A VISCOSITY SOLUTION APPROACH TO THE ASYMPTOTIC ANALYSIS OF QUEUEING SYSTEMS

by

Paul Dupuis
Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA 01003

Hitoshi Ishii
Department of Mathematics
Chuo University
Tokyo 112, Japan

H. Mete Soner
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Research Report No. 88-24

July 1988

A VISCOSITY SOLUTION APPROACH TO THE ASYMPTOTIC ANALYSIS OF QUEUEING SYSTEMS

by

Paul Dupuis[1]
Hitoshi Ishii[2]

and

H. Metè Soner[3]

---

[1]Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts, 01003. This research was completed while visiting the Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island and was supported in part by NSF 8511470, AFOSR 85-0315, and DAAL 03-86-K-0171.

[2]Department of Mathematics, Chuo University, Tokyo 112, Japan. This research was completed while visiting the Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island, was supported in part by AFOSR 85-0315.

[3]Department of Mathematics, Carnegie Mellon University, Pittsburgh, Pennsylvania. This research was supported in part by NSF DMS-8702537.

1. Introduction

In this paper we consider an asymptotic analysis of a queueing system. Suppose the "state" of the queueing system at time t is given by the n-dimensional vector $X_t \in (Z^+)^n$, where $Z^+ = \{0,1,2,\ldots\}$. For small positive ϵ, the scaling of interest here is given by $X_t^\epsilon = \epsilon X_{t/\epsilon}$, corresponding to large time and large excursions. We shall assume that the original process $X_\cdot$ is modelled as a jump Markov process. Hence the rescaled process $X_\cdot^\epsilon$ is also a jump Markov process, with state space given by $S^\epsilon = \{y \in R^n : y/\epsilon \in (Z^+)^n\}$.

The problem we are interested in concerns the estimation of probabilities of certain rare events involving the original (unscaled) system. For example, take $n = 2$ and consider the event

(1.1)

$$A^\epsilon = \{x_t + y_t \geq M/\epsilon \text{ for some } 0 \leq t \leq T/\epsilon, \text{ given } x_0 = x/\epsilon,\ y_0 = y/\epsilon\}$$

where M,T are positive real numbers and $X_t = (x_t,y_t)$. In the rescaled system this event is given by

(1.2) $$\{x_t^\epsilon + y_t^\epsilon \geq M \text{ for some } 0 \leq t \leq T, \text{ given } x_0^\epsilon = x,\ y_0^\epsilon = y\}$$

The results of this paper give asymptotic ($\epsilon \downarrow 0$) estimates of $P(A^\epsilon)$ of the form $\exp((-I(x,y) + 0(1))/\epsilon)$, where the $0(1)$ term converges to zero uniformly for (x,y) in compact subsets of $\{(x,y) : x \geq 0, y \geq 0, x + y < M\}$, and where $I(x,y) = u(x,y,0)$, where u is the value function of a non-standard

deterministic optimal control problem. The formulation of this control problem can be found in Section 3, below.

The problem we have described is one of estimating the probability of an event corresponding to a large deviation of the scaled queueing system. In the general theory of large deviations for stochastic dynamical systems, one is given a process X_t^ϵ, defined for $0 \le t \le T$, with sample paths living in some space D and is asked to obtain a family of functionals $S(x,\cdot) : D \mapsto [0,\infty]$ such that (in addition to other properties)

(1.3) for any open set $A \subset D$,

$$\liminf_{\epsilon \downarrow 0} \epsilon \ell n P_x(X_\cdot^\epsilon \in A) \ge -\inf_{\phi \in A} S(x,\phi)$$

(1.4) for any closed set $G \subset D$

$$\limsup_{\epsilon \downarrow 0} \epsilon \ell n P_x(X_\cdot^\epsilon \in G) \le -\inf_{\phi \in G} S(x,\phi)$$

where P_x denotes the probability given $X_0^\epsilon = x$. We refer to Varadhan [27], and Stroock [26] for the precise properties required of S. The problem we are trying to solve is a special case of the full large deviations problem as described above, since we are interested in obtaining "rough" asymptotics of $P_x(X_\cdot^\epsilon \in A)$ (as given by (1.3) and (1.4)) only for a particular class of sets A.

The techniques employed in this paper are those of the theory of viscosity solutions to Hamilton-Jacobi equations. The application of such methods to problems concerning large deviations originated with the work of Evans and Ishii [8]. Further work in this area may be found in [1], [2], [9], [11], [12], [13],

[17], and [20]. For a general introduction to problems concerning large deviations of dynamical systems, the reader is referred to the book of Freidlin and Wentzell [14], where probabilistic (as opposed to analytical) techniques are employed. An example of how probabilistic methods may be used to estimate escape probabilities is in [7]. Also, in [18]-[19] some formal formulae were obtained for problems similar to the one described here, and applications were discussed in [23], [28].

The new features involved in developing a large deviations for processes of the type that arise from queueing systems result from the "boundaries" of the system. For simplicity consider a system of two queues (x_t, y_t) in which interarrival and service times are constants, and for which the relationships between the queues is as depicted in Fig. 1 below. Define the rescaled system $(x_t^\epsilon, y_t^\epsilon) = \epsilon(x_{t/\epsilon}, y_{t/\epsilon})$. If both x and y are strictly positive, then the conditional statistics of $((x_{t+\delta}^\epsilon, y_{t+\delta}^\epsilon) - (x_t^\epsilon, y_t^\epsilon))$ given $(x_t^\epsilon, y_t^\epsilon) = (x,y)$ are roughly independent of (x,y). However, as $x \wedge y \to 0$ (one or both queues empty) there is an abrupt change in the statistics of the small time increment, since the associated jump measure suffers a discontinuity. As we will see below, the nature of the stochastic process we deal with is such that this transition in the jump measure leads to a nonlinear boundary condition for the associated partial differential equation (PDE).

Since it is the treatment of the effects due to the "boundaries" that is novel, we devote the majority of the paper to the detailed development of a simple two-dimensional system that exhibits the essential new features. However, the arguments may be applied to queueing systems whose structure

routing schemes, etc.) is quite general, and we attempt to indicate this generality as well.

The outline of the paper is as follows. In Section 2, we define the logarithmic transformation of the probability of interest and show that it converges to a viscosity solution of an associated Hamilton-Jacobi equation, as e tends to zero. In Section 3, we obtain a representation for the solution of this equation in terms of the value function of a certain nonstandard optimal control problem. This suggests the form of the functional that would be correct if (1.3) and (1.4) were to hold. Sections 4, 5, and 6 prove the uniqueness of viscosity solutions satisfying a nonlinear boundary condition, which ensures that our two representations are in fact the same. These sections are of independent interest. We conclude in Section 7 with a discussion of extensions. In particular, Section 7.1.2 contains a summary of the main results of the paper, written for a system of interconnected queues. The appendix includes a brief discussion of a weaker formulation of the FDE.

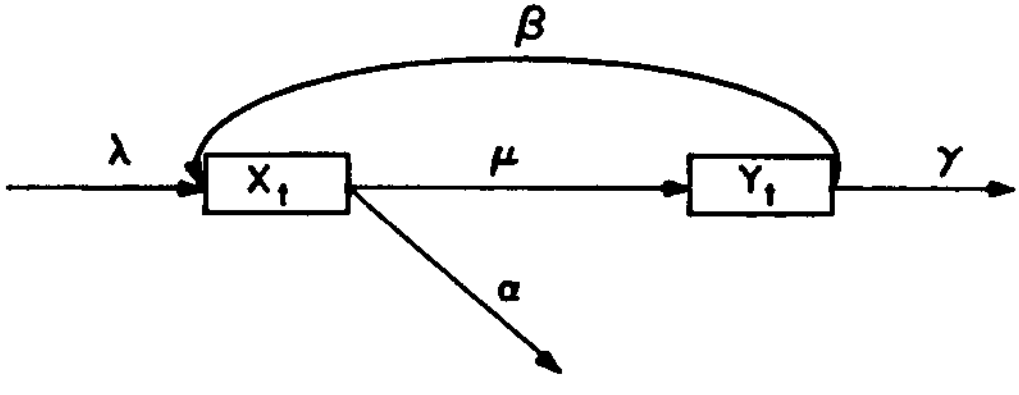

Figure 1

2. An Example

We return our attention to the queueing system depicted in Fig. 1, and consider the problem of determining the limiting behavior of

$$(2.1)\quad u^{\epsilon}(x,y,t) = -\epsilon\, \ell n\, P(x_s^{\epsilon} + y_s^{\epsilon} \geq M \quad \text{for some} \quad s \in [t,T] \mid x_t^{\epsilon} = x,\ y_t^{\epsilon} = y)$$

For the sake of notational simplicity, we take $M = 1$. The process corresponding to the queueing system depicted in Fig. 1 is a jump Markov process (x_t, y_t) whose jump measure is concentrated on the points $(1,0)$, $(1,-1)$, $(0,-1)$, $(-1,0)$, and $(-1,1)$ with intensities λ, β, γ, α, and μ respectively, unless a boundary is encountered. We assume that all the intensities are nonnegative. In order to obtain a nontrivial system we must also assume that $\lambda > 0$, $\mu > 0$, and that either $\gamma > 0$ or $\beta \wedge \alpha > 0$. When the process is on a boundary, only those jumps that do not lead to escape are allowed by the jump measure, and they retain the intensities that are in effect on the interior. We then use the definition $(x_t^{\epsilon}, y_t^{\epsilon}) = \epsilon(x_{t/\epsilon}, y_{t/\epsilon})$ to obtain the scaled system. See Figure 2.

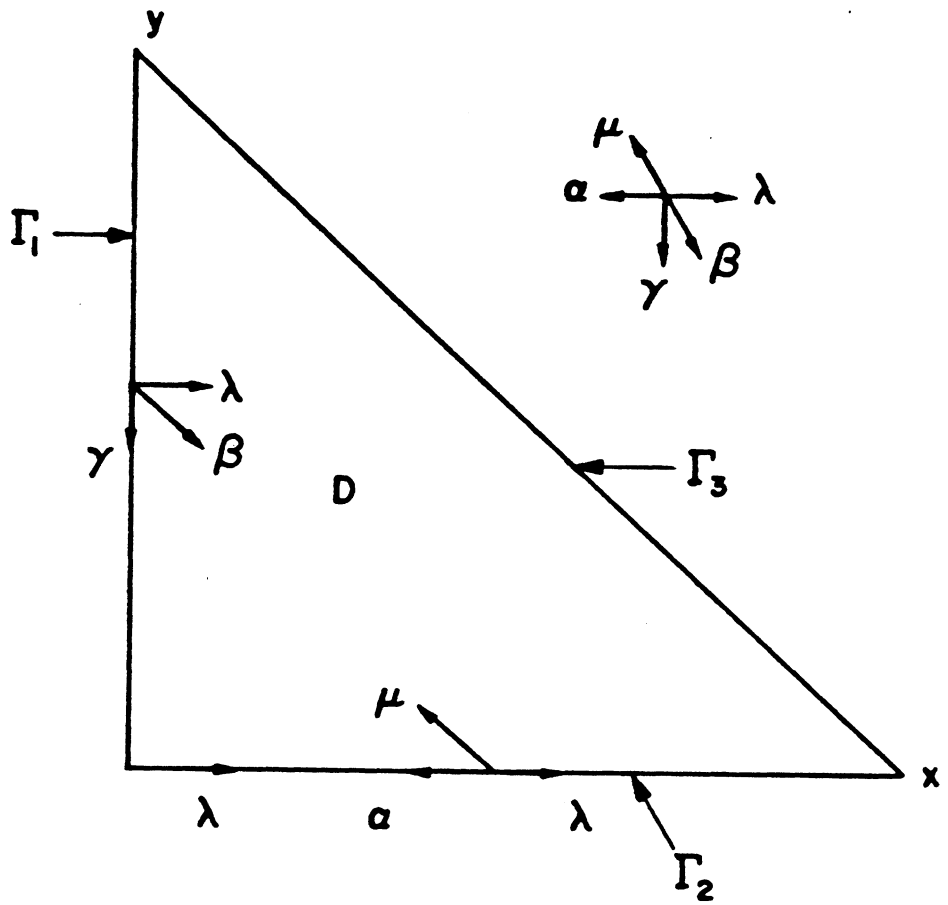

Figure 2

Define

$$S^\epsilon = \{\epsilon(i,j) : (i,j) \in Z^2\}$$

$$D = \{(x,y) : x > 0,\ y > 0,\ x + y < 1\}$$

$$\Gamma_1 = \{(0,y) : 0 < y < 1\}$$

$$\Gamma_2 = \{(x,0) : 0 < x < 1\}$$

$$\Gamma_3 = \{(x,y) : x \geq 0,\ y \geq 0,\ x + y = 1\}$$

$$D^\epsilon = D \cap S^\epsilon$$

$$\Gamma_i^\epsilon = \Gamma_i \cap S^\epsilon,\ i = 1,2,3.$$

Then for $u^\epsilon(x,y,t)$ defined by (2.1) the Chapman-Kolmogorov equations imply

$$(2.2a) \quad -\frac{\partial}{\partial t} u^\epsilon(x,y,t) + H^\epsilon(x,y,u^\epsilon(\cdot,\cdot,t)) = 0, \ (x,y,t) \in D^\epsilon \times (0,T),$$

$$(2.2b) \quad -\frac{\partial}{\partial t} u^\epsilon(x,y,t) + H^\epsilon_{\partial,i}(x,y,u^\epsilon(\cdot,\cdot,t)) = 0, \ (x,y,t) \in \Gamma^\epsilon_i \times (0,T), \ i = 1,2,$$

$$(2.2c) \quad -\frac{\partial}{\partial t} u^\epsilon(x,y,t) + H^\epsilon_c(x,y,u^\epsilon(\cdot,\cdot,t)) = 0, \ (x,y,t) \in \{(0,0)\} \times (0,T),$$

$$(2.3) \quad u^\epsilon(x,y,t) = 0, \ (x,y,t) \in \Gamma^\epsilon_3 \times (0,T),$$

$$(2.4) \quad u^\epsilon(x,y,T) = +\infty, \ (x,y) \in (\bar{D} \setminus \Gamma_3) \cap S^\epsilon,$$

(here ∂,i denotes boundary number i, and c denotes the corner).

