REGULARITY OF THE VALUE FUNCTION FOR A TWO-DIMENSIONAL SINGULAR STOCHASTIC CONTROL PROBLEM

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Research Report No. 88-26

August 1988

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*Work supported by the National Science Foundation under grant DMS-87-02537.

**Work supported by the Air Force Office of Scientific Research under grant AFQSR-85-0360.

Abstract

It is desired to control a two-dimensional Brownian motion by adding a (possibly singularly) continuous process to it so as to minimize an expected infinite-horizon discounted running cost. The Hamilton-Jacobi-Bellman characterization of the value function V is a variational inequality which has a unique twice continuously differentiable solution. The optimal control process is constructed by solving the Skorohod problem of reflecting the two-dimensional Brownian motion along a free boundary in the $-\nabla V$ direction.

Key words: Singular stochastic control, variational inequality, free boundary problem, Skorohod problem.

AMS subject classification: 93E20, 35R35

Abbreviated title: Regularity of the Value Function

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1. Introduction.

We study regularity of the solution of the variational inequality associated with a two-dimensional singular stochastic control problem with a convex running cost. The solution u of this variational inequality, which is the value function for the control problem, is shown to be of class C^2. We also study the regularity of the free boundary in $\mathbb{R}^2$ which divides the region where u satisfies a second order elliptic equation from the region where it does not. The free boundary is shown to be smooth, and this fact is instrumental in our construction of the optimal process for the stochastic control problem.

Previous work on the regularity of the value function in singular stochastic control has focussed on one-dimensional problems. Benes, Shepp & Witsenhausen (1980) suggested that the value function for these problems should be of class C^2 and used this so-called "principle of smooth fit" to determine some otherwise free parameters which arose in the solution of their problems. It has been used in the same way by Harrison (1985), Harrison & Taylor (1978), Harrison & Taksar (1983), Karatzas (1981, 1983), Lehoczky & Shreve (1986), Shreve, Lehoczky & Gaver (1984) and Taksar (1985). See also Chow, Menaldi & Robin (1985). An important question is whether the principle of smooth fit can be expected to apply to multi-dimensional singular control problems, or is it strictly a one-dimensional phenomenon. The paper Karatzas & Shreve (1986) suggests that it might apply in higher dimensions. These authors study the singular control of a one-dimensional Brownian motion under a constraint on the total variation of the control process (a "finite-fuel" constraint). The fuel remaining constitutes a second state variable, and the value function for this problem was found to be of class C^2 jointly in both state variables. One should observe, however, that the second state variable observe, however, that the second state variable in this problem is not a diffusion; indeed, the fuel remaining is constant until control is exercised, at which time it decreases an amount equal to the displacement caused by the control.

The present paper concerns the control of a two-dimensional Brownian motion, and control can cause displacement in any direction. Thus, the discovery of a C^2 value function provides strong support for belief in a widely applicable principle of smooth fit. Nonetheless, the argument of this paper depends heavily on the fact that only two dimensions are involved (see Remark 6.2), and we have not found a way to obtain a similar result in higher dimensions.

This paper is organized as follows. Section 2 defines the underlying stochastic control problem, and Section 3 relates it to a free boundary problem, the so-called Hamilton-Jacobi-Bellman (HJB) equation. Section 4 constructs a $C^{1,1}$, nonnegative convex solution u to the HJB equation and proves its uniqueness. Sections 5-10 upgrade the regularity of u to C^2. The key idea here is to use the gradient flow of u to change to a more convenient pair of coordinates. This is a generalization of the device used by many authors in one-dimensional problems of differentiating the Bellman equation so as to obtain a more standard free boundary problem. In Section 11 the free boundary is shown to be of class $C^{2,\alpha}$ for any $\alpha \in (0,1)$. In Section 12 we return to the stochastic control problem, which now reduces to the Skorohod problem of finding a Brownian motion reflected along the free boundary in the $-\nabla u$ direction. The established regularity of u and the free boundary allow us to assert the existence and uniqueness of a solution to the Skorohod problem and finally complete the proof begun in Section 3 that u is the value function for the stochastic control problem of Section 2.

2. The singular stochastic control problem

Let $\{W_t, \mathcal{F}_t;\ 0 \le t < \infty\}$ be a standard, two-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}_t\}$ be the augmentation of the filtration generated by W (see Karatzas & Shreve (1987, p. 89). The state process for our control problem is

$$(2.1) \qquad X_t \triangleq x + \sqrt{2}\ W_t + \int_0^t N_s\, d\zeta_s, \quad 0 \le t < \infty,$$

where $x \in \mathbb{R}^2$ is the initial condition and the control process pair $\{(N_t, \zeta_t);\ 0 \le t < \infty\}$ is $\{\mathcal{F}_t\}$-adapted and satisfies the conditions:

$$(2.2) \qquad |N_t| = 1, \quad \forall\ 0 \le t < \infty, \quad \text{a.s.},$$

where $|\cdot|$ denotes the Euclidean norm, and

(2.3) ζ is nondecreasing, left-continuous, and $\zeta_0 = 0$ a.s.

The process N gives the direction and ζ gives the intensity of the "push" applied by the controller to the state X.

Given control processes N and ζ, we define the corresponding cost

$$(2.4) \qquad V_{N,\zeta}(x) \triangleq E^x \int_0^\infty e^{-t}\,[h(X_t)dt + d\zeta_t],$$

where $h: R^2 \to \mathbb{R}$ is a strictly convex function satisfying, for appropriate positive constants C_0, c_0 and q:

(2.5) $$h \in C^{2,1}_{\ell oc}(\mathbb{R}^2),$$

(2.6) $$0 \leq h(x) \leq C_0(1 + |x|^q) \ \forall\, x \in \mathbb{R}^2,$$

(2.7) $$|\nabla h(x)| \leq C_0(1 + h(x)) \quad \forall\, x \in \mathbb{R}^2,$$

(2.8) $$c_0|y|^2 \leq D^2h(x)y \cdot y \leq C_0|y|^2(1 + h(x)) \ \forall\, x \in \mathbb{R}^2,\ y \in \mathbb{R}^2.$$

Without loss of generality, we also assume that

(2.9) $$0 = h(0) \leq h(x) \qquad \forall\, x \in \mathbb{R}^2.$$

For $x \in \mathbb{R}^2$, we define the value function

(2.10) $$V(x) \triangleq \inf_{N,\zeta} V_{N,\zeta}(x).$$

3. The Hamilton-Jacobi-Bellman equation

We shall show that the value function V of (2.10) is characterized by the Hamilton-Jacobi-Bellman (HJB) equation

$$\max\{u - \Delta u - h,\ |\nabla u|^2 - 1\} = 0. \tag{3.1}$$

The following theorem gives a partial description of the relationship between V and the HJB equation. More definitive results are proved in Section 12.

3.1 Theorem. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a convex, C^2 solution of (3.1). Then $u \le V$. For a given $x \in \mathbb{R}^2$, suppose there exists a control process pair (N,ζ) such that $V_{N,\zeta}(x) < \infty$ and the corresponding state process (2.1) satisfies

$$u(X_t) - \Delta u(X_t) - h(X_t) = 0 \quad \forall\ t \in (0,\infty), \text{ a.s.}, \tag{3.2}$$

$$\int_0^t 1_{\{N_s = -\nabla u(X_s)\}} d\zeta_s = \zeta_t \quad \forall\ t \in [0,\infty), \text{ a.s.}, \tag{3.3}$$

$$u(X_t) - u(X_{t+}) = \zeta_{t+} - \zeta_t \quad \forall\ t \in [0,\infty), \text{ a.s.} \tag{3.4}$$

Then

$$u(x) = V(x) = V_{N,\zeta}(x),$$

i.e., (N,ζ) is optimal at x.

Proof: Let $x \in \mathbb{R}^2$ and any control process pair (N,ζ) be given. Applying Itô's rule for semi-martingales (Meyer (1976), pp. 278, 301) to $e^{-t}u(X_t)$, adjusting the result to account for the fact that ζ is left-continuous rather than right-continuous, and observing that $|\nabla u| \le 1$ so $E\int_0^t e^{-s}\nabla u(X_s)dW_s = 0$, we obtain for $t \ge 0$:

$$(3.5) \qquad u(x) = E\, e^{-t}u(X_t) + E\int_0^t e^{-s}[u(X_s) - \Delta u(X_s) - h(X_s)]ds$$

$$+ E\int_0^t e^{-s}h(X_s)ds + E\int_0^t [-e^{-s}\nabla u(X_s) \cdot N_s]d\zeta_s$$

$$+ E \sum_{0 \le s < t} e^{-s}[u(X_s) - u(X_{s+}) + \nabla u(X_s) \cdot N_s(\zeta_{s+} - \zeta_s)].$$

The second and fifth terms on the right-hand side of (3.5) are nonpositive because of (3.1) and the convexity of u, respectively. Because $|\nabla u| \le 1$, the fourth term is dominated by $E\int_0^t e^{-s}d\zeta_s$, and thus we have

$$(3.6) \qquad u(x) \le E\, e^{-t}u(X_t) + E\int_0^t e^{-s}[h(X_s)ds + d\zeta_s].$$

We wish to let $t \to \infty$ in (3.6) to obtain

$$(3.7) \qquad u(x) \le E\int_0^\infty e^{-s}[h(X_s)ds + d\zeta_s] = V_{N,\zeta}(x).$$

Assume $E\int_0^\infty e^{-s}h(X_s) < \infty$, for otherwise (3.7) is obviously true. This implies that

$$\lim_{t\to\infty} E\, e^{-t}h(X_t) = 0.$$

Now (2.8), (2.9) and the inequality $|\nabla u| \leq 1$ (from (3.1)) imply that $\forall\, y \in \mathbb{R}^2$,

$$u(y) \leq u(0) + |y| \leq u(0) + 1 + |y|^2 \leq u(0) + 1 + \frac{2}{c_0}\, h(y), \tag{3.8}$$

so

$$\lim_{t\to\infty} E\, e^{-t}u(X_t) = 0.$$

We may therefore pass to the limit in (3.6) along a sequence $\{t_n\}_{n=1}^\infty$ such that $E\, e^{-t_n}u(X_{t_n}) \to 0$ as $t_n \to \infty$, and (3.7) follows. Since (N,ζ) is an arbitrary control process pair, we have $u(x) \leq V(x)$.

If (3.2)-(3.4) are satisfied, then the second and fifth terms on the right-hand side of (3.5) are zero, and the fourth term is $E\int_0^t e^{-s}d\zeta_s$. It follows that equality holds in (3.6), and hence also in (3.7), i.e.,

$$u(x) \leq V(x) \leq V_{(N,\zeta)}(x) = u(x). \qquad \square$$

3.2 Remark. Equation (3.1) is similar but not equivalent to a problem arising in elastic-plastic torsion (Ting (1966, 1967), Duvant & Lanchon (1967), Brezis & Sibony (1971)). The elastic-plastic problem is posed on a bounded domain $\Omega \subset \mathbb{R}^n$, and is to minimize

$$J(v) \triangleq \int_\Omega \frac{1}{2} |\nabla v|^2 + \frac{1}{2} v^2 - vh$$

over $K \triangleq \{v \in H_0^1(\Omega);\ \|\nabla v\|_\infty \leq 1\}$. Equivalently, one seeks $u \in K$ satisfying

$$\int (h - u)(v - u) - \int \nabla u \cdot (\nabla v - \nabla u) \leq 0 \quad \forall\, v \in K.$$

If u solves the elastic-plastic torsion problem, then

$$(u - \Delta u - h)(|\nabla u| - 1) = 0,$$

but $u - \Delta u - h$ may be positive. In the special case that h is a nonnegative constant function, a solution to the elastic-plastic problem also satisfies (3.1) (see Evans (1979), Section 6), but such an h is not interesting in the control problem.

4. Solution of the Hamilton-Jacobi-Bellman equation

The existence of a $W7^{2fCo}_{\ell oc}$ solution to the HJB equation (3.1) follows from a modification of Evans (1979) (see also Ishii and Koike (1983)), who treated a bounded domain and general h and space dimension. We need to refer to this construction in the next section, so we provide it here.

Let j3 : R -* R be a C°° function satisfying

(4.1i) J3(r) =0 V r € (-oo, 0],

(4.1ii) /?(r) > 0 V r € (0, <*>),

(4.1iii) |3(r) = r - 1 V r € [2,«>),

(4.1iv) j3'(r) $\geq$ 0, P"(r) $\geq$ 0 V r € R.

For each *e* > 0, we form the penalization function

(4.2) $j3_{\varepsilon}(r) \triangleq P\left(\frac{r-1}{\varepsilon}\right)$ V r € R,

and we consider the penalized equation

(4.3) $u^e - Au^e + P_e(|vu^e|^2) = h.$

The following lemma is proved in the appendix.

4.1 Lemma. For every e € (0,1), there exists a nonnegative, convex, $<J^2$ solution u^e to (4.3). There exist positive constants CL_1, CL_2 and p,

independent of ϵ, such that $\forall\ \epsilon \in (0,1)$, $\forall\ x \in \mathbb{R}^2$:

(4.4) $$0 \le u^\epsilon(x) \le C_1(1 + |x|^p),$$

(4.5) $$|\nabla u^\epsilon(x)| \le C_1(1 + |x|^p),$$

and for every $y \in \mathbb{R}^2$,

(4.6) $$0 \le D^2u^\epsilon(x)y\cdot y \le C_2|y|^2(1 + u^\epsilon(x)).$$

<u>4.2 Definition</u>. We define a norm on the vector space of 2×2 matrices by

$$\|A\| \triangleq \sqrt{\mathrm{trace}(AA^T)}\ .$$

If A is symmetric with eigenvalues λ_1 and λ_2, then

(4.7) $$\|A\| = \sqrt{\lambda_1^2 + \lambda_2^2}\ .$$

<u>4.3 Theorem</u>. The HJB equation (3.1) has a nonnegative, convex solution $u \in W^{2,\infty}_{\ell oc}$ satisfying

(4.8) $$\|D^2u(x)\| \le C_3\ (1 + |x|^m),\ \text{a.e.}\quad x \in \mathbb{R}^2,$$

for some $C_3 > 0$ and $m \in \mathbb{N}$.

<u>Proof</u>: Because D^2u^ϵ is locally bounded, uniformly in $\epsilon \in (0,1)$, we may

choose a decreasing sequence $\{\epsilon_n\}_{n=1}^{\infty}$ with limit zero such that $\{u^{\epsilon_n}\}_{n=1}^{\infty}$ and $\{\nabla u^{\epsilon_n}\}_{n=1}^{\infty}$ converge uniformly on compact sets, and $\{D^2 u^{\epsilon_n}\}_{n=1}^{\infty}$ converges in the $L^{\infty}_{\ell oc}$ - weak* topology. Define $u = \lim_{n\to\infty} u^{\epsilon_n}$, so that $\nabla u = \lim_{n\to\infty} \nabla u^{\epsilon_n}$ and the weak* limit of $\{D^2 u^{\epsilon_n}\}_{n=1}^{\infty}$ is $D^2 u$. Passage to the limit in (4.3) gives (3.1). □

<u>Lemma 4.4</u>. Let $u \in W^{2,\infty}_{\ell oc}$ be a nonnegative, convex solution to the HJB equation (3.1), and define

$$\mathscr{C} \triangleq \{x \in \mathbb{R}^2; \ |\nabla u(x)|^2 < 1\}. \tag{4.9}$$

Then for every unit vector ν,

$$u_{\nu\nu} \triangleq (D^2 u)\, \nu \cdot \nu > 0 \quad \text{on } \mathscr{C}, \tag{4.10}$$

$\mathscr{C}$ is bounded, and u attains its unique minimum over $\mathbb{R}^2$ inside $\mathscr{C}$.

<u>Proof</u>: We have

$$u - \Delta u = h \quad \text{on } \mathscr{C}, \tag{4.11}$$

and $h \in C^{2,1}_{\ell oc}(\mathbb{R}^2)$, so $u \in C^{4,\alpha}(\mathscr{C})$ $\forall\, \alpha \in (0,1)$. Differentiating (4.11), we obtain

$$u_{\nu\nu} - \Delta\, u_{\nu\nu} = h_{\nu\nu} \quad \text{on } \mathscr{C},$$

and since $h_{\nu\nu} > 0$, relation (4.10) holds. Equation (4.11) also implies that $u \geq h$ on $\mathscr{C}$, and since $|\nabla u| \leq 1$ on $\mathbb{R}^2$ but h grows at least quadratically (see (2.8)), $\mathscr{C}$ must be bounded.

