CLASSIFICATION OF BALANCED SETS AND CRITICAL POINTS OF EVEN FUNCTIONS ON SPHERES

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1. INTRODUCTION. This paper is concerned with Lyusternik-Schnirelman type principles for the determination of critical points of even functionals on spheres. The actual motivation for this work is the study of isoperimetric problems in the calculus of variations, in this paper however we deal only with a finite dimensional analogue.

Let a be a smooth even function on the sphere S^N. By a "Lyusternik-Schnirelman type principle" we mean a principle of one of the forms

$$(1.1) \qquad v_n = \min\{\max_{x\in B} a(x) : T(B) \geq n\}, \quad n = 1,2,\ldots,N+1,$$

or

$$(1.2) \qquad i)_n = \max\{\min_{x\in B} a(x) : T(B) \geq n\}, \quad n = 1,2,\ldots,N+1.$$

Here *r* should be an integer valued function whose domain is the class of closed balanced subsets of S^N, it should be non-decreasing with respect to set inclusion and distinguish spheres of different dimensions (and fulfill some additional more technical conditions). We define such a class of functions, which we shall call "types", in such a way that for any type *r* the principles (1.1) and (1.2) determine critical values of a. The best known and most commonly used example of a "type" is the "genus" defined by Krasnosel'skii,[7],[8]. Another is one whose definition is due to Yang,[12], and whose use in the calculus of variations occurred first in [5] and later in [3]; this appears at present to be the ideal choice. Contrary to what one might first expect there are infinitely many "types" *r* which can serve in (1.1) or (1.2) for the determination of critical values. In this paper we shall study the entire class of these functions in some detail. To motivate such a study after we have already tentatively identified an ideal one we observe that such more detailed knowledge seems still to contribute to a more complete understanding of the critical point problem, see for example Theorem 12.1. The attempt to fully understand the properties of the standard types, such as Krasnosel'skii's genus, leads naturally to the notion of the "dual" of a type which then enlarges the class of functions under consideration. Finally, the more "types" one has at one's disposal the more tools one has for the estimation of critical values, this is particularly relevant in the infinite dimensional case. A similar point of view underlies much of the work in [1].

Our study is directed primarily to the answering of three basic questions. The first of these questions concerns the relation between principles of the form (1.1) and principles of the form (1.2). Specifically, when do a principle of the first sort and a principle of the second sort determine the same critical values for an even function a on S^N with only the index order reversed? This question leads naturally to the notion of "duality" of types, and it was in order that this duality could be defined that a "type" had to be defined in a less restrictive way than might have been suggested by the most commonly used examples. Thus a type may not take the same value on two balanced sets even though there is an odd homeomorphism between them; the type of a set being possibly also dependent on its relative position in S^N (a similar situation holds for Lyusternik-Schnirelman category). Also, type is not necessarily preserved under an immersion $S^N \subseteq S^{N+1}$. This notion of duality was anticipated but not formally defined or investigated in [4], see p. 432. Also relevant is the work of Heinz, [6], who introduced a very similar notion for the Lyusternik-Schnirelman category. The analogue of his idea in our context would lead to a "co-type" (analogous to co-dimension) rather than a "dual type"; this was in fact necessary in his infinite dimensional context. We have chosen to deal with "duality" instead because of the resultant unification of the theory; for the infinite dimensional theory these results can be translated into the language of "co-type".

Our second question is suggested by one that is raised by Ambrosetti and Rabinowitz in [1]. It is the very basic question, Supposing the even functional a on S^N to have only isolated critical points, under what conditions on τ does (1.1) necessarily determine N+1 distinct critical values? We identify the condition on a type that is necessary and sufficient in order that this be the case.

The last question concerns the correlation between the Morse index of a critical point on a critical level determined by (1.1) and the index n (in (1.1)) of the critical level on which it lies. Suppose a admits only non-degenerate critical points and has exactly two on any critical level (any C^1 even functional can be approximated in the C^1-norm by such functions). As shown in [3], if τ is the Krasnosel'skii genus then a critical point on the level $a(x) = \mu_n$ has Morse index $\geq$ n-1 while if it is the function mentioned above that is due to Yang then a critical point on that level has Morse index = n-1. A major aim when this research began was to determine whether the strict inequality could hold for Krasnosel'skii's genus. The answer was found to be in the affirmative, we in fact find necessary and sufficient conditions both for the inequality and for equality. Here also the notion of duality plays an important role. The question described in this paragraph has been studied also by Bahri and Lions,[2].

Sections 2 through 7 deal with the basic theory of "types", sections 8 and 9 introduce particular properties which types can have and which are especially relevant to our three main questions. Section 10 defines some specific types in addition to the examples already given in section 3 and proceeds to discuss (although not completely) the properties which these individual types possess or don't possess. Section 11 gives the applications of the preceding material to questions (specifically but not only the three above) concerning critical points. Finally in section 12 we identify a certain property, which can be regarded as a stability property, which characterizes those critical values of an even function a on S^N which will be determined by the principle (1.1) for <u>some</u> type τ.

We emphasize the indebtedness of this work to the work of Conner and Floyd,[4]. Many of the deeper and more interesting properties of some of the particular examples of types, as listed in section 10, are from [4]. Specifically, the answer to our question concerning the Morse index inequality when the Krasnosel'skii genus is used in (1.1) depends on some results from there. It is shown in [4] that each principle ideal domain L gives rise to a type τ, here we have only treated that associated with Z_2 (which coincides with the Yang function referred to earlier) and very briefly that associated with Z. It might be interesting to pursue the question of what special properties are possessed by types associated with other such rings.

<u>2. RELATIVE TYPE</u>. By $\mathbf{B}_N$ we denote the class of closed balanced subsets of $S^N \subseteq R^{N+1}$ (B is balanced if $x \in B$ implies $-x \in B$); H_N denotes the class of odd homeomorphisms of S^N onto itself. By a standard imbedding $i:S^n \mapsto S^N$ we shall understand the restriction to S^n of an isometric linear transformation from R^{n+1} to R^{N+1}, thus $i(S^n)$ is the intersection of S^N with an (n+1)-dimensional hyperplane through the origin in R^{N+1}.

For $B \in \mathbf{B}_N$, $C \in \mathbf{B}_M$, $\mathbf{U}(B)$ denotes the set of balanced neighborhoods of B in S^N, **and** $\mathbf{S}(B,C)$ denotes the set of odd continous maps $f:B \mapsto C$.

Definition 2.1. A relative type is a non-negative integer-valued function τ that is defined on $\sum_{N=1}^{\infty} \mathbf{B}_N$ and satisfies, for $B,\ C \in \mathbf{B}_N$

1) $\tau(B) \leq \tau(C)$ when $B \subseteq C$;

2) if $h \in H_N$ then $\tau(h(B)) = \tau(B)$;

3) every B has a neighborhood U with $\bar{U} \in \mathbf{B}_N$ and $\tau(\bar{U}) = \tau(B)$;

4) $\tau(\phi) = 0$ and if $i:S^n \mapsto S^N$ is a standard imbedding then

$$\tau(i(S^n)) = n + 1. \tag{2.1}$$

The set of all relative types will be denoted by **T**.

Remarks. 1. Condition 4) requires that a "type" distinguish spheres of different dimension. The requirement (2.1) is a normalization, some sources, e.g. [4],[12],[13],[14], that consider such functions use instead of (2.1) the normalization $\tau(i(\dot{S}^n)) = n$; we shall refer to this as the "topologists normalization".

2. We emphasize that in the presence of a standard imbedding $i:S^M \mapsto S^N$ we need not have $\tau(B) = \tau(i(B))$ for $B \in \mathbf{B}_M$.