The Hamiltonians are given by

$$\begin{aligned}(2.5a) \quad H^\epsilon(x,y,\phi(\cdot,\cdot)) = {} & \lambda\left[\exp\left[-\frac{1}{\epsilon}[\phi(x+\epsilon,y) - \phi(x,y)]\right] - 1\right] \\ & + \beta\left[\exp\left[-\frac{1}{\epsilon}[\phi(x+\epsilon,y-\epsilon) - \phi(x,y)]\right] - 1\right] \\ & + \gamma\left[\exp\left[-\frac{1}{\epsilon}[\phi(x,y-\epsilon) - \phi(x,y)]\right] - 1\right] \\ & + \alpha\left[\exp\left[-\frac{1}{\epsilon}[\phi(x-\epsilon,y) - \phi(x,y)]\right] - 1\right] \\ & + \mu\left[\exp\left[-\frac{1}{\epsilon}[\phi(x-\epsilon,y+\epsilon) - \phi(x,y)]\right] - 1\right]\end{aligned}$$

(2.5b) $H^{\epsilon}_{1}(x,y,\cdot(\cdot,\cdot)) = H^{\epsilon}(x,y,\phi(\cdot,\cdot))$

$$- a\left[\exp\left[-\tfrac{1}{\epsilon}\phi^{*}(x-e,y) - \phi^{*}(x,y)\right]\right] - 1\Big]$$

$$- \mu\left[\exp\left[-\frac{1}{\epsilon}[\phi(x-e,y+e) - \phi(x,y)]\right] - 1\right]$$

(2.5c) $H^{\epsilon}_{2}(x,y,\cdot(\cdot,\cdot)) = H^{\epsilon}(x,y,\phi^{*}(-,-))$

$$- \beta\left[\exp\left[-\tfrac{1}{\epsilon}(x+e,y-e) - \phi(x,y)\right]\right] - 1\Big]$$

$$- \tau\left[\exp\left[-\frac{1}{\epsilon}[\phi(x,y-e) - \phi(x,y)]\right] - 1\right]$$

(2.5d) $H^{\epsilon}(x,y,\phi^{*}(\cdot,\cdot)) = \alpha[\exp[-\tfrac{1}{\epsilon}(x+e,y) - \phi(x,y)]] - 1]$.

It follows that if $\phi \in C(\bar{D})$, then

(2.6) $$\lim_{\epsilon \downarrow 0} H^{\epsilon}(x,y,\phi(\cdot,\cdot)) = H(x,y,\phi^{*}(x,y))$$

(2.7a) $$\lim_{\epsilon \downarrow 0} H^{\epsilon}_{i}(x,y,\phi(\cdot,\cdot)) = H_{i}(x,y,\phi^{*}(x,y)) \quad i = 1,2.$$

(2.7b) $$\lim_{\epsilon \downarrow 0} H^{\epsilon}_{c}(x,y,\phi(\cdot,\cdot)) = H_{c}(x,y,\phi^{*}(x,y))$$

uniformly in (x,y), where

$$(2.8a)\quad H(p,q) = \lambda[e^{-p} - 1] + \beta[e^{q-p} - 1] + \gamma[e^{q} - 1] + \alpha[e^{p} - 1] + \mu[e^{p-q} - 1]$$

$$(2.8b)\quad H_{\partial,1}(p,q) = H(p,q) - \alpha[e^{p} - 1] - \mu[e^{p-q} - 1]$$

$$(2.8c)\quad H_{\partial,2}(p,q) = H(p,q) - \beta[e^{q-p} - 1] - \gamma[e^{q} - 1]$$

$$(2.8d)\quad H_{c}(p,q) = \lambda[e^{-p} - 1] \ .$$

Lemma 2.1. *For each* $T' < T$, *there is a constant* $K(T')$ *independent of* ϵ *such that*

$$|u^{\epsilon}(x,y,t)| \leq K(T')$$

for all $t \leq T'$.

The result is a simple consequence of the fact that $\lambda > 0$, and the easy proof is omitted.

Following [2], we next define $\bar{u}$ and $\underline{u}$ as follows:

$$(2.9a)\quad \bar{u}(x,y,t) = \overline{\ell im}_{\substack{(x^{\epsilon},y^{\epsilon},t^{\epsilon})\to(x,y,t)\\(x^{\epsilon},y^{\epsilon},t^{\epsilon})\in\bar{D}\cap S^{\epsilon}\times[0,T)\\ \epsilon\to 0}} u^{\epsilon}(x^{\epsilon},y^{\epsilon},t^{\epsilon})$$

$$(2.9b)\quad \underline{u}(x,y,t) = \underline{\ell im}_{\substack{(x^{\epsilon},y^{\epsilon},t^{\epsilon})\to(x,y,t)\\(x^{\epsilon},y^{\epsilon},t^{\epsilon})\in\bar{D}\cap S^{\epsilon}\times[0,T)\\ \epsilon\to 0}} u^{\epsilon}(x^{\epsilon},y^{\epsilon},t^{\epsilon}) \ .$$

Theorem 2.1. *Suppose* $\phi \in C^1(\bar{D} \times (0,T))$, *and that* (x_0,y_0,t_0) (*with* $t_0 < T$) *satisfies*

$$(\bar{u} - \phi)(x_0,y_0,t_0) = \max(\bar{u} - \phi) \qquad [(\underline{u} - \phi)(x_0,y_0,t_0) = \min(\underline{u} - \phi)] .$$

If:

i) $(x_0,y_0) \in D$, *then*

$$(2.10a) \qquad -\frac{\partial}{\partial t}\phi\,(x_0,y_0,t_0) + H(\nabla\phi(x_0,y_0,t_0)) \leq 0 \quad [\geq 0] .$$

ii) $(x_0,y_0) \in \Gamma_i$, $\quad i = 1,2$, *then*

$$(2.10b) \qquad \min_{[\max]} \left\{ -\frac{\partial}{\partial t}\phi\,(x_0,y_0,t_0) + H(\nabla\phi(x_0,y_0,t_0)); \right.$$

$$\left. -\frac{\partial}{\partial t}\phi\,(x_0,y_0,t_0) + H_{\partial,i}(\nabla\phi(x_0,y_0,t_0)) \right\} \leq 0 \quad [\geq 0]$$

iii) $(x_0,y_0) = (0,0)$ *then*

$$(2.10c) \qquad \min_{[\max]} \left\{ -\frac{\partial}{\partial t}\phi + H(\nabla\phi); -\frac{\partial}{\partial t}\phi + H_{\partial,1}(\nabla\phi); \right.$$

$$\left. -\frac{\partial}{\partial t}\phi + H_{\partial,2}(\nabla\phi); -\frac{\partial}{\partial t}\phi + H_c(\nabla\phi) \right\} \leq 0 \quad [\geq 0] .$$

Proof. Without loss of generality, we may assume that any maximum or minimum holds in the strict sense (by simply replacing ϕ by $\phi^{\delta}(x,y,t) = \phi(x,y,t) \pm \delta|(x,y,t) - (x_0,y_0,t_0)|^2$ and using that $\phi_x = \phi^{\delta}_x, \phi_y = \phi^{\delta}_y$, and $\phi_t = \phi^{\delta}_t$ at (x_0,y_0,t_0)). We prove ii) for the case of a maximum and for $i = 1$. All other cases are proved in a similar way.

From the definition of $\overline{u}$ there exist $(x^{\epsilon},y^{\epsilon},t^{\epsilon}) \in (D \cup \Gamma_1) \cap S^{\epsilon} \times [0,T)$ such that

$$(u^{\epsilon} - \phi)(x^{\epsilon},y^{\epsilon},t^{\epsilon}) = \max[(u^{\epsilon} - \phi)] \tag{2.11a}$$

$$\ell im(x^{\epsilon},y^{\epsilon},t^{\epsilon}) = (x_0,y_0,t_0) \tag{2.11b}$$

Owing to (2.11b), we may assume that either $(x^{\epsilon},y^{\epsilon}) \in D^{\epsilon}$ or $(x^{\epsilon},y^{\epsilon}) \in \Gamma_1^{\epsilon}$. If $(x^{\epsilon},y^{\epsilon}) \in D^{\epsilon}$ then by (2.2a)

$$\begin{aligned} 0 &= -\frac{\partial}{\partial t}u^{\epsilon}(x^{\epsilon},y^{\epsilon},t^{\epsilon}) + H^{\epsilon}(x^{\epsilon},y^{\epsilon},u^{\epsilon}(\cdot,\cdot,t^{\epsilon})) \\ &\geq -\frac{\partial}{\partial t}\phi(x^{\epsilon},y^{\epsilon},t^{\epsilon}) + H^{\epsilon}(x^{\epsilon},y^{\epsilon},\phi(\cdot,\cdot,t^{\epsilon})) \end{aligned} \tag{2.12}$$

which implies

$$\begin{aligned} 0 &\geq \overline{\underset{\epsilon\downarrow 0}{\ell im}} \left[-\frac{\partial}{\partial t}\phi(x^{\epsilon},y^{\epsilon},t^{\epsilon}) + H^{\epsilon}(x^{\epsilon},y^{\epsilon},\phi(\cdot,\cdot,t^{\epsilon}))\right] \\ &= -\frac{\partial}{\partial t}\phi(x_0,y_0,t_0) + H(\nabla\phi(x_0,y_0,t_0)) \ . \end{aligned} \tag{2.13}$$

If $(x^e, y^e) \in T_j^\epsilon$, then

$$(2.14) \qquad 0 = \lim_{\epsilon \downarrow 0} [- \Lambda_u{}^{fc}(x^e, y^e, t^e) + H\Lambda_A(x^e.y^e.u^e(-.\bullet, t^e))]$$

$$\geq \overline{\lim_{\epsilon \downarrow 0}} \left[- \frac{\partial}{\partial t} \phi(x^\epsilon, y^\epsilon, t^\epsilon) + H^\epsilon_{\partial,1}(x^\epsilon.y^\epsilon.\phi(\cdot,\cdot,t^\epsilon)) \right]$$

$$= - \frac{\partial}{\partial t} \phi(x_0.y_0.t_0) + H_{\partial,1}(\nabla\phi(x_0.y_0.t_0))$$

Then (2.13) and (2.14) give (2.10b). D

The Limiting Equation

We have shown that in a certain sense (which we now make precise) that $\bar{u}$ and $\underline{u}$ satisfy the equation

$$(2.15) \qquad - \Lambda\text{-}u + H(vu) = 0 \quad \text{in} \quad D \times (0,T)$$

with appropriate boundary conditions. First we give a definition. This a straightforward generalization of the definitions given in [3], [5]. See also [15], [21].

