; 0 \leq nk \leq T$$

where

$$\bar{r} = \begin{cases} 2 & \text{if } r=3 \\ r & \text{otherwise} \end{cases}$$

and C depends on $\|(\frac{\partial}{\partial t})^j u\|_r$; see Elliott & Zheng [1986].

We will assume that

(1.6) $$|u_0^h - u(\cdot, 0)|_0 \leq Ch^{\bar{r}}$$

where C is independent of h,k.

We also need the subspace inequality

(1.7) $$|\chi|_{0,\infty} \le \frac{C}{h} |\chi|_0 \quad \forall \chi \in \overset{\circ}{S}{}^r_h$$

where C depends on r and α.

Error bounds for the continuous in time Galerkin approximation of the Cahn-Hilliard equation based on the spaces $\overset{\circ}{S}{}^r_h$ were obtained in Elliott & Zheng [1986]. Numerical experiments based on the Crank-Nicolson time discretization (1.4) are reported on in Elliott & French [1987]. Also in this latter paper an error analysis was carried out based on the assumptions that (A) (1.4) has a unique solution for k sufficiently small *and* (B) $|U^n|_{0,\infty}$ is uniformly bounded independently of h and k. It is the purpose of this note to justify (B) and to show that (1.4) has a unique solution in the ball $B = \{\chi \in \overset{\circ}{S}{}^h_r : |\chi|_{0,\infty} \le M+1\}$ for h and k sufficiently small. We also analyse an iterative method to solve (1.4) which is similar but slightly different to that used in Elliott & French [1987].

Section 3 has the main Theorem of this note which is an optimal order error estimate for the fully discrete Crank-Nicolson method (1.4) where the solution is obtained at each time step by a fixed point iteration procedure. Section 2 has several auxiliary results for (1.1) in the special case where φ' is bounded. The main idea here is that because we are interested in computing with unbounded $\varphi'(\cdot)$ the error analysis of (1.4) requires an L^∞ bound on U^n. Unfortunately it is not clear how to obtain an $\bar{a}$ priori bound simply by studying (1.4). However since (1.4) is a perturbation of (1.1), it is possible to use the boundedness of $u(x,t)$ to derive the fact that there is a unique solution $U^n \in B$ for h and k sufficiently small.

2. Bounded $\varphi'(\cdot)$

In this section we study the fully discrete approximation of (1.1) under the assumption that φ' is bounded. These results are used in section 3 in the analysis of the interesting case of (1.4) with φ' unbounded.

Proposition 2.1. Let $\varphi(\cdot)$ be such that

(2.1) $$\varphi \in C^1(\mathbb{R}) \quad ; \quad \sup_{s \in \mathbb{R}} |\varphi'(s)| \le \lambda$$

There exists a $k_0 > 0$, sufficiently small such that for

(2.2) $$k < k_0$$

there is a unique sequence $\{U^n\}_{0 \le nk \le T}$ solving (1.4) and

(2.3) $$|U^n - u(\cdot, nk)|_0 \le C(u)\,(h^{\bar{r}} + k^2)$$

(2.4) $$|U^n - u(\cdot, nk)|_{0,\infty} \le C(u)(h^{\bar{r}-1} + k^2/h).$$

Furthermore for each $n > 0$ and $U_0^n \in \overset{\circ}{S}{}_h^r$ the sequence $\{U_j^n\}_{j=0}^{\infty}$ defined by:

(2.5) $$(U_j^n, \chi) + \frac{\gamma k}{2}(D^2 U_j^n \;, D^2\chi) = k(\varphi(\tfrac{1}{2}(U_{j-1}^n + U^{n-1})), D^2\chi) + (U^{n-1}, \chi) - \frac{\gamma k}{2}(D^2 U^{n-1}, D^2\chi) \;\; \forall\, \chi \in \overset{\circ}{S}{}_h^r$$

converges to U^n at the rate.