3. When furnished with the Hausdorff metric, $\mathbf{B}_N$ becomes a compact metric space, property 3) is equivalent, in the presence of property 1), to the upper semi-continuity of τ on this metric space.

4. Let P be a triangulation of S^N that is invariant under the involution $x \mapsto -x$ (such triangulations will be referred to simply as symmetric triangulations) and let $P^{(k)}$ denote the k^{th} baricentric subdivision of P. It follows from properties 1) and 3) that τ is determined by its values on the balanced complexes of P and all of the $P^{(k)}$.

Definition 2.2. A topological type is a relative type which in place of 2) satisfies the stronger condition:

2') if $B \in \mathbf{B}_N$, $C \in \mathbf{B}_M$ and $\mathbf{S}(B,C) \neq \phi$ then $\tau(B) \leq \tau(C)$.

The subset of T that consists of topological types will be denoted by $\mathbf{T}_t$.

3. EXAMPLES. We begin by defining two examples of relative types. Let the non-negative integer N be given and for $1 \leq n \leq N+1$ let i_n denote a standard imbedding of S^{n-1} in S_N. Then we define τ_1 and τ_2 on $\mathbf{B}_N$ by:

(3.1) $\tau_1(B) = \min\{n\colon \forall\ U \in \mathbf{U}(i_n(S^{n-1}))\ \exists\ h \in H_N \text{ with } h(B) \subseteq U\}, \qquad B \in \mathbf{B}_N,$

(3.2) $\tau_2(B) = \max\{n\colon \forall\ U \in \mathbf{U}(B)\ \exists\ h \in H_N \text{ with } h(i_n(S^{n-1})) \subseteq U\}, \qquad B \in \mathbf{B}_N;$

it is clear that these definitions do not depend on the choices of i_n.

It is trivial to verify that τ_1 and τ_2 satisfy conditions 1), 2), 3), of Definition 2.1. To verify condition 4) it suffices to show that not every balanced neighborhood of $i_n(S^{n-1})$ contains an odd homeomorph of S^k if $k > n-1$. To this end we first note that we can choose the balanced neighborhood U of $i_n(S^{n-1})$ so that $i_n(S^{n-1})$ is a retract of U; it can be assumed that the retraction mapping is odd. The assertion then follows from the Borsuk-Ulam

Theorem (which in fact is equivalent to the nonvacuity of T).

Two examples of topological types are:

$$(3.3) \qquad \tau_3(B) = \min\{n : \mathbf{S}(B,S^{n-1}) \neq \phi\}, \qquad B\in\mathbf{B}_N,$$

$$(3.4) \qquad \tau_4(B) = \max\{n : \forall\ U\in\mathbf{U}(B),\ \mathbf{S}(S^{n-1},U) \neq \phi\}, \qquad B\in\mathbf{B}_N.$$

That (3.3) and (3.4) define types follows readily from the Borsuk-Ulam Theorem; that they are topological is obvious; it is immediate that the non-vacuity of T_t is equivalent to the Borsuk-Ulam Theorem.

Remarks. 1. τ_3 is the genus defined by Krasnosel'skii, [7],[8], τ_4 is suggested by the index defined in [4] (it is necessary to modify the definition in [4] in order that 3) of Definition 2.1 hold for arbitrary $B \in \mathbf{B}_N$). Other examples can be found in [4]; with the "topologist's normalization" used in [4] the type of a standardly imbedded sphere would agree with its dimension; cf. (2.1).

2. It is fairly easy to see that for $B \in \mathbf{B}_2$ one has $\tau_1(B) = \tau_2(B)$, and thus the restrictions to $\mathbf{B}_2$ of all types coincide. On $\mathbf{B}_3$ the situation is different. Consider a set which is "linked" but not connected, for example, out of (a standard imbedding of) S^1 in S^3 cut a balanced set consisting of two short open arcs. Then construct a balanced set $\tilde{B}$ by attaching at each cut one of a pair of small linked circles to one component and the other to the other component. Then $\tau_3(\tilde{B}) = 1$ while $\tau_1(\tilde{B}) = 2$. On the other hand if the set $B \in \mathbf{B}_3$ is a simple closed curve with small overhand knots at each of a pair of antipodal points then $\tau_2(B) = 1$ while $\tau_4(\tilde{B}) = 2$. Further considerations of

this sort, and with the introduction of many links or many knots, lead to the conclusion that the set of restrictions of types to B_3 is infinite. As we shall see in section 10 the restrictions of the topological types T_3 and r_4 to B_4 do not coincide. This is the case also for B_3; the components of the "linked" set $\tilde{B}$ described above must admit separation by a set C € B_3 with 7^*_3(C) = 3 but such a set cannot have T_4(C) = 3.

4. LATTICE STRUCTURE. The set T has a natural lattice structure with

(4.1) T >_T' if and only if T(B) ≥ T'(B) for every B € B^

and

TVT'(B) = max(r(B),T'(B)), TAT'(B) = min(r(B),r'(B)), B € B^.

Proposition 4.1 T i^a complete lattice with respect to the partial ordering (4.1).

Proof. Let S C T, then T_Q = inf{r:r € S} is given by

(4.2) T_Q(B) = min{T(B):T € S}, B € B^;

note that the set {T(B):T € S} is finite for any given B. To see that (4.2) indeed defines a relative type we need only verify 3) of Definition 2.1, as the other conditions of that definition are obviously satisfied by T_Q. However for any given B € B^ there is a r € S such that T_Q(B) = r(B) and for that choice of T there is a U € U(B) with T($\bar{U}$) = r(B). With *r* and U so chosen,

$$\tau_0(B) \le \tau_0(\bar{U}) \le \tau(\bar{U}) = \tau(B) = \tau_0(B).$$

For $S \subseteq \mathbf{T}$ we define $\tau_0 = \sup\{\tau : \tau \in S\}$ by

$$\tau_0(B) = \inf\{\max_{\tau \in S} \tau(\bar{U}) : U \in \mathbf{U}(B)\}, \quad B \in \mathbf{B}_N.$$

Proposition 4.2 *We have*

$$(4.3) \qquad \tau_1 = \sup\{\tau : \tau \in T\}, \qquad \tau_2 = \inf\{\tau : \tau \in T\}.$$

$\mathbf{T}_t$ *is a complete sublattice of* $\mathbf{T}$ *and*

$$(4.4) \qquad \tau_3 = \sup\{\tau : \tau \in T_t\}, \qquad \tau_4 = \inf\{\tau : \tau \in T_t\}.$$

Proof. It is clear from (3.1), (3.2) and Definition 2.1 that if $\tau \in T$ then $\tau_2(B) \le \tau(B) \le \tau_1(B)$ for $B \in \mathbf{B}_N$. The second assertion is obvious and (4.4) is immediate from (3.3), (3.4) and 2') of Definition 2.2.

5. DUALITY.

Given a relative type τ we define its *dual* τ^* by

$$(5.1) \quad \tau^*(B) = \min\{n : \exists\, C \in B_N \text{ with } \tau(C) = N-n+1 \text{ and } C \cap B = \phi\}, \qquad B \in B_N.$$

The following is a useful equivalent formulation of this definition.

Lemma 5.1 *Let* $\tau \in \mathbf{T}$ *and let* τ^* *be defined by* (5.1), *then for*

$B \in \mathbf{B}_N$,

(5.2) $$\max\{\tau(C):C \in \mathbf{B}_N,\ C\cap B = \phi\} = N + 1 - \tau^*(B).$$

<u>Proposition 5.2</u> <u>The dual τ^* of a relative type τ is a relative type</u>.