Definition 2.1. We say that the upper (lower) semicontinuous function u is a *viscosity* sub (super) solution to the equation

(2.16) $$-\frac{\partial}{\partial t}u + H(\nabla u) = 0 \quad (\text{on } D \times (0,T))$$

together with the boundary conditions

(2.17a) $$-\frac{\partial}{\partial t}u + H(\nabla u) = 0 \quad \text{or} \quad -\frac{\partial}{\partial t}u + H_{\partial,i}(\nabla u) = 0 \quad (\text{on } \Gamma_i \times (0,T)),\ i = 1,2$$

(2.17b) $$-\frac{\partial}{\partial t}u + H(\nabla u) = 0, \text{ or } -\frac{\partial}{\partial t}u + H_{\partial,1}(\nabla u) = 0, \text{ or}$$

$$-\frac{\partial}{\partial t}u + H_{\partial,2}(\nabla u) = 0, \text{ or } -\frac{\partial}{\partial t}u + H_c(\nabla u) = 0 \quad (\text{on } \{(0,0)\} \times (0,T))$$

(2.17c) $$u = 0 \qquad (\text{on } \Gamma_3 \times (0,T))$$

and with infinite terminal data at time T if for any $\phi \in C^1(\bar{D} \times (0,T))$ and point $(x,y,t) \in \bar{D} \times (0,T)$ such that $(u - \phi)(x,y,t) = \max(\min)[u - \phi]$, we have (at the point (x,y,t))

(2.18) $$-\frac{\partial}{\partial t}\phi + H(\nabla\phi) \leq 0 \ (\geq 0) \text{ whenever } (x,y) \in D\ ,$$

(2.19a) $$\min_{(\max)} \left[-\frac{\partial}{\partial t}\phi + H(\nabla\phi);\ -\frac{\partial}{\partial t}\phi + H_{\partial,i}(\nabla\phi)\right] \leq 0 \ (\geq 0)$$

whenever $(x,y) \in \Gamma_i$, $i = 1,2$,

(2.19b)
$$\min_{(\max)} \left[-\frac{\partial}{\partial t}\phi + H(\nabla\phi);\ -\frac{\partial}{\partial t}\phi + H_{\partial,1}(\nabla\phi);\right.$$
$$\left. -\frac{\partial}{\partial t}\phi + H_{\partial,2}(\nabla\phi);\ -\frac{\partial}{\partial t}\phi + H_c(\nabla\phi)\right] \leq 0 \quad (\geq 0)$$

whenever $(x,y) = (0,0)$,

(2.19c)
$$u \leq 0 \ (u \geq 0) \text{ whenever } (x,y) \in \Gamma_3 ,$$

and if $u(x,y,t) \to +\infty$ as $t \uparrow T$, for all $(x,y) \in \bar{D}\backslash\Gamma_3$.

Remark. There is an obvious analog for the equation with finite terminal data.

Lemma 2.2. *$\bar{u}$ and $\underline{u}$, defined by (2.9), are respectively sub and super solutions of (2.16) and (2.17) with infinite terminal data.*

Definition 2.2. We say a function u is a *viscosity solution* of (2.16) and (2.17) if its u.s.c. and l.s.c. envelopes

(2.20)
$$u^*(x,y,t) = \overline{\ell im}_{(\bar{x},\bar{y},\bar{t})\to(x,y,t)} u(\bar{x},\bar{y},\bar{t}),\ u_*(x,y,t) = \underline{\ell im}_{(\bar{x},\bar{y},\bar{t})\to(x,y,t)} u(\bar{x},\bar{y},\bar{t})$$

are sub and supersolutions, respectively.

It will follow from the results of Sections 2 and 3, together with the uniqueness results of Section 5, that $\bar{u}$ and $\underline{u}$ are both equal to the unique continuous viscosity solution. In the next section we give an alternative representation of this solution.

3. A Second Representation

Let $L(w,v)$, $L_{\theta, X}(w.v)$, and $L_c(w,v)$ be the Legendre transforms of $H(p,q)$, $H_{a>1}(p,q)$, and $H^{\wedge}p.q)$:

$$L(w,v) = \sup_{p.q} [-wp - vq - H(p,q)] \tag{3.1a}$$

$$L_{a>1}(w,v) = \sup_{p.q} [-wp - vq - H_{dtl}(p,q)] \tag{3.2b}$$

$$L_c(w,v) = \sup_{p.q} [-wp - vq - H_c(p,q)] \tag{3.3c}$$

As is well known, the Legendre transform defines a function that is convex and lower semicontinuous in the dual variables (w,v). Moreover, the above functions can be expressed almost explicitly by using the Legendre transform $h(t)$ of $e^{-s} - 1$, i.e.

$$h(t) \begin{cases} t\,fc\,t - t + 1 & ; \quad t \geq 0 \\ +\infty & ; \quad t < 0. \end{cases} \tag{3.2}$$

Then, we have the following alternative expressions for L, $L_{\theta, X}$, $L_{\theta,2}$ and L_c.

$$L(w.v) = \inf\{\lambda h(t_1) + \beta h(t_2) + \gamma h(t_3) + \alpha h(t_4) + \mu h(t_5) : \tag{3.3a}$$

$$Xt^{\wedge}l.O) + Pt_2(l,-1) + nrt_3(0,-l) + at_4(-l,0) + \mu t_5(-1.1) = (w.v)\}$$

(3.3b) $$L_{\partial,1}(w,v) = \inf\{\lambda h(t_1) + \beta h(t_2) + \gamma h(t_3) \ ;$$

$$\lambda t_1(1,0) + \beta t_2(1,-1) + \gamma t_3(0,-1) = (w,v)\}$$

(3.3c) $$L_{\partial,2}(w,v) = \inf\{\lambda h(t_1) + \alpha h(t_4) + \mu h(t_5) \ :$$

$$\lambda t_1(1,0) + \alpha t_4(-1,0) + \mu t_5(-1,1) = (w,v)\}$$

(3.3d) $$L_c(w,v) = \begin{cases} \lambda h(w/\lambda) & ; \ v = 0 \\ +\infty & ; \ v \neq 0 \end{cases}$$

Remarks. These expressions may be interpreted as a manifestation of the "contraction principle" [27] and the fact that our process may be thought of as being the sum of several independent Poisson processes. Owing to our assumptions on the jump rates, $L(w,v)$ is finite for all values of (w,v). However, $L_{\partial,1}$ and $L_{\partial,2}$ are finite only on certain convex cones, and the cones themselves depend on which of the jump rates are positive. For example, if $\gamma = 0$, and if λ and β are positive then $L_{\partial,1}$ is finite only on the (open) cone generated by $(1,0)$ and $(1,-1)$.

We continue by defining a "cost" that is appropriate for each of the boundaries Γ_1, Γ_2:

(3.4a) $$\ell_{\partial,1}(w,v) = \begin{cases} \inf\{\rho L(\bar{w},\bar{v}) + (1-\rho)L_{\partial,1}(\tilde{w},\tilde{v}) : \rho \in [0,1] , \\ \qquad\qquad \rho(\bar{w},\bar{v}) + (1-\rho)(\tilde{w},\tilde{v}) = (0,v)\} & ; \ w = 0 \\ L(w,v) & ; \ w > 0 \\ +\infty & ; \ w < 0 \end{cases}$$

$$(3.4b)\qquad \ell_{\partial,2}(w,v) = \begin{cases} \inf\{\rho L(\bar{w},\bar{v}) + (1-\rho)L_{\partial,2}(\tilde{w},\tilde{v}) : \rho \in [0,1], \\ \qquad\qquad \rho(\bar{w},\bar{v}) + (1-\rho)(\tilde{w},\tilde{v}) = (w,0)\} & ; v = 0 \\ L(w,v) & ; v > 0 \\ +\infty & ; v < 0 \end{cases}$$

Remark. The parameter ρ appearing in (3.4) has an interesting and natural large deviations interpretation. In the probabalistic approach to proving lower large deviation bounds, one typically considers a change of measure such that under the new measure (which we denote by $\bar{P}^\epsilon$) the process $x_\cdot^\epsilon$ 'centers' on a given deterministic path ϕ (in the sense that $x_\cdot^\epsilon \Rightarrow \phi$ under $\bar{P}^\epsilon$). One then obtains a lower bound from the formula (P^ϵ is the original measure)

$$P^\epsilon(A) = \int_A \frac{dP^\epsilon}{d\bar{P}^\epsilon}\, d\bar{P}^\epsilon ,$$

where the set A constains a neighborhood of ϕ. Under the 'optimal' change of measure that centers on ϕ (largest asymptotic lower bound for $-\epsilon\, \ell og\, P^\epsilon(A)$) the dominant term in $dP^\epsilon/d\bar{P}^\epsilon$ is of the form $\exp - S(x,\phi)/\epsilon$, where S is the functional appearing in (1.3) and (1.4). Now consider our process $x_\cdot^\epsilon$, and a path ϕ that lies on Γ_1. For simplicity take $\phi(t) = t(0,v)$. For our process we may consider a change of measure as being equivalent to changing the jump rates. Suppose we consider $\rho \in [0,1]$, $(\bar{w},\bar{v})$ and $(\tilde{w},\tilde{v})$ such that $\rho(\bar{w},\bar{v}) + (1-\rho)(\tilde{w},\tilde{v}) = (0,v)$. Consider a change of measure (change of jump rate) that centers the process on $(\bar{w},\bar{v})$ while in D^ϵ, and on $(\tilde{w},\tilde{v})$ while in Γ_1^ϵ. It is easy to prove in this case that under $\bar{P}^\epsilon$ (and as $\epsilon \to 0$) the relative

proportion of time the process $x_\cdot^\epsilon$ spends in D^ϵ to the time spent in Γ_1^ϵ is $\rho/(1-\rho)$. If we separately choose the jump rates to correspond to $L(\bar{w},\bar{v})$ in D^ϵ and $L_{\partial,1}(\tilde{w},\tilde{v})$ in Γ_1^ϵ, then the dominant term under this change of measure is

$$\exp[-T(\rho L(\bar{w},\bar{v}) + (1-\rho)L_{\partial,1}(\tilde{w},\tilde{v}))/\epsilon] \ .$$

It follows that the 'tightest' lower bound (which should also give the form of the upper bound) is

$$\exp[-T \inf\{\rho L(\bar{w},\bar{v})+(1-\rho)L_{\partial,1}(\tilde{w},\tilde{v}) : \rho(\bar{w},\bar{v})+(1-\rho)(\tilde{w},\tilde{v})=(0,v),\ \rho\in[0,1]\}/\epsilon] \ .$$

This suggests the form of $\ell_{\partial,1}$ given by (3.4a).

Finally we must define a cost for the corner point (0,0). We set

(3.5)

$$\ell_c(w,v) = \begin{cases} \inf\{\rho_1 L(w_1,v_1) + \rho_2 L_{\partial,1}(w_2,v_2) + \rho_3 L_{\partial,2}(w_3,v_3) + \rho_4 L_c(w_4,v_4) : \\ \sum_{i=1}^4 \rho_i(w_i,v_i) = (0,0) \ , \ \rho_i \geq 0 \ , \ \sum_{i=1}^4 \rho_i = 1\} & ; w = v = 0 \\ L(w,v) & ; w > 0,\ v > 0 \\ \ell_{\partial,1}(w,v) & ; w = 0,\ v > 0 \\ \ell_{\partial,2}(w,v) & ; w > 0,\ v = 0 \\ +\infty & ; w < 0 \ \text{ or } \ v < 0 \end{cases}$$

Then, our candidate for a continuous viscosity solution is

$$(3.6)\quad u(x.y.t) = \inf_{f(\cdot)\in A_{x.y.t}} \int_0^t [L(f(s))1_{\{f(s)\in D\}} + \kappa_c(f(.))1_{\{f(s)=(0,0)\}} + \sum_{i=1}^{2} \xi_{\sigma,i}(\xi(s))1_{\{\xi(s)\in\Gamma_i\}}]ds$$

where 1_A is the indicator of the Borel set A, and

$$(3.7)\quad A_{x\,y\,t} = \{f : [t.G] \to \bar{D} : f(t) = (x,y)\ ,\ \varphi(9) \in r_y\ 6 \le T, \text{ and}$$

f is absolutely continuous}

Theorem 3.1. *The* uaiue *function defined by* (3.6) *and* (3.7) Is a *continuous viscosity solution to* (2.16), (2.17).

