COMBINATORS HEREDITARILY OF ORDER TWO

by

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Research Report No. 88-33 2

August 1988

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Introduction

In [4] we introduced the combinators hereditarily of order one (HOO) and showed that the word problem for HOO combinations is ℓog space complete for polynomial time. In this note we shall introduce the combinators hereditarily of order two (HOT). We shall show that every partial recursive function is representable by a HOT combinator, although HOT combinators form a hierarchy by definitional level (and consequently are not combinatorially complete). In particular, the word problem for HOT combinations has Turing degree $0'$.

HOT

We shall think of members of HOT as atoms with associated reduction rules. Some care will be needed since the resulting conversion relation does not coincide with]5 conversion of the corresponding X terms. Consequently, for negative results we shall switch to the A calculus and]3 conversion. HOT and —> are defined simultaneously by induction as follows.

If 3C is a combination of x's and y's then X defined by the reduction rule Xxy -» 9C belongs to HOT. If 3C is a —> normal combination of members of HOT, x's, and y's then X defined by reduction rule Xxy -» 9C belongs to HOT. In each case we write X = Xxy 3t.

Examples

L s Xxy x(yy)

U = Xxy y(xxy)

0 = Xxy y(xy)

H_O s L

H_x s Xxy x(yL))

$H_{n+2} \equiv \lambda xy\ x(yH_{n+1})$

Encoding Data Types in HOT

Booleans:

T = K

$F \equiv K_*$

D = Axy xyT

$\neg \equiv (\lambda xy\ yFT)T$

Integers: Barendregt numerals

$$\underline{0} \equiv K_* K_*$$

$$\text{succ} \equiv \lambda xy \quad yFx$$

$$\text{pred} \equiv (\lambda xy \quad yF)T$$

$$\text{zero} \equiv (\lambda xy \quad yT)T$$

$$\text{sg} \equiv (\lambda xy \quad yT\underline{0}\ \underline{1})T$$

$$\overline{\text{sg}} \equiv (\lambda xy \quad yT\underline{1}\ \underline{0})T$$

A Hierachy

If $X \equiv \lambda xy\ \mathfrak{X}$ set $\text{rnk}(X) = 1 + \max\{\text{rnk}(Y)\colon\ Y$ appears in $\mathfrak{X}\}$. So, for example, $\text{rnk}(H_n) = n + 1$.

Below $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}\ldots$ range over arbitrary combinations of HOT combinators and variables. $\mathfrak{X}^\lambda$ is the λ term which results from $\mathfrak{X}$ by repeatedly replacing HOT combinators by their λ definitions and $\mathfrak{X}^{\lambda\,CL} \equiv \mathfrak{X}$.

We note here that $\twoheadrightarrow$ is a regular left normal combinatory reduction system ([3]) and consequently satisfies the Church-Rosser and standization theorems.

We shall show that H_n^λ does not β convert to an applicative combination of HOT^λ combinators of ranks $\leq n$. For this we need some preliminaries.

Def$\underline{^n}$ M is head secure if for some $\mathfrak{X} \ni y$, $Mxy \twoheadrightarrow x\mathfrak{X}$

Def$^{\underline{n}}$ $\mathfrak{X}$ unrolls to $\mathcal{Y}$ if there are HOT combinators $Y_1, \dots, Y_n$ $(n \geq 0)$ such that

$$\mathfrak{X} \underset{\ell}{\rightarrow} Y_1(\dots(Y_n\mathcal{Y})\dots)$$

(see [1] pg. 327)

Lemma:

Suppose that M is a head secure combination of HOT combinators of rank $\leq n$ and $Mxy \underset{\text{head}}{\rightarrow} \mathfrak{X}$. Then $\mathfrak{X}$ is a combination of y's, HOT combinators of rank $< n$, and $\mathcal{Y}$ s.t. Mx unrolls to $\mathcal{Y}$.

Proof:

The conclusion is true of Mxy, so suppose that the conclusion is true of $\mathfrak{X}_1$ and $\mathfrak{X}_1 \underset{\text{head}}{\rightarrow} \mathfrak{X}_2$.

