EXPLICIT EXPRESSIONS FOR THE LYAPUNOV EXPONENTS OF CERTAIN MARKOV PROCESSES

by

Arie Leizarowitz
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Research Report No. 88-22 2

July 1988

Explicit expressions for the Lyapunov exponents of certain Markov processes

By Arie Leizarowitz

Abstract

A method for the explicit computation of the Lyapunov exponents of certain Markov processes is developed. Its utility is demonstrated by an application to two-dimensional random evolution differential equations. Our approach exploits the relation between the Lyapunov exponent and the p-moment Lyapunov exponents, as was first observed and studied by Arnold [1]. The p-moment Lyapunov exponent is characterized by the domain in which the Laplace transform of t -*$E|x(t)|^P$ is finite.

We apply our results to the random harmonic oscillator and derive an explicit expression for the Lyapunov exponent. In a simple case it is computed by quadratures.

Key words-* Lyapunov exponents, p-moment Lyapunov exponents, random evolution differential equations, random harmonic oscillator.

AMS(MOS) subject classification: Primary 93E15, 34F05; Secondary 60J25, 70L05.

Explicit expressions for the Lyapunov exponents of certain stochastic processes

1. Introduction. The asymptotic behavior of a system which is described by a differential equation can be characterized, to a large extent, by its Lyapunov exponent and its p-moment Lyapunov exponents. It is hence obviously desirable to be able to compute these numbers explicitly. As mentioned by Wihstutz in his survey paper [20] there are not many results in this direction. In fact all the explicit expressions refer to two-dimensional systems, in particular to the random harmonic oscillator. Moreover, the framework in these results is such that the system depends on some small parameter ϵ and one computes the asymptotic behavior of the Lyapunov exponent in the limit where $\epsilon \to 0$. (See Arnold, Papanicolaou and Wihstutz [5], Pinsky [17], Pardoux and Wihstutz [15] and Kleimann [12].)

The object of this work is to study the Lyapunov exponents of a class of Markov processes and their relationship with the p-moment Lyapunov exponents. As a byproduct of this study we derive explicit formulas for the Lyapunov exponents of certain two dimensional processes. Part of the discussion holds in quite a general context. We will, however, demonstrate our method by applying it to a certain type of two-dimensional random evolution differential equation. This class of equations contains the random harmonic oscillator.

Let $t \to x(t;x_0)$ be a Markov process which satisfies $P(x(0;x_0) = x_0) = 1$. The real number λ is the Lyapunov exponenet of x if almost surely (a.s.)

$$\lim_{t\to\infty} \frac{1}{t} \log|x(t;x_0)| = \lambda$$

for almost every initial value x_0.

Another quantity which is of interest for us is the p-moment Lyapunov exponent $g(p;x_0)$, which is given whenever it is defined for a real number p by the expression

(1.1) $$g(p;x_0) = \lim_{t\to\infty} \frac{1}{t} \log E|x(t;x_0)|^p.$$

The relation between the Lyapunov exponent and the p-moment Lyapunov exponents $g(p;x_0)$ was first observed and studied by Arnold (see Arnold [1] and Arnold, Oeljeklaus and Pardoux [4]). In [1] Arnold considered the linear differential euqation $\dot{x}(t) = A(\xi_t)x(t)$, where ξ_t is a nice ergodic process on a smooth manifold. He showed that $g(p;x_0)$ does not depend on x_0, that the limit in (1.1) exists for every real p, that $p \to g(p)$ is a convex function, $g(0) = 0$ and $g(p)/p$ is nondecreasing. Moreover $g(\cdot)$ is differentiable at $p = 0$ and

(1.2) $$\lambda = g'(0).$$

We will show that these properties hold for a general class of Markov processes.

In order to use (1.2) to compute λ we need to know the asymptotic behavior of $E|x(t)|^p$ as $t \to \infty$ for small values of $|p|$. Our method is thus to study a differential equation which is satisfied by the function

$$(1.3) \qquad \varphi(t,x_0) = E|x(t,x_0)|^p.$$

This method was used in [13] to obtain bounds and estimates for the Lyapunov exponenet λ.