Proof. The continuity of u follows from the boundedness of L on compact sets. Suppose that $\$ \in C^1(\bar{D} \times [0,T])$ and that

$$(3.9)\qquad (u - f K v V V^{=} "\wedge t^{11} " *]$$

We may assume without loss that the maximum is zero. If $(\wedge' Yo*\wedge) \wedge D^x (0\text{»}T)$, then the standard proof [21] works. (Alternatively the reader can glean the proof from the development below.) Next assume that $(XQ\wedge YQ\wedge Q) \in r_i^x (\circ\text{»}^T)$- Then dynamic programming [10] yields that for any $(w,v) \in R^+ \times R$ and $A > 0$,

(3.10)

$$u(0,y_0,t_0) = \phi(0,y_0,t_0) \le \int_{t_0}^{t_0+\Delta} [L(w,v)1_{\{w>0\}} + \ell_{\partial,1}(0,v)1_{\{w=0\}}]ds$$

$$+ \phi(x_0 + w\Delta, y_0 + v\Delta, t_0 + \Delta).$$

It follows that (see (3.4a))

(3.11) $$- \frac{\partial}{\partial t}\phi(0,y_0,t_0) - \ell_{\partial,1}(w,v) - \langle (w,v), \nabla\phi(0,y_0,t_0)\rangle \le 0$$

for $(w,v) \in R^+ \times R$ and hence

(3.12) $$- \frac{\partial}{\partial t}\phi(0,y_0,t_0) + \sup_{w\ge 0,v} [-\ell_{\partial,1}(w,v) - \langle (w,v), \nabla\phi(0,y_0,t_0)\rangle] \le 0$$

Using the definition of $\ell_{\partial,1}$, we rewrite (3.12) as

(3.13) $$\max\Big\{- \frac{\partial}{\partial t}\phi + \sup_{w>0,v}{}^{*} [-L(w,v) - \langle (w,v), \nabla\phi\rangle] ;$$

$$\sup[\rho\Big[- \frac{\partial}{\partial t}\phi - L(\bar{w},\bar{v}) - \langle (\bar{w},\bar{v}), \nabla\phi\rangle\Big] +$$

$$+ (1-\rho)\Big[- \frac{\partial}{\partial t}\phi - L_{\partial,1}(\tilde{w},\tilde{v}) - \langle (\tilde{w},\tilde{v}), \nabla\phi\rangle\Big] : \rho \in [0,1],$$

$$\rho\bar{w} + (1-\rho)\tilde{w} = 0]\Big\} \le 0 \quad \text{at } (0,y_0,t_0)$$

Now assume that

$$(3.14) \qquad -\frac{\partial}{\partial t}\phi + H_{\partial,1}(\nabla\phi) > 0 \quad \text{at} \quad (0,y_0,t_0)$$

The fact that $H_{\partial,1}$ is the Legendre transform of $L_{\partial,1}$, and continuity properties of $L_{\partial,1}$, imply there is (w^*,v^*) with $w^* > 0$ such that

$$(3.15) \qquad -\frac{\partial}{\partial t}\phi + [-L_{\partial,1}(w^*,v^*) - \langle(w^*,v^*),\nabla\phi\rangle] > 0 \quad \text{at } (0,y_0,t_0).$$

Also, (3.13) gives

$$(3.16) \qquad -\frac{\partial}{\partial t}\phi + \sup_{w>0,v}[-L(w,v) - \langle(w,v),\nabla\phi\rangle] \le 0 \quad \text{at } (0,y_0,t_0)$$

and by taking $(\tilde{w},\tilde{v}) = (w^*,v^*)$ in (3.13) we obtain that at $(0,y_0,t_0)$

(3.17)

$$-\frac{\partial}{\partial t}\phi + \sup[-L(w,v) - \langle(w,v),\nabla\phi\rangle : (w,v) = -\frac{(1-\rho)}{\rho}(w^*,v^*) + \frac{1}{\rho}(0,\bar{v}),$$

$$\rho \in [0,1],\ \bar{v} \in R] \le 0$$

Combining yields

$$(3.18) \qquad -\frac{\partial}{\partial t}\phi + \sup_{w,v}[-L(w,v) - \langle(w,v),\nabla\phi\rangle]$$

$$= -\frac{\partial}{\partial t}\phi + H(\nabla\phi) \le 0 \quad \text{at } (0,y_0,t_0).$$

Recall that we have assumed (3.14) in proving (3.18). We have thus proved:

$$\min\left\{-\frac{\partial}{\partial t}\phi + H(\nabla\phi),\ -\frac{\partial}{\partial t}\phi + H_{\partial,1}(\nabla\phi)\right\} \le 0 \quad \text{at } (0,y_0,t_0). \tag{3.19}$$

Now suppose that for $(x_0,y_0) \in \Gamma_1$ we have

$$(u - \phi)(x_0,y_0,t_0) = \min[u - \phi] = 0. \tag{3.20}$$

Using dynamic programming arguments (as in the proof of Theorem 2.1 of [25]), and the form of $\ell_{\partial,1}$, we can show that there exist $w \ge 0$, v such that

$$-\frac{\partial}{\partial t}\phi(0,y_0,t_0) + [-\ell_{\partial,1}(w,v) - \langle(w,v),\nabla\phi(0,y_0,t_0)\rangle] \ge 0 \tag{3.21}$$

If $w > 0$, $\ell_{\partial,1}(w,v) = L(w,v)$, and obviously $-\frac{\partial}{\partial t}\phi + H(\nabla\phi) \ge 0$ at $(0,y_0,t_0)$. Now suppose that $w = 0$. Then, the definition of $\ell_{\partial,1}$ yields.

$$\begin{aligned}\sup\Big[\rho\Big[-\frac{\partial}{\partial t}\phi - L(\bar{w},\bar{v}) - \langle(\bar{w},\bar{v}),\nabla\phi\rangle\Big] \\ + (1-\rho)\Big[-\frac{\partial}{\partial t}\phi - L_{\partial,1}(\tilde{w},\tilde{v}) - \langle(\tilde{w},\tilde{v}),\nabla\phi\rangle\Big] : \rho \in [0,1], \\ \rho\bar{v} + (1-\rho)\tilde{v} = v,\ \ \rho\bar{w} + (1-\rho)\tilde{w} = 0\Big] \ge 0 \quad \text{at } (0,y_0,t_0)\end{aligned} \tag{3.22}$$

Clearly in either case, we obtain

$$\max\left\{-\frac{\partial}{\partial t}\phi + H(\nabla\phi),\ -\frac{\partial}{\partial t}\phi + H_{\partial,1}(\nabla\phi)\right\} \ge 0 \quad \text{at } (0,y_0,t_0) \tag{3.23}$$

Exactly the same arguments work if $(XQ»YQ)$ $^{er}{}_{2}$* ^{Fina}lly ^{we} consider the point $(x_Q, y_0) = (0,0)$. Assume that the maximum of $(u - \langle f \rangle)$ is achieved at $(0,0,t_U^{\sim})$. Dynamic programming arguments give

$$(3.24) \qquad -\wedge + \sup_{w\geq 0, v\geq 0} [-\alpha_c(w.v) - \langle (w,v), v^* \rangle] \leq 0 \quad \text{at} \quad (0,0,t_Q),$$

which we rewrite as

$$(3.25) \qquad \max\Big/ -\frac{\partial}{\partial t}\phi + \sup_{w\geq 0, v\geq 0} [-L(w.v) - \langle (w.v).\nabla\phi \rangle] ,$$

$$-\frac{\partial}{\partial t}\phi + \sup_{w=0,v>0} [-\epsilon_{\partial.1}(w.v) - \langle (w,v), v\alpha \rangle] ,$$

$$-\frac{\partial}{\partial t}\phi^* + \sup_{w>0,v=0} t\text{-}e\, {}_{\circ,\tilde{Z}}(w.v) - \langle (w,v), v^* \rangle] ,$$

$$\sup\Big[p_i\Big[\frac{\partial}{\partial t}\phi^* \;''\; L(^{w}r^{v}i) \;''\; \langle (^{w}r^{v}1)\alpha \nabla\phi \rangle\Big]$$

$$+ P\Big[2\text{fol}^* - L_{\partial.1}(w_2, v_2) - \langle (w_2, v_2).\nabla\phi \rangle\Big]$$

$$+ p_3\Big[-\frac{\partial}{ST^*} - L_{a,2^*}{}^{W3'V3>} - \langle (w_3, v_3), \nabla\phi \rangle\Big]$$

$$+ p_4\Big[-\frac{\partial}{\partial t}\phi - L_c(w_4, v_4) - \langle (w_4, v_4).\nabla\phi \rangle\Big] :$$

$$\sum_{i=1}^{4} P\wedge Wj.vp = (0.0).\ p\wedge O, \quad \sum_{i=1}^{4} P_i = 1\,j\,| \leq 0 \quad \text{at} \quad (0,0,t_Q)$$

We assume at $(0,0,t_0)$

$$(3.26)\qquad \begin{cases} -\frac{\partial}{\partial t}\phi + H(\nabla\phi) > 0 \\ -\frac{\partial}{\partial t}\phi + H_{\partial,i}(\nabla\phi) > 0 \quad , \quad i = 1,2 \\ -\frac{\partial}{\partial t}\phi + H_c(\nabla\phi) > 0 \end{cases}$$

Then there exist (w_1^*,v_1^*), (w_2^*,v_2^*), (w_3^*,v_3^*), (w_4^*,v_4^*) such that $w_2^* > 0$, $v_3^* > 0$, $w_4^* > 0$, and $v_4^* = 0$, and further satisfying

$$(3.27)\qquad \left.\begin{array}{l} -\frac{\partial}{\partial t}\phi + [-L(w_1^*,v_1^*) - \langle(w_1^*,v_1^*),\nabla\phi\rangle] > 0 \\ -\frac{\partial}{\partial t}\phi + [-L_{\partial,1}(w_2^*,v_2^*) - \langle(w_2^*,v_2^*),\nabla\phi\rangle] > 0 \\ -\frac{\partial}{\partial t}\phi + [-L_{\partial,2}(w_3^*,v_3^*) - \langle(w_3^*,v_3^*),\nabla\phi\rangle] > 0 \\ -\frac{\partial}{\partial t}\phi + [-L_c(w_4^*,v_4^*) - \langle(w_4^*,v_4^*),\nabla\phi\rangle] > 0 \end{array}\right\} \text{ at } (0,0,t_0)$$

Arguing as before the third term in (3.25) gives

$$(3.28)\qquad -\frac{\partial}{\partial t}\phi + [-L(w,v) - \langle(w,v),\nabla\phi\rangle] \leq 0 \text{ at } (0,0,t_0)$$

for (w,v) of the form

$$- \frac{(1-\rho)}{\rho}(w_3^*, v_3^*) + \frac{1}{\rho}(0, \bar{v})$$

for $\rho \in [0,1]$ and $\bar{v} > 0$, while the second term in (3.25) gives (3.28) for all (w,v) of the form

$$- \frac{(1-\rho)}{\rho}(w_2^*, v_2^*) + \frac{1}{\rho}(\bar{w}, 0)$$

for $\rho \in [0,1]$, $\bar{w} > 0$. Combined with the first term in (3.25), this gives (3.28) for the shaded portion in fig. 3. Region I is the open convex cone generated by (0,1) and (1,0), region II is the half-open convex cone generated by (1,0) and $-(w_2^*, v_2^*)$, while region III is generated by (0,1) and $-(w_3^*, v_3^*)$. From the fourth term in (3.25) we obtain (3.28) for the closed cone generated by $-(w_2^*, v_2^*)$, $-(w_3^*, v_3^*)$, and $-(w_4^*, v_4^*)$, which contradicts (3.26). Hence, at $(0,0,t_0)$

$$(3.29) \quad \min\left\{ -\frac{\partial}{\partial t}\phi + H(\nabla\phi),\ -\frac{\partial}{\partial t} + H_c(\nabla\phi),\ \min_{i=1,2} -\frac{\partial}{\partial t}\phi + H_{\partial,i}(\nabla\phi) \right\} \leq 0 .$$

The case where a minimum is achieved at (0,0) is handled in a similar fashion. We have thus proved: u *defined by* (3.6) *and* (3.7) *is a viscosity solution of* (2.16) *and* (2.17).

