COMBINATORS AND THE THEORY OF PARTITIONS

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Abstract

We show that the unification problem for untyped combinations of B and I under $\beta\eta$ (or just β) conversion is undecidable. The proof depends on a bijection between B,I combinations and integer partitions. The bijection yields as a corollary an old counting result for integer partitions with "triangle number" m.

B,I Combinations

We shall consider combinations of B and I under $\beta\eta$ conversion. The use of η lends a certain elegance but is not essential. As noted by Curry ([3]) combinations form a monoid with identity element I and $x \circ y \equiv Bxy$.

Examples: $B_n \equiv \underbrace{B(\ldots(BB)\ldots)}_{n}$

$B_n^m \equiv \underbrace{B_n \circ \cdots \circ B_n}_{m}$

clearly, $BI =_{\beta\eta} I$. Curry observed that

(1) $$B(x \circ y) =_{\beta} Bx \circ By$$

but these identities do not yield a complete set of combinatory axioms for β or $\beta\eta$ conversion. We have

(2) $$B \circ Bx =_{\beta} B^2x \circ B$$

It follows immediately that each B,I combinaition $\beta\eta$ converts to either I or a combination of the form

$$B_{n_k}^{m_k} \circ \cdots \circ B_{n_0}^{m_0}$$

where $n_k > \cdots > n_0 > 0$ and $m_k, \ldots, m_0 > 0$. Such a combination is said to be in partition normal form with $n = n_0 + \cdots + n_k$ and $m = m_0 + \cdots + m_k$. We shall presently prove that partition normal forms are unique, so (2) supplies the missing combinatory axiom.

$\beta\eta$ normal form		partition normal form
$\lambda x_1 \ldots x_n yz.$	$x_1 \ldots x_n(yz)$	B_n
$\lambda yzx_1 \ldots x_n$	$y(zx_1 \ldots x_n)$	B^n
$\lambda x_1 \ldots x_n yz.$	$x_1(\ldots(x_n(yz))\ldots)$	$B_n \circ B_{n-1} \circ \cdots \circ B_1$

We record here the following obvious facts ([4]). Every B,I combination $\beta\eta$ converts to a proper combinator without selective permutative or duplicative effect. Moreover, each such proper combinator $\beta\eta$ converts to a B,I combination. An algorithm for computing the partition normal form from the $\beta\eta$ normal form will be provided later.

Every B,I combination is obviously left $\beta\eta$ invertible. By Dezani's theorem ([3]) only I is right $\beta\eta$ invertible. Nevertheless, we shall later observe that the monoid satisfies the right cancellation law.

Proposition 1. Partition normal forms are unique.

Proof: Suppose $M \equiv B_{n_k}^{m_k} \circ \cdots \circ B_{n_0}^{m_0}$ and $N \equiv B_{p_\ell}^{q_\ell} \circ \cdots \circ B_{p_0}^{q_0}$ are partition normal forms. Put $\#M = n_0m_0 + \cdots + n_km_k$ and $\#N = p_0q_0 + \cdots + p_pq_p$. Observe that $M \underset{\beta\eta}{\neq} I$ since

$$M \underbrace{I \ldots I}_{n_k-1} \underset{\beta\eta}{=} B^{m_k}$$

$$M \underbrace{I \ldots I}_{n_k-1} B \underset{\beta\eta}{=} B_{m_k+1}$$

$$M \underbrace{I \ldots I}_{n_k-1} B \underbrace{I \ldots I}_{m_k} \underset{\beta\eta}{=} B \underset{\beta\eta}{\neq} I.$$

