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ZERO INVESTMENT IN A HIGH YIELD ASSET CAN BE OPTIMAL

by

Gan-lin Xu
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

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Gan-lin Xu*
Department of Mathematics
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Abstract

In a market with one stock and one bond, a risk averse agent would normally follow the principle of holding a positive amount of stock if and only if its mean rate of return is strictly larger than the interest rate of the bond We provide an example to show that in the latter case, it may be optimal not to invest in the stock.

§ 1 Introduction. This paper presents a counterexample to a commonly held belief concerning single-agent consumption/investment decision problems. In this example, an agent is endowed with some initial wealth and has available two assets in which he may invest. The assets of the example are special; nevertheless, they have the characteristics of very general ones. One of them is riskless, a bond with zero interest rate. The other is a stock, a risky asset with random but observable mean rate of return. The agent has a strictly increasing, strictly concave utility function U, and he attempts to invest in the two assets so as to maximize the expected utility from terminal wealth $E\,U(X_T)$.

Because of the concavity of his utility function, the agent will be risk averse, that is, he will not invest in the risky asset unless it has a more favorable mean rate of return. Moreover, in a continuous-time trading model, it is plausible that whenever the mean rate of return on the stock is positive (i.e., strictly greater than the interest rate of the bond), the optimal portfolio does hold a positive amount of the stock. In our model, the agent can sell the assets short, and so one might also expect that whenever the mean rate of return on the stock is negative, the optimal portfolio holds a short position in the stock. We call this description of the form of the optimal portfolio the naive principle. The validity of this principle can be observed in the work of **Merton** [2], who treated models with constant mean rate of return and a restricted class of utility functions. The verification of the principle for more general utility functions, but still with constant mean rates of return, can be found in **Karatzas & Shreve** [1], p.387.

In our model, the utility function is quite simple, belonging to the class considered by Merton. However, the mean rate of return of the stock is a stochastic process. Because the mean rate of return is observable, one might hypothesize that the agent should follow the naive principle. This is in fact not the case. The unique optimal portfolio we find sometimes holds a neutral position in the stock when the mean rate of return is positive and also sometimes holds a neutral position when the mean rate of return is negative.

In section 2, we give the mathematical description of our model, and in section 3, the mean rate of return of the stock is defined. Finally, in section 4, we obtain the explicit solution to the problem stated in section 2.

§ 2 The Model. In order to model the uncertainty of return, we assume that there is a complete probability space (Ω , $\Im$, P), on which the process $W=\{ w_t , \Im_t , 0 \le t < \infty \}$ is a standard one- dimensional Brownian motion relative to the filtration $\{ \Im_t \}$. Let us consider that the assets are traded continuously on the fixed time-horizon [0 , 4]. The price of the bond is static, i.e.,

$$p_0(t) \equiv 1, \qquad 0 \le t \le 4 . \tag{2.1}$$

The price of the stock evolves according to the integral equation

$$p_1(t) = 1 + \int_0^t \theta_s \, p_1(s) \, ds + \int_0^t p_1(s) dw_s , \quad 0 \le t \le 4 . \tag{2.2}$$

Here the process $\theta = \{ \theta_t , \Im_t , 0 \le t \le 4 \}$, a bounded, adapted process defined in section 3, is called the mean rate of return of stock.

Definition 2.1. A portfolio process $\pi = \{ \pi_t , \Im_t , 0 \le t \le 4 \}$ is a measurable, adapted, and real valued process for which

$$E \int_0^4 \pi_t^4 \, dt < \infty . \tag{2.3}$$

We envision now an agent who starts with an initial endowment $x = e^3$ and invests in the two assets described above. If the process $\pi = \{ \pi_t , \Im_t , 0 \le t \le 4 \}$ is the portfolio process he chooses, then his wealth at time t, denoted by X(t), satisfies the differential equation (see[1], p.372)

$$dX_t = \pi_t \theta_t dt + \pi_t d w_t , \quad 0 \le t \le 4 , \tag{2.4}$$

$$X_0 = e^3 .$$

Definition 2.2 A portfolio process is said to be admissible if the wealth process X of (2.4) satisfies

$$X_t \geq 0, \quad 0 \leq t \leq 4, \text{ a.s.} \tag{2.5}$$

We denote the set of admissible portfolio process by D.