<u>Proof</u>. The dual of a relative type τ obviously satisfies 1), 2) of Definition 2.1; 3) also is easily verified for if $B \in \mathbf{B}_N$ is given and $C \in \mathbf{B}_N$ with $C\cap B = \phi$ and $\tau(C) + \tau^*(B) = N + 1$ then we need only choose the balanced neighborhood U so that $\bar{U} \subseteq S^N\backslash C$ in order that $\tau^*(\bar{U}) = \tau^*(B)$. It remains to verify 4). Let N,n be given and let i be as in 4), then it suffices to show that

(5.3) $$\max\{\tau(C):C \in \mathbf{B}_N,\ C\cap(i(S^n)) = \phi\} = N - n.$$

It is clear that there is a standard imbedding $j(S^{N-n-1})$ in S^N that does not intersect $i(S^n)$ and thus

(5.4) $$\max\{\tau(C):C \in \mathbf{B}_N,\ C\cap(i(S^n)) = \phi\} \geq N - n.$$

With the standard imbedding $i(S^n)$ given take $j(S^{N-n-1})$ to be a standard imbedding in S^N that is orthogonal to $i(S^n)$. Any point $z \in S^N$ can then be represented in the form

(5.5) $$z = x\cos\theta + y\sin\theta$$

with $x \in i(S^n)$, $y \in j(S^{N-n-1})$, $0 \leq \theta \leq \pi/2$; (here θ is uniquely determined by z, and x and y are uniquely determined except when $\theta = 0$ or $\pi/2$). If $C \in \mathbf{B}_N$

and $i(S^n) \cap C = \phi$ then there is a $\delta > 0$ such that for $z \in C$ we have $\pi/2 \geq \theta \geq \delta$ in the representation (5.5). It follows easily that if U is any neighborhood of $j(S^{N-n-1})$ then there is an $h \in H_N$ such that $h(C) \subseteq U$. We conclude that $\tau(B) \leq N - n$ and thus the inequality opposite to (5.4) holds and (5.3) is proved.

Proposition 5.3 For $\tau \in \mathbf{T}$,

$$\tau^{**} = \tau. \tag{5.6}$$

Proof. For $B, C \in \mathbf{B}_N$ with $B \cap C = \phi$ it follows from (5.2) that

$$\tau^*(C) \leq N + 1 - \tau(B).$$

If $B \in \mathbf{B}_N$ is given and $U \in \mathbf{U}(B)$ is chosen so that $\tau(\bar{U}) = \tau(B)$ then

$$\tau(B) = \max\{\tau(B'):B' \in \mathbf{B}_N,\ B' \cap (S^N \backslash U) = \phi\},$$

and thus $\tau^*(S^N \backslash U) = N + 1 - \tau(B)$. It follows that

$$\max\{\tau^*(C):C \in \mathbf{B}_N,\ C \cap B = \phi\} = N + 1 - \tau(B),$$

and thus by Lemma 5.1, $\tau^{**}(B) = \tau(B)$. Since N and $B \in \mathbf{B}_N$ were arbitrary, (5.6) follows.

Proposition 5.4 Let $\tau, \tau' \in T$. Then

$$\tau \geq \tau' \quad \underline{\text{if and only if}} \quad \tau^* \leq \tau'^*,$$

<u>and</u>

$$(\tau \vee \tau')^* = \tau^* \wedge \tau'^*, \qquad (\tau \wedge \tau')^* = \tau^* \vee \tau'^*.$$

<u>Proposition 5.5</u> <u>We have</u>

$$\tau_1^* = \tau_2.$$

<u>Proof</u>. This follows from (4.3) and Proposition 5.4.

<u>Remark</u>. τ_3 and τ_4 are not duals of one another, what amounts to the same, neither τ_3^* nor τ_4^* is topological, as we shall see in section 10.

<u>6. INJECTION</u>. A standard imbedding $i: S^N \mapsto S^{N+1}$ induces a mapping $i: \mathbf{B}_N \to \mathbf{B}_{N+1}$ which in turn induces a mapping $i: \mathbf{T} \mapsto \mathbf{T}$, the latter being defined by

$$(i\tau)(B) = \tau(iB), \qquad \text{for } B \Rightarrow \mathbf{B}_N.$$

Obviously the induced map on $\mathbf{T}$ does not depend on the choice of the imbedding i. We will say that a type τ is <u>stable under injection</u> if

$$i\tau = \tau.$$

It is clear that topological types are stable under injection, we shall show that the converse is also true.

Theorem 6.1 For $\tau \in \mathbf{T}$ the following are equivalent:

i) $\tau \in \mathbf{T}_t$,

ii) $i\tau = \tau$.

Proof. That i) implies ii) is obvious, so it only is required to show that ii) implies i). Suppose that τ is injection-stable and let the non-negative integers N,M, the set $B \in \mathbf{B}_N$ and $f \in \mathbf{S}(B,S^M)$ be given. By increasing M if necessary we can assume that f has an extension (not to be distinguished notationally) $f \in \mathbf{S}(S^N,S^M)$. As in the proof of Proposition 5.2 we take standard imbeddings $i(S^N)$ and $j(S^M)$ in S^{N+M+1} that are orthogonal to one-another and represent a point $z \in S^{N+M+1}$ as

$$(6.1) \qquad z = x\cos\theta + y\sin\theta$$

with $x \in i(S^N)$, $y \in j(S^M)$, $0 \leq \theta \leq \pi/2$. Consider now the sets $B_\theta \in \mathbf{B}_{N+M+1}$ defined by

$$(6.2) \qquad B_\theta = \{x\cos\theta + f(x)\sin\theta : x \in i(B)\}, \qquad 0 \leq \theta \leq \pi/2.$$

It is easy to see that for any two values $\theta,\theta' \in (0,\pi/2)$ there is an $h \in H_{N+M+1}$ such that $h(B_\theta) = h(B_{\theta'})$ and thus $\tau(B_\theta)$ is independent of θ on $(0,\pi/2)$. Moreover, if U is any neighborhood of $B_{\pi/2}$ then there is a $\delta > 0$ such that $B_\theta \subseteq U$ for $0 < \pi/2 - \theta < \delta$ and thus $\tau(B_\theta) \leq \tau(B_{\pi/2})$ for $0 < \theta < \pi/2$.

Next we want to exhibit an $h \in H_{N+M+1}$ such that $h(B_0) = h(B_\delta)$ for some (small) $\delta > 0$. To this end we put

$$Y(x,y,\theta) = \|y\sin\theta + \alpha(\theta)f(x)\|^{-1}(y\sin\theta + \alpha(\theta)f(x)),$$

where the continuous function $\alpha(\theta)$ remains to be determined, take

$$\mu(x,y,\theta) = \sin^{-1}(\|y\sin\theta + \alpha(\theta)f(x)\|),$$

and then define

$$h(z) = x\cos\mu + Y\sin\mu,$$

for z given by (6.1) and with $Y = Y(x,y,\theta)$, $\mu = \mu(x,y,\theta)$. We take $\alpha(\theta)$ to be defined on $[0,\pi/2]$ and to be non-negative, identically zero except in the neighborhood of $\theta = 0$ and to satisfy

$$|\alpha(\theta) - \alpha(\theta')| < |\sin\theta - \sin\theta'|,$$

for $\theta,\theta' \in [0,\pi/2]$ unless $\theta = \theta'$. It can then be verified that h is continuous and injective and thus a homeomorphism of S^{N+M+1} onto itself. We thus have

$$h(B_0) = h(B_{\alpha(0)}),$$

and since we can take $\alpha(0) > 0$ we conclude that

$$\tau(B_0) = \tau(B_{\alpha(0)}) \leq \tau(B_{\pi/2}).$$

However since τ is injection-stable we have

$$\tau(B) = \tau(B_0) \quad \text{and} \quad \tau(f(B)) = \tau(B_{\pi/2}).$$

and thus

$$\tau(B) \leq \tau(f(B)).$$

It follows that τ is topological.