(2.6) $$|U_j^n - U^n|_0^2 \le \left(k\,\frac{\lambda^2}{4\gamma}\right)^j \; |U_0^n - U^n|_0^2$$

and

(2.7) $$|U_j^n - U^n|_{0,\infty}^2 \le C\,\frac{\lambda}{\gamma}\left(k\,\frac{\lambda^2}{4\gamma}\right)^{j-1/2} |U_0^n - U^n|_0^2 \,.$$

Proof: For fixed $n \ge 1$, given $U_0^n \in \overset{\circ}{S}{}_h^r$ it is clear that (2.5) generates a well defined sequence $\{U_j^n\}$ since the bilinear form $(\cdot,\cdot) + \frac{\gamma k}{2}(D^2\cdot \;, D^2\cdot)$ is strictly coercive on $\overset{\circ}{S}{}_h^r$. Set $e_j = U_{j+1}^n - U_j^n$. It follows by subtraction that for each $\chi \in \overset{\circ}{S}{}_h^r$ we have

(2.8) $$(e_j, \chi) + \frac{k\gamma}{2}(D^2 e_j, D^2\chi) = k([\varphi(\tfrac{1}{2}(U_j^n + U^{n-1})) - \varphi(\tfrac{1}{2}(U_{j-1}^n + U^{n-1}))], D^2\chi)$$

Taking $\chi = e_j$ in (2.8) yields

$$|e_j|_0^2 + \frac{\gamma k}{2}|e_j|_2^2 = \frac{k}{2}(\varphi'(\eta_j^n)e_{j-1}, D^2 e_j)$$
$$\leq \frac{k\lambda}{2} |e_{j-1}|_0 |e_j|_2$$
$$\leq \frac{k}{2}\left[\frac{\lambda^2}{2\gamma} |e_{j-1}|_0^2 + \frac{\gamma}{2} |e_j|_2^2\right]$$

so that

(2.9a) $$|e_j|_0^2 + \frac{\gamma k}{4}|e_j|_2^2 \leq \frac{k\lambda^2}{4\gamma}|e_{j-1}|_0^2$$

(2.9b) $$|e_j|_0^2 \leq \left(\frac{k\lambda^2}{4\gamma}\right)^j |e_0|_0^2$$

Taking k sufficiently small, $\{U_j^n\}$ is a Cauchy sequence with limit $U^n \in \overset{o}{S}{}_h^r$ and, by the continuity of φ, passing to the limit in (2.5) yields (1.4). A similar argument yields the uniqueness of U^n solving (1.4). Furthermore setting $E_j = U_j^n - U^n$ and subtracting (1.4) from (2.5) we obtain the following analogue of (2.9)

(2.10a) $$|E_j|_0^2 + \frac{\gamma k}{4}|E_j|_2^2 \leq \frac{k\lambda^2}{4\gamma} |E_{j-1}|_0^2$$

(2.10b) $$|E_j|_0^2 \leq \left(\frac{k\lambda^2}{4\gamma}\right)^j |E_0|_0^2 \quad .$$

Note that (2.10b) proves (2.6).

In order to prove (2.7) note that taking $\chi=1$ in (1. 4) and (2.5) yields $(E_j,1) = 0$ so that by the Poincaré inequality

$$|v|_0 \leq C\ (|v|_1 + |(v,1)|) \quad \forall\ v \in H^1(I)\ ,$$

the inequality

$$|v|_1 \leq |v|_2^{1/2}\ |v|_0^{1/2} \quad \forall\ v \in H_E^2(I)\ ,$$

and the Sobolev inequality

$$|v|_{0,\infty} \leq C\ \|v\|_1 \ \ \forall\ v \in H^1(I)$$

we obtain

$$|E_j|_{0,\infty} \leq C\ |E_j|_0^{1/2}\ |E_j|_2^{1/2}\ ;$$

applying (2.10) yields (2.7).

We now turn to the proof of (2.3) and (2.4) To prove (2.3) we use standard techniques (see Thomeė [1984]). For completeness, parts of the proof from Elliott

& French [1987] are included below.

Let $t_n = nk$, $P_h u^n \equiv P_h^n(u\cdot,t_n)$ and $t_{n+1/2} = \frac{1}{2}(t_n + t_{n+1})$.