Let $\delta \in (0, \frac{1}{2})$ be given, and choose $x^\delta \in \mathbb{R}^2$ such that

$$u(x^\delta) \leq u(x) + \delta \quad \forall\, x \in \mathbb{R}^2.$$

Define

$$\psi_\delta(x) \triangleq u(x) + \delta\, |x - x^\delta|^2 \quad \forall\, x \in \mathbb{R}^2,$$

and note that ψ_δ attains its minimum over $\mathbb{R}^2$ at some point y^δ. In particular,

$$0 = \nabla\psi_\delta(y^\delta) = \nabla u(y^\delta) + 2\delta(y^\delta - x^\delta). \tag{4.12}$$

But also

$$u(y^\delta) + \delta|y^\delta - x^\delta|^2 = \psi_\delta(y^\delta) \leq \psi_\delta(x^\delta) = u(x^\delta) \leq u(y^\delta) + \delta.$$

It follows that $|y^\delta - x^\delta| \leq 1$, and returning to (4.12), we see that $|\nabla u(y^\delta)| \leq 2\delta < 1$. Therefore, $y^\delta \in \mathscr{C}$ $\forall\, \delta \in (0, \frac{1}{2})$, and the sequence $\{y^{\frac{1}{n}}\}_{n=3}^\infty$ accumulates at some $y^0 \in \overline{\mathscr{C}}$. From (4.12) we have $\nabla u(y^0) = 0$, so $y^0 \in \mathscr{C}$, and the convexity of u on $\mathbb{R}^2$ implies that u attains its minimum at y_0. This minimum is unique because of (4.10). □

4.6 Theorem. There is only one nonnegative, convex solution $u \in W^{2,\infty}_{\ell oc}$ to the HJB equation (3.1).

Proof: Let u_1 and u_2 be two nonnegative, convex solutions to (3.1), and let y^0 be the point where u_2 attains its minimum. Given $\delta > 0$, define

$$\varphi_\delta(x) \triangleq u_1(x) - u_2(x) - \delta\, |x - y^0|^2 \quad \forall\, x \in \mathbb{R}^2.$$

The function φ_δ attains its maximum at some $x^\delta \in \mathbb{R}^2$, and $0 = \nabla\varphi_\delta(x^\delta) = \nabla u_1(x^\delta) - \nabla u_2(x^\delta) - 2\delta(x^\delta - y^0)$.

Consequently,

$$1 \geq |\nabla u_1(x^\delta)|^2 = |\nabla u_2(x^\delta)|^2 + 4\delta^2\, |x^\delta - y^0|^2 + 4\delta\, \nabla u_2(x^\delta) \cdot (x^\delta - y^0).$$

Because u_2 is convex, $\nabla u_2(x^\delta) \cdot (x^\delta - y^0) \geq 0$, so either $|\nabla u_2(x^\delta)|^2 < 1$ or $x^\delta = y^0$. This last equality would imply that $\nabla u_2(x^\delta) = 0$, so in any event, $|\nabla u_2(x^\delta)|^2 < 1$. From (3.1) we have

$$\Delta\, u_2(x^\delta) = u_2(x^\delta) - h(x^\delta).$$

Because φ attains its maximum at x^δ, we have from the Bony maximum principle (Bony (1967), Lions (1983))

$$0 \geq \liminf_{x \to x^\delta} \operatorname{ess} \Delta \varphi(x)$$

$$= \liminf_{x \to x^\delta} \operatorname{ess} [\Delta u_1(x) - \Delta u_2(x) - 4\delta]$$

$$\geq u_1(x^\delta) - u_2(x^\delta) - 4\delta.$$

It follows that $\forall x \in \mathbb{R}^2$,

$$u_1(x) - u_2(x) = \varphi_\delta(x) + \delta |x - y^0|^2$$

$$\leq \varphi_\delta(x^\delta) + \delta |x - y^0|^2$$

$$\leq \delta (4 + |x - y^0|^2).$$

Letting $\delta \downarrow 0$, we obtain $u_1 \leq u_2$. The reverse inequality is proved by interchanging u_1 and u_2. □

<u>4.6 Remark</u>. Throughout the remainder of the paper, u will denote the unique nonnegative, convex solution in $W^{2,\infty}_{\ell oc}$ to equation (3.1). The set $\mathscr{C}$ will be given by (4.9), and $y^0 \in \mathscr{C}$ will denote the unique minimizer of u. We shall prove that $u \in C^{2,\alpha}_{\ell oc}(\mathbb{R}^2)$ $\forall \alpha \in (0,1)$ (Theorem 10.3), $\partial\mathscr{C}$ is of class $C^{2,\alpha}$ $\forall \alpha \in (0,1)$ (Corollary 11.3), and $n(x) \cdot \nabla u(x) \geq \sigma \ \forall x \in \partial\mathscr{C}$, where $n(x)$ is the outward normal to $\mathscr{C}$ at x and σ is a positive constant (Lemma 12.3).

5. An obstacle problem

Let us return to the construction of u in the proof of Theorem 4.3 as the limit of a sequence of functions $\{u^{\epsilon_n}\}_{n=1}^{\infty}$, where each u^{ϵ_n} satisfies (4.3). Define $w^{\epsilon_n} \triangleq |\nabla u^{\epsilon_n}|^2$ and compute the product of ∇u^{ϵ_n} with the gradient of both sides of (4.3) to obtain

$$w^{\epsilon_n} - \frac{1}{2}\Delta w^{\epsilon_n} + 2\beta'_{\epsilon_n}(w^{\epsilon_n})(D^2u^{\epsilon_n})\nabla u^{\epsilon_n} \cdot \nabla u^{\epsilon_n} = H^{\epsilon_n} \tag{5.1}$$

where

$$H^{\epsilon_n} \triangleq \nabla h \cdot \nabla u^{\epsilon_n} - \|D^2u^{\epsilon_n}\|^2.$$

Along a subsequence, which we also call $\{\epsilon_n\}_{n=1}^{\infty}$, $\{H^{\epsilon_n}\}_{n=1}^{\infty}$ converges to

$$\bar{H} \triangleq \nabla h \cdot \nabla u - \chi, \tag{5.2}$$

where χ is the limit of $\|D^2u^{\epsilon_n}\|^2$ in the weak* topology on $L^{\infty}_{\ell oc}$. We will show that

$$w = |\nabla u|^2 = \lim_{n\to\infty} |\nabla u^{\epsilon_n}|^2$$

solves an obstacle problem involving $\bar{H}$, and we will then obtain $W^{2,p}_{\ell oc}$ regularity for w by invoking the theory of variational inequalities.

For $r > 0$ chosen so that $B_r(0) \triangleq \{x \in \mathbb{R}^2;\ |x| < r\}$ contains $\mathscr{C}$, define

$$K_r \triangleq \{v \in W^{1,2}(B_r);\ \ 0 \le v \le 1 \text{ on } B_r \text{ and } v - 1 \in W_0^{1,2}(B_r)\}.$$

We pose the problem of finding $\varphi \in K_r$ such that

$$(5.3)\qquad \frac{1}{2}\int_{B_r(0)} \nabla\varphi \cdot \nabla(v - \varphi) \ge \int_{B_r(0)} (H - w)(v - \varphi) \quad \forall\, v \in K_r .$$

<u>5.1 Lemma</u>. The function $w = |\nabla u|^2$ solves (5.3).

<u>Proof: Let $v \in K_r$</u> be given. From (5.1) we have

$$(5.4)\qquad \int_{B_r(0)} \left(w^{\varepsilon_n} - \frac{1}{2}\Delta w^{\varepsilon_n} - H^{\varepsilon_n}\right)(v - w^{\varepsilon_n})$$

$$= -\int_{B_r(0)} 2\beta'_{\varepsilon_n}(w^{\varepsilon_n})(D^2 u^{\varepsilon_n})\nabla u^{\varepsilon_n} \cdot \nabla u^{\varepsilon_n}(v - w^{\varepsilon_n}) .$$

The function u^{ε_n} is convex, $\beta'_{\varepsilon_n}(w^{\varepsilon_n}) = 0$ whenever $w^{\varepsilon_n} < 1$, and $v - w^{\varepsilon_n} < 0$ whenever $w^{\varepsilon_n} > 1$. Therefore, the right-hand side of (5.4) is nonnegative, and integration by parts yields

$$(5.5)\qquad \frac{1}{2}\int_{\partial B_r(0)} (v - w^{\varepsilon_n})\nabla w^{\varepsilon_n} \cdot n + \frac{1}{2}\int_{B_r(0)} \nabla w^{\varepsilon_n} \cdot \nabla(w - w^{\varepsilon_n})$$

$$\ge \int_{B_r(0)} (H^{\varepsilon_n} - w^{\varepsilon_n})(v - w^{\varepsilon_n}),$$

where n is the outward normal on $\partial B_r(0)$. Now $w^{\epsilon_n} \to v$ uniformly on $\partial B_r(0)$, $w^{\epsilon_n} \to w$ uniformly on $B_r(0)$, and $H^{\epsilon_n} \to \bar{H}$, $\nabla w^{\epsilon_n} \to \nabla w$, both the latter convergences being weak* in $L^\infty(B_r(0))$. Because the weak* limit of $|\nabla w^{\epsilon_n}|^2$ dominates $|\nabla w|^2$, we may pass to the limit in (5.5) to obtain

$$(5.6) \qquad \frac{1}{2}\int_{B_r(0)} \nabla w \cdot (\nabla v - \nabla w) \geq \int_{B_r(0)} (\bar{H} - w)(v - w) \quad \forall\, v \in K_r.$$

□

<u>5.2 Theorem</u>. For every $p \in (1,\infty)$, $w \triangleq |\nabla u|^2 \in W^{2,p}_{\ell oc}$.

<u>Proof</u>: This is a classical result. See, for example, Lemma 5.1 and Theorem 3.11, p. 29 of Chipot (1984).

□

<u>5.3 Corollary</u>. We have $w \in C^{1,\alpha}(\mathbb{R}^2)$ for any $\alpha \in (0,1)$.

<u>Proof</u>: This follows from Sobolev imbedding (Gilbarg & Trudinger, Theorem 7.17, p. 163). □

<u>5.4 Remark</u>. Integration by parts allows us to rewrite (5.6) as

$$\int_{B_r(0)} (w - \frac{1}{2}\Delta w - \bar{H})(v - w) \geq 0 \quad \forall \quad v \in K_r,$$

for all sufficiently large r, and so

$$\max \{w - \frac{1}{2} \Delta w - \bar{H},\ w - 1\} = 0. \tag{5.7}$$

Now χ appearing in (5.2) dominates $\|D^2 u\|$, and so $\bar{H}$ is dominated by

$$H \triangleq \nabla h \cdot \nabla u - \|D^2 u\|^2. \tag{5.8}$$

But let $x^0 \in \mathcal{C}$ be given and choose $\epsilon > 0$ such that the closed disk $\overline{B_{2\epsilon}(x_0)}$ is contained in $\mathcal{C}$. Choose a positive integer N such that

$$|\nabla u^{\epsilon_n}(x)| < 1 \quad \forall \quad n \geq N,\ x \in \overline{B_{2\epsilon}(x^0)}.$$

From (4.1i), (4.2) and (4.3), we see that

$$u^{\epsilon_n} - \Delta\, u^{\epsilon_n} = h \quad \text{on} \quad \overline{B_{2\epsilon}(x^0)}.$$

According to Theorem 4.6, p. 60 of Gilbarg & Trudinger (1983), for every $\alpha \in (0,1)$, $|u^{\epsilon_n}|_{C^{2,\alpha}(B_\epsilon(x^0))}$ is bounded uniformly in $n \geq N$. Thus, on $B_\epsilon(x^0)$, $D^2 u^{\epsilon_n}$ is continuous and converges uniformly to $D^2 u$, $\chi = \|D^2 u\|^2$, and $\bar{H} = H$. We conclude that (5.7) remains valid if $\bar{H}$ is replaced by H, i.e.,

$$\max\{w - \frac{1}{2} \Delta w - H,\ w - 1\} = 0. \tag{5.9}$$

6. D^2u inside $\bar{\mathscr{C}}$

Inside the set $\mathscr{C}$ defined by (4.9), u satisfies the elliptic equation $u - \Delta u = h$, and is therefore smooth (at least $C^{4,\alpha}$ $\forall\ \alpha \in (0,1)$ because h is $C^{2,1}$). In this section, we describe the behavior of D^2u as $\partial\mathscr{C}$ is approached from inside $\mathscr{C}$.

6.1 Lemma. Let $z \in \partial\mathscr{C}$ be given. As $x \in \mathscr{C}$ approaches z, $D^2u(x)$ approaches the matrix

$$A(z) \triangleq (u(z) - h(z)) \begin{bmatrix} u_2^2(z) & -u_1(z)u_2(z) \\ -u_1(z)u_2(z) & u_1^2(z) \end{bmatrix},$$

where u_i denotes the i-th partial derivative of u.

Proof: Because $w = |\nabla u|^2 = 1$ on $\partial\mathscr{C}$, A(z) can be characterized as the unique 2×2 positive semidefinite matrix with eigenvalues zero and $u(z) - h(z)$, and with $\nabla u(z)$ an eigenvector corresponding to the eigenvalue zero. Let ν be a unit vector orthogonal to the unit vector $\nabla u(z)$. It suffices to show that

(6.1) $$\lim_{\substack{x \to z \\ x \in \mathscr{C}}} D^2u(x)\ \nabla u(z) = 0$$

(6.2) $$\lim_{\substack{x \to z \\ x \in \mathscr{C}}} D^2u(x)\nu = (u(z) - h(z))\nu.$$

Because $w = |\nabla u|^2$ attains its maximum value of 1 at z, and ∇w is continuous (Corollary 5.3), we have

$$0 = \nabla w(z) = \lim_{\substack{x \to z \\ x \in \mathscr{C}}} \nabla w(x) = \lim_{\substack{x \to z \\ x \in \mathscr{C}}} D^2u(x)\ \nabla u(x).$$

Since ∇u is continuous and $D^2u \in L^\infty_{\ell oc}$, (6.1) follows.

Let $0 = \lambda_1(x) \leq \lambda_2(x)$ denote the eigenvalues of $D^2u(x)$. Then $u(x) - h(x) = \Delta u(x) = \lambda_1(x) + \lambda_2(x)\ \forall\ x \in \mathscr{C}$, and (6.1) shows that $\lim_{\substack{x \to z \\ x \in \mathscr{C}}} \lambda_1(x) = 0$. Consequently,

$$\text{(6.3)} \qquad \lim_{\substack{x \to z \\ x \in \mathscr{C}}} \lambda_2(x) = u(z) - h(z),$$

which is thus nonnegative. If $u(z) - h(z) = 0$, then $D^2u(x)$ approaches the zero matrix and (6.2) holds. If $u(z) - h(z) > 0$, then (6.1) implies that any unit eigenvector corresponding to $\lambda_1(x)$ must, as $x \in \mathscr{C}$ approaches z, approach colinearity with $\nabla u(z)$. Hence, any unit eigenvector corresponding to $\lambda_2(x)$ approaches colinearity with ν, and (6.2) follows from (6.3). □

<u>6.2 Remark</u>. The characterization of $A(z)$ used in the proof of Lemma 6.1 makes critical use of the fact that our problem is posed in two dimensions. The two-dimensional nature of the problem also plays a fundamental role in Lemma 8.1, and together these lemmas provide the basis for Section 10, where the existence of a continuous version of D^2u on $\mathbb{R}^2$ is established.

<u>6.3 Theorem</u>. For every $\alpha \in (0,1)$, $u \in C^{2,\alpha}(\bar{\mathscr{C}})$, i.e., D^2u restricted to $\mathscr{C}$ has an α-Hölder continuous extension to $\bar{\mathscr{C}}$.

<u>Proof</u>: Because $|\nabla u| = 1$ on $\partial\mathscr{C}$, we can choose an open set $G \subset \mathscr{C}$ such that $|\nabla u|$ is bounded away from zero on $\mathscr{C}\backslash G$. Elliptic regularity implies the Hölder continuity of D^2u on $\bar{G}$, so it suffices to prove uniform Hölder continuity of D^2u on $\mathscr{C}\backslash G$.