Remark. The contraction used in the above proof was suggested by the so-called mapping cylinder, [11].

We remarked in section 3 that the non-vacuity of T implies the Borsuk-Ulam theorem, we are now in a position to demonstrate this. The following proposition is independent of the Borsuk-Ulam theorem.

Proposition 6.2. If T is non-empty then so is T_t.

Proof. Let $\tau \in T$. If τ is injection-stable then by Theorem 6.1 it is topological and there is nothing to prove. Otherwise,

$$\lim_{n\to\infty} \sup\{i^n\tau, i^{n+1}\tau, i^{n+2}\tau, \ldots\}, \quad \lim_{n\to\infty} \inf\{i^n\tau, i^{n+1}\tau, i^{n+2}\tau, \ldots\},$$

are injection-stable and hence topological.

Corollary 6.3. The non-vacuity of T implies the Borsuk-Ulam Theorem.

7. SUSPENSION.

The Freudenthal suspension, [11], $\sigma: B_n \to B_{n+1}$ determines a map $\sigma: T \to T$ by $(\sigma\tau)(B) = \tau(\sigma B) - 1$, $B \in B_n$; we say τ is suspension-stable if $\sigma\tau = \tau$.

The suspension σ can be represented as follows. Let $i:S^N \mapsto S^{N+1}$ be a standard imbedding and let $y_0 \in S^{N+1}$ be orthogonal to $i(S^N)$. The general element $z \in S^{N+1}$ then has the representation

$$z = x\cos\theta + y_0\sin\theta, \qquad x \in i(S^N), \quad |\theta| \leq \pi/2. \tag{7.1}$$

and for $B \in \mathbf{B}_N$,

$$\sigma B = \{x\cos\theta + y_0\sin\theta : x \in i(B),\ |\theta| \leq \pi/2\}.$$

If $B \in \mathbf{B}_N$, $C \in \mathbf{B}_M$ and $f \in \mathbf{S}(B,C)$ are given then f has a suspension $\sigma f \in \mathbf{S}(\sigma B, \sigma C)$. For $z \in i(B)$ and of the form (7.1)

$$(\sigma f)(z) = i'f(i^{-1}(x))\cos\theta + y_0'\sin\theta,$$

here i' is a standard imbedding of S^M in S^{M+1}, and $y_0' \in S^{M+1}$ is orthogonal to $i'(S^M)$. If $h \in H_N$ then $\sigma h \in H_{N+1}$; this latter fact is needed in the verification that the suspension of a type is a type.

<u>Proposition 7.1</u> <u>For</u> $\tau \in \mathbf{T}$ <u>we</u> <u>have</u>

$$\sigma(\tau^*) = (i\tau)^*.$$

<u>Proof</u>. First we note that if $B, C \in \mathbf{B}_N$ with $B \cap C = \phi$ then

$$\tau^*(\sigma B) + \tau(i(C)) \leq N + 2,$$

which implies

$$(7.2) \qquad (\sigma(\tau^*))(B) + (i\tau)(C) \leq N + 1, \quad \text{for } B,C \in \mathbf{B}_N \text{ with } B \cap C = \phi.$$

If $B \in \mathbf{B}_N$ is given then, by Lemma 5.1, there is a $C' \in \mathbf{B}_{N+1}$ such that $(\sigma B) \cap C' = \phi$ and

$$(7.4) \qquad \tau^*(\sigma B) + \tau(C') = N + 2.$$

We can assume that C' is of the form

$$(7.5) \qquad C' = \{x\cos\theta + y_0 \sin\theta : x \in C_1, \ |\theta| \leq \pi/2 - \delta\}$$

where $0 < \delta < \pi/2$ and $C_1 \subseteq i(S^N)$. We then have $C_1 \subseteq C'$ and thus

$$\tau(C') = \tau(C_1),$$

(since given any $U \in \mathbf{U}(C_1)$ there is an $h \in H_{N+1}$ such that $h(C') \subseteq U$). If we take $C'' = i^{-1}(C_1) \in \mathbf{B}_N$ then $B \cap C'' = \phi$ and

$$(\sigma(\tau^*))(B) + (i\tau)(C'') = N + 1.$$

Combining this with (7.2) yields

$$\max\{(i\tau(C)) : C \in B_N, \ B \cap C = \phi\} = N + 1 - (\sigma(\tau^*))(B),$$

and the assertion follows from Lemma 5.1.

Proposition 7.1 implies the following result.

Theorem 7.2. _Let $\tau \in \mathbf{T}$, then τ^* is suspension-stable if and only if τ is topological_.

If $B \in \mathbf{B}_N$ and $\mathbf{S}(B, S^{n-1}) \neq \phi$ then clearly $\mathbf{S}(\sigma B . S^n)$ is also non-empty since the suspension of a mapping in the former set belongs to the latter. Thus we see that

$$\sigma\tau_3 \leq \tau_3, \tag{7.6}$$

and similarly,

$$\sigma\tau_4 \geq \tau_4, \tag{7.7}$$

it follows from results of [4] however that equality holds in neither case. As in [4] we can form the stabilizations $\Sigma\tau_3$, $\Sigma\tau_4$ as follows,

$$\Sigma\tau_3(B) = \lim_{n\to\infty} \sigma^n\tau_3(B), \quad \Sigma\tau_4(B) = \lim_{n\to\infty} \sigma^n\tau_4(B), \quad B \in \mathbf{B}_N;$$

the duality (7.8) was also noted in [4].

Proposition 7.3 _The types $\Sigma\tau_3$ and $\Sigma\tau_4$ are respectively the greatest and least elements in the set $\mathbf{T}_t \cap (\mathbf{T}_t)^*$. Moreover_

$$\Sigma\tau_4 = (\Sigma\tau_3)^*. \tag{7.8}$$

Proof. It is clear that σ is order-preserving on $\mathbf{T}$ and also that $\sigma(\mathbf{T}_t) \subseteq \mathbf{T}_t$. Thus if $\tau' \in \mathbf{T}_t \cap (\mathbf{T}_t)^*$ then by (4.4)

$$\sigma^n\tau^4 \le \tau' \le \sigma^n\tau_3, \qquad \text{for } n = 1,2,\ldots,$$

and thus the first assertion follows. Given that, (7.8) follows readily from Proposition 5.4.

8. MORE ON DUALITY. In this section we consider various properties which a relative type might possess and establish what the duals of these properties are. We understand by the dual (P*) of a property (P) a property that is possessed by τ^* if and only if τ has (P); as shown in section 7, suspension stability and membership in $\mathbf{T}_t$ are dual properties. In this and the next section we formulate several properties and their duals.

First there is the subadditive property:

(S) $$\tau(B_1 \cup B_2) \le \tau(B_1)+\tau(B_2), \quad \text{for } B_1, B_2 \in \mathbf{B}_N,$$

whose dual is the *intersection property*:

(S*) $$\tau(B_1 \cap B_2) \ge \tau(B_1)+\tau(B_2)-N-1, \quad \text{for } B_1, B_2 \in \mathbf{B}_N.$$

Proposition 8.1 *If* τ *is subadditive then* $\tau \ge \tau^*$.

If τ *and* τ' *are subadditive so is* $\tau \vee \tau'$ *and* $\tau \wedge \tau'$ *has the intersection property if both* τ *and* τ' *do*.

The following property is self-dual.

(F) *If* $B, Q \in \mathbf{B}_N$, Q *is finite and* $\tau(C) < \tau(B)$ *whenever* $C \in \mathbf{B}_N$ *and* $C \subseteq B \setminus Q$ *then contained in every open set* U *with* $B\setminus Q \subseteq U$ *there is a* $D \in \mathbf{B}_N$ *with* $\tau(D) = \tau(B)-1$.