Case 1; $\mathfrak{X}_1 \equiv \mathcal{Y}\, \mathcal{Y}_1 \dots \mathcal{Y}_m$ where Mx unrolls to $\mathcal{Y}$. Since Mxy is headsecure $\mathcal{Y}$ is not an atom. If $\mathcal{Y}$ begins with a head redex, the conclusion clearly holds for $\mathfrak{X}_2$. The last remaining case is $\mathcal{Y} \equiv Y\, \mathcal{Y}_0$. By definition Mx unrolls to $\mathcal{Y}_0$. Moreover, if Z appears in the r.h.s. of the defining reduction rule for Y, then $\mathrm{rnk}(Z) < n$. Thus $\mathfrak{X}_2$ satisfies the conclusion.

Case 2; $\mathcal{X}_1 \equiv Y\, \mathcal{Y}_1 \ldots \mathcal{Y}_m$ where $\mathrm{rnk}(Y) < n$. As above, $\mathcal{X}_2$ satisfies the conclusion.

Corollary:

If M is head secure then Mx has a normal form

$$Y_1(\ldots(Y_n x)\ldots)$$

Proof:

If M is head secure then for some $\mathcal{X} \ni y$, $Mxy \underset{\text{head}}{\twoheadrightarrow} x\mathcal{X}$. Since Mx does not unroll to $x\mathcal{X}$, Mx unrolls to x.

Proposition:

There is no combination M of HOT combinators of rank $\leq n$ such that

$$M^{\lambda} \underset{\beta}{=} H_n^{\lambda}$$

Proof:

The proof is by induction on n. The basis case, when $n = 0$ is trivial. Suppose $n > 0$ and $M^{\lambda} \underset{\beta}{=} H_n^{\lambda}$. Then $M^{\lambda}xy \underset{\beta}{\twoheadrightarrow} x(y\, H_{n-1}^{\lambda})$. By the standardization theorem there is a reduction

$$M^{\lambda}xy \underset{\beta\ \text{head}}{\twoheadrightarrow} x\mathcal{X} \underset{\substack{\beta\ \text{head} \\ \text{reduction of} \\ \mathcal{X}}}{\twoheadrightarrow} x(y\mathcal{Y}) \underset{\beta\ \text{internal}}{\twoheadrightarrow} x(y\, H_{n-1}^{\lambda})$$

Thus we have

$$Mxy \underset{\text{head}}{\longrightarrow} x\mathfrak{X}^{CL} \underset{\substack{\text{head}\\ \text{reduction of}\\ \mathfrak{X}^{CL}}}{\longrightarrow} x(y\ \mathfrak{Y}^{CL}).$$

In particular, M is head secure so Mx has a normal form $X_1(\ldots(X_m x)\ldots)$. We have then (*)

$$X_1(\ldots(X_m x)\ldots)y \underset{\text{head}}{\longrightarrow} x\mathfrak{U} \underset{\substack{\text{head}\\ \text{reduction of}\\ \mathfrak{U}}}{\longrightarrow} x(y\mathfrak{Z})$$

where $\mathfrak{Z}^{\lambda} \underset{\beta}{\longrightarrow} H^{\lambda}_{n-1}$. Now $\mathfrak{Z}$ is combination of HOT combinators of rank < n, y's, and terms

$$X_i(\ldots(X_m x)\ldots)$$

for $i = 1, \ldots, m + 1$. Put $N \equiv [\Omega/x, \Omega/y]\mathfrak{Z}$. Then as above

$$Nxy \longrightarrow x(yy) \quad \text{if} \quad n = 1$$

$$Nxy \longrightarrow x\mathfrak{X}_1 \longrightarrow x(y\mathfrak{Y}_1) \text{ where } \mathfrak{Y}^{\lambda}_1 \underset{\beta}{\longrightarrow} H^{\lambda}_{n-2} \quad \text{when} \quad n > 1.$$

In either case N is head secure so Nx has a normal form $Y_1(\ldots(Y_k x)\ldots)$. When $n = 1$, since each HOT combinator in N appears only in interated function position applied to Ω, we have $k = 0$ and

$Nx \twoheadrightarrow x$. This a contradiction. When $n > 1$, by similar reasoning $rnk(Y_j) < n$ for $j = 1 \ldots k$. By standarization there is a reduction

$$Y_1(\ldots(Y_k x)\ldots)y \underset{\text{head}}{\twoheadrightarrow} x\,\mathcal{U}_1 \underset{\substack{\text{head}\\ \text{reduction of}\\ \mathcal{U}_1}}{\twoheadrightarrow} x(y\,\mathcal{Z}_1)$$

where $\mathcal{Z}_1^{\lambda} \underset{\beta}{\twoheadrightarrow} H_{n-2}^{\lambda}$. This reproduces (*) at one lower rank.