Let a unit vector ν be given, and define on $\mathscr{C}\backslash G$, $\eta \triangleq \frac{\nabla u}{|\nabla u|}$, $z \triangleq \nu - (\nu\cdot\eta)\eta$,

$$\gamma \triangleq \begin{cases} \dfrac{z}{|z|} & \text{if } z \neq 0, \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \eta & \text{if } z = 0. \end{cases}$$

Observe that $\eta \cdot \gamma = 0$ and $|\eta| = |\gamma| = 1$. Therefore,

$$\Delta u = (D^2u)\,\eta\cdot\eta + (D^2u)\gamma\cdot\gamma \quad \text{on } \mathscr{C}\backslash G.$$

Direct calculation shows that on $\mathscr{C}\backslash G$,

$$(D^2u)\,\nu\cdot\nu = (D^2u)\,z\cdot z + 2(\nu\cdot\eta)(D^2u)\,\eta\cdot z + (\nu\cdot\eta)^2(D^2u)\,\eta\cdot\eta$$

$$= |z|^2(\Delta u - (D^2u)\eta\cdot\eta) + 2(\nu\cdot\eta)(D^2u)\eta \cdot (\nu - (\nu\cdot\eta)\eta)$$

$$+ (\nu\cdot\eta)^2(D^2u)\eta\cdot\eta.$$

Since $\Delta u = u-h$ and $2(D^2u)\eta = \frac{\nabla w}{|\nabla u|}$ on $\mathscr{C}\backslash G$, we have

$$(6.4) \qquad (D^2u)\nu\cdot\nu = |\nu - \frac{(\nu\cdot\nabla u)}{|\nabla u|^2}\nabla u|^2 \, (u - h - \frac{1}{2}\frac{(\nabla w\cdot\nabla u)}{|\nabla u|^2})$$

$$+ \frac{(\nu\cdot\nabla u)}{|\nabla u|^2}\nabla w \cdot \nu - \frac{(\nu\cdot\nabla u)^2}{2|\nabla u|^4}\nabla w \cdot \nabla u \quad \text{on} \quad \mathscr{C}\setminus G.$$

All the terms appearing on the right-hand side of (6.4) are uniformly Hölder continuous in $\mathscr{C}\setminus G$ (recall Corollary 5.3). □

7. The gradient flow

Recalling Remark 4.6, we let y^0 € ^ denote the unique minimizer of u. Using the strict convexity of u in ^ (Lemma 4.4), we choose 6 > 0, *p.* > 0 such that

(7.1) $B_{26}(y°)C<6,$

(7.2) $D^2u(x)y \cdot y \geq n|y|^2$ V x € $B_{25}(y°)$,

(7.3) *VL* < $|vu(x)|^2 \leq _2 I$ V x € *dBJy°),*

(7.4) vu(y° + 69) • 6 > *p.* V 6 € Sj,

where Sj = dB^O) is the set of unit vectors in IR^Ω. For 0 € Sj, we define the gradient flow ^(t,0) to be the unique solution to the differential equation

(7.5) $\frac{d}{dt}$"^*'6) = vu(x'/(t'0))« t >- °.

with the initial condition

(7.6) ^(0,0) = y° + 60.

We will find it convenient to use *\fj* to change coordinates in R^2. The following theorem justifies this.

7.1 Theorem. The map *>//* is a homeomorphism from [0,«>) x S_1 onto IR^2\B^(y^0).

<u>Proof</u>: Let us for the moment fix $0 \in S_1$ and define

$n(t) \overset{\Delta}{=}$ >Kt,9) - y° V $t \geq 0$. Because $|vu| \leq 1$, we have $|n(t)| \leq t + 5$, and

$y^0 + \frac{\delta\Lambda t}{|}$ $n(t) \in B_{25}(y^0)$ V $t > 0$. We conclude from the convexity of u on IR^2

and from (7.2) that for $t > 0$:

(7.7) $$\frac{\pm}{dt}|n(t)|^2 = 2\, vu(y^\circ + n(t)) \bullet n(t)$$

$$= 2[vu(y^\circ + n(t)) - vu(y^\circ + \frac{\delta\Lambda t}{t} n(t))] \bullet n(t)$$

$$+ 2\, [u(y^\circ + \wedge n(t)) - vu(y^\circ)] \bullet n(t)$$

$$\geq 2 \int_0^{\frac{6At}{t}} D^2u(y^\circ + -m(t))n(t) \bullet n(t)di$$

$$\geq 2\mu(1 \wedge f)\ |n(t)|^2.$$

Since $|n(0)|^2 = 6^2$, we can integrate (7.7) to obtain the inequality

(7.8) $$|*(t.e) - y^\circ|^2 \wedge (1 V \backslash) ** e^{\wedge tA6}> V\ t \geq 0,\ 0 \in S_1.$$

One consequence of (7.8) is that

(7.9) $$k(s,9) - *(0.<*>)l > 0\ \ V\ s > 0.\ 6 \in S_r\ \ y \in S_r$$

Now let $s,t \in [0,\infty)$ and $0,\varphi \in S_1$ be given. Again using the convexity of u, we may write

$$(7.10)\qquad |\psi(t+s,\theta) - \psi(t,\varphi)|^2 = |\psi(s,\theta) - \psi(0,\varphi)|^2 + 2\int_0^t [\nabla u(\psi(\tau+s,\theta)) - \nabla u(\psi(\tau,\varphi))] \cdot [\psi(\tau+s,\theta) - \psi(\tau,\varphi)]d\tau \geq |\psi(s,\theta) - \psi(0,\varphi)|^2.$$

If θ, φ are in S_1 and t_1, t_2 are in $[0,\infty)$ and $t_1 \neq t_2$, then (7.9), (7.10) imply that $\psi(t_1,\theta) \neq \psi(t_1,\varphi)$. If $t_1 = t_2$ but $\theta \neq \varphi$, then the uniqueness of solutions to (7.5) implies that $\psi(t_1,\theta) \neq \psi(t_2,\varphi)$. This concludes the proof that ψ is injective.

It is clear from its definition that ψ is continuous. Define

$$D \triangleq \psi([0,\infty) \times S_1) \subset \mathbb{R}^2 \setminus B_\delta(y^0)$$

to be the range of ψ. Let $x \in D$ and $\epsilon > 0$ be given. It follows from (7.8) that there exists $T > 0$ such that

$$D \cap B_\epsilon(x) \subset \psi([0,T] \times S_1).$$

But an injective, continuous map on a compact set has a continuous inverse, so ψ^{-1} is continuous at x.

It remains to show that $D = \mathbb{R}^2 \setminus B_\delta(y^0)$. There is a function $\hat{\psi}\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}^2$ such that

$$\hat{\psi}(t,\beta) = \psi(t, (\cos\beta, \sin\beta)) \quad \forall\ (t,\beta) \in [0,\infty) \times \mathbb{R},$$

and $\hat{\psi}$ is continuous and locally injective. It follows from Deimling (1985), Theorem 4.3, p. 23 that

$$D \cap (\mathbb{R}^2 \setminus \overline{B_\delta(y^0)}) = \hat{\psi}((0,\infty) \times \mathbb{R})$$

is open. On the other hand, if $\{x^n\}_{n=1}^{\infty} \subset D$ is a sequence with limit $x^0 \in \mathbb{R}^2$, then (7.8) shows that $\{\psi^{-1}(x^n)\}_{n=1}^{\infty}$ is bounded and thus has an accumulation point $(t^0,\theta^0) \in [0,\infty) \times S_1$. The continuity of ψ implies that $x^0 = \psi(t^0,\theta^0)$, so D is closed. It follows that $D = \mathbb{R}^2 \setminus B_\delta(y^0)$. □

<u>7.2 Corollary</u>. For $\theta \in S_1$ and $\gamma \in [\frac{1}{2}, 1]$, define

(7.11) $$T_\gamma(\theta) \triangleq \inf\{t \geq 0;\ \ |\nabla u(\psi(t,\theta))|^2 \geq \gamma\}.$$

Then

$$\sup_{\substack{\frac{1}{2}\leq\gamma\leq 1 \\ \theta\in S_1}} T_\gamma(\theta) \leq \sup_{\theta\in S_1} T_1(\theta) < \infty.$$

<u>Proof</u>: According to Lemma 4.4, $\mathscr{C}$ is bounded. We can use (7.8) to choose $t^* \in (0,\infty)$ such that

$$\mathscr{C} \subset \psi([0,t^*] \times S^1).$$ □

<u>7.3 Theorem</u>. The homeomorphism ψ is Lipschitz continuous on compact subsets of $[0,\infty) \times S_1$, and ψ^{-1} is Lipschitz continuous on all of $\mathbb{R}^2 \setminus B_\delta(y^0)$.

<u>Proof</u>: It follows immediately from (7.5) that $|\frac{d}{dt}\psi(t,\theta)| \le 1$ $\forall\ (t,\theta) \in [0,\infty) \times S_1$. Now let $T > 0$ be given and use Theorem 4.3 to choose a Lipschitz constant C for ∇u on $\psi([0,T] \times S_1)$. For $\theta,\varphi \in S_1$ and $t \in [0,T]$, we have

$$|\psi(t,\theta) - \psi(t,\varphi)| \le |\psi(0,\theta) - \psi(0,\theta)|$$

$$+ \int_0^t |\nabla u(\psi(\tau,\theta)) - \nabla u(\psi(\tau,\varphi))| d\tau$$

$$\le \delta\ |\theta - \varphi| + C \int_0^t |\psi(\tau,\theta) - \psi(\tau,\varphi)| d\tau.$$

Gronwall's inequality gives

$$|\psi(t,\theta) - \psi(t,\varphi)| \le \delta\ e^{CT}\ |\theta - \varphi|,$$

and the local Lipschitz continuity of ψ is proved.

To prove the global Lipschitz continuity of ψ^{-1}, we let $x^1,x^2 \in \mathbb{R}^2 \setminus B_\delta(y^0)$ be given and define $(t_1,\theta_1) = \psi^{-1}(x^1)$, $(t_2,\theta_2) = \psi^{-1}(x^2)$. Assume without loss of generality that $|x^1 - x^2| \le 1$ and that $t_1 \ge t_2$. Set $s = t_1 - t_2$. According to (7.10) and (7.8),

$$(7.12)\qquad |x^1 - x^2| \geq |\psi(s,\theta_1) - \psi(0,\theta_2)|$$

$$\geq |\psi(s,\theta_1) - y^0| - |y^0 - \psi(0,\theta_2)|$$

$$\geq \delta(1 \vee \tfrac{s}{\delta})^{\mu\delta}\, e^{\mu(s\wedge\delta)} - \delta$$

$$\geq \delta\mu(1 \vee \tfrac{s}{\delta})^{\mu\delta}(s\wedge\delta).$$

If $0 \leq s \leq \delta$, then (7.12) yields

$$(7.13)\qquad |t_1 - t_2| \leq \frac{1}{\delta\mu}\, |x^1 - x^2|.$$

If $s \geq \delta$ and $\mu\delta \geq 1$, (7.12) again yields (7.13). Finally, if $s \geq \delta$ and $0 < \mu\delta < 1$, (7.12) yields $|x^1 - x^2| \geq \mu\, \delta^{1-\mu\delta} s^{\mu\delta}$, so

$$(7.14)\qquad |t_1 - t_1| \leq (\mu\, \delta^{1-\mu\delta})^{\frac{1}{\mu\delta}}\, |x^1 - x^2|^{\frac{1}{\mu\delta}} \leq (\mu\delta^{1-\mu\delta})^{\frac{1}{\mu\delta}}\, |x^1 - x^2|.$$

Relations (7.14) and (7.15) imply the global Lipschitz continuity of the first component of ψ^{-1}, i.e., there exists a constant $L > 0$ such that

$$(7.15)\qquad |t_1 - t_2| \leq L|\psi(t_1,\theta_1) - \psi(t_2,\theta_2)|$$

$$\forall\ (t_1,\theta_1),(t_2,\theta_2) \in [0,\infty) \times S_1.$$

Now let $x^1,x^2 \in \mathbb{R}^2\setminus B_\delta(y^0)$ be given, and define (t_1,θ_1), (t_2,θ_2) and

$s = t_1 - t_2 \geq 0$ as before. From (7.10), (7.5) and (7.6), we have

$$|x^1 - x^2| \geq |\psi(s,\theta_1) - \psi(0,\theta_2)|$$

$$\geq - |\psi(s,\theta_1) - \psi(0,\theta_1)| + |\psi(0,\theta_1) - \psi(0,\theta_2)|$$

$$\geq - s + \delta\, |\theta_1 - \theta_2|.$$

Relation (7.15) gives us

$$|\theta_1 - \theta_2| \leq \frac{1}{\delta}\, |t_1 - t_2| + \frac{1}{\delta}\, |x^1 - x^2| \leq \frac{1}{\delta}\, (1 + L)|x^1 - x^2|.$$

□

7.4 Remark. In much of what follows, we will use the coordinates $(t,\theta) \in [0,\infty) \times S_1$ rather than the coordinates $x \in \mathbb{R}^2\backslash B_\delta(y^0)$. We may identify S_1 with the unit circle, and let $[0,\infty) \times S_1$ have the product of Lebesgue measure and arc length measure. An important consequence of Theorem 7.3 is that ψ maps measure zero subsets of $[0,\infty) \times S_1$ onto Lebesgue measure zero subsets of $\mathbb{R}^2\backslash B_\delta(y^0)$. Likewise, ψ^{-1} preserves measure zero sets.

8. $W^{2,\infty}$ regularity for the obstacle problem

The purpose of this section is to show that the function $w = |\nabla u|^2$ is in $W^{2,\infty}_{\ell oc}$. This improves the regularity result of Theorem 5.2.

8.1 Lemma. We have

$$(D^2u)\nabla u = 0, \quad \|D^2u\| = \Delta u \quad \text{a.e.} \quad \text{on } \mathbb{R}^2\backslash\mathscr{C}. \tag{8.1}$$

Proof: By the definition of $\mathscr{C}$, w attains its maximum value of 1 at every point in $\mathbb{R}^2\backslash\mathscr{C}$, so $\nabla w = 0$ everywhere on $\mathbb{R}^2\backslash\mathscr{C}$. But $\nabla w = 2(D^2u)\nabla u$ almost everywhere on $\mathbb{R}^2$, and the first part of (8.1) follows. Since D^2u is singular almost everywhere on $\mathbb{R}^2\backslash\mathscr{C}$, the second part of (8.1) also holds.

□

8.2 Remark. Because D^2u is positive definite on $\mathscr{C}$ and positive semidefinite almost everywhere on $\mathbb{R}^2$, and since (recalling Remark 7.4)

$$\frac{d}{dt} w(\psi(t,\theta)) = 2D^2u(\psi(t,\theta))\nabla u(\psi(t,\theta)) \cdot \nabla u(\psi(t,\theta)) \tag{8.2}$$

$$\text{a.e. } (t,\theta) \in [0,\infty) \times S^1,$$

the function $t \mapsto w(\psi(t,\theta))$ is nondecreasing for almost every $\theta \in S^1$. In particular, with $T_1(\theta)$ defined by (7.11), we have

$$w(\psi(t,\theta)) \equiv 1 \quad \forall\ t \geq T_1(\theta), \text{ a.e. } \theta \in S^1. \tag{8.3}$$

8.3 Theorem. The function $w = |vu|^2$ is in VT^{2}>a>.

Proof: Recall that w satisfies (5.9), where V a € (0,1), H ≜ vh • vu - IID²ull² is of class <j²>a inside ᶜ6, and H is defined up to almost everywhere equivalence on K²\6. We define

$$\text{(8.4)} \qquad \hat{H}(x) \dot{S} \begin{cases} vh(x) \bullet vu(x) - IID^2u(x)ll^2 \quad V \quad x € <€ \\ vh(x) \bullet vu(x) - [(u(x) - h(x))^+]^2 \quad \text{if} \quad x € \mathbb{R}^2\backslash\mathscr{C}. \end{cases}$$

Now u - h = Au ≥ 0 on %, so u - h ≥ 0 on 6*6. Theorem 6.3 and Lemma 6.1 then show that H is locally Hölder continuous with exponent a for any a € (0,1). Because of (3.1) and Lemma 8.1,

$$u - h \le Au = IID^2ull \quad \text{a.e. on } K\text{^}VB.$$

But Au ≥ 0 a.e. on K², so

$$[(u - h)^+]^2 \le IID^2ull^2 \quad \text{a.e. on} \quad \mathbb{R}^2\backslash\mathscr{C}.$$

Therefore $\hat{H} \geq H$ a.e. on *tP*.* and $\hat{H} = H$ on <6, so (5.9) yields

$$\text{(8.5)} \qquad \max\{w - |\, Aw - \hat{H}, \; w - 1\} = 0.$$

With the aid of (8.5) and the Hölder continuity of H, we can obtain the $W^{2,\infty}$ regularity of w from the theory of variational inequalities. More precisely, choose r so that KB (0) and observe that the Dirichlet r

problem

$$\varphi - \frac{1}{2}\Delta\varphi = \hat{H} \quad \text{on} \quad B_r(0),$$

$$\varphi = 0 \quad \text{on} \quad \partial B_r(0),$$

has a solution φ which is in $C^{2,\alpha}(\overline{B_r(0)})$ for any $\alpha \in (0,1)$ (Ladyzhenskaya & Ural'tseva (1968), Theorem 3.1.3, p. 115). Set $\bar{w} \triangleq w - \varphi$, so that $\bar{w} \in W^{2,p}(\overline{B_r(0)})$ for any $p \in (1,\infty)$, and

$$\max\{\bar{w} - \frac{1}{2}\Delta\bar{w},\ \bar{w} - 1 + \varphi\} = 0 \quad \text{in} \quad B_r(0), \tag{8.6}$$

$$\bar{w} = 1 \qquad \text{on} \quad \partial B_r(0). \tag{8.7}$$

Define

$$L_r \triangleq \{v \in W^{1,2}(B_r(0));\ -\varphi \le v \le 1-\varphi \text{ on } B_r(0) \text{ and } v-1 \in W_0^{1,2}(B_r)\},$$

and note from (8.6), (8.7) that $\bar{w} \in L_r$ and

$$\frac{1}{2}\int_{B_r(0)} \nabla\bar{w} \cdot (\nabla v - \nabla\bar{w}) \ge -\int_{B_r(0)} \bar{w}(v - \bar{w}) \quad \forall \quad v \in L_r.$$

It follows from Chipot (1984), Theorem 3.25, p. 49, that $\bar{w} \in W^{2,\infty}(B_r(0))$, so also $w \in W^{2,\infty}(B_r(0))$. On $\mathbb{R}^2 \setminus B_r(0)$, $w \equiv 1$. □

<u>8.4 Corollary</u>. We have $D^2u \in W^{1,\infty}(\bar{\mathscr{C}})$.