Proposition 8.2 Property (S) implies property (F) and if τ is topological and satisfies (F) then $\sigma\tau \leq \tau$.

9. COMPLEXES. Let P be a symmetric triangulation of S^N, we will denote the set of balanced subcomplexes of P by $\mathbf{B}_N(P)$.
A triangulation is to be understood here in the sense of [11,p. 113]. We will make an additional assumption concerning the triangulations under consideration. We assume that each simplex of P is the radial projection on S^N of the convex hull (in R^{N+1}) of its vertices (so that 0 is implicitly assumed not to belong to that convex hull). Alternatively we may deal directly with the polyhedron whose simplices are the convex hulls of the vertices of the simplices of P; the radial projection of this polyhedron onto S^N is a homeomorphism. In particular we shall consider the formation of convex combinations within the simplices of P to be well-defined.

By the dimension of a P-complex K (i.e. a subcomplex K of a triangulation P of S^N) we understand the maximum of the dimensions of the simplices of K. We will denote the k^{th} baricentric subdivision of P by $P^{(k)}$. The relative interior of a simplex σ will be denoted by $\overset{\circ}{\sigma}$.

Lemma 9.1. Let P be a symmetric triangulation of S^N, and let K be a subcomplex of P. Then there exists a unique maximal subcomplex K^* of $P^{(1)}$ such that $K \cap K^* = \phi$. For any $\tau \in \mathbf{T}$,

$$\tau(K) + \tau^*(K^*) = N + 1. \tag{9.1}$$

If an (N-n+1)-simplex of K^* has non-empty intersection with a simplex σ of P then it has non-empty intersection with a face of σ of dimension $\leq$ n-1.

Proof. The complex K^* can be characterized as follows. If $\sigma_1, \sigma_2, \ldots, \sigma_n$ are simplices of P such that for $k = 1, \ldots, n-1$, σ_k is a proper face of σ_{k+1} and σ_1 is not a simplex of K then the baricenters of the $\{\sigma_k\}$ are the vertices of an (n-1)-simplex of K^*, conversely any simplex of K^* has such a representation. Except for (9.1) the assertions of the lemma obviously follow from this characterization.

In order to prove (9.1) we first observe that any point x of the polyhedron P can be uniquely represented in the form

$$x = ty + (1-t)z, \qquad t \in [0,1], \quad y \in K, \; z \in K^*,$$

with y and z belonging to the lowest dimensional simplex of $P^{(1)}$ that contains x. It follows that if B is a compact set, V is open and $B \subseteq S^N \backslash K$, $K^* \subseteq V$ then there is an $h \in H_N$ which leaves K and K^* pointwise fixed and maps B into V, i.e. for which $h(B) \subseteq V$; a similar statement holds with the roles of K and K^* reversed. The assertion (9.1) clearly follows.

Proposition 9.2. Let P be a symmetric triangulation of S^N. Let $K \in \mathbf{B}_N(P)$ and let $\dim K = n$. Then for any $\tau \in \mathbf{T}$, $\tau(K) \leq n+1$. If K is the n-skeleton of P, i.e. the union of all the n-simplices of P then for any type τ, $\tau(K) = n+1$.

Proof. Let the symmetric triangulation P and $\tau \in \mathbf{T}$ be given. We first prove that if K is the n-skeleton of P then $\tau(K) \leq n+1$. It follows from elementary linear algebra that there is an (N-n)-dimensional hyperplane through the origin in R^{N+1} that does not intersect any of the n-simplices of P. In other words there is a standard imbedding of S^{N-n-1} in S^N that does not intersect the n-skeleton of P; the asserted inequality follows.

Now suppose that K is as above, i.e. the n-skeleton of P, then the complex K* is a complex of dimension N-n. From what we have just proved we must then have $\tau^*(K^*) \leq N-n+1$. It follows then from (9.1) that $\tau(K) \geq n+1$.

We return to the discussion of properties of types and their duals. We consider the <u>local dimension property.</u>'

(D) <u>lejt</u> K € BjyfP) <u>with</u> r(K) = n, <u>if</u> a <u>is a simplex of</u> K <u>of dimension</u> < n-1 and σ i£ <u>not a proper face of any simplex of K</u> t<u>hen r</u>(K\(oU-cr)) = T(K),

whose dual is:

(D*) <u>if</u>. K € B^P) <u>is</u>. <u>a simplicial complex with</u> T(K) = n <u>and</u> K^"1"1^ <u>denotes the</u> (m-1)-<u>skeleton</u> <u>o</u>£ K <u>then</u> r(K^m -1)) = m f<u>or m</u> = 1,2. . . . ,n.

<u>Proof</u> g£ <u>the duality</u> o<u>f.</u> (D) <u>and</u> (D*). We first show that *r* has property (D) if r* has property (D*). Let K € B_N,(P) and let a be a simplex of K of dimension < n-1 which is not a proper face of a simplex of K. Let

$$K' = K\backslash(\mathring{o}U\text{-}\mathring{a}).$$

We shall show that if m = r(K') < n then T(K) < n. By (9.2) the complex K'^* has $\tau(K'^*)$ = N-m+1. The assumption of (D*) for τ^* implies that the (N-n+1)-skeleton K" of K'* has T*(K") = N-n+2. If K" fl K = <*p* we are finished. Otherwise K" PI K <u>C</u> ($\mathring{o}$U-$\mathring{a}$) and since dim *a* < n-1 there is an h 6 L_N which agrees with the identity except on a neighborhood of

$K'' \cap K$, such that $K'' \cap h(K) = \phi$. This yields $\tau(K) = \tau(h(K)) < n$ as was to be proved.

Suppose now that τ has property (D). Let $K \in \mathbf{B}_N(P)$ with $\tau^*(K) = n$ and suppose that for some m, $1 \leq m \leq n$, $\tau^*(K^{(m-1)}) < m$. Form the complex $K'_m \in \mathbf{B}_N(P^{(1)})$ by deleting from $K^{(m-1)}*$ (the relative interior of) every simplex of dimension $< N-m+1$ that is not the face of an $(N-m+1)$-simplex of $K*$. In view of our assumption concerning $K^{(m-1)}$ it follows from (9.1) that $\tau(K^{(m-1)})*) > N-m+1$ and thus since τ has property (D), $\tau(K'_m) > N-m+1$. It follows from the last assertion of Lemma 9.1 and the construction of K'_m that if K'_m has non-empty intersection with K then it also has non-empty intersection with $K^{(m-1)}$ which is a contradiction. Thus we must have $\tau(K^{(m-1)}*) = N-m+1$ and $\tau^*(K^{(m-1)}) = m$.

<u>Corollary 9.3</u> <u>Suppose that</u> $\tau \in \mathbf{T}$ <u>has property</u> (D) <u>and let</u> $B \in \mathbf{B}_N$ <u>with</u> $\tau(B) = n$. <u>Let</u> $x \in B$ <u>have a neighborhood</u> V <u>such that</u> $V \cap B$ <u>is a</u> C^2-<u>hypersurface of dimension</u> $< n-1$. <u>Then for any sufficiently small neighborhood</u> V′ <u>of</u> x, $\tau(B\backslash(V' \cup -V') = n$.

<u>Proof</u>. The proof is similar to the first part of that of the duality assertion above.

<u>Lemma 9.4</u> <u>The properties</u> (D) <u>and</u> (D*) <u>are preserved under injection and suspension</u>.

<u>Lemma 9.5</u>. <u>If</u> τ <u>has property</u> (F) <u>then can</u> (D^*) <u>be formulated</u>; <u>if</u> $K \in \mathbf{B}_N(P)$ <u>and</u> $\tau(K) = n$ <u>then</u> $\tau(K^{(n-1)}) = n$.