A large portion of the recent work about Lyapunov exponents of solutions of differential equations was concerned with linear differential equations of various types. We mention just a few studies out of the vast literature on the subject. In the context of white driving noise we mention the works of Has'minskii [10], [11], Pinsky [15] and Arnold, Oeljeklaus and Pardoux [4]. In the context of real driving noise we mention the work of Arnold [1], Arnold, Kleimann and Oeljeklaus [3], Arnold and Kleimann [2], Crauel [8] and Kleimann [12]. In the context of driving noise processes of jump type we mention the work of Arnold, Papanicolaou and Wihstutz [15], Blankenship and Loparo [7], Pinsky [17] and Blakenship and Li [6].

We will consider the class of homogeneous differential equations, which contains the class of linear differential equations. Namely, we will consider equations of the form

$$\dot{x}(t) = f(x,\omega)$$

where f has the property $f(sx,\omega) = sf(x,\omega)$ for every scalar $s > 0$, every x and every ω. It will then follow that the functions $(t,x) \to \varphi(t,x)$ in (1.3) satisfy a differential equation with a certain homogeneity property. This will serve to reduce the number of independent variables. In the case of two dimensional systems this leads to a study of an ordinary differential equation which yields the explicit expression for λ.

The paper is organized as follows. In section 2 we introduce a definition of the exponents $g(p)$ which is different from the one given in (1.1) and which may be applied for a large class of Markov proceses. The two definitions coincide whenever the limit in (1.1) exists. We will derive some properties of the function $p \to g(p)$, in particular we will show that it is convex and satisfies a version of the relation (1.2). This last is derived in section 3.

In section 4 we will consider solutions of random evolution homogeneous differential equations. We will derive systems of partial differential equations which are satisfied by the functions $(t,x) \to \varphi(t,x)$ in (1.3) and by their Laplace transforms $(s,x) \to \psi(s,x)$. A key fact in our method is that $\psi(s,x)$ is finite for $s > g(p)$ and diverge as $s \downarrow g(p)$. In this section we will show how the homogeneity enables to reduce by one the number of independent variables. In section 5 we consider two-dimensional homogeneous differential equations. In this case the study is reduced to that of an ordinary (deterministic) differential equation with periodic coefficients. We proceed to study the nonsingular case, where the coefficients of the first derivatives never vanish. It then follows that $p \to g(p)$ is differentiable at $p = 0$ and the Lyapunov exponent satisfies (1.2). We use this to derive an explicit expression for the Lyapunov exponent in terms of the fundamental solution of the above mentioned ordinary differential equation with periodic coefficients.

In section 6 we verify that the random harmonic oscillator (with sufficiently small driving noise coefficients) is described by a nonsingular system. In cases where the perturbing noise depends on a small parameter ϵ we show how one can obtain the asymptotic expansion of the Lyapunov exponent in powers of ϵ by quadratures. We also use quadratures to compute the Lyapunov exponent in the special case where the driving Markov chain has two states.

2. The p-moment Lyapunov exponent.

We consider a Markov process $t \to x(t,\omega)$ in the state space $X = \bigcup_{i=1}^{\nu} X_i$ where $\nu \geq 1$ is an integer and each X_i is a copy of R^n such that $X_i \cap X_j = \{0\}$ whenever $i \neq j$. Thus X is a finite dimensional cone. We denote by $|x|$ the norm of x as an element of R^n.

The underlying probability space is $(\Omega,\mathcal{F},P)$ and we assume that $t \to x(t)$ is a space homogeneous process in the sense that

$$P_{\alpha x_0}(x(t) \in \alpha A) = P_{x_0}(x(t) \in A) \tag{2.1}$$

for every $x_0 \in X$, $\alpha > 0$, $t > 0$ and a Borel set $A \subset X$.

Remark 2.1. The homogeneity assumption (2.1) is not necessary for most of the discussion in this section (though it simplifies some of the notations). We assume it, however, since the development in the subsequent sections is concerned only with homogeneous processes.

We are interested in the growth properties of $|x(t)|$ as $t \to \infty$, thus we consider, for fixed real s and p, the following random variable

$$Y_p(s) = \int_0^\infty e^{-st} |x(t)|^p dt. \tag{2.2}$$

It follows from the positivity of the integrand in (2.2) that the random variable $Y(s)$ is well defined (the value $+\infty$ not excluded). We assume the following.

Assumption A. There is an s_0 such that $Y_p(s_0)$ is a.s. finite and in $L^1(\Omega)$. There is an s_1 such that $Y_p(s_1)$ is not in $L^1(\Omega)$.