<u>Proof</u>: Use the $W^{1,\infty}$ regularity of ∇w in (6.4). □

9. Lipschitz continuity of T_γ

Recall the mappings $T_\gamma : S_1 \mapsto [0,\infty)$ defined by (7.11) for each $\gamma \in [\frac{1}{2},1]$. The continuity of $\nabla u \circ \psi$ implies the lower semicontinuity of each T_γ. In this section we prove that for each $\gamma \in [\frac{1}{2},1]$, T_γ is, in fact, Lipschitz continuous.

9.1 Lemma. We have

(9.1)
$$K \triangleq \sup_{\nu\in S_1, x\in\mathscr{C}} \frac{|\nabla w(x)|}{D^2u(x)\nu\cdot\nu} < \infty.$$

Proof: Let $\nu,\eta \in S_1$ be given and set $f \triangleq (D^2u)\nu\cdot\nu$ and $g \triangleq \nabla w\cdot\eta$. Then in $\mathscr{C}$,

$$f - \Delta f = (D^2h)\nu\cdot\nu \geq c_0, \quad g - \Delta g = 2\ \nabla H\cdot\eta - g,$$

where $c_0 > 0$ is the constant in (2.8), and H, defined by (5.8), is in $W^{1,\infty}(\bar{\mathscr{C}})$ because of Corollary 8.4. Furthermore, $g = 0 \leq f$ on $\partial\mathscr{C}$. Therefore the maximum principle implies that $g - Kf \leq 0$ in $\mathscr{C}$, where

$$K \triangleq \frac{1}{c_0}(2\ \|\nabla H\|_{L^\infty(\bar{\mathscr{C}})} + \|\nabla w\|_{L^\infty(\bar{\mathscr{C}})}).$$

In other words, $\nabla w \cdot \eta \leq K\ (D^2u)\nu\cdot\nu$. □

<u>9.2 Theorem</u>. For each $\gamma \in [\frac{1}{2},1]$, the mapping $T_\gamma : S_1 \mapsto [0,\infty)$ is Lipschitz continuous with a Lipschitz constant which is independent of γ.

<u>Proof</u>: For each $\gamma \in [\frac{1}{2},1]$, define

$$\mathcal{C}_\gamma \triangleq \{\psi(t,\theta);\ 0 \le t < T_\gamma(\theta)\} \cup B_\delta(y^0)$$

(with ψ, δ and y^0 as in (7.1) - (7.6)). Each $\mathcal{C}_\gamma$ is open, $w < \gamma$ on $\mathcal{C}_\gamma$ and $w = \gamma$ on $\partial\mathcal{C}_\gamma$. For $\gamma \in [\frac{1}{2},1)$, we also have $\mathcal{C}_\gamma \subset \mathcal{C}$. Because of (4.10), ∇w dos not vanish on $\mathcal{C}$, so for fixed $\gamma \in [\frac{1}{2},1)$ and $z \in \partial\mathcal{C}_\gamma$, the outward normal to $\mathcal{C}_\gamma$ exists and is

$$n(z) \triangleq \frac{\nabla w(z)}{|\nabla w(z)|} = \frac{2\, D^2u(z)\ \nabla u(z)}{|\nabla w(z)|} .$$

In fact $D^2 w$ is continuous in $\mathcal{C}$ and bounded in $\mathbb{R}^2$ (Theorem 8.3), so for every $\gamma \in [\frac{1}{2},1)$, $\partial\mathcal{C}_\gamma$ has bounded curvature, i.e., there are constants $\epsilon > 0$, $K_\gamma > 0$ such that for every $z \in \partial\mathcal{C}_\gamma$, and for every $x \in B_\epsilon(z)$:

$$(9.2) \qquad (x - z) \cdot n(z) \ge K_\gamma\ |x - z|^2 \Rightarrow x \in \mathbb{R}^2 \setminus \mathcal{C}_\gamma .$$

We may use the local boundedness of $\frac{d^2}{dt^2}\psi(t,\theta) = \frac{1}{2}\nabla w(\psi(t,\theta))$ and the Lipschitz continuity of ψ to choose a constant $K_2 > 0$ such that for every $\gamma \in [\frac{1}{2},1)$, every $\beta \in [0,1]$, and every $\theta, \varphi \in S_1$:

$$(9.3) \qquad |\psi(T_\gamma(\theta) + \beta,\theta) - \psi(T_\gamma(\theta),\theta) - \beta\ \nabla u(\psi(T_\gamma(\theta),\theta))| \le K_2\beta^2,$$

$$(9.4) \qquad |\psi(T_\gamma(\theta) + \beta,\theta) - \psi(T_\gamma(\theta) + \beta,\varphi)| \le K_2|\theta - \varphi|.$$

With K as in (9.1), choose $L > \max\{\frac{1}{2} K K_2, 1\}$. Let $\theta,\varphi \in S_1$ be given with $|\theta - \varphi| \le \frac{1}{L}$, and set

$$\beta = L\ |\theta - \varphi|,\ z = \psi(T_\gamma(\theta),\theta),\ x = \psi(T_\gamma(\theta) + \beta,\varphi).$$

Then (9.3), (9.4) imply the existence of vectors $\upsilon,\eta \in B_1(0)$ such that

$$x = z + \beta\nabla u(z) + K_2\beta^2\upsilon + K_2\ |\theta - \varphi|\eta.$$

We calculate

$$\begin{aligned}(x-z) \cdot n(z) &= \frac{2\beta D^2u(z)\nabla u(z)\cdot\nabla u(z)}{|\nabla w(z)|} + K_2\beta^2 n(z)\cdot\upsilon + K_2\ |\theta - \varphi|\ n(z)\cdot\eta \\ &\ge \frac{2\beta}{K} - K_2\beta^2 - K_2\ |\theta - \varphi| \\ &= (\frac{2L}{K} - K_2)\ |\theta - \varphi| - K_2L^2\ |\theta - \varphi|^2,\end{aligned}$$

and

$$\begin{aligned}K_\gamma|x - z|^2 &= K_\gamma|\beta\ \nabla u(z) + K_2\beta^2\upsilon + K_2|\theta - \varphi|\eta\ |^2 \\ &\le 9K_\gamma\ (L^2 + K_2^2L^4 + K_2^2)\ |\theta - \varphi|^2.\end{aligned}$$

It is clear that for $|\theta - \varphi|$ sufficiently small, $\nu \in B_\epsilon(z)$ and

$$(x - z) \cdot n(z) \geq K_\gamma \; |x - z|^2,$$

from which we conclude (see (9.2)) that $x \in \mathbb{R}^2 \backslash \mathscr{C}_\gamma$, i.e.,

$$T_\gamma(\varphi) \leq T_\gamma(\theta) + \beta = T_\gamma(\theta) + L|\theta - \varphi|.$$

Interchanging the roles of θ and φ, we obtain

$$|T_\gamma(\theta) - T_\gamma(\theta)| \leq L \; |\theta - \varphi|$$

for all $\theta, \varphi \in S_1$ such that $|\theta - \varphi|$ is sufficiently small.

For each $\theta \in S_1$, the mapping $t \mapsto w(\psi(t,\theta))$ is strictly increasing on $[0,T_1(\theta)]$ (see (8.2) and (4.10)). Therefore, the mapping $\gamma \mapsto T_\gamma(\theta)$ is continuous on $[\frac{1}{2},1]$. The Lipschitz continuity of T_1 follows from the uniform Lipschitz continuity of T_γ for $\gamma \in [\frac{1}{2},1)$. □

<u>9.3 Corollary</u>. With ψ, δ and y^0 as in (7.1) - (7.6), we have

$$\mathscr{C} = \{\psi(t,\theta);\; \theta \in S^1,\; t \in [0,T_1(\theta))\} \cup B_\delta(y^0). \tag{9.5}$$

<u>Proof</u>: Define $\tilde{\mathscr{C}}$ to be the set on the right-hand side of (9.5). It is clear that $\tilde{\mathscr{C}} \subset \mathscr{C}$, and because of (8.3) and Remark 7.4, the Lebesgue measure of $\mathscr{C}\backslash\tilde{\mathscr{C}}$ is zero. Let $x \in \mathscr{C}\backslash\tilde{\mathscr{C}}$ be given, and define $(t,\theta) \triangleq \psi^{-1}(x)$. Then $t \geq T_1(\theta)$, but because $w(T_1(\theta),\theta) = 1$, we must in fact have $t > T_1(\theta)$. The continuity of T_1 and w allows us to choose an open neighborhood of (t,θ) contained in $\mathscr{C}\backslash\tilde{\mathscr{C}}$, and this contradicts the Lebesgue negligibility of $\mathscr{C}\backslash\tilde{\mathscr{C}}$. □

10. D^2u outside <6

We saw in Lemma 8.1 that D^2u is singular almost everywhere in R^2V6. Indeed

(10.1) $$u_{11}u_1 + u_{12}u_2 = 0, \quad u_{12}u_1 + u_{22}u_2 = 0 \quad \text{a.e. on } \mathbb{R}^2\backslash\mathscr{C},$$

and because $u_1^2 + u_{\wedge}^2 = 1$ on IR^2Y6, we have

(10.2) $$D^2u = Au \begin{bmatrix} u_2^2 & \sim u_1u_2 \\ -u_1u_2 & u_1^2 \end{bmatrix} \quad \text{a.e. on } \mathbb{R}^2\backslash\mathscr{C}.$$

Because u has continuous first partial derivatives on K^2, the proof of continuity of D^2u on R^2V6 reduces to a search for a continuous version of Au on this set. In order for D^2u to be continuous across $\&€$, we must also have $Au = u\text{-}h$ on $\&€$ (see Lemma 6.1).

We shall construct the desired continuous version of Au in the $(t,8)$ variables. Indeed, if we set

$$x(t,e) = \Delta u(\psi(t,\theta)) \quad \forall e c S^1, \; t \geq T_1(\theta).$$

then a formal calculation relying on (10.2) and the constancy of w on R^2V8 leads to

(10.3) $$\frac{d}{dt}\lambda(t,\theta) = \frac{1}{2}\,\Delta w(\psi(t.G)) \;\text{"}\; \|D^2u0Kt,9))\|^2$$

$$= -A^2(t,6) \quad \forall\, 9 \in S^1, \; t \geq T_1(\theta).$$

Integrating this equation and invoking the condition $\Delta u = u-h$ on $\partial\mathscr{C}$, we obtain

$$(10.4)\qquad \lambda(t,\theta) = \frac{u(\psi(T_1(\theta),\theta)) - h(\psi(T_1(\theta),\theta))}{1 + (t-T_1(\theta))[u(\psi(T_1(\theta),\theta)) - h(\psi(T_1(\theta),\theta))]}$$

$$\forall\ \theta \in S_1,\quad t \geq T_1(\theta).$$

The task before us is to show that with λ defined by (10.4), the function $\lambda \circ \psi^{-1}$ is a version of Δu on $\mathbb{R}^2\backslash\mathscr{C}$. This is essentially a justification of the formal differentiation in (10.3), which involved third-order derivatives of u.

Let $\rho\colon \mathbb{R}^2 \to [0,\infty)$ be a C^∞ function with support in $B_1(0)$ and satisfying $\int_{\mathbb{R}^2} \rho = 1$. For $n = 1,2,\ldots$, we define mollifications of u by

$$(10.5)\qquad u^{(n)}(x) \triangleq \int_{\mathbb{R}^2} u(x - \tfrac{1}{n}\,\xi)\ \rho(\xi)d\xi = n^2 \int_{\mathbb{R}^2} u(\xi)\ \rho(n(x-\xi))\ d\xi.$$

Then $\nabla u^{(n)}$ and $D^2u^{(n)}$ are locally bounded, uniformly in n, and $u^{(n)} \to u$, $\nabla u^{(n)} \to \nabla u$ and $D^2u^{(n)} \to D^2u$ in $L^1_{\ell oc}$. By passing to subsequences if necessary, we assume that these convergences occur almost everywhere. We define for $(t,\theta) \in [0,\infty) \times S^1$:

$$(10.6)\qquad \ell^{(n)}(t,\theta) \triangleq \Delta u^{(n)}(\psi(t,\theta)),\quad n = 1,2,\ldots,$$

$$(10.7)\qquad \ell\ (t,\theta) \triangleq \Delta u(\psi(t,\theta)),$$

and observe that $\ell^{(n)}(t,\theta) \to \ell(t,\theta)$ for almost every $(t,\theta) \in [0,\infty) \times S^1$

(Remark 7.4).

10.1 Lemma. The functions

$$\dot{\ell}^{(n)}(t,\theta) = \nabla \Delta u^{(n)}(\psi(t,\theta)) \cdot \nabla u(\psi(t,\theta)) \tag{10.8}$$

are locally bounded, uniformly in n.

Proof: Observe first of all that

$$\dot{\ell}^{(n)} = \nabla \Delta u^{(n)} \cdot \nabla u^{(n)} + \nabla \Delta u^{(n)} \cdot (\nabla u - \nabla u^{(n)})$$

$$= \frac{1}{2} \Delta(|\nabla u^{(n)}|^2) - \|D^2 u^{(n)}\|^2 + \nabla \Delta u^{(n)} \cdot (\nabla u - \nabla u^{(n)}),$$

where $\dot{\ell}^{(n)}$ is evaluated at (t,θ), and the right-hand side is evaluated at $\psi(t,\theta)$. It suffices to obtain uniform local bounds on $\Delta(|\nabla u^{(n)}|^2)$ and $\nabla \Delta u^{(n)} \cdot (\nabla u - \nabla u^{(n)})$.

Define for $i \in \{1,2\}$ the functions

$$F_{ii}^{(n)}(x) \triangleq \int_{\mathbb{R}^2} (|\nabla u(x - \tfrac{1}{n}\xi)|^2)_{i,i}\, \rho(\xi)d\xi$$

$$= n \int_{\mathbb{R}^2} (|\nabla u(x - \tfrac{1}{n}\xi)|^2)_i\, \rho_i(\xi)d\xi$$

$$= 2n \int_{\mathbb{R}^2} \nabla u_i(x - \tfrac{1}{n}\xi) \cdot \nabla u(x - \tfrac{1}{n}\xi)\, \rho_i(\xi)d\xi,$$

$$n = 1,2,\ldots,$$

and note that these functions are uniformly bounded in n (Theorem 8.3). Then

$$(|\nabla u^{(n)}(x)|^2)_{ii} = 2n^6 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla u(\xi) \cdot \nabla u(\eta)[\rho_{ii}(n(x-\xi))\rho(n(x-\eta))$$

$$+ \rho_i(n(x-\xi))\rho_i(n(x-\eta))]d\xi d\eta$$

$$= 2n^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla u(x - \frac{1}{n}\xi) \cdot \nabla u(x - \frac{1}{n}\eta)[\rho_{ii}(\xi)\rho(\eta)$$

$$+ \rho_i(\xi)\rho_i(\eta)]d\xi d\eta$$

$$= 2n \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla u_i(x - \frac{1}{n}\xi) \cdot \nabla u(x - \frac{1}{n}\eta)\rho_i(\xi)\rho(\eta)d\xi d\eta$$

$$+ 2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla u_i(x - \frac{1}{n}\xi)\, \nabla u_i(x - \frac{1}{n}\eta)\rho(\xi)\rho(\eta)d\xi d\eta.$$

The last term is locally bounded in x, uniformly in n. The next to last term is

$$F_{ii}^{(n)}(x) + 2n \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla u_i(x - \frac{1}{n}\xi) \cdot [\nabla u(x - \frac{1}{n}\eta) - \nabla u(x - \frac{1}{n}\xi)]$$

$$\rho_i(\xi)\rho(\eta)d\xi d\eta,$$

which is also locally bounded in x, uniformly in n, because $\forall\ \xi,\eta \in B_1(0)$,

$$|\nabla u(x - \frac{1}{n}\eta) - \nabla u(x - \frac{1}{n}\xi)| \le \frac{2}{n} \sup_{B_1(x)} \|D^2u\|.$$

This provides a uniform local bound on $\Delta(|\nabla u^{(n)}|^2)$.