Now suppose that $M \underset{\beta\eta}{=} N$. We prove by induction on $\#M + \#N$ that $M \equiv N$. Let r be smallest such that $n_r \geq 2$ and s smallest such that $p_s \geq 2$; so $1 \geq r,s \geq 0$. Then $P \equiv B^{m_k}_{n_k-1} \circ \cdots \circ B^{m_r}_{n_r-1} \underset{\beta\eta}{=} MI \underset{\beta\eta}{=} NI \underset{\beta\eta}{=} B^{q_\ell}_{p_\ell-1} \circ \cdots \circ B^{q_s}_{p_s-1} \equiv Q$ are in partition normal form. Thus by induction hypothesis, or the above remark, $P \equiv Q$. In particular, $k - r = \ell - s$ and $n_k,\ldots,n_r = p_\ell,\ldots,p_s$; $m_k,\ldots,m_r = q_\ell,\ldots,q_s$. Put $t = n_k m_k + \ldots + n_r m_r$. We have $B_{n_k+1+\#M} \circ M \underset{\beta\eta}{=} MB_{n_k+1} \underset{\beta\eta}{=} NB_{n_k+1} \underset{\beta\eta}{=} B_{n_k+1+\#N} \circ N$ by (2). Thus $B_{1+\#M} \underset{\beta\eta}{=} (B_{n_k+1+\#M} \circ M) \underbrace{I \ldots I}_{n_k} \underset{\beta\eta}{=} (B_{n_k+1+\#N} \circ N) \underbrace{I \ldots I}_{n_k} \underset{\beta\eta}{=} B_{1+\#N}$. Hence $\#M = \#N$ and $\#M - t = \#N - t$. This completes the proof.

Corollary: If M_0 and M_1 are B,I combinations such that $M_0 \underset{\beta\eta}{\neq} M_1$ then there are B,I combinations $N_1,\ldots,N_n$ such that

$$M_i N_1 \ldots N_n \underset{\beta\eta}{=} B$$

$$M_{1-i} N_1 \ldots N_n \underset{\beta\eta}{=} I$$

for some i.

Consequently, a model of the B,I fragement of the λ calculus either contains the free model or has B = I. In the latter case, application is associative and the model is just a monoid.

Examples:

products	partition normal forms
$B_p \circ (B_{n_k}^{m_k} \circ \cdots \circ B_{n_0}^{m_0})$	$B_{n_k+1}^{m_k} \circ \cdots \circ B_{n_r+1}^{m_r} \circ B_p \circ B_{n_{r-1}}^{m_{r-1}} \circ \ldots$ where r is smallest s.t. $n_r > p$
$(B_{n_k}^{m_k} \circ \cdots \circ B_{n_0}^{m_0}) \circ B_p$	$B_{n_k}^{m_k} \circ \cdots \circ B_{p+s} \circ B_{n_r}^{m_r} \circ \cdots$ where r is largest such that $p + n_0 m_0 + \cdots + n_{r-1} m_{r-1} > n_r$ and $s = n_0 m_0 + \cdots + n_r m_r$

Proposition 2. $M \circ P \underset{\beta\eta}{=} N \circ P \Rightarrow M \underset{\beta\eta}{=} N.$

Proof: The proposition is clear if anyone of the 3 M, N, or P beta eta converts to I. So we may assume that we have partition normal forms

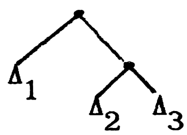
$$M =_{\beta\eta} B_{n_k} \circ \cdots \circ B_{n_0} \qquad n_k \geq \cdots \geq n_0 \geq 1$$
$$N =_{\beta\eta} B_{m_\ell} \circ \cdots \circ B_{m_0} \qquad m_\ell \geq \cdots \geq m_0 \geq 1$$
$$P =_{\beta\eta} B_{r_s} \circ \cdots \circ B_{r_0} \qquad r_s \geq \cdots \geq r_0 \geq 1.$$

Note immediately that $k = \ell$. The proof is by induction on s, and the induction step follows from associativity. Thus we may assume $s = 0$. Let t be the largest t s.t. $B_{n_t} \not\equiv B_{m_t}$. Suppose we have the partition normal forms

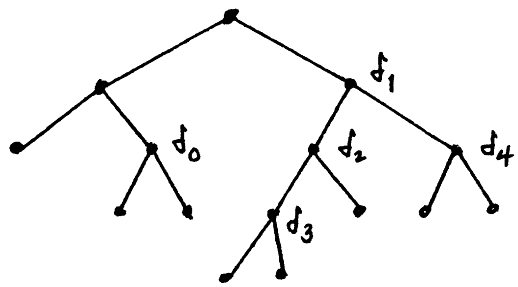
$$M \circ B_{r_0} =_{\beta\eta} B_{n_k} \circ \cdots \circ B_{n_i} \circ B_{r_0+i} \circ B_{n_{i-1}} \circ \cdots \circ B_{n_0}$$
$$N \circ B_{r_0} =_{\beta\eta} B_{m_k} \circ \cdots \circ B_{m_j} \circ B_{r_0+j} \circ B_{m_{j-1}} \circ \cdots \circ B_{m_0}$$

where $k + 1 \geq i,j \geq 0$. Wℓog we may assume $j > i$. Since $B_{r_0+j} \not\equiv B_{m_{j-1}}$ we have $t = j$. In particular, $B_{n_t} \equiv B_{r_0+t}$, $B_{n_{t-1}} \equiv B_{m_t}$, . . . , $B_{n_i} \equiv B_{m_{i+1}}$, $B_{r_0+i} \equiv B_{m_i}$. But since $t > i$ $r_0 + i > m_i$. This is a contradiction and completes the proof.