□

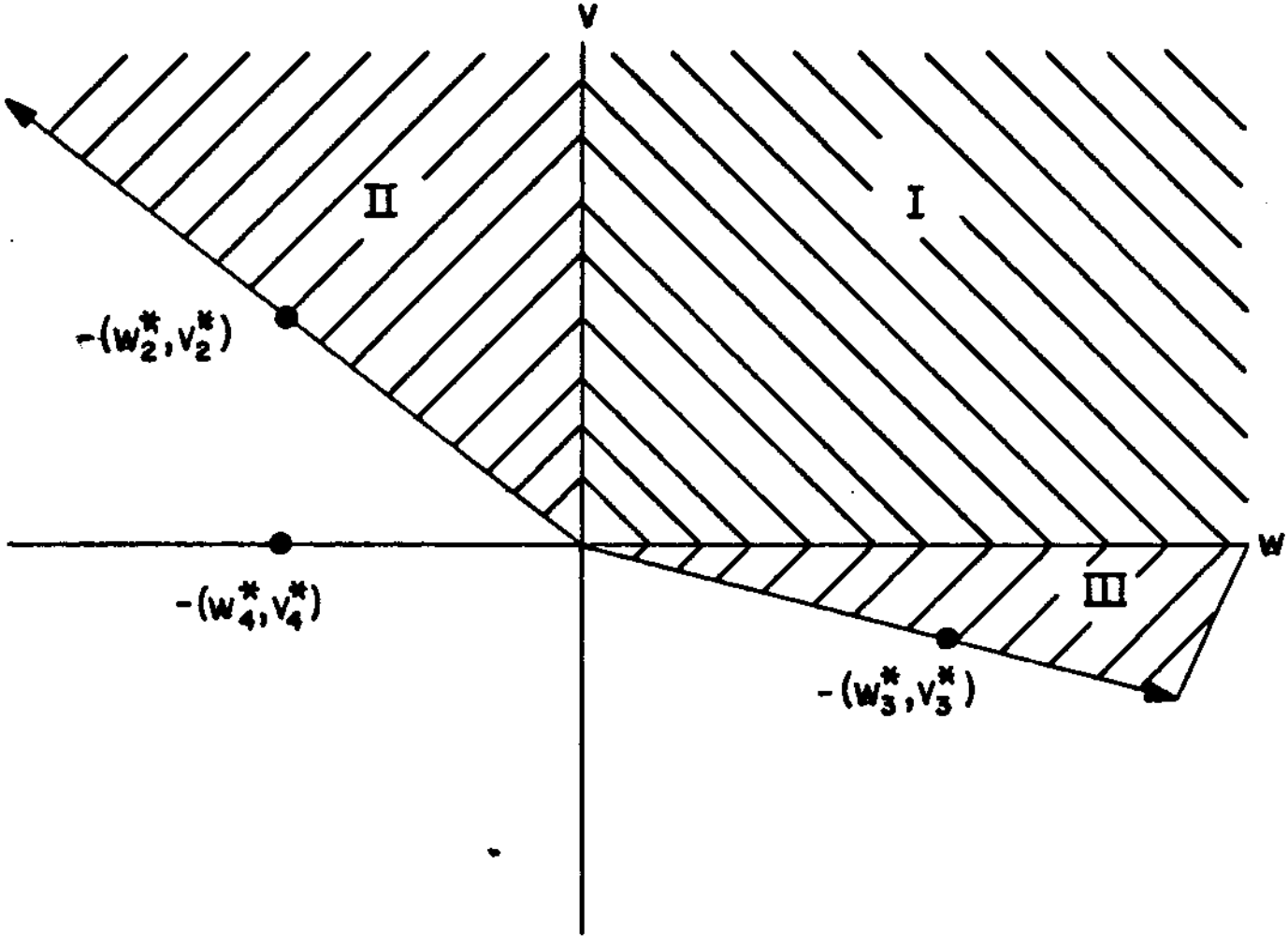

Figure 3

4. The Uniqueness Theorem

In this section and in the two that follow we prove uniqueness for the viscosity solutions to the equations (2.16) and (2.17). In particular we will prove that $\bar{u}$, $\underline{u}$ (c.f. (2.9)) are sub and super viscosity solutions to (2.16), (2.17). Hence, an immediate consequence of the comparison result (Theorem 4.2)is the uniform convergence of the sequence u^{ϵ} to the unique solution of (2.16), (2.17), which is equal to both $\bar{u}$ and $\underline{u}$.

To simplify the exposition, the problem and notations of the preceding sections are retained. However, the methods used are applicable to more general problems, some of which we describe in Section 7. To simplify, we switch from (x,y), (p,q), etc., notation to (x_1,x_2), (p_1,p_2), etc. notation.

In order to compare viscosity solutions we require suitable test functions ϕ which will force the interior equations (and not the boundary equations) to hold at maximizing points of $\tilde{u}(x_1,x_2,t) - \underset{\sim}{u}(y_1,y_2,t) - \phi(x_1 - y_1, x_2 - y_2)$, for any two viscosity solutions $\tilde{u}$, and $\underset{\sim}{u}$. Naturally, the form of the test functions ϕ depends on the boundary conditions. Although the use of such functions ϕ in this fashion is now standard ([2], [22], [25]), our construction of the test functions is quite different from constructions that exist in the literature.

In this section we assume the existence of such a sequence of test functions, and relegate the construction to the two following sections.

Assumption A.4.1.

For each $\delta > 0$, *there exist test functions* $\{\phi_{\epsilon,\delta}\} \subset C^1(R^2)$, $0 < \epsilon < 1$, *satisfying*

$$(4.1)\qquad \begin{cases} H_{\partial,i}(\nabla\phi_{\epsilon,\delta}(x_1,x_2)) \geq H(\nabla\phi_{\epsilon,\delta}(x_1,x_2)) - \delta \ , & \text{for } x_i \leq 0 \\ H_{\partial,i}(\nabla\phi_{\epsilon,\delta}(x_1,x_2)) \leq H(\nabla\phi_{\epsilon,\delta}(x_1,x_2)) + \delta \ , & \text{for } x_i \geq 0 \end{cases}$$

$i = 1,2$, *and*

$$(4.2)\qquad \begin{cases} \phi_{\epsilon,\delta}(0,0) = 0 \\ \phi_{\epsilon,\delta}(x_1,x_2) \geq \frac{1}{\epsilon} \quad \text{for} \quad \epsilon \leq (x_1^2 + x_2^2)^{1/2} \leq \text{diam } D. \end{cases}$$

Theorem 4.1. *Let* $\tilde{u}$ *(resp.* $\underset{\sim}{u}$*) be an u.s.c. viscosity subsolution (resp. ℓ.s.c. viscosity supersolution) of* (2.16) *and* (2.17) *but with finite terminal data at* $t = T$. *Assume A.4.1, and that* $\tilde{u}(x_1,x_2,T) \leq \underset{\sim}{u}(x_1,x_2,T)$ *for* $(x_1,x_2) \in \bar{D}$. *Then* $\tilde{u} \leq \underset{\sim}{u}$ *on* $\bar{D} \times [0,T]$.

Proof. Fix $0 < \tau < T$, and set $Q = \bar{D} \times (\tau,T]$, $x = (x_1,x_2)$, $y = (y_1,y_2)$. For $\delta > 0$, define $U,\bar{U}$ by

$$(4.3)\qquad \begin{cases} U(x,t) = \tilde{u}(x,t) - 2\delta\dfrac{(T-\tau)^2}{(t-\tau)} \\ \bar{U}(x,t) = \underset{\sim}{u}(x,t) + 2\delta\dfrac{(T-\tau)^2}{(t-\tau)} \end{cases}$$

Observe that to prove the conclusion of the theorem, it suffices to show that

$$(4.4)\qquad U \leq \bar{U} \quad \text{on} \quad Q$$

for all $6, T > 0$. Let us assume that

$$\sup_{Q}(U - \bar{U}) > 0 \tag{4.5}$$

Finally, for $0 < e < 1$, and $0 < p < \sup_{Q}(U - \bar{U})$ consider the auxilary function

$$*(x,y,t,s) := U(x,t) - \bar{U}(y,t) - \hat{}\,(x - y) - \pm\{t - s)^2 - p \tag{4.6}$$

where $(x,t),(y,s) \in Q$ and $\$_x$ is as in A.4.1. Note that $\$(x,y,t,s)$ tends to $-^{00}$ uniformly when either t or s approaches T. Therefore, using the semicontinuity of $\$$ we conclude that $\$$ attains its maximum on Q, say at $(\bar{x}.\bar{y}.\bar{t}.\bar{s}) \in Q$. Moreover,

$$*(x,\bar{y},\bar{t}.\bar{s}) \geq \sup_{Q}(U - \bar{U}) - p > 0 \tag{4.7}$$

Also, we claim that neither $(\bar{x},\bar{t})$ nor $(\bar{y},\bar{s})$ belongs to $2 = T_3 \times [T.T] \cup \bar{D} \times \{T\}$; the part of the boundary on which the Dirichlet condition is imposed. Indeed

$$*(x,y,t,s)\ i\ \tilde{u}(x,t) - \underset{\sim}{u}(y.s) - \hat{}_{fi}(x - y) - i(t - s)^2 - p \tag{4.8}$$

Recall that $\tilde{u} \pounds \underset{\sim}{u}$ on 2, that $\tilde{u}$ and $-\underset{\sim}{u}$ are upper semicontinuous, and $\bullet_{e\,6}(x - y) \geq 1/e$ if $|x - y| \geq e$. Using the fact that $i(7 - \bar{s})^2$ and

$\phi_{\epsilon,\delta}(\bar{x} - \bar{y})$ must be bounded independently of ϵ, together with (4.2), we conclude that $|\bar{x} - \bar{y}|$ and $|\bar{t} - \bar{s}|$ tend to zero as ϵ approaches zero. Thus, if $(x,t) \in \Sigma$, then (y,s) is near Σ (for small ϵ) and conversely. Hence, for small enough ϵ, $\Phi(x,y,t,s) \leq 0$ whenever $(x,t) \in \Sigma$ or $(y,s) \in \Sigma$. This together with (4.7) gives

$$(\bar{x},\bar{t}),(\bar{y},\bar{s}) \notin \Sigma. \tag{4.9}$$

We continue by using the equations (2.16), (2.17). First, observe that the map

$$(x,t) \mapsto \tilde{u}(x,t) - \left[\phi_{\epsilon,\delta}(x - \bar{y}) + \frac{2\delta(T - \tau)^2}{t - \tau} + \frac{(t - \bar{s})^2}{\epsilon}\right] \tag{4.10}$$

attains its maximum at $(\bar{x},\bar{t})$. Since u is a viscosity subsolution of (2.16), (2.17) and $(\bar{x},\bar{t}) \notin \Sigma$, this observation yields that either

$$2\delta\frac{(T - \tau)^2}{(\bar{t} - \tau)^2} - 2\frac{(\bar{t} - \bar{s})}{\epsilon} + H(\nabla\phi_{\epsilon,\delta}(\bar{x} - \bar{y})) \leq 0, \tag{4.11}$$

or one of the other inequalities appearing in (2.17) holds. But we claim that in each case the following inequality holds:

$$\delta - 2\frac{(\bar{t} - \bar{s})}{\epsilon} + H(\nabla\phi_{\epsilon,\delta}(\bar{x} - \bar{y})) \leq 0. \tag{4.12}$$

Clearly (4.11) implies (4.12). To handle the other cases, we use the assumed

properties of $\phi_{\epsilon,\delta}$ (c.f. (4.1)), which imply the following (for $i = 1,2$).

$$(4.13)\quad \bar{x}_i = 0 \Rightarrow \bar{x}_i - \bar{y}_i \le 0 \Rightarrow H_{\partial,i}(\nabla\phi_{\epsilon,\delta}(\bar{x} - \bar{y})) \ge H(\nabla\phi_{\epsilon,\delta}(\bar{x} - \bar{y})) - \delta$$

$$(4.14)\quad \bar{y}_i = 0 \Rightarrow \bar{x}_i - \bar{y}_i \le 0 \Rightarrow H_{\partial,i}(\nabla\phi_{\epsilon,\delta}(\bar{x} - \bar{y})) \le H(\nabla\phi_{\epsilon,\delta}(\bar{x} - \bar{y})) + \delta$$

It is now straightforward to obtain (4.12) from (4.13) and (2.17). Indeed, suppose that instead of (4.11) we have

$$(4.15)\qquad 2\delta\frac{(T - \tau)^2}{(\bar{t} - \tau)^2} - 2\frac{(\bar{t} - \bar{s})^2}{\epsilon} + H_{\partial,i}(\nabla\phi_{\epsilon,\delta}(\bar{x} - \bar{y})) \le 0$$

for $i = 1$ or 2. This may happen only if $\bar{x}_i = 0$. Consequently (4.13) holds, and (4.13) together with (4.15) gives (4.12). In the case $\bar{x} = (0,0)$ we use the identity $H_c = H_{\partial,1} + H_{\partial,2} - H$ together with (4.13).

Similarly, since $\underset{\sim}{u}$ is a viscosity supersolution to (2.16), (2.17), we obtain

$$(4.16)\qquad -\delta - 2\frac{(\bar{t} - \bar{s})}{\epsilon} + H(\nabla\phi_{\epsilon,\delta}(\bar{x} - \bar{y})) \ge 0$$

In this argument, we use (4.14) instead of (4.13).

Now, subtract (4.16) from (4.12) to obtain that $2\delta < 0$. By contradiction, (4.4) is true.

□

To extend this uniqueness result to the case of infinite terminal data we

adapt the ideas of [12] (see also [6]), and make use of two facts which hold in our problem:

(4.16) $$C_2 := \inf\{H(p) : p \in R^2\} > -\infty$$

(4.17) there exists a viscosity solution u of (2.16), (2.17) which belongs to $C(\overline{D} \times (0,T))$ and which tends to $+\infty$ as $t \uparrow T$, uniformly on compact subsets of $\overline{D} \setminus \Gamma_3$ (c.f. Theorem 3.1).

We start with a lemma. For the remainder of this section u denotes the continuous function described in (4.17).

Lemma 4.1. *For* $x = (x_1, x_2) \in \overline{D}$, $u(x,t) - C_2 t$ *is a nondecreasing function of* t.

Proof. Fix $(x_0, t_0) \in D \times (0,T)$, and $0 < \tau < t_0$. Choose $\epsilon > 0$ and $A < \infty$ so that

$$B(x_0, \epsilon) = \{x : |x - x_0| < \epsilon\} \subset D$$

and

(4.18) $$u(x,t) \leq u(x_0, t_0) + \epsilon + A|x - x_0|^2$$

whenever $(x,t) \in \partial B(x_0, \epsilon) \times [\tau, t_0] \cup B(x_0, \epsilon) \times \{t_0\}$. Define $\tilde{u}$ by

$$\tilde{u}(x,t) = u(x_o,t_Q) + e + A|x - x_Q|^2 - C_2(t - t_Q)$$

Then

(4.19) $$- g\tilde{u} + H(v\tilde{u}) \geq 0 \quad \text{in} \quad B(x_Q,e) \times [T,t_Q]$$

and by the comparison principle [5] and (4.18) $u \leq \tilde{u}$ on $B(x_{QI}e) \times [T,t_Q]$. In particular, $u(x_Q,t) \leq u(x_o,t_o) + e + C_2(t - t_Q)$, which implies the conclusion of the lemma.