On the other hand,

$$\nabla \Delta u^{(n)}(x) \cdot (\nabla u(x) - \nabla u^{(n)}(x))$$

$$= n^5 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Delta u(\xi)[\nabla u(x) - \nabla u(\eta)] \cdot \nabla\rho(n(x-\xi))\rho(n(x-\eta))d\xi d\eta$$

$$= n \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Delta u(x - \frac{1}{n}\xi)[\nabla u(x) - \nabla u(x - \frac{1}{n}\eta)] \cdot \nabla\rho(\xi)\rho(\eta)d\xi d\eta,$$

and the boundedness of this expression follows from the local Lipschitz continuity of ∇u. □

Because of Lemma 10.1, a subsequence of $\{\dot{\ell}^{(n)}\}_{n=1}^{\infty}$ converges in the $L^{\infty}_{\ell oc}$-weak* topology to a function $\zeta \in L^{\infty}_{\ell oc}([0,\infty) \times S_1)$. We assume without loss of generality that the full sequence converges. For each nonnegative integer k, choose a number $t_k > k$ such that $\{\ell^{(n)}(t_k,\theta)\}_{n=1}^{\infty}$ converges for a.e. $\theta \in S_1$, and define $\lambda_k(t_k,\theta)$ to be this limit. (Whereas $\ell(\cdot,\cdot)$ is defined up to almost everywhere equivalence on $[0,\infty) \times S_1$, the functions $\lambda_k(t_k,\cdot)$ are defined up to almost everywhere equivalence on S_1.) We insist furthermore that t_0 be chosen so that $\psi(t_0,\theta) \in \mathscr{C}\ \forall\ \theta \in S_1$. Then $\Delta u(\psi(t_0,\cdot))$ is defined pointwise on S_1 because Δu is continuous on $\mathscr{C}$, and so we may require that

$$\lambda_0(t_0,\theta) = \Delta u(\psi(t_0,\theta)) \quad \forall\ \theta \in S_1.$$

For each $k = 0,1,\dots,$ define $\lambda_k\colon [0,\infty) \times S^1 \mapsto \mathbb{R}$ by

$$\lambda_k(t,\theta) \triangleq \lambda_k(t_k,\theta) + \int_{t_k}^{t} \zeta(s,\theta)ds,$$

so that any two versions $\hat{\lambda}_k$ and $\tilde{\lambda}_k$ of this function have the property that the set $\{\theta \in S_1 \mid \exists\ t \in [0,\infty) \text{ with } \hat{\lambda}_k(t,\theta) \neq \tilde{\lambda}_k(t,\theta)\}$ has measure zero.

We now relate the functions λ_k, $k = 0,1,\dots,$ to the function ℓ of (10.7). Let φ be a continuous, real-valued function on $[0,\infty) \times S_1$, and define

$$\Phi(t,\theta) \triangleq \int_0^t \varphi(s,\theta)\ ds \quad \forall\ (t,\theta) \in [0,\infty) \times S_1.$$

For $k = 0,1,\dots,$

$$\int_{S_1} \int_0^{t_k} \lambda_k(s,\theta)\varphi(s,\theta)\ dsd\theta$$

$$= \int_{S_1} [\lambda_k(t_k,\theta)\Phi(t_k,\theta) - \int_0^{t_k} \zeta(s,\theta)\varphi(s,\theta)ds]d\theta$$

$$= \lim_{n\to\infty} \int_{S_1} [\ell^{(n)}(t_k,\theta)\Phi(t_k,\theta) - \int_0^{t_k} \dot{\ell}^{(n)}(s,\theta)\varphi(s,\theta)ds]d\theta$$

$$= \lim_{n\to\infty} \int_{S_1} \int_0^{t_k} \ell^{(n)}(s,\theta)\varphi(s,\theta)dsd\theta$$

$$= \int_{S_1} \int_0^{t_k} \ell(s,\theta)\varphi(s,\theta)dsd\theta.$$

It follows that $\lambda_k = \ell$ a.e. on $[0,t_k] \times S_1$. In particular, for any two nonnegative integers k and m, λ_k and λ_m agree almost everywhere on $[0,t_k \wedge t_m] \times S_1$, and hence almost everywhere on $[0,\infty) \times S_1$. In particular,

$$\lambda_0(t,\theta) = \Delta u(\psi(t,\theta)), \quad \text{a.e. } (t,\theta) \in [0,\infty) \times S^1, \tag{10.9}$$

and for a.e. $\theta \in S_1$,

$$\lambda_0(t,\theta) = \Delta u(\psi(t_0,\theta)) + \int_{t_0}^{t} \zeta(s,\theta)ds \quad \forall\ t \in [0,\infty). \tag{10.10}$$

<u>10.2 Lemma</u>. Almost everywhere on the set

$$\psi^{-1}(\mathbb{R}^2 \backslash \mathscr{C}) = \{(t,\theta) \in [0,\infty) \times S_1;\ t \geq T_1(\theta)\},$$

the function ζ appearing in (10.10) is equal to $-\lambda_0^2$.

<u>Proof</u>: From (10.8) we have

$$\dot{\ell}^{(n)} \circ \psi^{-1} + (\ell \circ \psi^{-1})(\ell^{(n)} \circ \psi^{-1})$$

$$= \nabla\, \Delta u^{(n)} \cdot \nabla u + \Delta u\, \Delta u^{(n)}$$

$$= (u_{12}^{(n)} u_2 + u_{11}^{(n)} u_1)_1 + (u_{12}^{(n)} u_1 + u_{22}^{(n)} u_2)_2$$

$$+ u_{11}^{(n)} u_{22} + u_{11} u_{22}^{(n)} - 2u_{12}^{(n)} u_{12}.$$

Now $u^{(n)}_{11}u_{22} + u_{11}u^{(n)}_{22} - 2u^{(n)}_{12}u_{12}$ is locally bounded, uniformly in n, and converges almost everywhere to $2 \det D^2 u$, which is zero on $\mathbb{R}^2\setminus\mathscr{C}$. It follows from (10.1) that for any function $\varphi \in C^1_0(\mathbb{R}^2\setminus\bar{\mathscr{C}})$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^2\setminus\bar{\mathscr{C}}} [\dot{\ell}^{(n)}\circ\psi^{-1} + (\ell\circ\psi^{-1})(\ell^{(n)}\circ\psi^{-1})]\,\varphi$$

$$= \lim_{n\to\infty}\int_{\mathbb{R}^2\setminus\bar{\mathscr{C}}} [(u^{(n)}_{12}u_2 + u^{(n)}_{11}u_1)_1 + (u^{(n)}_{12}u_1 + u^{(n)}_{22}u_2)_2]\,\varphi$$

$$= -\lim_{n\to\infty}\int_{\mathbb{R}^2\setminus\bar{\mathscr{C}}} (u^{(n)}_{12}u_2 + u^{(n)}_{11}u_1)\varphi_1 + (u^{(n)}_{12}u_1 + u^{(n)}_{22}u_2)\varphi_2$$

$$= 0.$$

Because the functions $\dot{\ell}^{(n)}\circ\psi^{-1} + (\ell\circ\psi^{-1})(\ell^{(n)}\circ\psi^{-1})$ are locally bounded, uniformly in n, we can show that for every $\varphi \in L^1(\mathbb{R}^2\setminus\bar{\mathscr{C}})$,

(10.11) $$\lim_{n\to\infty}\int_{\mathbb{R}^2\setminus\bar{\mathscr{C}}} [\dot{\ell}^{(n)}\circ\psi^{-1} + (\ell\circ\psi^{-1})(\ell^{(n)}\circ\psi^{-1})]\,\varphi = 0.$$

Now let $\gamma \in L^1(\psi^{-1}(\mathbb{R}^2\setminus\bar{\mathscr{C}}))$ be given so that $(\gamma\circ\psi^{-1})|J\psi^{-1}| \in L^1(\mathbb{R}^2\setminus\bar{\mathscr{C}})$, where $|J\psi^{-1}|$ is the bounded (Theorem 7.3) determinant of the Jacobian of ψ^{-1}. From (10.11) it follows that

$$\lim_{n\to\infty}\int_{\psi^{-1}(\mathbb{R}^2\setminus\bar{\mathscr{C}})} (\dot{\ell}^{(n)} + \ell\ell^{(n)})\gamma$$

$$= \lim_{n\to\infty}\int_{\mathbb{R}^2\setminus\bar{\mathscr{C}}} [\dot{\ell}^{(n)}\circ\psi^{-1} + (\ell\circ\psi^{-1})(\ell^{(n)}\circ\psi^{-1})](\gamma\circ\psi^{-1})|J\psi^{-1}|$$

$$= 0.$$

On the other hand, $\dot{\ell}^{(n)} + \ell\ell^{(n)}$ converges in the $L^{\infty}_{\ell oc}$-weak* topology on $[0,\infty) \times S_1$ to $\zeta + \ell^2 = \zeta + \lambda_0^2$ a.e., and the lemma follows. □

<u>10.3 Theorem</u>. There is a Lipschitz continuous version of D^2u on $\mathbb{R}^2$.

<u>Proof</u>: For $\theta \in S_1$ and $0 \leq t < T_1(\theta)$, define

$$\lambda(t,\theta) \triangleq \Delta u(\psi(t,\theta)), \tag{10.12}$$

where, of course, we mean the Lipschitz continuous version of Δu inside $\mathscr{C}$ (Corollary 8.4). For $\theta \in S_1$ and $t \geq T_1(\theta)$, define $\lambda(t,\theta)$ by (10.4), which gives us a Lipschitz function. At $t = T_1(\theta)$, the Lipschitz continuity of λ follows from (10.4), Lemma 6.1, and the equality $|\nabla u|^2 = 1$ on $\partial\mathscr{C}$. The Lipschitz continuity of ψ^{-1} implies the Lipschitz continuity of $\lambda \circ \psi^{-1}$.

It remains to show that $\lambda \circ \psi^{-1}$ is a version of Δu, or equivalently,

$$\lambda(t,\theta) = \Delta u(\psi(t,\theta)), \text{ a.e. } (t,\theta) \in [0,\infty) \times S_1. \tag{10.13}$$

In light of (10.9) and (10.12), we need only show that for a.e. $\theta \in S_1$,

$$\lambda(t,\theta) = \lambda_0(t,\theta) \quad \forall \quad t \geq T_1(\theta). \tag{10.14}$$

But (10.10) shows that for a.e. $\theta \in S_1$, the function $t \mapsto \lambda_0(t,\theta)$ is absolutely continuous on $[0,\infty)$; in particular,

(10.15)
$$\lambda_0(T_1(\theta),\theta) = \lim_{t\uparrow T_1(\theta)} \lambda_0(t,\theta)$$

$$= \lim_{t\uparrow T_1(\theta)} \Delta u(\psi(t,\theta))$$

$$= \lim_{t\uparrow T_1(\theta)} [u(\psi(t,\theta)) - h(\psi(t,\theta))]$$

$$= u(\psi(T_1(\theta),\theta)) - h(\psi(T_1(\theta),\theta)).$$

Equation (10.10) and Lemma 10.2 imply that for a.e. $\theta \in S_1$,

(10.16)
$$\dot{\lambda}_0(t,\theta) = -\lambda_0^2(t,\theta), \quad \text{a.e. } t \geq T_1(\theta).$$

Equations (10.15) and (10.16) imply (10.14). □

11. Regularity of the free boundary

In this section we apply known regularity results for free boundaries to show that the boundary of $\mathcal{C}$ is of class $C^{2,\alpha}$ for all $\alpha \in (0,1)$. In order to apply these results, we recall that $w = |\nabla u|^2$ is a $W^{2,\infty}$ function (Theorem 8.3) which satisfies (see (5.9)) $1-w \geq 0$ on $\mathbb{R}^2$ and

$$\frac{1}{2}\Delta(1-w) = H-w \quad \text{on} \quad \mathcal{C}, \tag{11.1}$$

where we recall that $H \triangleq \nabla h \cdot \nabla u - \|D^2u\|^2$. We shall establish the strict positivity of the forcing term $H-w$ on $\partial\mathcal{C}$. Recall that

$$w - \frac{1}{2}\Delta w - H \leq 0 \quad \text{on} \quad \mathbb{R}^2,$$

and $w = 1$, $\Delta w = 0$ on $\mathbb{R}^2 \backslash \bar{\mathcal{C}}$, so

$$H-w = H-1 \geq 0 \quad \text{on} \quad \mathbb{R}^2 \backslash \mathcal{C}. \tag{11.2}$$

<u>11.1 Lemma</u>. The function H is locally Lipschitz continuous, and $H > 1$ on $\partial\mathcal{C}$.

<u>Proof</u>: The local Lipschitz continuity of H follows from Theorem 10.3. To prove that $H > 1$ on $\partial\mathcal{C}$, we assume that there exists a point on $\partial\mathcal{C}$ where $H = 1$. Without loss of generality, we take this point to be the origin $(0,0)$, and we take $\nabla u(0,0) = (-1,0)$.

We first obtain an upper bound on H near $(0,0)$. Inside $\mathcal{C}$, H is differentiable and

$$(11.3) \qquad \nabla H \cdot \nabla u = (D^2h)\nabla u \cdot \nabla u + (D^2u)\nabla u \cdot \nabla h - \nabla(\|D^2u\|^2) \cdot \nabla u.$$

Let υ^1 and υ^2 be unit eigenvectors for D^2u, and let λ_1 and λ_2 denote their respective (nonnegative) eigenvalues. Then

$$(11.4) \qquad \nabla(\|D^2u\|^2) \cdot \nabla u = \mathrm{tr}\ (D^2w\ D^2u) - 2\ \mathrm{tr}\ [(D^2u)^3]$$

$$= \lambda_1(D^2w)\upsilon^1\cdot\upsilon^1 + \lambda_2(D^2w)\upsilon^2\cdot\upsilon^2 - 2(\lambda_1^3 + \lambda_2^3)$$

$$\leq 2\ \|D^2u\|_{L^\infty(\mathscr{C})} \sup_{\upsilon\in S_1} (D^2w)\upsilon\cdot\upsilon.$$

Applying Theorem 1 and the remark following it from Caffarelli (1977) to the function $1-w$, we have that for some positive constants C and ϵ,

$$(11.5) \quad \sup_{\upsilon\in S_1} D^2w(x,y)\upsilon\cdot\upsilon \leq C\ |\log(\mathrm{dist}((x,y),\ \partial\mathscr{C}))|^{-\epsilon} \quad \forall\ (x,y) \in \mathscr{C}.$$

Combining (11.3)-(11.5), we conclude that

$$(11.6)\ \nabla H(x,y) \cdot \nabla u(x,y) \geq D^2h(x,y)\nabla u(x,y) \cdot \nabla u(x,y) + \frac{1}{2} \nabla w(x,y) \cdot \nabla h(x,y)$$

$$- 2\ \|D^2u\|_{L^\infty(\mathscr{C})}\ C\ |\log(\mathrm{dist}((x,y),\ \partial\mathscr{C}))|^{-\epsilon} \quad \forall\ (x,y) \in \mathscr{C}.$$

As (x,y) approaches $(0,0) \in \partial\mathscr{C}$, $|\nabla u(x,y)|$ approaches 1 and $\nabla w(x,y)$ approaches 0. Using (2.8) and (11.6), we can choose $\tilde{\epsilon} > 0$ such that

(11.7) $$\nabla H(x,y) \cdot \nabla u(x,y) \geq \frac{c_0}{2} \quad \forall\ (x,y) \in [-\tilde{\varepsilon},\tilde{\varepsilon}]^2 \cap \mathscr{C}.$$

Let $\theta_0 \in S_1$ be such that $\psi(T_1(\theta_0),\theta_0) = (0,0)$. For $t \in (0,T_1(\theta_0))$ chosen so that $\psi(t,\theta_0) \in [-\tilde{\varepsilon},\tilde{\varepsilon}]^2$,

$$\frac{d}{dt} H(\psi(t,\theta_0)) = \nabla H(\psi(t,\theta_0)) \cdot \nabla u(\psi(t,\theta_0)) \geq \frac{c_0}{2} .$$