<u>Integer partitions</u>

A binary tree Δ is said to be <u>righteous</u> if $\Delta =$  for some Δ_1, Δ_2, and Δ_3. The $\beta\eta$ normal forms of B,I combinations and the righteous binary trees are in obvious 1 - 1 correspondence. If δ is an internal node of Λ let $\#\delta$ = the number of leaves of Λ which lie properly to the left of δ. $\#\Lambda = \sum_{\delta \in \Lambda} \#\delta$. Enumerate the internal nodes of Λ with nonzero #: $\delta_d \ldots \delta_0$ from right to left and bottom to top. Clearly, $\#\delta_d \geq \cdots \geq \#\delta_0$.

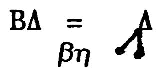
Example

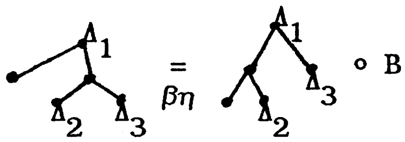


Let $\Lambda^{\beta\eta}$ be the $\beta\eta$ normal form corresponding to Λ.

Observation 1: $B_{\#\delta_d} \circ \cdots \circ B_{\#\delta_0} \underset{\beta\eta}{=} \Lambda^{\beta\eta}$. Proposition 1 and Observation 1 establish a 1 - 1 correspondence between integer partitions and righteous binary trees. The correspondence amounts to computing partition normal forms from $\beta\eta$ normal forms and vice versa.

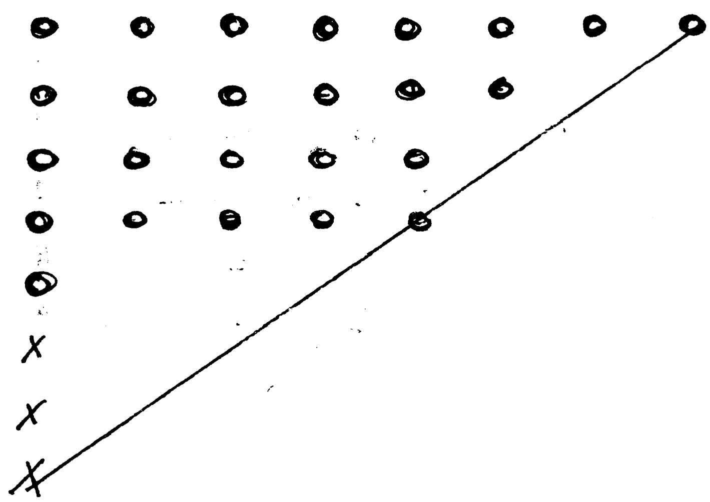
<u>Sample Computations</u>





The correspondence between partitions and righteous binary trees with m internal nodes works out the following way. Define the <u>triangle number</u> of the partition $n_k + \cdots + n_1$ to be the least integer $k + \ell$ such that for $i = 1 \ldots k$ $n_i \leq i + \ell$ where $\rho \geq 0$. The righteous binary trees with m internal nodes are in $1 - 1$ correspondence with partitions with triangle number $m - 1$. This can also be seen by the "walk above the diagonal" construction from elementary combinatorics, and appears to be due to L. Carlitz.