Proof. It suffices to show that, given (F), for any $K \in \mathbf{B}_N(P)$ we have

$$(9.2) \qquad \tau(K^{(m)}) \leq \tau(K^{(m-1)}) + 1, \qquad m = 1,\dots,N.$$

To verify this we apply the definition of (F) with $B = K^{(m)}$, Q the set that consists of the baricenters of the m-dimensional simplices of $K^{(m)}$ and with $U = S^N \setminus K^{(m-1)*}$. Using the homeomorphism that occurs in the proof of Lemma 9.1 we see that if $C \in \mathbf{B}_N$ and $C \subseteq U$ then $\tau(C) \leq \tau(K^{(m-1)})$. The inequality (9.2) readily follows.

10. PROPERTIES OF PARTICULAR TYPES. We next consider some of the particular types introduced earlier as well as some new ones and indicate which of the above properties they possess or don't possess.

In [4] Conner and Floyd introduced a cohomology co-index, in fact, given any principal ideal domain L, their construction gives rise to a function which, after an additive renormalization i.e. addition of 1, becomes a topological type. The types that result in this way are all subadditive. We will be interested in particular in those that result when L = Z and when L = Z_2; we shall denote these by τ_5 and τ_6 respectively. We list several of the fundamental properties of τ_5 and τ_6 that are proved in [4]; these results are reformulated in the terminology and with the normalization used here.

Proposition 10.1 *The topological type* τ_5 (which results from adding 1 to the Z-cohomology co-index defined in [4]) *is subadditive and suspension-stable. If* $K \in \mathbf{B}_N(P)$, *where* P *is some triangulation of* S^N, *and* $\tau_3(K) = 3$ *then* $\tau_5(K) = 3$.

The topological type τ_6 (which results from adding 1 to the Z_2-cohomology co-index) is subadditive, suspension-stable and self-dual, i.e. $\tau_6^* = \tau_6$.

See [4], pp. 426–432, specifically p. 430 for the equality $\tau_3(K) = \tau_5(K)$ when $\tau_3(K) = 3$, see also p. 433.

Remark. τ_6 (with the topologist's normalization) was defined and its fundamental properties determined already by C.-T. Yang, [12], [14].

Proposition 10.2. The type τ_6 has property (D), hence also (D*).

See [3] for a proof.

We next list some results from [4] concerning τ_3 and τ_4.

Proposition 10.3. Neither τ_3 nor τ_4 is suspension-stable. However if $K \in \mathbf{B}_N$ is a balanced complex with $\tau_3(K) = 1+\dim K$ then $(\sigma\tau_3)(K) = \tau_3(K)$.

See [4], pp. 433–434; note that a finite simplicial complex with fixed point free simplicial involution can always be assumed to be a subcomplex of some (triangulated) S^N. A simple example showing that τ_4 is not suspension stable follows from consideration of the set $\tilde{B}$ considered in the remarks at the end of section 3. It can be shown that there does not exist an odd continuous map of S^2 into the complement of that set, consequently, $\tau_4^*(\tilde{B}) = 2$ and hence τ_4^* is not topological and τ_4 is not suspension-stable.

Corollary 10.4 The type τ_3 (Krasnosel'skii genus) has properties (S) and (D) but does not have property (D*); τ_4 does not have property (F).

Proof. For proofs that τ_3 has (S) and (D) see [10] and [3] respectively..

To see that τ_3 does not have property (D*) observe that by Proposition

10.3 there are complexes K such that $(\sigma\tau_3)(K) < \tau_3(K)$, while if $n = \tau_3(K) = \tau_3(K^{(n-1)})$ then $(\sigma\tau_3)(K) = (\sigma\tau_3)(K^{(n-1)}) = n$. Thus there must be complexes K with $n = \tau_3(K) > \tau_3(K^{(n-1)})$, i.e. τ_3 fails to have property (D*).

By Proposition 8.2 a topological type τ with property (F) must have $\sigma\tau \leq \tau$. Thus in view of (7.7) τ_4 would have to be suspension-stable if it had property (F). We conclude therefore from Proposition 10.3 that τ_4 does not have property (F).

<u>Proposition 10.5</u> τ_4 <u>has</u> (D*) <u>but</u> <u>neither</u> (S) <u>nor</u> (S*). τ_5 <u>does</u> <u>not</u> <u>have</u> (S*) (and thus is not self-dual). $\Sigma\tau_3$ <u>has</u> (D) <u>and</u> $\Sigma\tau_4$ <u>has</u> (D*).

<u>Proof</u>. For a balanced complex K, $\tau_4(K)$ can be defined as in [4] by

$$\tau_4(K) = \max\{n : S(S^{n-1}, K) \neq \phi\},$$

since an element of the set $S(S^{n-1}, K)$ can be assumed to be simplicial with respect to some symmetric triangulation of S^{n-1} it readily follows that τ_4 has (D*). Since (F) is self-dual either (S) or (S*) implies (F), thus τ_4 can have neither.

To prove the assertion concerning τ_5 we note that there is an example due to Yang, [13], of a set which can be realized as a complex K_0 in S^4 and such that $\tau_3(K_0) = 3$ while $\tau_6(K_0) = 2$. We have then $3 = \tau_6(K_0*) \leq \tau_3(K_0*)$ and thus the existence of a balanced subcomplex K' of K_0* such that $K' \cap K_0 = \phi$ and $\tau_3(K') = 3$. By Proposition 10.1, $\tau_5(K_0) = \tau_5(K') = 3$ and thus τ_5 does not have (S*).

Finally, since τ_3 has (D) it follows from Lemma 9.4 that $\Sigma\tau_3$ has (D) as well; the assertion concerning $\Sigma\tau_4$ then follows from the duality (7.8).

Remark. It is clear from the above results that $\tau_3, \tau_4, \tau_5, \tau_6$ are all distinct. We know (Proposition 7.3) that $\tau_5 \leq \Sigma\tau_3$ however we do not have a proof that the latter two types are distinct.

11. CRITICAL POINTS. A_N denotes the set of even C^2 functions on S^N with only isolated critical points; A_N^∞ denotes the subset consisting of C^∞ functions with just one pair of antipodal critical points, and those non-degenerate, on each critical level. We observe that C^k even functions on S^N can be approximated uniformly, along with their derivatives up to order k, by functions in A_N^∞, [9], see also [3]. For $a \in A_N$ let

$$(11.1) \qquad \mu_n = \min\{\max_{x \in B} a(x) : B \in \mathbf{B}_N, \tau(B) \geq n\}, \qquad n = 1,2,\ldots,N+1,$$

$$(11.2) \qquad \nu_n = \max\{\min_{x \in B} a(x) : B \in \mathbf{B}_N, \tau(B) \geq n\}, \qquad n = 1,2,\ldots,N+1;$$

$\{\mu_n\}$ and $\{\nu_n\}$ will be called the τ-min-max and τ-max-min values of a. It is immediate from the definitions (11.1) and (11.2) that

$$(11.3) \qquad \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N+1},$$

and

$$(11.4) \qquad \nu_1 \geq \nu_2 \geq \cdots \geq \nu_{N+1} .$$

Theorem 11.1. *Let $\tau \in \mathbf{T}$ and let $a \in A_N$ then the τ-min-max and τ^*-max-min values of a are critical values of a and are related by*

$$(11.5) \qquad \mu_n = \nu^*_{N-n+2}, \qquad n = 1,\ldots,N+1.$$

Proof. The fact that (11.1) and (11.2) define critical values of a follows from standard results as are found e.g. in [10], although the theorem proved there is for a particular type it is quite clear that the same proof applies for any type τ. The proof makes use of the existence, which we shall require below, of a "push-down" operator for a, i.e. a $g \in H_N$ such that $a(g(x)) \leq a(x)$ with equality only if x is a critical point of a in which case $g(x) = x$.