We use the standard error decomposition:

$$U^n - u(\cdot\,, t_n) = [U^n - P_h u^n] + [P_h u^n - u(\cdot\,, t_n)] = \theta^n + \rho^n$$

Using (1.4) we have

$$(\partial\theta^n, \chi) + \gamma(D^2\theta^{n+1/2}, D^2\chi) = (\varphi(U^{n+1/2}) - \varphi(u(\cdot,t_{n+1/2})), D^2\chi)$$

$$- (\partial P_h u^n - u_t, \chi) - \gamma(D^2[\tfrac{1}{2}u(\cdot, t_{n+1}) + \tfrac{1}{2}u(\cdot, t_n) - u(\cdot, t_{n+1/2})], D^2\chi).$$

Setting $\chi = \theta^{n+1/2}$, the above becomes

$$(2.11)\qquad (\partial\theta^n, \theta^{n+1/2}) + \gamma|\theta^{n+1/2}|_2^2 \le [|\varphi(U^{n+1/2}) - \varphi(u(\cdot,t_{n+1/2}))|_0 +$$

$$|\partial P_h u^n - u_t(\cdot, t_{n+1/2})|_0 + \gamma|\tfrac{1}{2}u(\cdot, t_{n+1}) + \tfrac{1}{2}u(\cdot, t_n) - u(\cdot, t_{n+1/2})|_2]|\theta^{n+1/2}|_2$$

Using the Cauchy-Schwarz inequality, we obtain

$$(2.12)\qquad (\partial\theta^n, \theta^{n+1/2}) \le C(I_1 + I_2 + I_3)^2\ ,$$

where the I's are the three terms in the first factor on the right-hand side of (2.11). Estimate I_3:

$$I_3 = \gamma\left|\int_{t_{n+1/2}}^{t_{n+1}} (s - t_{n+1/2})u_{tt}(\cdot, s)\ ds - \int_{t_n}^{t_{n+1/2}} (s - t_{n+1/2})u_{tt}(\cdot, s)\ ds\right|_0^2$$

$$\le \gamma k^{\frac{3}{2}} \left(\int_{t_n}^{t_{n+1}} \|u_{tt}(\cdot, s)\|_2^2\ ds\right).$$

Estimate I_1 :

$$I_1 = |\varphi'(\eta)[U^{n+1/2} - u(\cdot, t_{n+1/2})]|_0 \le \lambda\ |U^{n+1/2} - u(\cdot, t_{n+1/2})|_0\ ,$$

$$\le C[|\theta^{n+1/2}|_0 + \tfrac{1}{2}(|\rho^{n+1}|_0 + |\rho^n|_0 + |\tfrac{1}{2}u(\cdot, t_{n+1}) + \tfrac{1}{2}u(\cdot, t_n) - u(\cdot, t_{n+1/2})|_0\ ].$$

The last term in the above is estimated in the same way as I_3.

Finally we consider I_2. Let $\partial u(\cdot,t_n) = k^{-1}(u(\cdot,t_{n+1}) - u(\cdot,t_n))$.

$$\begin{aligned}
I_2 \le\ & |\partial P_h u^n - \partial u(\cdot,t_n)|_0 + |\partial u(\cdot,t_n) - u_t(\cdot,t_{n+1/2})|_0 \\
=\ & k^{-1}|\rho^{n+1} - \rho^n|_0 + |k^{-1}(u(\cdot,t_{n+1}) - u(\cdot,t_n)) - u_t(\cdot,t_{n+1/2})|_0 \\
\le\ & k^{-1/2}\left(\int_{t_n}^{t_{n+1}} |\rho_t(\cdot,s)|_0^2\, ds\right)^{1/2} \\
&+ k^{-1}\left(\left|\int_{t_{n+1/2}}^{t_{n+1}} (s - t_{n+1/2})^2 u_{ttt}(\cdot,s)\, ds - \int_{t_n}^{t_{n+1/2}} (s - t_{n+1/2})^2 u_{ttt}(\cdot,s)\, ds\right|_0\right) \\
\le\ & k^{-1/2}\left(\int_{t_n}^{t_{n+1}} |\rho_t(\cdot,s)|_0^2\, ds\right)^{1/2} + k^{\frac{3}{2}}\left(\int_{t_n}^{t_{n+1}} |u_{ttt}(\cdot,s)|_0^2\, ds\right)^{1/2}
\end{aligned}$$