Primative Recursive Functions

Put $\hat{succ} \equiv \lambda xy(\lambda uv\ \ vFu)(xy)$ and

$$\oplus \equiv (\lambda uv\ \ v\underline{0}UU(\lambda xy\ \ yT(K_* K_*)(\hat{succ}(x(yF))))v)T.$$

Then $\oplus$ is a normal HOT combination and $\oplus^{\lambda}$ has a β normal form. Moreover,

$$\oplus\underline{n} = \underbrace{\oplus(\lambda xy\ yT(K_* K_*)(\hat{succ}(x(\text{pred } y))))}_{\Phi}\ \underline{n}$$

$$= \underline{n}\ T(K_* K_*)(\hat{succ}(\Phi(\text{pred } \underline{n})))$$

$$= \underbrace{\hat{succ}(\ldots(\hat{succ}}_{n}\ \underline{0})\ldots).$$

Thus

$$\oplus\underline{n}\ \underline{m} = \underbrace{\text{succ}(\dots(\text{succ}}_{n}\ \underline{m})\dots) \equiv \underline{n + m}.$$

Put $\hat{\oplus} \equiv \lambda xy\ \ xy\underline{O}UU(\lambda uv\ \ vT(K_{*}K_{*})(\widehat{\text{succ}}(u(vF))))(xy)y$

$$\otimes \equiv (\lambda uv\ \ v\underline{O}UU(\lambda xy\ \ yT(K\underline{O})(\hat{\oplus}(x(yF))))v)T$$

Then $\otimes$ is a normal HOT combination and $\otimes^{\lambda}$ has a β normal form. Moreover,

$$\otimes\underline{n} = \underbrace{\Theta(\lambda xy\ \ yT(K\underline{O})(\hat{\oplus}(x(\text{pred } y))))}_{\Psi}\ \underline{n}$$

$$= \underline{n}T(K\underline{O})(\hat{\oplus}(\Psi(\text{pred } \underline{n})))$$

$$= \underbrace{\hat{\oplus}(\dots(\hat{\oplus}}_{n}(K\underline{O}))\dots)$$

Thus

$$\otimes\underline{n}\ \underline{m} = \underbrace{\oplus(\dots(\oplus(\oplus}_{n}(K\underline{O}\ \underline{m})\underline{m})\underline{m})\dots)\underline{m} = \underline{n\cdot m}$$

Set $\widehat{\text{pred}} \equiv \lambda xy\ \ xyT\ \underline{O}(xyF)$

$$\theta \equiv (\lambda uv\ \ v\underline{O}UU(\lambda xy\ \ yT\underline{O}(\widehat{\text{pred}}(x(yF))))v)T$$

Then θ is a normal HOT combination and θ^{λ} has a β normal form.

Moreover,

$$\theta\underline{n} = \underbrace{\theta(\lambda xy\ \ y T \underline{0}(\hat{pred}(x(pred\ y))))\ \underline{n}}_{\chi}$$

$$= \underline{n}\ T\ \underline{0}(\hat{pred}(\chi(pred\ \underline{n})))$$

$$= \underbrace{\hat{pred}(\ldots(\hat{pred}}_{n}\ \underline{0})\ldots)$$

Thus

$$\theta\underline{n}\ \underline{m} = \begin{cases} \underbrace{pred(\ldots(pred}_{n}\ \underline{m})\ldots) & \text{if } m \geq n \\ \underline{0} & \text{else} \end{cases}$$

so $\theta\underline{n}\ \underline{m} = \underline{m \dot{-} n}$.