Example:



triangle number of $8 + 6 + 5 + 5 + 1 = 8$

Corollary. The number of partitions with trianble number $m - 1$ is

$$\frac{1}{m+1}\binom{2m}{m} - \frac{1}{m}\binom{2m-2}{m-1}$$

Unification

Let us put $\underline{0} \equiv I$ and $\underline{n} \equiv B^n$, and set $\mathrm{Int} = \{\underline{n} : n \in \omega\}$

(3) $$M \in \mathrm{Int} \Leftrightarrow B \circ M \underset{\beta\eta}{=} M \circ B$$

For, suppose $M \equiv B^{m_k}_{n_k} \circ \cdots \circ B^{m_0}_{n_0}$ is partition normal with $n_k > 1$. Then $B \circ M \underset{\beta\eta}{=} B^{m_k}_{n_k+1} \circ N$ for some N so $B \circ M \underset{\beta\eta}{\neq} M \circ B$. Let

$\mathrm{Parts}(x,y,z) \Leftrightarrow x = By \circ z \wedge \mathrm{Int}(z)$ and $\mathrm{Conj}(x,y) \Leftrightarrow \exists z\ x = z \circ B \wedge$
$\mathrm{Parts}(z \circ B_2, y, z)$

(4) $$\mathrm{Conj}(\underline{n},M) \Leftrightarrow M \underset{\beta\eta}{=} B_n$$

For, suppose $\underline{n} \underset{\beta\eta}{=} N \circ B$. Then $N \underset{\beta\eta}{=} \underline{n-1}$ and $N \circ B_2 \underset{\beta\eta}{=} B_{n+1} \circ \underline{n-1}$.
Hence if $\mathrm{Parts}(N \circ B_2, P, Q)$ then $P \underset{\beta\eta}{=} B_n$ and $Q \underset{\beta\eta}{=} \underline{n-1}$.
Define $\mathrm{ap}(x,y,z) \Leftrightarrow \exists u\ \mathrm{Conj}(B\cdot x,u) \wedge$

$z \circ u = u \circ z \wedge$

$\exists v\ \mathrm{Conj}(B^2 \circ x \circ y, v) \wedge$

$z \circ Bu = v \circ z$

(5) $$\mathrm{ap}(\underline{m},\underline{n},M) \Leftrightarrow M \underset{\beta\eta}{=} B^n_{m+1} \underset{\beta\eta}{=} \underline{m}\ \underline{n}$$

For, suppose $\mathrm{Conj}(\underline{m+1},N_1)$. Then by (4) $N_1 \underset{\beta\eta}{=} B_{m+1}$. Similarly, if $\mathrm{Conj}(\underline{m+n+2},N_2)$ then $N_2 \underset{\beta\eta}{=} B_{m+n+3}$ which is absurd. Suppose that M has the partion normal form

$$B^{m_k}_{n_k} \circ \cdots \circ B^{m_0}_{n_0}$$

If $m + 1 < n_k$ then the lrgest part in the partition normal form of $B_{m+1} \circ M$ is B_{n_k+1} but the largest part in the partition normal form of

$M \circ B_{m+1}$ is either B_{n_k} or B_r for some $r > n_k + 1$. If $n_0 < m + 1$ then the largest part in the partition normal form of $M \circ B_{m+1}$ is B_r for some $r > m + 1$ while the largest part in the partition normal form of $B_{m+1} \circ M$ is B_{m+1}. Hence $M \underset{\beta\eta}{=} B_{m+1}^{m_k}$. Now $M \circ B_{m+2} \underset{\beta\eta}{=} B_{m+2+m_k} \circ M$ so by right cancellation $B_{m+2+m_k} \underset{\beta\eta}{=} B_{m+2+n}$ and $m_k = n$.

Define $Ap(x,y,z) \Leftrightarrow \exists ab \exists uv\ Parts(x,a,b)$

$\land\ Parts(y,u,v)$

$\exists w_1 w_2\ ap(b,v,w_1) \land$

$b \circ Bu = w_2 \circ b \land$

$z = x \circ w_2 \circ w_1$

(6) $$Ap(M,N,P) \Leftrightarrow P \underset{\beta\eta}{=} MN.$$

First note that if $M \underset{\beta\eta}{=} BM_0 \circ \underline{m}$ and $N \underset{\beta\eta}{=} BN_0 \circ \underline{n}$ then $MN \underset{\beta\eta}{=} M_0 \circ \underline{m}(BN_0) \circ \underline{m}\ \underline{n}$. Moreover, $\underline{m} \circ BN_0 \underset{\beta\eta}{=} \underline{m}(BN_0) \circ \underline{m}$ so $Ap(M,N,MN)$. Conversly, if $Parts(M,M_0,M_1)$, $Parts(N,N_0,N_1)$ and $ap(M_1,N_1,P)$, then by (3), $M_1 \underset{\beta\eta}{=} \underline{m}$ and $N_1 \underset{\beta\eta}{=} \underline{n}$ for some m and n, and $P \underset{\beta\eta}{=} \underline{m}\ \underline{n}$ by (5).