With these estimates (2.12) becomes

$$\tfrac{1}{2}k^{-1}(|\theta^{n+1}|_0^2 - |\theta^n|_0^2) \le C(|\theta^{n+1/2}|_0^2 + w_n),$$

where

$$w_n = |\rho^{n+1}|_0^2 + |\rho^n|_0^2 + k^3\left(\int_{t_n}^{t_{n+1}} \|u_{tt}(\cdot,s)\|_2^2 + |u_{ttt}(\cdot,s)|_0^2\, ds\right) + k^{-1}\int_{t_n}^{t_{n+1}} |\rho_t(\cdot,s)|_0^2\, ds,$$

which yields

$$|\theta^{n+1}|_0^2 \le \frac{1 + Ck}{1 - Ck}|\theta^n|_0^2 + Ckw_n.$$

Iterating this inequality, we have for $k_0 < 1/2\, C$

$$|\theta^n|_0^2 \le C(T)\left(|\theta^n|_0^2 + Ck\sum_{j=0}^{n-1} w_j\right).$$

Noting that

$$|\theta^0|_0^2 \le |\rho^0|_0^2 + |U^0 - u_0|_0^2,$$

we obtain (2.3) by recalling the bounds (1.5).

It remains to prove (2.4). We note that there exists $I_h u(\cdot,t_n) \in \overset{0}{S}{}_h^r$, an interpolant of $u(\cdot,t_n)$, which satisfies

$$|I_h u(\cdot,t_n) - u(\cdot,t_n)|_{0,\infty} \le C(u)h^{\bar r}$$

so that, using the inverse norm inequality (1.7),

$$\begin{aligned}
|U^n - u(\cdot,t_n)|_{0,\infty} \le\ & C(u)h^r + |I_h u(\cdot,t_n) - U^n|_{0,\infty} \\
\le\ & C(u)h^r + Ch^{-1}|I_h u(\cdot,t_n) - U^n|_0 \\
\le\ & C(u)h^r + Ch^{-1}[|I_h u(\cdot,t_n) - u(\cdot,t_n)|_0 \\
&+ |u(\cdot,t_n) - U^n|_0]
\end{aligned}$$

and so (2.4) now follows from (2.3).

Corollary 2.1: There exists constants $h_0 > 0$ and $\varepsilon_0 > 0$ independent of h and k such that if $h \le h_0$, $k^2/h \le \varepsilon_0$ and

(2.13) $\quad |U_0^n|_{0,\infty} \le K \quad$ for $0 \le nk \le T$

for some constant $K > 0$ then

$$|U_j^n|_{0,\infty} \le M + 1 \quad \text{for } 0 \le nk \le T$$
$$\text{and } j = 1, 2, \ldots .$$

Proof:

From the estimate (2.4) and the given bound (1.2) on u we have

$$|U^n|_{0,\infty} \le M + C(u)\,(h^{\bar{r}-1} + \frac{k^2}{h}) .$$

(2.7) gives

$$|U_j^n|_{0,\infty} \le |U^n|_{0,\infty} + (ck)^{1/2}\,|U_0^n - U^n|_0$$

$$\le (1 + (ck)^{1/2})\,|U^n|_{0,\infty} + (ck)^{1/2}\,|U_0^n|_{0,\infty}$$

$$\le (M + C(u)\,(h^{\bar{r}-1} + \frac{k^2}{h}))\,(1 + (ck)^{1/2}) + (ck)^{1/2}\,K.$$

Clearly for ε_0 and h_0 sufficiently small the right hand side is less than $M + 1$.

3 Unbounded $\varphi'(\,)$

We now consider (1.1) and (1.4) without requiring that $\varphi'(\cdot)$ be bounded; this is the case for the Cahn-Hilliard equation where

$$\varphi(u) = 3u^2 - \beta^2 .$$

In order to obtain error bounds for this case we analyse problem $(\tilde{P}_{h,k})$ which has φ replaced by $\tilde{\varphi} \in C^1(\mathbb{R})$ such that

(3.1a) $$\tilde{\varphi}(s) = \varphi(s) \qquad \text{for } |s| \le M + 1$$

and

(3.1b) $$\sup_{s \in \mathbb{R}} |\tilde{\varphi}'(s)| \le \lambda .$$

In this case we know that, because of the assumptions (1.2), the initial boundary-value problem (1.1) has the same unique solution when φ is replaced by $\tilde{\varphi}$.