Clearly if s_0 is as in Assumption A then $Y(s)$ is a.s. finite for every $s \geq s_0$.

Example 2.2. We consider a random evolution linear differential equation

$$\frac{dy(t)}{dt} = A_{j_t} y(t) \quad , \quad y(0) = y_0 \tag{2.3}$$

where $\{j_t\}_{t\geq 0}$ is a Markov process with a finite state space $\{1,\ldots, \nu\}$, $y(t) \in R^n$, for $t \geq 0$ and some integer $n \geq 1$, and A_i is an $n \times n$ matrix for $1 \leq i \leq \nu$. We thus consider the Markov process

(2.4) $$x(t) = (y(t), j_t) \quad , \quad t \geq 0$$

with the state space $X = \bigcup_{i=1}^{\nu} X_i$, each X_i being a copy of R^n. It follows from (2.3) that there is a constant $c > 0$ such that $e^{-ct}|y_0| \leq |y(t)| \leq e^{ct}|y_0|$ for every y_0, ω and $t \geq 0$. Thus Assumption A holds with $s_0 > |p|c$ and $s_1 < -|p|c$.

<u>Example 2.3</u>. Generalizing the situation in Example 2.2 we consider the random evolution homogeneous system

(2.5) $$\dot{y}(t) = f_{j_t}(y(t)) \quad , \quad y(0) = y,$$

where $\{j_t\}_{t\geq 0}$ is a Markov process with a finite state space $\{1, \ldots, \nu\}$, $y(t) \in R^n$ for some integer n and all $t \geq 0$, and $f_i : R^n \to R^n$ is a homogeneous Lipschitz continuous function for $1 \leq i \leq \nu$. We consider the Markov process $t \to x(t)$ in (2.4) and it follows from the relation

$$\frac{d}{dt}|y(t)| = \left[\frac{y(t)}{|y(t)|} \cdot f_{j_t}\left[\frac{y(t)}{|y(t)|}\right]\right]|y(t)|,$$

which is valid whenever $y(t) \neq 0$, that there is a constant $c > 0$ such that

$$e^{-ct}|y_0| \leq |y(t)| \leq e^{ct}|y_0|$$

for every $y_0 \neq 0, \omega$ and $t \geq 0$. Thus Assumption A holds with the choice $s_0 > |p|c$ and $s_1 < -|p|c$.

It follows from Assumption A that

$$\inf\{s : EY_p(s) < \infty\}$$

is finite. It may, however, depend on the initial condition y_0, and we want to exclude this possibility. We thus consider the projected process

$$\theta(t) = x(t)/|x(t)|$$

which is a stochastic process in the state space

$$S = \bigcup_{i=1}^{\nu} S_i$$

where S_i is a copy of the unit sphere in R^n. The process $t \to \theta(t)$ is well defined provided that a.s. $x(t) \neq 0$ for all $t \geq 0$. Then it follows that $t \to \theta(t)$ is a Markov process on S with the transition probability function

$$Q_{s_0}(t,B) = P_{x_0}(t,A)$$

where $P_{x_0}(t,A)$ is the transition probability function of the process $t \to x(t)$, the set A is given by $A = \bigcup_{\alpha>0} \alpha B$ and $x_0 = \beta s_0$ for some

P > 0 (we employ the notation /3(y_Q,i) = (Py_Q,i) for /? > 0 and (y,i) € X). It follows from (2.1) that the definition of Q_{g_O} in (2.5) does not depend on the choice of /5 for x_Q. We want t -» G(t) to have an irreducibility property so that the following Assumption holds:

<u>Assumption B</u>. The process t -» x(t) is such that a.s. x(t) ? 0 for all t ≥ 0 whenever x(0) ? 0 a.s. Moreover, the Markov process t -» 6(t) on S is such that for every s_Q € S

(2.6) Q_{s_0}(G(t) € B) > 0 for some t > 0

whenever B C S is a Borel set with positive Lebesgue measure.

<u>Definition 2.5</u>. Let t ->x(t) be a space homogeneous Markov process satisfying Assumptions A and B. Then we call x(#) <u>a homogeneous irreducible process</u>.

<u>Remark 2.6</u>. It is not hard to give conditions which imply the validity of Assumption B. We will do this for the special cases considered in the subsequent sections.