D

Theorem 4.2. Let $\tilde{u}$ (resp. $\underset{\sim}{u}$) be *an* u.s.c. uiscoslty subsoiution *(resp. £.s.c.* uiscoslty supersolutton) *of* (2.16), (2.17). Assume A.4.1. Then $\tilde{u} \leq \underset{\sim}{u}$ *on* $\bar{D} \times (0,T)$.

Proof. We prove that $\tilde{u} \leq u$ and $u \leq \underset{\sim}{u}$ on $\bar{D} \times (0,T)$, where u a continuous viscosity solution to (2.16), (2.17). First note that by the definition of a viscosity solution with infinite terminal data and by Lemma 4.1, $u(x,t)$ tends to $+\infty$ as $t \mid T$, uniformly on compact subsets of $\bar{D}\backslash I_\delta$. Hence, for each $e > 0$, there is $0 < \delta_Q < e$ such that for any $x \in \bar{D}$ and $0 < \delta < \delta_Q$

(4.20) $$\underset{\sim}{u}(x,T - e) - e \leq u(x,T - \delta)$$

By Theorem 4.1, for any $x \in \bar{D}$, $e < t \leq T$, and $0 < 6 < \delta_Q$

(4.21) $$\underset{\sim}{u}(x,t - e) - e \leq u(x,t - 6)$$

Letting δ go to zero, and replacing $t - \epsilon$ by t we obtain

$$\underset{\sim}{u}(x,t) \leq u(x,t + \epsilon) + \epsilon \tag{4.22}$$

for any $x \in \overline{D}$, and $0 < t \leq T - \epsilon$. Letting ϵ go to zero

$$\underset{\sim}{u}(x,t) \leq u(x,t) \tag{4.23}$$

for $(x,t) \in \overline{D} \times (0,T)$. A very similar argument (again exploiting the continuity of u) gives $u \leq \underset{\sim}{u}$ on $\overline{D} \times (0,T)$.

□

5. Construction of the test functions

In this section we show how to construct $\{\phi_{\epsilon,\delta}\}$ satisfying A.4.1 for the case where all of $\lambda,\beta,\gamma,\alpha$, and μ are strictly positive. Cases where one (or more) of these is zero are considered in the next section.

The basis of the construction is an interesting use of the Legendre transform. Define

$$H^c_{\theta,i} = H - H_{\theta,i}$$

for $i = 1,2$. For now we look for a function $\phi(\cdot)$ which satisfies only (4.1) with $\delta = 0$. Define (for i = 1,2)

$$C_i^+ = \{(p_1,p_2) : H^c_{\theta,i}(p_1,p_2) \geq 0\}$$

(5.1)

$$C_i^- = \{(p_1,p_2) : H^c_{\theta,i}(p_1,p_2) \leq 0\}$$

(Refer to Fig. 4)

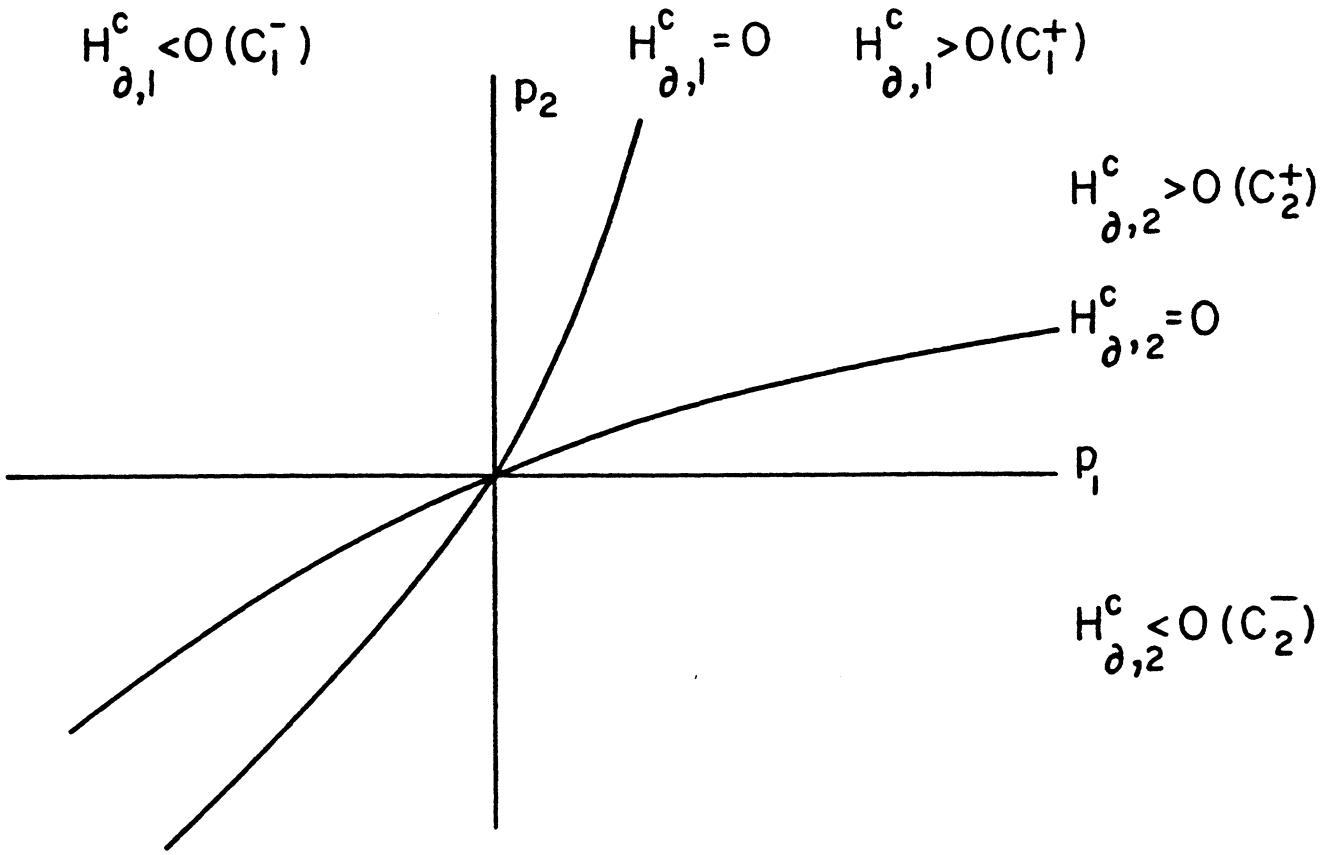

Figure 4

Then to satisfy (4.1) with $\delta = 0$, we require

(5.2) $\nabla\phi(x_1,x_2) \in C_i^+$(resp. C_i^-) whenever $x_i \geq 0$ (resp. $x_i \leq 0$), $i = 1,2$.

Assume that it is possible to find a strictly convex, finite valued function $R(p_1,p_2)$ such that $R(0,0) = 0$, $(0,0) \in \partial R(0,0)$, (where ∂ denotes the set of subdifferentials of a convex function [24]), and

(5.3) $\wedge$-$_i$R(P$_r$P$_2$) * ° (^{res}P- * °) ^{if} (Pi-P2>$^{€c}$i$^{+(resp.}$ C$_i^-$), i = 1,2.

(I.e. vR partitions the (p_i,p_2)-plane in the same way as (H°$_{\partial,}$ $\wedge$.U$\overset{c}{g}_{,2}$).) Define

(5.4) $$\wedge(Xj.x\wedge = \sup_{P_r P_2} [x_1p_1 + x_2p_2 - R(p_1.p_2)]$$

(this differs slightly from our previous definition of the Legendre transform). By conjugate duality [24]

(5.5) $$(P_rP_2) \in 3<Kx_rx_2) +-* (x_rx_2) \in 3R(p_rp_2)$$

Let (Pj.P$_2$) ^{e} S$\wedge$(x_{lf}x$_2$). By (5.3) and (5.5), Xj *I* 0 (resp. Xj ≤ 0) if and only if (P$_i$1.P2) $^€$ C-$_i^+$ (resp. C$_i^-$). A similar result holds regarding x$\wedge$. Since R(0,0) = 0 we obtain <*t*>_> 0, and (0,0) € 3R(0,0) implies *(0,0) = R(0,0) = 0. In particular, if + is differentiable it satisfies (4.1) with 6 = 0. Moreover, the differentiability of • follows from the uniform convexity of R. More precisely, suppose that R satisfies the following

(5.6) $$\text{«im inf}_{SHC0}\{R(p) : |p| = s\}/s = +\infty.$$

(5.7) for every L > 0 there is e = e(L) > 0 such that R(p) - e|p|$^{\swarrow}$ is a convex function of p, on |p| ≤ L. In other words, R is uniformly convex on bounded subsets of R$^{\Omega}$.

Then, $\phi \in C^1(R^2)$, [24].

We continue by constructing R having the properties (5.3), (5.6), and (5.7). To obtain (5.3), we look for $r_1(p)$, $r_2(p) > 0$ such that

$$\frac{\partial}{\partial p_i} R(p) = r_i(p) H^c_{\partial,i}(p) \quad , \quad i = 1,2 \quad , \quad p \in R^2. \tag{5.8}$$

A necessary and sufficient condition that (5.8) hold for some $R \in C^2(R^2)$ is

$$\frac{\partial}{\partial p_1}(r_2 H^c_{\partial,2}) = \frac{\partial}{\partial p_2}(r_1 H^c_{\partial,1}). \tag{5.9}$$

In the present case it is relatively simple to guess a form for r_1 and r_2 so that (5.9) holds. By taking

$$r_1(p_1,p_2) = A \exp(ap_1 + bp_2)$$

and

$$r_2(p_1,p_2) = B \exp(cp_1 + dp_2),$$

we obtain as sufficient conditions for (5.9)

$$\begin{cases} A/B = \beta/\mu \\ c = \beta\alpha/q \quad , \quad b = \mu\tau/q \\ a = c - 1 = -(\tau\alpha + \mu\tau)/q \\ d = b - 1 = -(\tau\alpha + \beta\alpha)/q \\ q = \tau\alpha + \mu\tau + \beta\alpha. \end{cases} \tag{5.10}$$

Integrating, and choosing the constants of integration in such a way that $R(0,0) = 0$, $\nabla R(0,0) = \nabla R(0,0)$, we obtain

$$(5.11) \qquad R(p_1,p_2) = q\Big[(\mu/\alpha[\exp(\alpha\beta p_1 - \alpha(\beta + \tau)p_2)/q - 1]$$

$$+ [\exp(\alpha\beta p_1 + \mu\tau p_2)/q - 1]$$

$$+ (\beta/\tau)[\exp(-(\mu + \alpha)\tau p_1 + \mu\tau p_2)/q - 1]\Big]$$

This function, in addition to (5.3), satisfies (5.7) and (5.8). Hence, its Legendre transform ϕ satisfies (4.1) with $\delta = 0$. Finally, we obtain the sequence $\{\phi_{\epsilon,0}\}$ by appropriately rescaling ϕ.

Lemma 5.1. *Let ϕ be the Legendre transform of the function R given by (5.11). Assume $\lambda,\alpha,\beta,\mu,\tau > 0$. Then, there is a function $\rho(\epsilon) > 0$ such that the sequence*

$$\phi_\epsilon(x) = \rho(\epsilon)\phi(x/\rho(\epsilon))$$

satisfies (4.1) with $\delta = 0$, and (4.2).

Proof. Since $\nabla\phi_\epsilon(x) = \nabla\phi(\frac{x}{\rho(\epsilon)})$, clearly ϕ_ϵ satisfies (4.1) with $\delta = 0$, for any choice of $\rho(\epsilon) > 0$. The finiteness of R implies the existence of $\theta(L)$ such that

$$R(p) \leq \tfrac{1}{2}\theta(L)|p|^2 \quad \text{whenever} \quad |p| \leq L$$

which in turn gives

$$\phi(x) \geq \frac{|x|^2}{2\theta(L)} \quad \text{whenever} \quad |x| \leq L\theta(L) \tag{5.12}$$

Set

$$L_\epsilon = \frac{2 \text{ Diam } D}{\epsilon^3} \quad , \quad \rho(\epsilon) = \frac{\epsilon^3}{2\theta(L_\epsilon)} \tag{5.13}$$

We now calculate directly that

$$\phi_\epsilon(x) \geq \frac{|x|^2}{2\rho(\epsilon)\theta(L_\epsilon)} = \frac{|x|^2}{\epsilon^3}$$

whenever $|x| \leq L_\epsilon\theta(L_\epsilon)\rho(\epsilon) = \text{Diam } D$. Hence, ϕ_ϵ satisfies (4.2).