It follows that for some $\tau > 0$,

(11.8) $$H(\psi(T_1(\theta_0) - t,\theta_0)) \leq H(\psi(T_1(\theta_0),\theta_0)) - \frac{1}{2} c_0 t$$

$$= 1 - \frac{1}{2} c_0 t \quad \forall \quad t \in (0,\tau).$$

But also

(11.9) $$|\psi(T_1(\theta_0) - t,\theta_0) - (t,0)| = |\psi(T_1(\theta_0) - t,\theta_0) - \psi(T_1(\theta_0),\theta_0)$$

$$+ t\ \nabla u(\psi(T_1(\theta_0),\theta_0))|$$

$$\leq t^2 \|D^2 u\|_{L^\infty(\mathscr{C})} \quad \forall \quad t \in (0,T_1(\theta_0)).$$

Let $\beta > 0$ be a Lipschitz constant for H in a sufficiently large neighborhood of $(0,0)$. From (11.8), (11.9), we have for all $t \in (0,\tau)$:

$$H(t,0) \le H(\psi(T_1(\theta) - t,\ \theta_0)) + |H(t,0) - H(\psi(T_1(\theta_0) - t,\ \theta_0))|$$

$$\le 1 - \frac{1}{2} c_0 t \ + \ \beta t^2 \ \|D^2 u\|_{L^\infty(\mathscr{C})} .$$

Choosing τ smaller, if necessary, we have $H(t,0) \le 1 - \frac{1}{3} c_0 t$ for all $t \in (0,\tau)$. Again using the Lipschitz continuity of H, we obtain the desired upper bound

$$(11.10) \qquad H(x,y) \le 1 - \frac{1}{3} c_0 x + \beta |y| \quad \forall \ (x,y) \in [0,\tau] \times [-\tau,\tau].$$

We next construct a function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ such that for appropriate $\rho,\sigma \in (0,\tau)$,

$$(11.11) \qquad \varphi - \frac{1}{2} \Delta\varphi \ge H \quad \text{on} \quad [0,\rho] \times [-\sigma,\sigma],$$

$$(11.12) \qquad \varphi \ge 1 \quad \text{on} \quad \partial([0,\rho] \times [-\sigma,\sigma]),$$

$$(11.13) \qquad \varphi(0,0) = 1.$$

For this purpose, choose $0 < \rho < \min\{\tau, \dfrac{c_0}{6\sqrt{2}\,\beta}\}$ such that

$$(11.14) \qquad (1 - \frac{\rho^2}{4}) \sinh \sqrt{2}\,\rho \ge \sqrt{2}\,\rho .$$

Then define

(11.15) $$a \triangleq \min\{1, \frac{\pi^2}{4\sqrt{2}}\}$$

(11.16) $$A \triangleq \frac{c_0}{3}\left(1 - \frac{\sqrt{2}\, p}{\sinh\sqrt{2}\, p \, \cosh\sqrt{2}\, a}\right)^{-1}.$$

$$\varphi(x,y) \triangleq 1 + Pa\left(2 - \frac{\cosh\sqrt{2}\, y}{\cosh\sqrt{2}\, a}\right)\left(1 - \frac{\sinh\sqrt{2}\,x + \sinh\sqrt{2}\,(p-x)}{\sinh\sqrt{2}\, p}\right)$$

$$+ Ap\left(1 - \frac{\cosh\sqrt{2}\, y}{\cosh\sqrt{2}\, a}\right)\left\{-\frac{x}{p} + \frac{\sinh\sqrt{2}\, x}{\sinh\sqrt{2}\, p}\right\} \quad \forall (x,y) \in \mathbb{R}^2.$$

Then

$$\varphi(0,y) = \varphi(p,y) = 1 \quad \forall\, y \in [-a,a],$$

$$\varphi(x,\pm a) = 1 + \beta a\left[1 - \frac{\sinh\sqrt{2}\, x + \sinh\sqrt{2}\,(p-x)}{\sinh\sqrt{2}\, p}\right] \geq 1 \quad \forall\, x \in [0,p]$$

because

(11.17) $$\sinh a + \sinh b \leq \sinh a \cosh b + \sinh b \cosh a$$

$$= \sinh(a+b) \quad \forall\, a,b \in \mathbb{R}.$$

It remains to verify (11.11). Direct computation reveals

$$\varphi(x,y) - \frac{1}{2}\Delta\,\varphi(x,y) = 1 + 2\beta\sigma - Ax + A\rho\,\frac{\sinh\sqrt{2}\,x\,\cosh\sqrt{2}\,y}{\sinh\sqrt{2}\,\rho\,\cosh\sqrt{2}\,\sigma}$$

$$- \beta\sigma\,\frac{\cosh\sqrt{2}\,y}{\cosh\sqrt{2}\,\sigma}\left(\frac{\sinh\sqrt{2}\,x + \sinh\sqrt{2}\,(\rho-x)}{\sinh\sqrt{2}\,\rho}\right)$$

$$\geq 1 + \beta\sigma - Ax + A\rho\,\frac{\sqrt{2}\,x}{\sinh\sqrt{2}\,\rho\,\cosh\sqrt{2}\,\sigma}$$

$$\geq 1 - \left(1 - \frac{\sqrt{2}\,\rho}{\sinh\sqrt{2}\,\rho\,\cosh\sqrt{2}\,\sigma}\right) Ax + \beta\sigma$$

$$\geq 1 - \frac{1}{3}\,c_0 x + \beta\,|y|$$

$$\geq H(x,y) \quad \forall \quad (x,y) \in [0,\rho] \times [-\sigma,\sigma],$$

where we have used (11.17), the inequality $a \leq \sinh a \;\; \forall\, a \geq 0$, (11.16), and (11.10).

On the other hand, (5.9) implies that

$$w - \frac{1}{2}\Delta w \leq H \quad \text{on} \quad [0,\rho] \times [-\sigma,\sigma]$$

$$w \leq 1 \quad \text{on} \quad \partial([0,\rho] \times [-\sigma,\sigma]).$$

The maximum principle implies that $w \leq \varphi$ on $[0,\rho] \times [-\sigma,\sigma]$. In particular, $\forall\, x \in [0,\rho]$,

$$w(x,0) - w(0,0) = w(x,0) - 1 \leq \sigma(x,0) - 1 = \varphi(x,0) - \varphi(0,0),$$

and thus

$$0 = \frac{\partial}{\partial x} w(0,0) \le \frac{\partial}{\partial x} \varphi(0,0). \tag{11.18}$$

The final step in the proof is to show that $\frac{\partial}{\partial x} \varphi(0,0) < 0$, so (11.18) is contradicted, as well as the assumption that $H = 1$ at some point on $\partial\mathscr{C}$. We compute

$$\frac{\partial}{\partial x} \varphi(0,0) = \sqrt{2}\, \beta\sigma(2 - \frac{1}{\cosh \sqrt{2}\,\sigma})(\frac{\cosh \sqrt{2}\,\rho - 1}{\sinh \sqrt{2}\,\rho}) \tag{11.19}$$

$$- A(1 - \frac{1}{\cosh \sqrt{2}\,\sigma})(1 - \frac{\sqrt{2}\,\rho}{\sinh \sqrt{2}\,\rho}).$$

The first term on the right-hand side of (11.19) is bounded above by

$$2 \sqrt{2}\, \beta\sigma\, (\frac{\cosh \sqrt{2}\,\rho - 1}{\sinh \sqrt{2}\,\rho}) \le 2\beta\sigma\rho.$$

As for the second term, (11.14) and the inequality $\cosh \sqrt{2}\,\sigma - 1 \ge \sqrt{2}\,\sigma$ imply that

$$A(1 - \frac{1}{\cosh \sqrt{2}\,\sigma})(1 - \frac{\sqrt{2}\,\rho}{\sinh \sqrt{2}\,\rho})$$

$$= \frac{c_0}{3} [(1 - \frac{\sqrt{2}\,\rho}{\sinh \sqrt{2}\,\rho})^{-1} + (\cosh \sqrt{2}\,\sigma - 1)^{-1}]^{-1}$$

$$\ge \frac{c_0}{3} [\frac{4}{\rho^2} + \frac{1}{\sqrt{2}\,\sigma}]^{-1}$$

$$= \frac{c_0}{3} (\frac{\sqrt{2}\,\rho^2\sigma}{\rho^2 + 4\sqrt{2}\,\sigma}).$$

Therefore,

$$\frac{\partial}{\partial x}\varphi(0,0) \le \sigma[2\beta\rho - \frac{c_0}{3}(\frac{\sqrt{2}\,\rho^2}{\rho^2 + 4\sqrt{2}\,\sigma})],$$

and (11.15) and the choice of ρ show that

$$\frac{\partial}{\partial x}\varphi(0,0) \le \sigma[2\beta\rho - \frac{c_0}{3\sqrt{2}}] < 0. \qquad \square$$

<u>11.2 Theorem</u>. The free boundary $\partial\mathcal{C}$ is of class C^1, and w has continuous second partial derivatives inside $\mathcal{C}$ up to $\partial\mathcal{C}$.

<u>Proof</u>: Because T_1 is Lipschitz (Theorem 9.2), for evey $\bar{\theta} \in S_1$, the point $(T_1(\bar{\theta}),\theta)$ is a point of positive density with respect to the measure of Remark 7.4 for the set $\{(t,\theta) \mid \theta \in S^1,\ t \in (T_1(\theta),\infty)\} = \psi(\mathbb{R}^2\setminus\bar{\mathcal{C}})$. But ψ and ψ^{-1} are locally Lipschitz, so every point of $\partial\mathcal{C}$ is a point of positive Lebesgue density for $\mathbb{R}^2\setminus\mathcal{C}$. It follows from Theorem 2 of Caffarelli (1977) that $\partial\mathcal{C}$ is Lipschitz. Caffarelli's Theorem 3 can now be applied (with v in Caffarelli's Assumption (H1) equal to our $1-w$), and it yields the desired results. $\square$

<u>11. 3 Corollary</u>. The boundary $\partial\mathcal{C}$ is of class $C^{2,\alpha}$ for every $\alpha \in (0,1)$.

<u>Proof</u>: In light of Theorems 6.3 and 11.2 and equation (6.4), D^2u has a C^1 extension from $\mathcal{C}$ to $\bar{\mathcal{C}}$. Therefore, $H-w$ appearing on the right-hand side of (11.1) has a C^1 extension from $\mathcal{C}$ to $\bar{\mathcal{C}}$, and because $\partial\mathcal{C}$ is of class C^1,

$H-w$ has a C^1 extension to an open set containing $\bar{\mathscr{C}}$. (In Lemma 12.4, we explain in some detail how to construct a similar extension.) Lemma 11.1 and Theorem 11.2 permit us to apply a theorem of Kinderlehrer & Nirenberg (1977) (see also Theorem 1.1(i) of Friedman (1982), p. 129), to conclude that $\partial\mathscr{C}$ is of class $C^{1,\alpha}$ for every $\alpha \in (0,1)$.

Now observe that ∇w solves the problem

$$\nabla w - \frac{1}{2}\Delta\nabla w = \nabla H \quad \text{in} \quad \mathscr{C},$$

$$\nabla w = 0 \quad \text{on} \quad \partial\mathscr{C}.$$

Since ∇H is continuous up to $\partial\mathscr{C}$ and $\partial\mathscr{C}$ is $C^{1,\alpha}$, Theorem 8.34 of Gilbarg & Trudinger (1983), p. 211, implies that ∇w is of class $C^{1,\alpha}$ on $\mathscr{C}$ up to $\partial\mathscr{C}$. Inserting this regularity into (6.4), we conclude that D^2u, and hence $H-w$, are of class $C^{1,\alpha}$ on $\mathscr{C}$ up to $\partial\mathscr{C}$. We may again appeal to Theorem 1.1 of Friedman (1982) to conclude that $\partial\mathscr{C}$ is of class $C^{2,\alpha}$ for every $\alpha \in (0,1)$. □

<u>11.4 Remark</u>. The bootstrapping in Corollary 11.3 can be continued until the regularity of h is exhausted. If, in place of assumption (2.5), we assume that $h \in C^{k,\alpha}_{\ell oc}$ for some $k \geq 3$ and $\alpha \in (0,1)$, then the free boundary is of class $C^{k,\alpha}$, w is of class $C^{k,\alpha}$ inside $\mathscr{C}$ up to $\partial\mathscr{C}$, and u is of class $C^{k+1,\alpha}$ inside $\mathscr{C}$ up to $\partial\mathscr{C}$. This argument uses Theorem 1.1, p. 107, of Ladyzhenskaya & Ural'tseva (1968), to wit, if ∇H is of class $C^{k-3,\alpha}$ up to $\partial\mathscr{C}$ and $\partial\mathscr{C}$ is $C^{k-1,\alpha}$, then ∇w is of class $C^{k-1,\alpha}$ up to $\partial\mathscr{C}$.

12. Construction of the optimal control process

12.1 Definition. Let x € <6 be given. A control process pair {(N $_t$.f $_t$); 0 ≤ t < <»} as in Section 2 is called a solution to the Skorohod problem for reflected Brownian motion in ^ starting at x and with reflection direction - vu along ff€ provided that:

(a) f is continuous,

(b) the process X defined by (2.1) satisfies X_t € «, 0 <_t < °°, a.s., and

(c) for all 0 ≤ t < ∞,

$$(12.1)\qquad \zeta_t = \int_0^t 1_{\{X_s \in \partial\mathcal{G},\ N_s = -\nabla u(X_s)\}}\, d\zeta_s \qquad \text{a.s.}$$

For every x € <6, the Skorohod problem of Definition 2.1 has a solution starting at x. This follows from Theorem 4.3 of Lions & Sznitman (1984), provided that the following three conditions are satisfied:

(Cl) <6 has a C^1 boundary and satisfies a uniform exterior sphere condition,

(C2) 3 *a* > 0 such that vu(x) • n(x) ≥ *a* V x € 8<g, where n(x) is the outward normal vector for <6 at x,

(C3) vu on % has an extension to a C^2 function on an open set containing $\bar{\mathcal{G}}$.

Condition (Cl) is implied by Corollary 11.3. We establish (C2) and (C3).

12.2 Lemma. Condition (C2) is satisfied.

Proof: Let x € &€ be given. We construct a sequence $\{x_k\}_{k=1}^{\infty}$ in <6 such that x_k -> x and $\frac{\nabla w(x_k)}{|\nabla w(x_k)|}$ → n(x). With K as in Lemma 9.1, we have

$$\frac{\nabla w(x_k)\cdot\nabla u(x_k)}{|\nabla w(x_k)|} \geq \frac{2}{K} ,$$

and (C2) follows.

As for the construction of $\{x_k\}_{k=2}^{\infty}$, we choose $r > 0$ such that $B_r(x + rn(x)) \cap \mathscr{C} = \phi$. Define $\bar{x} = x + \frac{1}{2} rn(x)$, so $B_{r/2}(\bar{x}) \cap \mathscr{C} = \phi$ and $x \in \partial B_{r/2}(\bar{x})$. Given $k \geq 2$, we define $\mathscr{C}_k \triangleq \{x \in \mathbb{R}^2;\ w(x) < 1 - \frac{1}{k}\}$. We then translate $B_{r/2}(\bar{x})$ in the $-n(x)$ direction until it touches $\partial\mathscr{C}_k$, i.e., we define

$$\rho_k = \sup\{\rho > 0;\ \ B_{r/2}(\bar{x} - \rho n(x)) \cap \mathscr{C}_k = \phi\},$$

and we choose $x_k \in \overline{B_{r/2}(\bar{x} - \rho_k n(x))} \cap \partial\mathscr{C}_k$. Then $B_{r/2}(\bar{x} - \rho_k n(x))$ is an exterior sphere for $\partial\mathscr{C}_k$ at x_k, so the outward normal to $\mathscr{C}_k$ at x_k is

$$\frac{\nabla w(x_k)}{|\nabla w(x_k)|} = \frac{\bar{x} - \rho_k n(x) - x_k}{|\bar{x} - \rho_k n(x) - x_k|} .$$

As $k \to \infty$, we have $x_k \to x$ and $\rho_k \to 0$, so $\dfrac{\nabla w(x_k)}{|\nabla w(x_k)|} \to n(x)$. □

<u>12.3 Lemma</u>. Condition (C3) is satisified.

<u>Proof</u>: Given $\epsilon > 0$, we can find a finite set of open discs $\{B_k\}_{k=1}^{n}$, each with radius ϵ, such that $\bar{\mathscr{C}} \subset \bigcup_{k=1}^{n} B_k$, and we can find C^∞ functions $\gamma_k\colon \mathbb{R}^2 \to [0,1]$ such that $\overline{\text{supp}\ \gamma_k} \subset B_k$ for every k and $\sum_{k=1}^{n} \gamma_k = 1$ on $\mathscr{C}$.