To prove (11.5) suppose that $B \in \mathbf{B}_N$, $\tau(B) \geq n$ and $\max_{x \in B} a(x) = \mu_n$, then for any $C \in \mathbf{B}_N$ with $\tau^*(C) = N - n + 2$ we have $B \cap C \neq \phi$ and thus $\upsilon_{N-n+2} \leq \min_{x \in B} a(x) \leq \mu_n$. On the other hand, if $n > 0$ then $\tau\{x : a(x) < \mu_n - \epsilon\} = m < n$ and there exists C' with $\tau^*(C') = N - m + 1 \geq N - n + 2$ with $\min_{x \in C'} a(x) \geq \mu_n - \epsilon$ and thus $\mu_n - \epsilon \leq \upsilon^*_{N-m+1} \leq \upsilon^*_{N-n+2} \leq \mu_n$. Since $\epsilon > 0$ is arbitrary (11.5) follows.

Theorem 11.2. *Let $\tau, \tau' \in T$. Then $\tau \leq \tau'$ if and only if for every N and every $a \in A_N$ the τ-min-max values $\{\mu_n\}$ and the τ'-min-max values $\{\mu'_n\}$ of a satisfy $\mu_n \geq \mu'_n$, for $n = 1, 2, \ldots, N+1$.*

Proof. The "only if" part is obvious. To prove the converse suppose that for some N there is a set $B \in \mathbf{B}_N$ such that $n = \tau(B) > \tau'(B)$. Then one can construct an $a \in A_N$ with τ-min-max value μ_n and $\tau(\{x : a(x) \leq \mu_n\}) < n$ (see the example in section 4 of [3]) so that $\mu_n < \mu'_n$.

Theorem 11.3. *If $\tau \in \mathbf{T}$ has property (F) then it has property (D) if and only if for every N and every $a \in A^{\infty}_N$ the (two non-degenerate) critical points on the level $\{x : a(x) = \mu_n\}$ have Morse index $\geq n-1$ for $n = 1, 2, \ldots, N+1$.*

<u>Proof</u>. The sufficiency of (D) for the indicated Morse index inequality was proved in [3], (we must take into account Corollary 9.3).

To verify the necessity, suppose that τ does not have property (D). Then τ^* fails to have (D^*) so in view of Lemma 9.5 there exist integers N,n, $0 < n < N$, and for some symmetric triangulation P of S^N a complex $K \in \mathbf{B}_N(P)$ such that $\tau^*(K) = n$ while $\tau^*(K^{(n-1)}) < n$. We can assume further that there is a simplex σ of K of dimension $m \geq n$ such that $\tau^*(K') < n$, where $K' = K \setminus (\mathring{\sigma} \cup -\mathring{\sigma})$. Let $C \in \mathbf{B}_N$ be such that $K' \subseteq \mathrm{int}(C)$ and $\tau^*(C) = \tau^*(K')$. Let $a_0 \in A_N$ be chosen so that $a_0(x) > 0$ for $x \in K$, $a_0(x) \geq 0$ on σ and $a_0(x) < 0$ for $x \notin C \cup \sigma$. Next let $\{y\}$ be a local coordinate system centered at some point p in the relative interior of σ and let (Ay,y) be a non-degenerate quadratic form of index $N - m$ which is non-negative on σ. Let V,V' be a neighborhoods of p such that $V' \subseteq V \subseteq C$ and $V \cap K \subseteq \mathring{\sigma}$. Construct a smooth even function a that agrees with a_0 outside of V and with (Ay,y) in V', has $a(x) < 0$ for $x \notin C \cup \sigma$ and $a(x) \geq 0$ for $x \in K$ with the equality holding on K only at p,-p. Finally, after another alteration which does not affect a on V' and which we do not distinguish notationally, we can assume that in addition $a \in A_N^\infty$.

Next we show that $\nu_n^* = 0$. Indeed if $B \in \mathbf{B}_N$ and $\min_{x \in B} a(x) > 0$ then $B \subseteq (C \cup \sigma) \setminus \{p,-p\}$ and thus $\tau^*(B) < n$; consequently $\nu_n = 0$ as asserted. The Morse index of the corresponding critical points p,-p agrees with the index of (Ay,y), i.e. $N - m$. By Theorem 11.1 the $(N - n - 2)^{\mathrm{nd}}$ τ-min-max value $\mu_{N-n-2} = 0$ and since the corresponding critical points p,-p have Morse index $N - m \leq N - n < (N - n + 2) - 1$ the necessity is proved.

<u>Theorem 11.4</u>. <u>Let</u> $\tau \in \mathbf{T}$. <u>Then</u> τ <u>has property</u> (F) <u>if and only if for every</u> N <u>and every</u> $a \in A_N$ <u>the</u> τ<u>-min-max values of</u> a <u>satisfy</u> $\mu_n < \mu_{n+1}$ <u>for</u> $n = 1,\ldots,N$.

<u>Proof</u>. First we suppose that τ is given and has property (F) and show that the τ-min-max values must all be distinct for any $a \in A_N$. Let a and n be given and let g be a "push-down" operator for a. Consider the set $U = \{x: a(g(x)) < \mu_{n+1}\}$. U is open and $U \subseteq \{x: a(x) \leq \mu_{n+1}\}\backslash Q$ where Q denotes the set of critical points of a on the level $a(x) = \mu_{n+1}$ (we assume, as we may, that $\tau(\{x: a(x) \leq \mu_{n+1}\}) = n+1$). It follows that there is a set $B \in \mathbf{B}_N$ with $B \subseteq U$ and $\tau(B) = n$. If $B_1 = g(B)$ then $\tau(B_1) = n$ and $\max\{a(x): x \in B_1\} < \mu_{n+1}$ from which it follows that $\mu_n < \mu_{n+1}$.

Now suppose that τ fails to have property (F). Specifically, suppose for some N there exist $B, Q \in \mathbf{B}_N$ with Q finite, $Q \leq B$, $\tau(B) = n+1$ and there exists a balanced open set U such that $U \supseteq B\backslash Q$ and $\tau(C) < n$ for any $C \in \mathbf{B}_N$ with $C \subseteq U$; we can and shall assume $Q \cap U = \phi$.
We proceed to construct an $a \in A_N$ for which $\mu_n = \mu_{n+1}$. We do the construction first under the assumption that B is "locally homogeneous" at points of Q. That is to say we assume that at each point $q \in Q$ there is a neighborhood V of q, a local Euclidean coordinate system $\{y\}$ on $V = V_q$ with origin at q and a real number $r = r_q$ such that a point y of V with $\|y\| < r$ belongs to B if and only if $ty \in B$ for $0 < t < r/\|y\|$. Let W be an open set with $Q \subseteq W$ and such that for each q, when represented in terms of the given local coordinates, $W \cap V_q$ is contained in the ball of radius $r_q/2$. We can construct a function $a_0 \in A_N^\infty$ such that: $a_0 < 0$ on $B\backslash W$, $a_0 > 0$ on $S^N\backslash(U \cup W)$ and a_0 has no critical points on the level $a_0(x) = 0$. Given $q \in Q$ we now alter a_0 on V_q as follows. Let $r' = r_q' = 3r_q/4$ and let $\theta \in C^\infty(\mathbf{R})$ be chosen with $\theta(t) = 0$ for $t < 0$, $\theta(t) = 1$ for $t > r'$ and $\theta'(t) > 0$ for $0 < t < r'$. In terms of the local coordinates on V_q we put

$$a(y) = \theta(\|y\|)a_0(r'y/\|y\|), \qquad \text{for } \|y\| \leq r'. \tag{11.6}$$

We can assume that a_0 has no critical points on $\|y\| = r'$ and that where it vanishes on that sphere the gradient is not normal to the sphere. The function defined locally by (11.6) then has no critical points in the ball

$$(11.7) \qquad \beta_q = \{y : \|y\| \leq r'\},$$

($r' = r_q'$) except at $y = 0$, i.e. $x = q$. The function that results when a_0 is (symmetrically) so altered in each V_q will be denoted by a; clearly $a \in A_N$. From the construction we have $a(x) < 0$ for $x \in B \setminus Q$ and $a(q) = 0$ for $q \in Q$.