□

6. Construction of the test functions; continued

In this section we remove the restriction that all $\vartheta(3_9 7_t a_t$ and JJ. must be strictly positive. It turns out it is not interesting to consider $jit = 0$ or $X = 0$, since for these cases the problem becomes trivial. In order to fix the ideas and exhibit the method in a simple **way,** we consider only the case $/5 = 0$. The other cases may be handled in a similar fashion.

The results of Sections 2-4 remain valid in this case, except that we can not construct test functions satisfying (4.1) with $6 = 0$, as we did in Section 5. Indeed, if we consider R defined by (5.11) and take $j3 = 0$, we obtain

$$(6.1) \qquad R(p_r p_2) = q|^M/a)[\exp(-aTp_2/q) - 1] + \exp(-jrrp_2/q) - lj$$

where $q = mra + JLIT$, and this function is not uniformly convex and does not satisfy (5.7). Hence we take a different tack, which requires an approximation argument.

Let $R(P, p_1, p_2)$ be given by (5.11), where we make the dependence on $/3 > 0$ explicit. Define

$$(6.2) \qquad RO.Lj.L^Pj.Pg) = R(P, P_r P_2) + I(L_1.L_2; p_1.p_2).$$

where

$$(6.3)\qquad I(L_1,L_2;p_1,p_2) = \begin{cases} 0 & \text{if } |p_1| \le L_1 \text{ and } |p_2| \le L_2 \\ +\infty & \text{otherwise} \end{cases}$$

Then $R(\beta,L_1,L_2;\cdot,\cdot)$ is uniformly convex, and finite in a neighborhood of the origin, if $\beta,L_1,L_2 > 0$. We then define $\phi(\beta,L_1,L_2;x_1,x_2)$ to be the Legendre transform of $R(\beta,L_1,L_2;\cdot,\cdot)$ (c.f. (5.4)).

Lemma 6.1. *Assume that* $\lambda,\tau,\alpha,\mu > 0$, *and* $\beta = 0$. *Then, there exist functions* $L_1(\epsilon),L_2(\epsilon),\rho(\epsilon)$, *and* $\beta(\epsilon,\delta)$ *such that the family of functions*

$$(6.4)\qquad \phi_{\epsilon,\delta}(x) = \rho(\epsilon)\phi(\beta(\epsilon,\delta),L_1(\epsilon),L_2(\epsilon);\frac{x}{\rho(\epsilon)})$$

satisfies (4.1) *and* (4.2).

Proof. Pick $L_1,L_2 > 2$ diam D, so that the following is satisfied for every $\beta,\epsilon \in (0,1]$.

$$(6.5)\qquad \begin{cases} \text{(i)} & \frac{\partial}{\partial p_1} R(\beta,\frac{L_1}{\epsilon^3},p_2) \in C^+_{1,\beta} \text{ , for } |p_2| \le \frac{L_2}{\epsilon^3} \\ \text{(ii)} & \frac{\partial}{\partial p_1} R(\beta,-\frac{L_1}{\epsilon^3},p_2) \in C^-_{1,\beta} \text{ , for } |p_2| \le \frac{L_2}{\epsilon^3} \\ \text{(iii)} & \frac{\partial}{\partial p_2} R(\beta,p_1,\frac{L_2}{\epsilon^3}) \in C^+_{2,\beta} \text{ , for } |p_1| \le \frac{L_1}{\epsilon^3} \\ \text{(iv)} & \frac{\partial}{\partial p_2} R(\beta,p_1,-\frac{L_2}{\epsilon^3}) \in C^-_{2,\beta} \text{ , for } |p_1| \le \frac{L_1}{\epsilon^3} \end{cases}$$

Here. $C^{\dagger}_{1,r}$ fi are as in (5.1), where we make the dependence on $/3 > 0$ explicit. The existence of such L_1. and L_2 follow from elementary geometric considerations (refer to Fig. 4).

Set

$$L^{\wedge}e) = \frac{L_i}{e^3}, \quad i = 1,2 \tag{6.6}$$

$$L_{\epsilon} = (L_x \wedge L_2)/e^3 \quad 0(e.6) = 6 \exp(-L_{fc}). \tag{6.7}$$

As in Lemma 5.1, there is $8(L_{\epsilon})$ such that

$$R(P(e.6),p) \leq |e(L_{fc})|p|^2 \quad . \quad |p| \leq L_{\epsilon} \tag{6.8}$$

for all $e,6 \in (0,1]$. Finally, set

$$P(e) = \frac{3}{2(fc_{\epsilon})}. \tag{6.9}$$

By using (6.8), we directly calculate that

$$\bullet(P(e.6).L_1(e).L_2(e);x) \} \frac{I_x l2}{2\theta(L_{\epsilon})} \quad . \quad |x| \leq L_{\epsilon}\theta(L_{\epsilon})$$

Hence

$$\phi_{\epsilon,\delta}(x) \geq \frac{|x|^2}{2\rho(\epsilon)\theta(L_\epsilon)} = \frac{|x|^2}{\epsilon^3}$$

whenever

$$|x| \leq L_\epsilon \theta(L_\epsilon)\rho(\epsilon) = \frac{(L_1 \wedge L_2)}{2}.$$

Since $L_1 \wedge L_2 >$ diam D, (4.2) is satisfied by $\phi_{\epsilon,\delta}$.

We continue by verifying (4.1). By conjugate duality [24], a version of (5.5) still holds even if R is not finite valued. In our particular example, we have

$$p = \nabla\phi_{\epsilon,\delta}(x) \Leftrightarrow x = \nabla R(\beta(\epsilon,\delta),p) \text{ if } |p_i| < L_i(\epsilon) \tag{6.13}$$

Hence, whenever

$$\left|\frac{\partial}{\partial x_j}\phi_{\epsilon,\delta}(x)\right| < L_j(\epsilon) \quad , \quad \text{for} \quad j = 1,2 \tag{6.14}$$

the construction of $R(\beta,\cdot)$ yields that (making the dependence of $H^c_{\partial,i}$ on β explicit)

$$H^c_{\partial,i,\beta}(\nabla\phi_{\epsilon,\delta}(x)) \geq 0 \text{ (resp. } \leq 0), \text{ if } x_i \geq 0 \text{ (resp. } \leq 0). \tag{6.15}$$

Then, in the case when (6.14) holds, one proves (4.1) after observing that

$$H^c_{\partial,1,0}(p) = H^c_{\partial,1,\beta}(p) \tag{6.16}$$

and

$$H^{c}_{\partial,2,0}(p) = H^{c}_{\partial,2,\beta}(p) - \beta[e^{p_2-p_1} - 1] \tag{6.17}$$

We therefore obtain (4.1) by using (6.7). Since by construction $|\frac{\partial}{\partial x_j}\phi_{\epsilon,\delta}(x)| \leq L_j(\epsilon)$, to complete the proof we have to consider the boundary cases.

First, suppose that $x_1 \geq 0$ and $\frac{\partial}{\partial x_1}\phi_{\epsilon,\delta}(x) = L_1(\epsilon)$. Then, (6.5) (i) together with (6.16) yields the desired result. Next, suppose that $x_1 \geq 0$, $\frac{\partial}{\partial x_2}\phi_{\epsilon,\delta}(x) = L_2(\epsilon)$, and $\frac{\partial}{\partial x_1}\phi_{\epsilon,\delta}(x) < L_1(\epsilon)$. In this case, definition of $\phi_{\epsilon,\delta}$ implies that

$$x_1 = \rho(\epsilon)\frac{\partial}{\partial p_1} R(\beta(\epsilon,\delta), \nabla\phi_{\epsilon,\delta}(x))$$

Hence, the construction of $R(\beta,\cdot)$ together with (6.16) yields the result.

All the other cases can be proved similarly.

□

7. Extensions and Comments

7.1. Extensions.

The techniques and ideas used in the analysis of our particular queueing model are in fact applicable to analogous problems for a very broad class of queue models, some of which will appear elsewhere. We will content ourselves in this subsection with describing only those extensions of the model and problem considered so far for which the proofs involved are very close to those of sections 2-6.

7.1.1. Different Escape Sets.

Let G be any bounded open set in R^2 whose boundary is smooth. Then in place of D we can use $\bar{G} \cap \{(x_1,x_2) : x_1 \geq 0, x_2 \geq 0\}$. We can consider unbounded G as well if Lemma 2.1 continues to hold. Thus we can take $D = \{(x_1,x_2) : x_1 \geq 0, 0 \leq x_2 \leq M\}$, which allows one to estimate the probability that queue 2 exceeds M/ϵ by time T/ϵ.

7.1.2. Higher Dimensions.

We can consider a system of n interconnected queues. Label the queues 1 through n, and let $S = \{1, \ldots, n\}$. Let X_t^i denote the number of customers in the i^{th} queue, and $X_t = (X_t^1, \ldots, X_t^n)$. Define

X_{ij} = jump intensity from queue i to queue j,

$A_{i,Q}$ ss jump intensity from queue i to outside the system

X_{n} . = intensity of arrivals at j from outside the system,

$$e_{i,j} = (0.0. \ldots .-1,0. \ldots .1,0 \ldots)$$

(-1 at ith, 1 at jth)

$$e_{1,0} = (0,0. \ldots .-1,0, \ldots,)$$

(-1 at rth)

$$e_{0,j} = (0.0. \ldots, .1.0. \ldots)$$

(1 at jth)

for every point $x \in \{(x_1, \ldots x_n) \in R^n : x_i \geq 0, i \in S\} = D$ define $I(x) = \{i \in S : x_i = 0\}$. For a subset s of S, we define the Hamiltonian $H(s,p)$ by

$$H(s,p) = \sum_{i \in S \setminus s} \sum_{j \in S} X_{i,j}[\exp\langle p.e_{i,j}\rangle - 1] \tag{7.1}$$

and its dual

$$L(s.v) = \sup_{P}[\langle v.p\rangle - H(s,p)] \ . \tag{7.2}$$

Finally, we define a 'cost' for each $x \in D$. Let $J(x,v) = \{i \in I(x) : v_i = 0\}$,

and let $\bar{I}(x)$ and $\bar{J}(x,v)$ be the set of subsets of $I(x)$ and $J(x,v)$, respectively. Then

$$(7.3)\quad \ell(x,v) = \inf\{ \sum_{s\in\bar{J}(x,v)} \rho_s L(x,v_s) : \sum_{s\in\bar{J}(x,v)} \rho_s v_s = v,\ \bigwedge_{s\in\bar{J}(x,v)} \rho_s \geq 0\ ,\ \sum_{s\in\bar{J}(x,v)} \rho_s = 1\}$$

Let G be an open set in $(R^+)^n$ with smooth boundary, and define $\partial G'$ to be the closure of $\{x \in \partial G : I(x) = \emptyset\}$ (here $\emptyset$ denotes the empty set). We assume that the origin is interior to the convex hull spanned by $\{\lambda_{i,j}e_{i,j}, (i,j) \in \{0,1, \ldots, n\}^2\}$. This implies $L(\emptyset,v)$ is finite for all values of v, and that the function defined by (7.4) below is continuous. This assumption does not imply a loss of generality. When it is violated, it implies that the behavior of the system is in fact trivial or decoupled.

Under these assumptions we have the following theorem, where $x_t^\epsilon = \epsilon X_{t/\epsilon}$ gives the scaled queue system.

Theorem 7.1. *Consider the following equation, interpreted in the viscosity sense:*

$$(7.4)\quad \begin{cases} -\frac{\partial}{\partial t}u + H(\emptyset, \nabla u) = 0, & (x,t) \in G \times (0,T) \\ -\frac{\partial}{\partial t}u + H(s, \nabla u) = 0 \ \text{ for some } \ s \in \bar{I}(x), & (x,t) \in \partial G \backslash \partial G' \times (0,T) \\ u = 0\ , & (x,t) \in \partial G' \times (0,T) \\ u \to +\infty \ \text{ as } \ t \to T\ , & x \in \bar{G}\backslash\partial G'\ . \end{cases}$$

Then the following results are true.

(1) *The equation* (7.4) *has a unique solution* u *in* $C(\overline{G} \times [0,T))$.

(2) *We have*

$$(7.5) \qquad \lim_{\epsilon} -\epsilon \log P\{x_\tau^\epsilon \notin G \text{ for some } \tau \in [t,T] \mid x_t^\epsilon = x\} = u(x,t) ,$$

with the convergence uniform in compact subsets of $\overline{G}\backslash\partial G' \times [0,T)$.