We can decompose u on $\mathscr{C}$ as $\sum_{k=1}^{n} \gamma_k u$, so it suffices to show that each $u_k \triangleq \gamma_k u$ has a C^2 extension from $B_k \cap \mathscr{C}$ to B_k. For sufficiently small $\epsilon > 0$, in each B_k there is a C^2 change of coordinates which results in $B_k \cap \bar{\mathscr{C}} \subset \{(x,y) \mid x \le 0\}$ and $B_k \backslash \bar{\mathscr{C}} \subset \{(x,y) \mid x > 0\}$. Now u_k has a C^2 extension from $B_k \cap \mathscr{C}$ to $B_k \cap \bar{\mathscr{C}}$ (Theorem 6.3), and taking u_k to be zero on $\{(x,y) \mid x \le 0\} \backslash (B_k \cap \bar{\mathscr{C}})$, we have a C^2 function on the closed left half-plane. For $x > 0$, $y \in \mathbb{R}$, define

$$u_k(x,y) = 3\, u_k(0,y) - 3\, u_k(-x,y) + u_k(-2x,y).$$

It is easy to check that this extended u_k is C^2 on all of $\mathbb{R}^2$. □

<u>12.4 Theorem</u>. Let $x \in \mathbb{R}^2$ be given. If $x \in \bar{\mathscr{C}}$, then the solution to the Skorohod problem of Definition 12.1 is an optimal control process pair for the singular stochastic control problem with initial condition x posed in Section 2. If $x \notin \bar{\mathscr{C}}$, then there exists a unique pair $(t,\theta) \in [0,\infty) \times S_1$ such that $x = \psi(t,\theta)$. Define $\hat{x} \triangleq \psi(T_1(\theta),\theta)$ and let $(\hat{N},\hat{\zeta})$ be a solution to the Skorohod problem starting at $\hat{x}$. Then (N,ζ) is optimal for the control problem with initial condition x, where

(12.2)
$$N_t \triangleq \begin{cases} -\nabla u(\bar{x}) & \text{if } t = 0, \\ \hat{N}_t & \text{if } t > 0, \end{cases}$$

(12.3)
$$\zeta_t \triangleq \begin{cases} 0 & \text{if } t = 0 \\ \hat{\zeta}_t + |x - \hat{x}| & \text{if } t > 0. \end{cases}$$

In either case, we have that $u(x) = V(x)$, where u is the solution to the HJB equation (3.1) (see Theorem 4.6) and V is the value function for the control problem defined by (2.10).

Proof: The theorem follows immediately from Theorem 3.1 once we observe that in the case $x \notin \bar{\mathcal{C}}$, Lemma 8.1 implies that for all $s \geq T_1(\theta)$,

$$\begin{aligned}\nabla u(\psi(s,\theta)) &= \nabla u(\hat{x}) + \int_{T_1(\theta)}^{s} \frac{d}{d\tau} \nabla u(\psi(\tau,\theta))d\tau \\ &= \nabla u(\hat{x}) + \int_{T_1(\theta)}^{s} D^2u(\psi(\tau,\theta))\nabla u(\psi(\tau,\theta))d\tau \\ &= \nabla u(\hat{x}).\end{aligned}$$

Thus, when $x \notin \mathcal{C}$, the control process pair (N,ζ) of (12.2), (12.3) causes the state to jump from $X_0 = x$ to $X_{0^+} = \hat{x}$ and $u(x) - u(\hat{x}) = |x - \hat{x}|$. After this initial jump, the state is kept inside $\bar{\mathcal{C}}$ by reflection in the $-\nabla u$ direction along $\partial\mathcal{C}$. □

13. Appendix. Proof of Lemma 4.1.

For $\epsilon \in (0,1)$, $R > 0$, denote by $u^{\epsilon,R}$ the solution to

$$(13.1) \qquad u^{\epsilon,R} - \Delta u^{\epsilon,R} + \beta_\epsilon(|\nabla u^{\epsilon,R}|^2) = h \quad \text{on} \quad B_R(0),$$

$$(13.2) \qquad u^{\epsilon,R} = 0 \quad \text{on} \quad \partial B_R(0).$$

The existence of $u^{\epsilon,R} \in C^2(\overline{B_R(0)})$ follows from Ladyzhenskaya & Ural'tseva (1968), Theorem 4.8.3, p. 301; uniqueness follows from the following lemma.

13.1 Lemma. Suppose that φ is a subsolution and ψ is a supersolution to (13.1). Then $\forall\, x \in B_R(0)$:

$$(13.3) \qquad \varphi(x) - \psi(x) \le \sup_{y \in \partial B_R(0)} [\varphi(y) - \psi(y)]^+.$$

Proof: If $\varphi - \psi$ attains its maximum over $\overline{B_R(0)}$ at an interior point x^*, then $\nabla\varphi(x^*) = \nabla\psi(x^*)$ and $0 \ge \Delta\varphi(x^*) - \Delta\psi(x^*) = \varphi(x^*) - \psi(x^*)$. □

13.2 Lemma. Let $q > 0$ be as in (2.6). There exists a constant $C_1 > 0$, independent of ϵ and R, such that

$$(13.4) \qquad 0 \le u^{\epsilon,R}(x) \le C_1(1 + |x|^q) \quad \forall\, x \in B_R(x).$$

Proof: To prove the nonnegativity of $u^{\epsilon,R}$, take $\varphi \equiv 0$ and $\psi = u^{\epsilon,R}$ in Lemma 13.1. To obtain the upper bound on $u^{\epsilon,R}$, take $\varphi = u^{\epsilon,R}$ and

$$\psi(x) = E \int_0^{\tau_x} e^{-t}\, h(x + \sqrt{2}\, W_t)dt,$$

where $\tau_x \triangleq \inf \{t \geq 0;\ |x + \sqrt{2}\, W_t| \geq R\}$. Then $\psi - \Delta\psi = h$ on $B_R(0)$, $\psi = 0$ on $\partial B_R(0)$, and Lemma 13.1 and (2.6) imply that

$$u^{\epsilon,R}(x) \leq E \int_0^{\tau_x} e^{-t}\, h(x + \sqrt{2}\, W_t)dt$$

$$\leq E \int_0^{\infty} e^{-t}\, h(x + \sqrt{2}\, W_t)dt$$

$$\leq 2^q C_0\, E \int_0^{\infty} e^{-t}(|x|^q + |\sqrt{2}\, W_t|^q)dt$$

$$\leq C_1\ (1 + |x|^q)$$

for an appropriate constant C_1. □

<u>13.3 Lemma</u>. There exist constants $C > 0$ and $p > 0$, independent of ϵ and R, such that

$$\max_{x\in\partial B_R(0)} |\nabla u^{\epsilon,R}(x)| \leq C(1 + R^p)\ \forall\ \epsilon \in (0,1),\ R > 0. \tag{13.5}$$

<u>Proof</u>: Let N be a positive integer greater than $\frac{q}{2}$, and define $g,\ B\colon [0,\infty) \to \mathbb{R}$ by

$$g(r) = \sum_{k=0}^{N} \frac{r^{2k}}{4^k (k!)^2}, \quad B(r) = \sum_{k=0}^{\infty} \frac{r^{2k}}{4^k (k!)^2}.$$

Then

$$g(r) - \frac{1}{r} g'(r) - g''(r) = \frac{r^{2N}}{4^N (N!)^2},$$

and

(13.6) $$B(r) - \frac{1}{r} B'(r) - B''(r) = 0.$$

For $R > 0$, define

$$\psi_R(x) = 2C_0 + C_0 4^N (N!)^2 g(|x|)$$

$$- [2C_0 + C_0 4^N (N!)^2 g(R)] \frac{B(|x|)}{B(R)} \ \forall\, x \in \mathbb{R},$$

so

$$\psi_R(x) - \Delta\psi_R(x) = C_0(2 + |x|^{2N}) \geq h(x) \ \forall\, x \in B_R(0),$$

$$\psi_R(x) = 0 \qquad \forall\ x \in \partial B_R(0).$$

It follows from Lemma 13.1 that $u^{\epsilon,R} \leq \psi_R$ on $B_R(0)$, and because these functions agree on $B_R(0)$ and because $\nabla u^{\epsilon,R}$ on $\partial B_R(0)$ must point inward, where $u^{\epsilon,R}$ is nonnegative, we have

$$|\nabla u^{\varepsilon,R}(x)| \le |\nabla \psi_R(x)| \quad \forall\, x \in \partial B_R(0).$$

But on $\partial B_R(0)$,

$$|\nabla \psi_R(x)| = |C_0 4^N (N!)^2 g'(R) - [2C_0 + C_0 4^N (N!)^2 g(R)] \frac{B'(R)}{B(R)}|.$$

Equation (13.6) and the nonnegativity of B'' show that

$$0 \le B'(r) \le r\, B(r) \quad \forall\, r > 0,$$

so we may bound the growth of $\max\limits_{x \in \partial B_R(0)} |\nabla \psi_R(x)|$ by a constant times $(1 + R^{2m+1})$. □

<u>13.4 Lemma</u>. There exist constants $C > 0$, $p > 0$, $\lambda > 0$, independent of ε and R, such that

(13.7) $|\nabla u^{\varepsilon,R}(x)| \le \lambda u^{\varepsilon,R}(x) + C|x|^p + C \quad \forall\, x \in B_R(0),\ \varepsilon \in (0,1),\ R > 0.$

<u>Proof</u>: With $C \ge 1$ and $p \ge 2$ satisfying (13.5), and C_0 as in (2.7), define $\lambda \triangleq \max\{2, C_0\}$, $B \triangleq C p^{\wedge} + C_0$, and consider the auxiliary function

$$\varphi(x) = \nabla u^{\varepsilon,R}(x) \cdot v - \lambda u^{\varepsilon,R}(x) - C|x|^p - B,$$

where $\varepsilon \in (0,1)$, $R > 0$ are fixed, and v is a fixed unit vector. It suffices to show that $\varphi(x) \le 0 \quad \forall\, x \in B_R(0)$, so let x^* be a point at which φ attains its maximum over $\overline{B_R(0)}$. If $x^* \in \partial B_R(0)$, then (13.5) implies that

$\varphi(x^*) \leq 0$. Thus, we need only consider the case that $x^* \in B_R(0)$, for which we have

$$0 \geq \Delta\varphi(x^*) = \Delta \, \nabla u^{\epsilon,R}(x^*) \cdot \nu - \lambda \, \Delta u^{\epsilon,R}(x^*) - C\, p^2 \, |x^*|^{p-2}.$$

Using (13.1), we may rewrite this as

$$(13.8) \qquad 0 \geq \nabla u^{\epsilon,R}(x^*) \cdot \nu + 2\, \beta_\epsilon'(r^*) \, \nabla[\nabla u^{\epsilon,R}(x^*) \cdot \nu] \cdot \nabla u^{\epsilon,R}(x^*)$$

$$- \nabla h(x^*) \cdot \nu - \lambda u^{\epsilon,R}(x^*) - \lambda\beta_\epsilon(r^*) + \lambda \, h(x^*) - C\, p^2 |x^*|^{p-2},$$

where r^* denotes $|\nabla u^{\epsilon,R}(x^*)|^2$. Because of (2.7),

$$|\nabla h(x)| \leq C_0 + \lambda \, h(x) \qquad \forall \;\; x \in \mathbb{R}.$$

Furthermore,

$$Cp^2 \; |x|^{p-2} \leq C\, p^p \; |\tfrac{x}{p}|^{p-2}$$

$$\leq C|x|^p + Cp^p \qquad \forall \;\; x \in \mathbb{R}.$$

Adding these two inequalities, we see that

$$|\nabla h(x^*)| + Cp^2 |x^*|^{p-2} \leq \lambda h(x^*) + C|x^*|^p + B.$$

Substitution into (13.8) yields

$$(13.9) \qquad 0 \geq \varphi(x^*) + 2\beta_\epsilon'(r^*)\ \nabla[\nabla u^{\epsilon,R}(x^*) \cdot \nu] \cdot \nabla u^{\epsilon,R}(x^*) - \lambda\beta_\epsilon(r^*).$$

Because $\nabla\varphi(x^*) = 0$, we also have

$$(13.10) \qquad \begin{aligned} 0 &= \nabla\varphi(x^*) \cdot \nabla u^{\epsilon,R}(x^*) \\ &= \nabla\,[\nabla u^{\epsilon,R}(x^*) \cdot \nu] \cdot \nabla u^{\epsilon,R}(x^*) - \lambda r^* \\ &\quad - Cp\ |x^*|^{p-2}\, x^* \cdot \nabla u^{\epsilon,R}(x^*). \end{aligned}$$

Substitution of (13.10) into (13.9) results in the inequality

$$\varphi(x^*) \leq \lambda[\beta_\epsilon(r^*) - 2\,\beta_\epsilon'(r^*)r^*] - 2\,Cp\ |x^*|^{p-2}\beta_\epsilon'(r^*)x^* \cdot \nabla u^{\epsilon,R}(x^*).$$

Let us assume that $\varphi(x^*) > 0$. Then

$$\sqrt{r^*} \geq \nabla u^{\epsilon,R}(x^*) \cdot \nu \geq B \geq 2,$$

so $r^* \geq 4$ and for all $\epsilon \in (0,1)$,

$$\beta_\epsilon(r^*) = \frac{r^*-1}{\epsilon} - 1, \quad \beta_\epsilon'(r^*) = \frac{1}{\epsilon}.$$

Consequently,

$$0 < \varphi(x^*) \le -\frac{\lambda}{\epsilon}\left(|\nabla u^{\epsilon,R}(x^*)|^2 + 1 + \epsilon\right) - \frac{2Cp}{\epsilon}|x^*|^{p-2}\, x^* \cdot \nabla u^{\epsilon,R}(x^*)$$

$$\le -\frac{\lambda}{\epsilon}\left(|\nabla u^{\epsilon,R}(x^*)|^2 + 1 + \epsilon\right) + \frac{2Cp}{\epsilon}|x^*|^{p-1}|\nabla u^{\epsilon,R}(x^*)|,$$

which implies that

$$|\nabla u^{\epsilon,R}(x^*)| \le \frac{2Cp}{\lambda}|x^*|^{p-1} \le Cp^p\left|\frac{x^*}{p}\right|^{p-1} \le C\,|x^*|^p + B.$$

This inequality contradicts the assumption that $\varphi(x^*) > 0$. □

<u>13.5 Lemma</u>. For each $\epsilon \in (0,1)$, there is an increasing sequence $\{R_n\}_{n=1}^{\infty}$ of positive numbers converging to infinity and a function $u^\epsilon \in C^2(\mathbb{R}^2)$ such that $\{u^{\epsilon,R_n}\}_{n=1}^{\infty}$ and $\{\nabla u^{\epsilon,R_n}\}_{n=1}^{\infty}$ converge uniformly to u^ϵ and ∇u^ϵ, respectively, on compact sets. Furthermore, u^ϵ is a solution to (4.3) and satisfies (4.4), (4.5), with C_1 and p independent of ϵ.