Using the above techniques and constructions one can find HOT combinators sq, rt, Δ, quadres, P, and R, all of whose λ have a β normal form s.t.

$$\Delta\underline{n}\ \underline{m} = \underline{|n - m|}$$

$$sq\ T\ \underline{n} = \underline{n}^2$$

$$rt\ T\ \underline{n} = \underline{\lfloor\sqrt{n}\rfloor}$$

$$\text{quadres } T\ \underline{n} = \underline{n - \lfloor\sqrt{n}\rfloor^2}$$

$$P\ \underline{n}\ \underline{m} = \underline{((n + m)^2 + m)^2 + n}$$

$$RT\ \underline{n} = \text{quadres } T(rt\ T\ \underline{n}).$$

Note that $RT(P\ \underline{n}\ \underline{m}) = \underline{m}$ and quadres $T(P\ \underline{n}\ \underline{m}) = \underline{n}$.

Suppose now that M and N are normal HOT combinations s.t. M^λ, N^λ have β normal forms. Define

$$\text{Sum}(M,N) \equiv (\lambda yx\ \ x\underline{0}\ \oplus(x\underline{0}\ Mx)(x\underline{0}\ Nx))T$$

$$\text{Comp}(M,N) \equiv (\lambda yx\ \ x\underline{0}\ M(x\underline{0}\ Nx))T$$

$$\text{It}(M) \equiv (\lambda yx\ \ x\underline{0}\ UU(\lambda uv\ \ vT\underline{0}(y\underline{0}\ M(u(vF))))x)T$$

Then Sum(M,N), Comp(M,N), and It(M) are normal HOT combinations whose λ's have β normal forms and

$$\begin{aligned} \text{Sum}(M,N)\underline{n} &= \oplus(M\underline{n})(N\underline{n}) \\ \text{Comp}(M,N)\underline{n} &= M(N\underline{n}) \\ \text{It}(M)\underline{n} &= \underbrace{M(\ldots(M}_{n}\ \underline{0})\ldots) \end{aligned}$$

Thus by [2] pg. 93 we have proved the following.

Proposition:

Every primitative recursive unary function is representable by a normal HOT combination whose λ has a β normal form.

Partial Recursive Functions

Let t be the characteristic function of Kleene's T predicate i.e. $T(e,x,y) \leftrightarrow t(e,x,y) = 0$ and $\neg T(e,x,y) \leftrightarrow t(e,x,y) = 1$. Then by the previous proposition for each Gödel number the unary primitive recursive function $t(e, x \dot- \lfloor\sqrt{x}\rfloor^2, \lfloor\sqrt{x}\rfloor \dot- \lfloor\sqrt{\lfloor\sqrt{x}\rfloor}\rfloor^2)$ is representable by a normal HOT combination H_e s.t. H_e^λ has a β normal form. By [2] pgs. 83 and 84 $H_e(P\ \underline{n}\ \underline{m}) = \underline{t(e,n,m)}$. Define $G_e \equiv$

$$(\lambda vu.\ u\underline{0}UU(\lambda xy(y\underline{0}\ H_e y\ T)(y\underline{0}Ry)(x(y\underline{0}P(y\underline{0}Ly)$$

$$((y\underline{0}\ succ)(y\ \underline{0}\ Ry))))(u\ \underline{0}\ Pu\underline{0}))T$$

Then

$$G_e\underline{n} = \Theta\ \underbrace{(\lambda xy(y\underline{0}\ H_e\ y\ T)(y\ \underline{0}\ Ry)(x(y\underline{0}P(y\underline{0}Ly))((y\ \underline{0}\ succ)(y\ \underline{0}\ Ry)))}_{\Gamma}\ (P\ \underline{n}\ \underline{0})$$

$$= H_e(P\ \underline{n}\ \underline{0})T\ \underline{0}\ (\Theta\Gamma(P\ \underline{n}\ \underline{1})) = \ldots.$$

$$\underline{\mu m\ t(e,n,m) = 0.}$$

Thus if F is a normal HOT combination representing Kleene's result extracting function with F^λ having a β normal form

$$(\lambda yx \quad x\underline{0}F(x\underline{0}\ G_e x))T$$

is a normal HOT combination whose λ has a β normal form which represents the partial recursive function {e}. Thus we have proved the following theorem.

Theorem:

Every partial recursive function is representable by a normal HOT combination whose λ has a β normal form.

References

[1] Barendregt, *The Lambda Calculus*, North Holland, 1984.

[2] Peter, *Recursive Functions*, Academic Press, 1967.

[3] Klop, *Combinatory Reduction Systems*, Math. Centrum, Amsterdam, 1980.

[4] Statman, Combinators hereditarily of order one, Manuscript, July 1988.