In addition, if $\underline{m} \circ BN_0 \underset{\beta\eta}{=} Q \circ \underline{m}$ then $Q \underset{\beta\eta}{=} \underline{m}(BN_0)$. For, if $N_0 \underset{\beta\eta}{=} I$ then this is the case. Otherwise N_0 has a partition normal form $B_{n_k}^{m_k} \circ \cdots \circ B_{n_0}^{m_0}$ so $\underline{m} \circ BN_0 \underset{\beta\eta}{=} B_{n_k+1+m}^{m_k} \circ \cdots \circ B_{n_0+1+m}^{m_0} \circ \underline{m} \underset{\beta\eta}{=} \underline{m}(BN_0) \circ \underline{m}$ and $Q \underset{\beta\eta}{=} \underline{m}(BN_0)$ by right cancellation. Thus $P \underset{\beta\eta}{=} MN$.

Finally, define $\mathrm{Mult}(x,y,z) \Leftrightarrow \exists u\ \exists v\ \exists w\ \mathrm{Conj}(z,u) \wedge \mathrm{Conj}(y,w) \wedge u \circ v = yv \circ w \wedge v \circ Bu = xu \circ v \wedge \mathrm{Int}\ (z)$

(7) If $m \geq n \geq 2$ then $\mathrm{Mult}(\underline{m},\underline{n},M) \Leftrightarrow M \underset{\beta\eta}{=} \underline{m \cdot n}$.

Suppose $m \geq n \geq 2$. Put $v := B_{n(m-1)} \circ \cdots \circ B_n$. Then $B_{nm} \circ v \underset{\beta\eta}{=} B^n v \circ B_n$ and $v \circ B_{nm+1} \underset{\beta\eta}{=} B_{nm+m} \circ v \underset{\beta\eta}{=} B^m B_{nm} \circ v$. Conversly, suppose $B_k \circ N \underset{\beta\eta}{=} B^n N \circ B_n$ and $N \circ B_{k+1} \underset{\beta\eta}{=} B^m B_k \circ N$. Since $m \geq 2$ $N \underset{\beta\eta}{\neq} I$. Thus N has a partition normal form

$$B_{n_\ell}^{m_\ell} \cdot \cdots \cdot B_{n_0}^{m_0}$$

The largest part in the partition normal form of $B^n N \circ B_n$ is $B_{n_\ell+n}$ so, since $n \geq 2$, $k = np + n$. In particular, $B_{n_\ell+n} \circ B_{n_\ell}^{m_\ell} \circ \cdots \circ B_{n_0}^{m_0} \equiv B_{n_\ell+n}^{m_\ell} \circ \cdots \circ B_{n_0+n}^{m_0} \circ B_n$. Hence $N \underset{\beta\eta}{=} B_{n\ell} \circ B_{n(\ell-1)} \circ \cdots \circ B_n$. Now $N \circ B_{k+1} \underset{\beta\eta}{=} B_{k+1+\ell} \circ N$ so by right cancellation $\ell = m - 1$. Thus $k = n \cdot m$ as desired.

Suppose $\mathcal{X}_1, \mathcal{Y}_1, \ldots, \mathcal{X}_n, \mathcal{Y}_n$ are combination of B,I, and $x_1 \ldots x_m$. The corresponding <u>unification</u> problem is the problem of determining if there are B,I combinations $M_1, \ldots, M_m$ s.t. for $\Theta = [M_1/x_1, \ldots, M_m/x_m]$

$$\Theta\mathcal{X}_1 =_{\beta\eta} \Theta\mathcal{Y}_1$$

$$\vdots$$

$$\Theta\mathcal{X}_n =_{\beta\eta} \Theta\mathcal{Y}_n$$

By (3) (4) and (7) Hilbert's 10th problem can be encoded as a unification problem.

<u>Theorem 1</u>. Unification is undecidable.(5) and (6) give a stronger result. We can require that each $\mathcal{X}_i$ or $\mathcal{Y}_i$ has the form

$$\mathcal{Z}_1 \circ \cdots \circ \mathcal{Z}_k$$

where each $\mathcal{Z}_\ell$ is either a B,I combination or of the form

$$B^r x_s.$$

We leave the decision problem for monoid equations open.

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