<u>Proof</u>: Let $\epsilon \in (0,1)$ be fixed and let $r > 0$ be given. Then $u^{\epsilon,R}$ and $\nabla u^{\epsilon,R}$ are bounded on $B_{2r}(0)$, uniformly in R and ϵ (Lemmas 13.2, 13.4). Elliptic regularity implies Hölder continuity of $\nabla u^{\epsilon,R}$ on $B_r(0)$, uniformly in $R \in [2r,\infty)$ (Gilbarg & Trudinger, Theorem 3.9, p. 41), and by the Arzela-Ascoli Theorem, we can find a sequence $\{R_n\}_{n=1}^{\infty}$ along which $\{u^{\epsilon,R_n}\}_{n=1}^{\infty}$ and $\{\nabla u^{\epsilon,R_n}\}_{n=1}^{\infty}$ converge uniformly on $B_r(0)$. Indeed, by diagonalization we can select $\{R_n\}_{n=1}^{\infty}$ so that $\{u^{\epsilon,R_n}\}_{n=1}^{\infty}$ and $\{\nabla u^{\epsilon,R_n}\}_{n=1}^{\infty}$ converge uniformly on compact sets to limits u^ϵ and ∇u^ϵ, respectively, where $u^\epsilon \in C^{1,\alpha}\ \forall\, \alpha \in (0,1)$. Passing to the limit in (13.1), we see that Δu^ϵ exists in the distributional sense and is equal to $u^\epsilon + \beta_\epsilon(|\nabla u^\epsilon|^2) - h$, which

is a $C^{0,\alpha}$ function. Elliptic regularity implies that D^2u^ϵ in fact exists in the classical sense and u^ϵ is $C^{2,\alpha}$. (By bootstrapping, we could conclude that u^ϵ is $C^{4,\alpha}$ because h is $C^{2,1}$.) □

The convexity of u^ϵ will be established by representing u^ϵ as the value function of a stochastic control problem with convex cost functions. With β_ϵ defined by (4.2), we define a convex function $g_\epsilon: \mathbb{R}^2 \to \mathbb{R}$ and its (convex) Legendre transform $\ell_\epsilon: \mathbb{R}^2 \to \mathbb{R}$ by

$$(13.11) \qquad g_\epsilon(x) \triangleq \beta_\epsilon(|x|^2), \ \ell_\epsilon(y) \triangleq \sup_{x\in\mathbb{R}^2} \{x \cdot y - g_\epsilon(x)\}.$$

For every $y \in \mathbb{R}^2$,

$$(13.12) \qquad \ell_\epsilon(y) \geq \frac{\epsilon}{2} |y|^2 - g_\epsilon\left(\frac{\epsilon}{2} y\right) \geq \frac{\epsilon}{4} |y|^2.$$

Furthermore, the supremum in the definition of ℓ_ϵ is attained if x is related to y by $y = 2\beta_\epsilon'(|x|^2)\, x$, i.e.,

$$(13.13) \qquad \ell_\epsilon(2\beta_\epsilon'(|x|^2)x) = 2\beta_\epsilon'(|x|^2)|x|^2 - \beta_\epsilon(|x|^2) \ \forall\, x \in \mathbb{R}^2.$$

A <u>control process</u> is any two-dimensional, absolutely continuous process η adapted to the Brownian motion $\{W_t, \mathcal{F}_t;\ 0 \leq t < \infty\}$ and satisfying $\eta_0 = 0$ a.s. Given an initial state $x \in \mathbb{R}^2$, the corresponding <u>state process</u> is

$$(13.14) \qquad Y_t \triangleq x + \sqrt{2}\, W_t - \eta_t.$$

For each $R > 0$, we define the cost corresponding to η up to the exit from $B_R(0)$ as

$$v_\eta^{\epsilon,R}(x) \triangleq E^x \int_0^{\tau_R} e^{-t} [h(Y_t) + \ell_\epsilon(\dot{\eta}_t)]dt,$$

where $\tau_R \triangleq \inf \{t \geq 0;\ |Y_t| \geq R\}$, and $\dot{\eta}_t = \frac{d}{dt}\eta_t$. The value function up to the exit from $B_R(0)$ is

$$v^{\epsilon,R}(x) \triangleq \inf_\eta v_\eta^{\epsilon,R}(x).$$

It is clear that $v^{\epsilon,R}(x)$ is nondecreasing in R, and

$$\text{(13.15)} \qquad \lim_{R\to\infty} v^{\epsilon,R}(x) \leq v^\epsilon(x) \triangleq \inf_\eta E^x \int_0^\infty e^{-t}[h(Y_t) + \ell_\epsilon(\dot{\eta}_t)]dt,$$

where v^ϵ is the value function for a control problem on $\mathbb{R}^2$.

<u>13.6 Lemma</u>. For each $\epsilon \in (0,1)$, $R > 0$, the solution $u^{\epsilon,R}$ of (13.1), (13.2) agrees with $v^{\epsilon,R}$ on $B_R(0)$.

<u>Proof</u>: Itô's lemma implies that for a given control process η, $x \in B_R(0)$ and $t \geq 0$:

$$(13.16)\quad E^x e^{-t\wedge\tau_R} u^{\epsilon,R}(Y_{t\wedge\tau_R}) = u^{\epsilon,R}(x) + E^x \int_0^{t\wedge\tau_R} e^{-s}[\beta_\epsilon(|\nabla u^{\epsilon,R}(Y_s)|^2)$$

$$- h(Y_s) - \nabla u^{\epsilon,R}(Y_s) \cdot \eta_s)]ds$$

$$\geq u^{\epsilon,R}(x) - E^x \int_0^{t\wedge\tau_R} e^{-s}[h(Y_s) + \ell_\epsilon(\dot{\eta}_s)]ds.$$

Letting $t \to \infty$, we see that $v_\eta^{\epsilon,R}(x) \geq u^{\epsilon,R}(x)$ for all η, so $v^{\epsilon,R}(x) \geq u^{\epsilon,R}(x)$. However, if Y^R is the solution to

$$Y_t^R = x - \int_0^t 2\beta_\epsilon'(|\nabla u^{\epsilon,R}(Y_s^R)|^2)\nabla u^{\epsilon,R}(Y_s^R)ds + \sqrt{2}\, W_t, \quad 0 \leq t \leq \tau_R,$$

then the corresponding control process satisfies

$$\dot{\eta}_t^R = 2\, \beta_\epsilon'(|\nabla u^{\epsilon,R}(Y_t^R)|^2)\, \nabla u^{\epsilon,R}(Y_t^R), \quad 0 \leq t \leq \tau_R,$$

and equality holds in (13.16) because of (13.13), i.e.,

$$v_{\eta^R}^{\epsilon,R}(x) = u^{\epsilon,R}(x) \leq v^{\epsilon,R}(x),$$

and thus $u^{\epsilon,R}(x) = v^{\epsilon,R}(x)$. □

<u>13.7 Lemma</u>. For each $\epsilon \in (0,1)$, the function u^ϵ constructed in Lemma 13.5 agrees with the value function v^ϵ defined in (13.15).

<u>Proof</u>: We have immediately from (13.15) and Lemma 13.6 that $u^\epsilon \leq v^\epsilon$. For the

reverse inequality, let $x \in R^2$ be given and define Y^∞ (up to the time of a possible explosion) by

$$Y^\infty_t = x - \int_0^t 2p_\varepsilon'(|\nabla u^\varepsilon(Y^\infty_s)|^2)\nabla u^\varepsilon(Y^\infty_s)ds + \sqrt{2}\,W_t.$$

Imitating (13.16), we have from Itô's lemma and (13.13) that for every $R > 0$,

$$(13.17)\qquad u^\varepsilon(x) = E^x \int_0^{t\wedge\tau_R} e^{-s}[h(Y^\infty_s) + \ell_\varepsilon(h^\infty_s)]ds + E^x e^{-t\wedge\tau_R} u^\varepsilon(Y^\infty_{t\wedge\tau_R}),$$

where

$$h^\infty_t \triangleq 2p_\varepsilon'(|\nabla u^\varepsilon(Y^\infty_t)|^2)\nabla u^\varepsilon(Y^\infty_t),\quad \tau_R \triangleq \inf\{t \ge 0;\ |Y^\infty_t| \ge R\}.$$

Deleting the (nonnegative) second term on the right-hand side of (13.17) and letting $R \to \infty$, $t \to \infty$, we obtain

$$(13.18)\qquad u^\varepsilon(x) \ge E^x \int_0^{\tau_\infty} e^{-s}[h(Y^\infty_s) + \ell_\varepsilon(h^\infty_s)]ds,$$

where $\tau_\infty \triangleq \lim_{R\to\infty} \tau_R$ is finite if and only if Y^∞ explodes in finite time.

To see that $\tau_\infty = \infty$ a.s., observe that for all $t \ge 0$, $R > 0$,

$$|Y^\infty_{t\wedge\tau_R}|^2 \le \int_0^{t\wedge\tau_R} [illegible]\,ds - \int_0^{t\wedge\tau_R} [illegible]\,ds + \int_0^{t\wedge\tau_R} |h^\infty_{s\wedge\tau_R}|^2 ds.$$

Gronwall's inequality implies

$$\max_{0\le s\le t\wedge\tau_R} |\eta_s^\infty|^2 \le e^t \int_0^{t\wedge\tau_R} |\dot\eta_s^\infty|^2 ds \le \frac{4e^t}{\epsilon}\int_0^{t\wedge\tau_R} \ell_\epsilon(\dot\eta_s^\infty)ds,$$

where we have used (13.12). Letting $R \to \infty$ and taking expectations, we conclude that

$$E^x \sup_{0\le s<t\wedge\tau_\infty} |\eta_s^\infty|^2 \le E^x \frac{4e^t}{\epsilon}\int_0^{t\wedge\tau_\infty} \ell_\epsilon(\dot\eta_s^\infty)ds \le \frac{4e^{2t}}{\epsilon} u^\epsilon(x) < \infty, \quad \forall\ t \ge 0.$$

But

$$\sup_{0\le s<t\wedge\tau_\infty} |Y_s^\infty| \le x + \sup_{0\le s<t\wedge\tau_\infty} |\eta_s^\infty| + \sqrt{2}\max_{0\le s\le t} |W_s|$$

and $\sup_{0\le s<t\wedge\tau_\infty} |Y_s^\infty| < \infty$ on $\{\tau_\infty \le t\}$. It follows that $P^*\{\tau_\infty \le t\} = 0\ \ \forall\ t \ge 0$. Inequality (13.18) can now be restated as

$$u^\epsilon(x) \ge E^x \int_0^\infty e^{-s}[h(Y_s^\infty) + \ell_\epsilon(\dot\eta_s^\infty)]ds \ge v^\epsilon(x). \qquad \square$$

<u>13.8 Corollary</u>. For each $\epsilon \in (0,1)$, the function u^ϵ constructed in Lemma 13.5 is convex.

<u>13.9 Corollary</u>. For each $\epsilon \in (0,1]$, $\lim_{|x|\to\infty} u^\epsilon(x) = \infty$.

<u>Proof</u>: In light of (2.8), (2.9), (13.12), and (13.15), we have

$$u^\epsilon(x) \ge \inf_\eta E^x \int_0^\infty e^{-t}\left[\frac{c_0}{2}|Y_t|^2 + \frac{\epsilon}{4}|\dot\eta_t|^2\right] dt.$$

But the right-hand side is the value asociated with a linear-quadratic-Gaussian problem, which is easily computed to be $\frac{1}{2}\alpha\ |x|^2 + 2\alpha$, where α is the positive root of the quadratic equation $\frac{2}{\epsilon}\alpha^2 + \alpha - c_0 = 0$. □

<u>13.10 Lemma</u>. There is a constant C_2, independent of ϵ, such that for every $\epsilon \in (0,1)$, the function u^ϵ constructed in Lemma 13.5 satisfies (4.6).

<u>Proof</u>: Let ν be a unit vector and define $u^\epsilon_{\nu\nu} \triangleq (D^2u)\nu\cdot\nu$. It suffices to produce a constant C_2, independent of ϵ and ν, such that

$$u^\epsilon_{\nu\nu} \leq C_2(1 + u^\epsilon).$$

We begin by differentiating (4.3) to obtain

$$(13.19) \qquad h_{\nu\nu} = u^\epsilon_{\nu\nu} - \Delta u^\epsilon_{\nu\nu} + 2\beta'_\epsilon(|\nabla u^\epsilon|^2)(\nabla u^\epsilon_{\nu\nu} \cdot \nabla u^\epsilon + |(D^2u^\epsilon)\nu|^2)$$

$$+ 4\beta''_\epsilon(|\nabla u^\epsilon|^2)(D^2u^\epsilon\nabla u^\epsilon \cdot \nu)^2$$

$$\geq u^\epsilon_{\nu\nu} - \Delta u^\epsilon_{\nu\nu} + 2\beta'_\epsilon(|\nabla u^\epsilon|^2)\ \nabla u^\epsilon_{\nu\nu} \cdot \nabla u^\epsilon.$$

Let x^ϵ be a minimizing point for u^ϵ, choose $p > 0$ satisfying (4.4), (4.5), choose $C_0 > 0$ to satisfy (2.8), let $\delta > 0$ be given, and define the auxiliary function

$$\varphi_\delta(x) = u^\epsilon_{\nu\nu}(x) - C_0u^\epsilon(x) - \delta\ |x - x^\epsilon|^{p+2}.$$

This function attains its maximum at some point y^δ, where we have

$$(13.20)\quad 0 = \nabla\varphi_\delta(y^\delta) = \nabla u^\epsilon_{\nu\nu}(y^\delta) - C_0\ \nabla u^\epsilon(y^\delta) - \delta(p+2)|y^\delta - x^\epsilon|^p(y^\delta - x^\epsilon),$$

$$(13.21)\quad 0 \ge \Delta\varphi_\delta(y^\delta) = \Delta u^\epsilon_{\nu\nu}(y^\delta) - C_0\ \Delta u^\epsilon(y^\delta) - \delta(p+2)^2|y^\delta - x^\epsilon|^p.$$

Substituting (4.3) into (13.21) and using (13.19), we obtain

$$(13.22)\qquad \begin{aligned} 0 &\ge u^\epsilon_{\nu\nu}(y^\delta) + 2\beta'_\epsilon(|\nabla u^\epsilon(y^\delta)|^2)\ \nabla u^\epsilon_{\nu\nu}(y^\delta)\cdot\nabla u^\epsilon(y^\delta) \\ &\quad - h_{\nu\nu}(y^\delta) - C_0\ u^\epsilon(y^\delta) - C_0\beta_\epsilon(|\nabla u^\epsilon(y^\delta)|^2) \\ &\quad + C_0 h(y^\delta) - \delta\ (p+2)^2\ |y^\delta - x^\epsilon|^p \\ &= \varphi_\delta(y^\delta) + 2\beta'_\epsilon(|\nabla u^\epsilon(y^\delta)|^2)\ \nabla u^\epsilon_{\nu\nu}(y^\delta)\cdot\nabla u^\epsilon(y^\delta) \\ &\quad - h_{\nu\nu}(y^\delta) - C_0\beta_\epsilon(|\nabla u^\epsilon(y^\delta)|^2) + C_0 h(y^\delta) \\ &\quad - \delta(p+2)^2\ |y^\delta - x^\epsilon|^p + \delta|y^\delta - x^\epsilon|^{p+2} \\ &\ge \varphi_\delta(y^\delta) + 2\beta'_\epsilon(|\nabla u^\epsilon(y^\delta)|^2)\ \nabla u^\epsilon_{\nu\nu}(y^\delta)\cdot\nabla u^\epsilon(y^\delta) \\ &\quad - C_0(1 + h(y^\delta)) - C_0\beta_\epsilon(|\nabla u^\epsilon(y^\delta)|^2) + C_0 h(y^\delta) \\ &\quad - 2\ \delta\ p^{\frac{p}{2}}\ (p+2)^{\frac{p+2}{2}} \end{aligned}$$

because of (2.8) and the fact that

$$- \delta(p + 2)^2 r^p + \delta\, r^{p+2} \geq - 2\delta\, p^{\frac{p}{2}} (p + 2)^{\frac{p+2}{2}} \quad \forall\ r \geq 0.$$

But (13.20) implies that

$$(13.23)\quad \nabla u^{\epsilon}_{\nu\nu}(y^{\delta}) \cdot \nabla u^{\epsilon}(y^{\delta}) = C_0 |\nabla u^{\epsilon}(y^{\delta})|^2 + \delta(p+2)|y^{\delta} - x^{\epsilon}|^p (y^{\delta} - x^{\epsilon}) \cdot \nabla u^{\epsilon}(y^{\delta})$$

$$\geq C_0 |\nabla u^{\epsilon}(y^{\delta})|^2$$

because u^{ϵ} is convex and attains its minimum at x^{ϵ}. Substitution of (13.23) into (13.22) yields

$$(13.24)\quad 0 \geq \varphi_{\delta}(y^{\delta}) + 2C_0 \beta'_{\epsilon}(|\nabla u^{\epsilon}(y^{\delta})|^2)|\nabla u^{\epsilon}(y^{\delta})|^2 - C_0 \beta_{\epsilon}(|\nabla u^{\epsilon}(y^{\delta})|^2)$$

$$- C_0 - 2\delta\, p^{\frac{p}{2}} (p+2)^{\frac{p+2}{2}} .$$

The convexity of β_{ϵ} implies that

$$\beta'_{\epsilon}(r) r \geq \beta_{\epsilon}(r) - \beta_{\epsilon}(0) = \beta_{\epsilon}(r) \quad \forall \ \ r \geq 0,$$

so (13.24) reduces to

$$\varphi_{\delta}(x) \leq \varphi_{\delta}(y^{\delta}) \leq C_0 + 2\delta\, p^{\frac{p}{2}} (p+2)^{\frac{p+2}{2}} \quad \forall \ \ x \in \mathbb{R}^2.$$

Letting $\delta \downarrow 0$, we obtain

$$u^{\epsilon}_{\nu\nu}(x) \leq C_0(1 + u^{\epsilon}(x)) \quad \forall \ \ x \in \mathbb{R}^2. \qquad \square$$

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