Theorem 3.1: There exist positive constants h_0 and ε_0 such that for $h \le h_0$ and $k^2 \le \varepsilon_0 h$:

(i) (1.4) has a unique solution U_B^n in the ball

$$B = \{\chi \in S_h^r : |\chi|_{0,\infty} \le M + 1\} .$$

(ii) For every $U_0^n \in B$ the fixed point iteration (2.5) converges to U_B^n .

(iii) The error bound

$$|U_B^n - u(\cdot, t_n)|_0 \le C(u)\left(h^{\bar{r}} + k^2\right)$$

holds.

Proof: Let U^n and U^n_j denote solutions of $(\tilde{P}_{n,k})$ and (2.5) for fixed n with φ replaced by $\tilde{\varphi}$. Proposition 2.1 applies and we know that $\tilde{U}^n$ is unique and is the limit as $j \to \infty$ of the well defined sequence $\{\tilde{U}^n_j\}$. Furthermore it follows from the Proposition and Corollary 2.1 that $U^n \in B$ and $\tilde{U}^n_j \in B$ for each $n \geq 1$ and $j = 0,1,2\ldots$. Since φ is identical to $\tilde{\varphi}$ on the ball B , Theorem 3.1 is an immediate consequence of section 2 with

$$U^n_j = \tilde{U}^n_j \quad , \quad U^n_B = \tilde{U}^n \ .$$

Remark: Suitable choices for $U^n_0 \in B$ are
a) $U^n_0 = U^{n-1}$ b) $U^n_0 = 2U^{n-1} - U^{n-2}$.
In case b) the fact that $U_0{}^n \in B$ is a consequence of the fact that $2U^{n-1} - U^{n-2}$ approximates $2u(\cdot,t_{n-1}) - u(\cdot,t_{n-2})$ in L^∞ and the latter approximates $u(\cdot,t_n)$ to order k^2 .

Remark: We considered other fully discrete numerical methods for problem (1.1). The <u>backward Euler method</u> for (1.1), "time-lagging" the nonlinear term, is given by the problem : Find $\{U^n\}_{0\leq nk\leq T}$ such that $U^n \in \overset{o}{S}{}^r_h$

$$(3.2) \qquad \left(\partial\, U^n, \chi\right) + \gamma\left(D^2\, U^{n+1}\,,\, D^2\, \chi\right) = \left(\varphi(U^n)\,,\, D^2\, \chi\right)$$

for all $\chi \in \overset{o}{S}{}^r_h$ and $U^o_0 = U^h_0$. An optimal order linearized <u>Crank-Nicholson method</u> is defined as above with (3.2) replaced by the expression below

$$(3.3) \qquad \left(\partial U^n\,,\, \chi\right) + \gamma\left(D^2\, U^{n+1/2}, D^2\, \chi\right) = \left(\varphi\left(\tfrac{3}{2}U^n - \tfrac{1}{2}\, U^{n-1}\right),\, D^2\, \chi\,\right) .$$

Again the nonlinear term is "time lagged" . A Crank-Nicholson predictor-corrector method has two steps. The prediction function is found by (3.3). Labelling this solution W^{n+1} the correction function is found by the expression

(3.4) $$\left(\partial U^n , \chi\right) + \gamma \left(D^2 U^{n+1/2}, D^2\chi\right) = \left(\varphi\left(\frac{1}{2}\left(W^{n+1} + U^n\right)\right), D^2\chi\right)$$

as above.

On each time step all the methods discussed above define a unique selection U^{n+1} given U^n We can prove there exist constants C, ε_0 and h_0 such that

(i) If $\{U^n\}$ is given by the backward Euler method (3.2), $h \le h_0$ and $k \le \varepsilon_0 h$ then

(3. 5) $$\left|U - u(\cdot,t_n)\right|_0 \le C\,(h^{\bar{r}} + k)$$

(ii) If $\{U^n\}$ is given by either of the Crank-Nicholson methods, $h \le h_0$ and $k \le \sqrt{\varepsilon_0 h}$ then

(3.6) $$\left|U^n - u(\cdot,t_n)\right|_0 \le C(h^{\bar{r}} + k^2) \ .$$

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