(3) *We have*

$$(7.6) \qquad u(x,t) = \inf_{\xi \in A_{x,t}} \int_t^\theta \ell(\xi,\dot{\xi})ds ,$$

where

$A_{x,t} = \{\xi : [t,\theta] \to R^n : \xi(t) = x, \xi(\theta) \in \partial G', \theta \leq T, \text{ and } \xi \text{ is absolutely continuous}\}$

Remark. The inf used to define $\ell(x,v)$ through (7.3) may be simplified. In fact, it is sufficient to sum over only those subsets of $\overline{J}(x,v)$ having only zero or one element.

7.1.3. Containment Probabilities.

Another class of probabilities that may be estimated via viscosity solution techniques are containment probabilities. Let G be open (in $(\mathbb{R}^+)^n$) with nice boundary. In this case we are interested in the asymptotics of

$$u^\epsilon(x,t) = -\epsilon \log P_{x,t}\{X_\tau^\epsilon \in G \text{ for all } t \le \tau \le T\}. \tag{7.7}$$

The associated PDE (to be interpreted in the viscosity sense) for this case is (using the system and notation of the previous subsection)

$$-\frac{\partial u}{\partial t} + H(\emptyset, \nabla u) = 0 \quad (x,t) \in G \times (0,T)$$

$$-\frac{\partial u}{\partial t} + H(s, \nabla u) = 0 \quad \text{for some } s \in \bar{I}(x), (x,t) \in \partial G \backslash \partial G' \times (0,T)$$

$$u = 0 \qquad (x,t) \in G \times \{T\}$$

The PDE approach for calculating asymptotics for these types of probabilities was first considered in [8]. The form of the associated variational representation for the limiting value of $u^\epsilon(x,t)$ in this case is given by (7.6), except we now replace $A_{x,t}$ by

$$\bar{A}_{x,t} = \{\xi : [t,T] \to \mathbb{R}^n : \xi(t) = x,\ \xi(\tau) \in G \text{ for } \tau \in [t,T], \text{ and } \xi \text{ is absolutely continuous}\}$$

A theorem analogous to Theorem 7.1 holds. The proof uses the same test

functions as those used in the case of escape probabilities. We omit the proof, and instead refer the reader to [25]. This work treats the comparison result for the same type of problem, but with an euqation that does not require such complicated test functions. The proofs that $\overline{u}$ and $\underline{u}$ (defined by (2.9)) are respectively sub and super solutions, and that (7.6) (with $A_{x,I.}$ replaced by $\overline{A}_{x,t}$) defines a solution, are essentially the same as those for escape probabi1i ties.

7.2. OQ the Relationship of the Results to a Large Deviation Principle.

As mentioned in the introduction, the results presented in this paper concerning the limiting behavior of certain classes of probabilities are all special cases of the results that would be available if the process $x_{\#}^{\epsilon}$ satisfied a large deviation principle. It is an interesting fact that in a certain sense the converse is also true. To be more specific, it is possible to prove that if for a given process it can be shown that the normalized £ogs of the escape and containment probabilities (given by (7.5) and (7.7) respectively) have the representation (7.6) (with the inf over $A_{x., t}$ and $\overline{A}_{x., v}$ respectively), then under some regularity conditions on the form of the function B appearing in (7.6), the measures induced by the process $x_{\#}^{\epsilon}$ satisfy a large deviation principle in the sense of [14; Sect. 3.3]. The rate function is given by

$$S(x,\xi) = \begin{cases} \int_0^T \ell(\xi,\dot{\xi})ds & , \ \xi \text{ is absolutely continuous, } \xi(0) = x \\ +\infty & , \text{ otherwise} \end{cases} .$$

Actually a slightly more general form of the results with regard to containment probabilities is needed, in which we replace x^ϵ by $x^{\epsilon,f} = x^\epsilon + f$, where f is a C^∞ deterministic function. However, the same techniques that apply for the case $f \equiv 0$ easily adapt to this case as well.

We do not give a detailed proof of this assertion, since such a proof in a general setting will appear elsewhere. Nonetheless it is worth mentioning the basic steps involved. We first note that under compactness of the 'level sets'

$$\Phi(x,r) = \{\phi : S(x,\phi) \leq r\}$$

the estimates (1.3), (1.4) follow if we can prove:

(1) given $\phi \in C([0,T] ; \mathbb{R}^n)$ such that $\phi(0) = x$, and $\delta > 0$,

$$\underline{\lim_{\epsilon}}\ \epsilon \log P_x\{\sup_{0\leq t\leq T} |x^\epsilon(t) - \phi(t)| \leq \delta\} \geq -S(x,\phi) , \tag{7.8}$$

(2) given $s < \infty$, and $\delta > 0$

$$\overline{\lim_{\epsilon}}\ \epsilon \log P_x\{\inf_{\phi\in\Phi(x,r)} \sup_{0\leq t\leq T} |x^\epsilon(t) - \phi(t)| \geq \delta\} \leq -r . \tag{7.9}$$

Obtaining (7.9) is easily accomplished by using the 'escape' estimates. To obtain (7.8), it must first be shown that it is sufficient to consider only ϕ that are piecewise C^{∞}. This requires regularity conditions on $\ell(\cdot,\cdot)$, which turn out to be trivially satisfied for the functionals considered in this paper. We then can obtain (7.8) by using the 'containment' estimates and the Markov property.

Appendix: A weaker formulation

In this section we present a weaker formulation of the PDE given in Definition 2.1, in order to relate our definition to more standard ways of describing boundary conditions.

First note that (2.19) implies

$$\text{(A.1)} \qquad \min_{(\max)} \left[-\frac{\partial}{\partial t}\phi + H(\nabla\phi) \; ; \; -H(\nabla\phi) + H_{\partial,i}(\nabla\phi) \right] \leq 0 \; (\text{resp. } \geq 0)$$

$$\text{if } (x,y) \in \Gamma_i \quad , \quad i = 1,2$$

and

$$\text{(A.2)} \qquad \min_{(\max)} \left[-\frac{\partial}{\partial t}\phi + H(\nabla\phi) \; ; \; -H(\nabla\phi) + H_{\partial,1}(\nabla\phi) \; ; \right.$$

$$\left. -H(\nabla\phi) + H_{\partial,2}(\nabla\phi) \; ; \; -H(\nabla\phi) + H_c(\nabla\phi) \right] \leq 0 \quad (\text{resp. } \geq 0)$$

$$\text{if } (x,y) = (0,0).$$

Dropping the fourth term in (A.2) (the H_c term) leads to a statement that is equivalent to letting that term remain. This follows from the equality

$$\text{(A.3)} \qquad -H + H_c = (-H + H_{\partial,1}) + (-H + H_{\partial,2}).$$

Thus, (A.2) holds if and only if (A.4) below holds.

$$\text{(A.4)} \quad \min_{(\max)_{\eta}} \left[-j^{\wedge} + H(v^*) \; ; \; -H(v^*) + H_g \, ^{\wedge} v^* \right]. \quad i = 1,2 \leq 0 \quad (\text{resp. } \geq 0)$$

if $(x.y) = (0,0)$

Note that (A.I) and (A.4) do not imply (2.19a) and (2.19b). In this weaker form the PDE has nonlinear boundary conditions (interpreted in the viscosity sense). Although this formulation is familiar, it is inferior to that given by (2.18) and (2.19). This latter definition is more useful in many ways, such as in proving uniqueness of the solution and in proving that the value function of the associated control problem is a viscosity solution (see Sections 3, 4).

The correct interpretation of our original formulation ((2.18), (2.19)) requires that we view (2.19a) and (2.19b) not as boundary conditions, but rather as the correct equations that would be associated to this part of the domain if we interpret the problem as one involving a discontinuous Hamiltonian, i.e. the correct Hamiltonians for the regions $\{(x.y) : x > 0, y > 0\}$, $\{(x.y) : x \leq 0, y > 0\}$, $\{(x,y) : x > 0, y \leq 0\}$, $\{(x,y) : x < 0, y < 0\}$ are $H(-)$ $U_{gx}(-)$. H_{o} $o^{\wedge *})^{*}$ $^{\wedge}$ $(^{\#})$ respectively. Taking the upper semicontinuous and lower semicontinuous envelopes of this discontinuous Hamiltonian yields the system (2.18), (2.19). Obviously the techniques we have developed are equally well suited to the treatment of analogous problems where the discontinuities of Hamiltonian appear in the interior of the domain of interest G.

Now consider the special case /3 = 0. In this case, we have

(A.5) $$-H(\nabla\phi) + H_{\partial,2}(\nabla\phi) = \gamma[e^{(\partial/\partial y)\phi} - 1] = 0$$

or

$$\frac{\partial}{\partial y}\phi = 0$$

as the boundary condition on Γ_2. Moreover, it is easy to show that $\ell_{\partial,2}(w,0)$ defined by (3.4b) has the following form.

(A.6) $$\ell_{\partial,2}(w,0) = \inf[L(w,v) : v \leq 0]$$

This expression agrees with the form of the integrand obtained in previous work of P.-L. Lions [22], where the Hamilton-Jacobi equations with Neumann type boundary conditions were studied.

References

[1] M. Bardi, An asymptotic formula for the Green's function of an elliptic operator, to appear in *Ann. Norm. Sup. Pisa.*

[2] G. Barles and B. Perthame, Exit time problems in optimal control and vanishing viscosity method, preprint.

[3] M.G. Crandall, L.C. Evans and P.-L. Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **282** (1984), 487-501.

[4] M.G. Crandall, H. Ishii and P.-L. Lions, Uniqueness of viscosity solutions of Hamilton-Jacobi equations revisited, *J. Math. Soc. Japan*, **39** (1987), 581-596.

[5] M.G. Crandall and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans, Amer. Math. Soc.* **277** (1983), 1-42.

[6] M.G. Crandall, P.-L. Lions and P.E. Souganidis, Maximal solutions and universal bounds for some partial differential equations of evolution, preprint.

[7] P. Dupuis and H. Kushner, Minimizing escape probabilities; A large deviations approach, preprint.

[8] L.C. Evans and H. Ishii, A PDE approach to some asymptotic problems concerning random differential equations with small noise intensities, *Ann. Inst. H. Poincare Analyse Nonlineaire* **2** (1985), 1-20.

[9] L.C. Evans and P.E. Souganidis, A PDE approach to certain large deviation problems for systems of parabolic equations, preprint.

[10] W.H. Fleming and R. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, New York, 1975.

[11] W.H. Fleming and H.M. Soner, Asymptotic expansions for markov processes with Lévy generators, *Journal of Appl. Math. and Opt.*, to appear.

[12] W.H. Fleming and P.E. Souganidis, A PDE approach to asymptotic estimates for optimal exit probabilities, *Ann. Scuola Norm. Sup. Pisa*, **13** (1986), 171-192.

[13] W.H. Fleming and P.E. Souganidis, Asymptotic series and the method of vanishing viscosity, *Indiana U. Math. J.* **35** (1986), 425-447.

[14] M.I. Friedlin and A.D. Wentzell, Random *Perturbations of* Dynamical Systems, Springer, New York, 1984.

[15] H. Ishii, Perron's method for Hamilton-Jacobi equations, *Duke Math. J.* 55 (1987), 369-384.

[16] H. Ishii, A boundary value problem of the Dirichlet type for Hamil ton- Jacobi equations, Ann *Scuola* Worm. Sup. *Piza* (to appear).

[17] H. Ishii and S. Koike, Remarks on elliptic singular perturbation problems, preprint.

[18] C. Knessl, B. Matkowsky, Z. Schuss and C. Tier, An asymptotic theory of large deviations for Markov jump processes, SIAM *J.* Appl. Math, 96/6 (1985), 1006-1028.

[19]_____, Asymptotic analysis of a state-dependent single-server queues, SIAM J. Appi. *Hath.*, *UG/h* (1986), 657-697.

[20] S. Koike, An Asymptotic formula for solutions of Hamilton-Jacobi-Bellman equations, *Nonlinear* Analysis, *Theor, Math.* Appl., *11* (1987), 429-436.

[21] P.-L. Lions, Generalized *Solutions of Hamilton-Jacobi Equations,* Pitman, Boston, 1982.

[22] P.-L. Lions, Neumann type boundary conditions for Hamil ton-Jacobi equations, *Duke Math. J.* 52 (1985), 793-820.

[23] S. Parekh and J. Walrand, Quick simulation of excessive backlogs in networks of queues, *Proc. of* IMA, U. Minnesota, June 1986, W.H. Fleming and P.-L. Lions editors.

[24] T. Rockafellar, *Convex* Analysis, Princeton Univ. Press, 1970.

[25] H.M. Soner, Optimal control with state-space constraints, I. SIAM *J. Control Opt., 2k* (1986) 552-561; *II. 2h* (1986), 1110-1122.

[26] D. Stroock, An *Introduction to the Theory of Large Deviations,* Springer, New York, 1984.

[27] S.R.S. Varadhan, Large *deviations and applications, SIAM* (1984).

[28] A. Weiss, A new technique for analyzing large traffic systems, *Adv. Appl. Prob.,* 18 (1986). 506-532.