Suppose that $C \in \mathbf{B}_N$ and $a(x) \leq 0$ on C. Clearly then we must have $C \subseteq U \cup W$ and if $C \cap Q \neq \phi$ then $\max\{a(x) : x \in C\} = 0$. If on the other hand $C \cap Q = \phi$ then there is an $h \in H_N$ which agrees with the identity outside of $\cup\{\beta_q : q \in Q\}$, is a q-centered radial pushback in each of these balls and for which we have $h(C) \subseteq U$, (we have used here the fact that $a(x) \leq 0$ on C). In the latter case then we must have $\tau(C) < n$. It follows that $\mu_n = \mu_{n+1} = 0$.

Finally we justify the "local homogeniety" assumption concerning B. Suppose that, except for the local homogeniety of B at points of Q, N and the sets B,Q and U are otherwise as above. For simplicity we can assume that Q is a doubleton $\{q,-q\}$. Let V_q and the coordinate system $\{y\}$ be as above (in particular the origin of the latter is at q). It is easily seen that we can assume that there are positive real numbers r,r' such that

(*) <u>for</u> $r < \|y\| < r'$, $y \in B$ <u>iff</u> $ty \in B$ <u>for</u> $r < \|ty\| < r'$.

We can assume that (*) holds also for U. Let $B' \in \mathbf{B}_N$ be the set that is determined by the conditions: $B' \setminus (\beta_q \cup \beta_{-q}) = B \setminus (\beta_q \cup \beta_{-q})$ and a point $y \in \beta_q$ with $y \neq 0$ belongs to B' if and only if $r'y/\|y\| \in B'$; let the balanced open set U' with $\{q,-q\} \notin U'$ be constructed similarly. Choose $W \in \mathbf{B}_N$ to contain a neighborhood of $\{q,-q\}$ and so that $\tau(B') = \tau(B' \cup W)$. There is an $h \in H_N$ that agrees with the identity outside of $\beta_q \cup \beta_{-q}$ and is a q-centered radial pushdown in β_q with $h(B) \subseteq B' \cup W$ so that $\tau(B') = \tau(B' \cup W) \geq \tau(B) = n+1$. The set B' clearly has the local homogeniety property assumed in the above construction, it remains only to show that for any $C \in \mathbf{B}_N$ with $C \subseteq U'$ we have $\tau(C) < n$. But if such a C is given then there is a $h \in H_N$ which agrees with the identity outside of $\beta_q \cup \beta_{-q}$, is a q-centered radial pushback in each of these balls and for which we have $h(C) \subseteq U$, and thus $\tau(C) < n$.

12. <u>ESSENTIAL CRITICAL VALUES</u>. In this section we characterize those critical values of a function $a \in A_N$ which are τ-min-max values of a for some $\tau \in \mathbf{T}$ or $\mathbf{T}_t$.

<u>Definition 12.1</u>. A critical value μ of $a \in A_N$ will be called <u>[topologically] essential</u> if there does not exist an $h \in H_N$ [$h \in S(\{x:a(x) \leq \mu\}, \{x:a(x) \leq \mu\})$] <u>with</u> $h(\{x:a(x) \leq \mu\}) \subseteq \{x:a(x) < \mu\}$.

<u>Theorem 12.2</u>. <u>A critical value</u> μ <u>of</u> $a \in A_N$ <u>is [topologically] essential if and only if it is a</u> τ<u>-min-max value of</u> a <u>for some</u> $\tau \in T$ [T_t].

<u>Proof</u>. The "if" assertion is obvious. We prove the "only if" statement for the case of an essential critical value. Let N and $a \in A_N$ be given and let $\{\mu^1\}$ and $\{\mu^2\}$ denote respectively the τ_1 and τ_2-min-max values of a. Assume that μ is an essential critical value of a that does not coincide with any of the τ_1 or τ_2-min-max values.
Let $B = \{x:a(x) \leq \mu\}$ and suppose that $\tau_1(B) = n$ so that $\mu'_n < \mu$. It follows

from (3.2) that given $n > 0$ there exists a standard imbedding i_n and an $h \in H_N$ such that $h(i_n(S^{n-1})) \subseteq \{x : a(x) < \mu^2_{n+n}\} = U_\epsilon$. Since $\tau_1(B) = n$ it follows from (3.1) that there exists an $h' \in H_N$ with $h'(B) \subseteq h^{-1}(U_\epsilon)$ and $(hh')(B) \subseteq \{x : a(x) < \mu^2_n + \epsilon\}$. Since μ is essential and ϵ is arbitrary it then follows that $\mu < \mu^2_n$ and $\tau_2(B) < n$. Put

$$I(B) = \{C \in \mathbf{B}_N : \tau_2(C) \leq n \text{ and } \forall\ U \in \mathbf{U}(C)\ \exists\ h \in H_N \text{ with } h(B) \subseteq U\},$$

and define $\tau \in \mathbf{T}$ by:

$$\tau(C) = n \text{ if } C \in I(B) \quad \text{and} \quad \tau(C) = \tau_2(C) \text{ otherwise.} \tag{12.1}$$

It is easy to verify that (12.1) defines a type. It is immediate from the essential property of μ that if $C \in I(B)$ then $\max\{a(x) : x \in C\} \geq \mu$, and this is also the case if $\tau_2(C) = n$. Thus

$$\mu = \min\{\max_{x \in C} a(x) : C \in \mathbf{B}_N, \tau(C) \geq n\}.$$

REFERENCES

1. A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14**,(1973),349–381.

2. A. Bahri, P.-L. Lions, Remarque sur la théorie variationelle des points critiques et applications, *C.R. Acad. Sci. Paris* **301**(1985), Ser. I, 145–147.

3. C. Coffman, Lyusternik-Schnirelman theory: complementary principles and the Morse index, *Nonlinear Analysis*, **12** (1988), 507–529.

4. P. Conner, E. Floyd, Fixed point free involutions and equivariant maps, *Bull. Amer. Math. Soc.*, **66**(1960),416–441.

5. E. Fadell, P. Rabinowitz, Bifurcation for odd potential operators and an alternative topological degree, *J. Funct. Anal.*, **26**(1977),48–67.

6. H.-P. Heinz, Un principe de maximum-minimum pour les valeurs critiques d'une fonctionelle non lineaire, *C.R. Acad. Sci. Paris* **275**(1972), Ser. A, 1317-1318.

7. M. Krasnosel'skii, An estimate of the number of critical points of functionals, *Uspehi Mat. Nauk* **7**,(1952) 157-164.

8. M. Krasnosel'skii, Topological Methods in the Theory of Nonlinear Integral Equations, Macmillan, New York, 1964.

9. A, Marino,G. Prodi, Metodi perturbative nella teori di Morse, *Boll. Un. Mat Ital.*, **11**,(1975) Supl. Fasc. 3, 1-32.

10. P. H. Rabinowitz, Some aspects of nonlinear eigenvalue problems, *Rocky Mountain J. Math.*, **3**,(1973) 161-202.

11. E. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.

12. C.-T. Yang, On the theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson, I, *Annals of Math.* **60**(1954), 262-282.

13. C.-T. Yang, On the theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson, II, *Annals of Math.* **62**(1955), 271-283.

14. C.-T. Yang, Continuous functions from spheres to Euclidean spaces, *Annals of Math.* **62**(1955), 284-292.