Assume the utility from the terminal wealth is measured by $E\frac{3}{2} X_4^{2/3}$. Then the mathematical problem the agent faces is

$$\begin{aligned} &\text{maximize} \quad J(\pi) \triangleq \frac{3}{2} E X_4^{2/3} \\ &\text{subject to} \quad \pi \in D. \end{aligned} \tag{2.6}$$

§ 3 The Mean Rate of Return. Defining the process θ takes several steps. First let

$$a_s \triangleq \operatorname{sgn} w_s = \begin{cases} +1 & \text{if } w_s \geq 0, \\ -1 & \text{if } w_s < 0, \end{cases} \tag{3.1}$$

$$M_t \triangleq \int_0^t a_s \, dw_s . \tag{3.2}$$

Then $M=\{ M_t, \Im_t, 0 \leq t < \infty\}$ is a continuous square-integrable martingale. Let

$$\tau_1 \triangleq \inf\{ t \mid t \geq 0, M_t \leq -1 \}, \quad \tau_2 \triangleq \inf\{ t \mid t \geq 0, M_t + \frac{1}{2}\langle M\rangle_t \geq 1 \}, \tag{3.3}$$

where $\langle M\rangle_t$ is the quadratic variation process of M. From ([1], p.7), τ_1, τ_2 are stopping times, and so is $\tau \triangleq \tau_1 \wedge \tau_2$. Notice that $\langle M\rangle_t = \int_0^t |a_s|^2 ds = t$, a.s.P. We have

$$\tau \leq 4 \qquad \text{a.s.P.} \tag{3.4}$$

Now we define

$$A_t \triangleq a_t \times 1_{\{ t < \tau \}}, \qquad B_t \triangleq (3/\sqrt{2}) \times 1_{\{ \tau_1 < t \wedge \tau_2 \}}. \tag{3.5}$$

For any fixed t, $t \wedge \tau_2$ is a stopping time, $\{ \tau_1 < t \wedge \tau_2 \} \in \Im_{t\wedge\tau_2}$ and $\Im_{t\wedge\tau_2} \subset \Im_t$.

Therefore, A_t and B_t are $\Im_t$-adapted process. The mean rate of return of the stock is defined by

(3.6) $6_t \wedge A_t + \frac{1}{3} B_t$.

It is clear that the process 8 = { 6 t , 3 t > 0 £ t : £ 4 } is a bounded, measurable, and adapted process,

Remark 1: On the set { (t, to) e [0,4] x Q |t<x(CD) }, Gt= A_t takes both values +1 and -1, but the optimal portfolio process we find in the next section is identically zero on this set

§ 4. Solution of The Model. First we state an easily verified property of our utility function.

~~Lemma 4.1~~. For every x^O, y>0, the inequality ry $\sim^2$+ xy £?x ^ holds, and equality holds if and only if x=y $\sim^3$.

In order to describe the optimal portfolio process, we need to define processes Z = { Z_t, 3t , 0 ≤ a £ 4 } and Y = { Y_t, 3_t , 0 ≤ £ t £ 4 } by

(4.1) $$Z_t \triangleq \exp\{-\int_0^t e_s dw_s - \frac{1}{2}\int_0^t e_s^2 ds\},$$

(4.2) $$Y_t \wedge \exp(3 + \int_0^t B_s dw_s - \frac{1}{2}\int_0^t B_s^2 ds + \int_0^t B_s 6_S\, ds).$$

By Ittf's rule, we have

(4.3) $$dZ_t = -9_t Z_t\, dw_t,\ 0 \wedge t \leq 4,$$

$$Zo = l,$$

(4.4) $$dY_t = e_t B t Y_t dt + B_t Y_t dw_t,\ 0 \leq t \wedge 4,$$

$$Yo = e^3,$$

(4.5) $$d(Z_t Yt) = Z \wedge i Y_t + Y, dZ_t + d\langle Z, Y\rangle_t = Z_t Yrf\ B_t - 6t)\, dw_t,\ 0 \leq t \leq 4\ .$$

From the boundedness of B and 8 we know that { $Z_t Y_t$, 3_t, 0 ≤ t < 4 } is a martingale, and, in particular,

(4.6) $$E\ Z_tY_t = E\ Z_0Y_0 = e^3,\ \forall\ 0 \le t \le 4.$$

Lemma 4.2. The processes Z and Y satisfy

(4.7) $$Y_4 = Z_4^{-3}.$$

Proof: First notice that $\ln(Z_4^3Y_4)$

$$= -3\int_0^4 \theta_s dw_s - \frac{3}{2}\int_0^4 \theta_s^2\, ds + 3 + \int_0^4 B_s d w_s - \frac{1}{2}\int_0^4 B_s^2 ds + \int_0^4 B_s\, \theta_s\, ds$$

$$= -3 \int_0^4 A_s dw_s - \frac{3}{2}\times \int_0^4 A_s^2 ds - \frac{1}{3}\int_0^4 B_s^2 ds + 3$$

$$= \begin{cases} -3\times(-1) - \frac{3}{2}\times \tau_1 - \frac{1}{3}\times\frac{9}{2}(4-\tau_1) + 3 & \text{if } \tau_1 \le \tau_2 \\ -3 + 0 + 3 & \text{if } \tau'_1 > \tau_2 \end{cases}$$

$= 0.$

The first equality comes from (3.6), and the second one comes from (3.3) and (3.5). This means $Z_4^3Y_4 = 1$ a.s.; thus the lemma is proved.

From this lemma and (4.6) we immediately get

(4.8) $$EY_4^{2/3} = EZ_4^{-2} = EZ_4Y_4 = e^3.$$

We are ready for the main result of this paper.

Theorem. The process $\pi^* \triangleq \{ B_t Y_t, \mathfrak{I}_t, 0 \le t \le 4 \}$ is the unique optimal portfolio process.

Proof: First, for a given $\pi \in D$, let X_t be the corresponding wealth process defined by (2.4). From Itô's rule, we have

$$dX_tZ_t = Z_tdX_t + X_tdZ_t + d\langle Z,X\rangle_t = (\pi_t Z_t - \theta_t X_t Z_t) d\, w_t,$$

so $\{X_tZ_t, \mathfrak{I}_t, 0 \le t \le 4\}$ is a non-negative semi-martingale, hence a supermartingale (see[1], p.36). Therefore $E\ Z_4X_4 \le E\ Z_0X_0 = e^3$. From Lemma 4.1 we have

(4.9) $\frac{3}{2} E X_4^{2/3} \leq E Z_4 X_4 + \frac{1}{2} E Z_4^{-2} \leq e^3 + \frac{1}{2} e^3 = \frac{3}{2} e^3.$

Noticing that process $\pi^* \in D$ and the process Y is its wealth process, from (4.8) we have

(4.10) $J(\pi^*) = \frac{3}{2} EY_4^{2/3} = \frac{3}{2} e^3.$

This proves the optimality.

For uniqueness, define $\tilde{P}(A) = E1_A Z_4$ for $A \in \mathfrak{I}_4$. From Girsanov's theorem we have that P and $\tilde{P}$ are mutually absolutely continuous and under the probability measure $\tilde{P}$, the process $\tilde{W}_t \triangleq \int_0^t \theta_s ds + dw_s$, $0 \leq t \leq 4$, is an $\{\mathfrak{I}_t\}$-Brownian motion. Let π_t be another optimal portfolio process in D, and let X_t be the corresponding wealth process. From (4.9) and Lemma 4.1 we have $X_4 = Z_4^{-3}$, a.s.P, so

(4.11) $X_4 = Y_4,$ a.s.$\tilde{P}$.

We can rewrite equations (4.4) and (2.4) as

(4.12) $dY_t = B_t Y_t d\tilde{W}_t, \quad Y_0 = e^3,$

(4.13) $dX_t = \pi_t d\tilde{W}_t, \quad X_0 = e^3.$

Note that $\pi \in D$ implies

$$\tilde{E} \int_0^4 \pi_t^2 dt = E \int_0^4 Z_t \pi_t^2 dt \leq \sqrt{E \int_0^4 Z_t^2 dt} \sqrt{E \int_0^4 \pi_t^4 dt} < \infty,$$

and this implies $\tilde{E} \int_0^4 (\pi_t - B_t Y_t)^2 dt = \tilde{E}(X_4 - Y_4)^2 = 0$. Thus the theorem is proved.

<u>Remark 2</u>: On the set $\{ (t, \omega) \in [0,4] \times \Omega \mid t < \tau(\omega) \}$, $\pi_t^* = B_t Y_t = 0$. This proves the assertion made in Remark 1.

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