The following result asserts that the values s for which Y_p(s) in (2.2) is integrable are essentially independent of the initial value x_U.

<u>Proposition 2.7</u>. Let t -»x(t) be a homogeneous irreducible process with the initial value x_Q and let Y_p(s) be the corresponding random variable

defined in (2.2). Then there is a number $g(p)$ such that for almost every x in X we have $Y_p(s) \in L^1(\Omega)$ if $s > g(p)$ and $Y_p(s) \notin L^1(\Omega)$ if $s < g(p)$. (The exceptional set of values in X may depend on p.)

<u>Proof</u>: For every p and a nonzero $x_0 \in X$ (namely x_0 of the form (y_0,i) for some $y_0 \neq 0$) we define

$$g(p;x_0) = \inf\{s \in R^1 : EY_p(s) < \infty\}.$$

It follows from Assumption A that for every fixed p the mapping $x_0 \to g(p;x_0)$ is a bounded measurable function from S into R^1. We notice that the homogeneity of $t \to x(t)$ implies that for every nonzero x_0

$$g(p;x_0) = g(p ;\frac{x_0}{|x_0|}).$$

Thus our goal is to show that the mapping $x_0 \to g(p;x_0)$ is essentially constant on S (namely has the same value except, possibly, on a set of zero Lebesgue measure).

Assume that $x_0 \in S$ and $B \subset S$ are such that B has positive Lebesgue measure. It follows from the irredicibility property that for some $t_0 > 0$

$$Q_{x_0}(\theta(t_0) \in B) > 0. \tag{2.7}$$

Then

$$E\int_0^\infty e^{-st}|x(t;x_0)|^p dt = E[E\int_0^\infty e^{-st}|x(t;x_0)|^p dt\,|x(t_0;x_0)] \geq$$

$$\geq e^{-st_0}E[E\int_{t_0}^\infty e^{-s(t-t_0)}|x(t;x_0)|^p dt\,|x(t_0;x_0)] =$$

$$= e^{-st_0}\int_X E\Big[\int_0^\infty e^{-st}|x(t;y)|^p dt\Big]P_{x_0}(x(t_0)\in dy) \geq$$

$$\geq e^{-st_0}\int_A E\Big[\int_0^\infty e^{-st}|x(t;y)|^p dt\Big]P_{x_0}(x(t_0)\in dy)$$

where $A = \bigcup_{\alpha>0}\alpha B$ (and B is as in (2.7)). If s is such that $s <$ $g(p;y)$ for every $y \in B$ then $E\int_0^\infty e^{-st}|x(t;y)|^p dt = \infty$ for every $y \in A$. Since (2.7) implies that $P_{x_0}(x(t_0) \in A) > 0$ it follows that whenever $s < g(p;y)$ for every $y \in B$ then $E\int_0^\infty e^{-st}|x(t,x_0)|^p dt = \infty$, namely $s < g(p;x_0)$. Since x_0 in this argument is arbitrary nonzero element of X, it folows that there is a null set $N_p \subset X$ and a number $g(p)$ such that $g(p;y) = g(p)$ for every $p \in X\backslash N_p$, and $g(p;y) \geq g(p)$ for every $g \in N_p$. This concludes the proof of the Proposition.

□

<u>Remark 2.8</u>. For certain linear systems we may have the simplification that the value $g(p)$ is common to all the nonzero x_0, without an exceptional

null set (see Arnold [1].) Also in the two-dimensional applications studied in the following sections the $g(p;x_0)$ are independent of x_0 in S.

We consider the function

$$g : X\backslash\{0\} \to R^1$$

whose existence is guaranteed in Proposition 2.7 and establish the following property:

Theorem 2.9. The function $p \to g(p)$ is convex. Moreover, if Q is defined by

$$Q = \{(p,s) \in R^2 : s \geq g(p)\} \tag{2.8}$$

then Q is a convex set and the function

$$(p,s) \to \log E \int_0^\infty e^{-st} |x(t;x_0)|^p dt$$

is convex.

Proof: We will show that Q is a convex subset of R^2, from which the convexity of $g(\cdot)$ follows. Let $s_1 > g(p_1)$, $s_2 > g(p_2)$ and $p = \alpha p_1 + (1 - \alpha)p_2$, $s = \alpha s_1 + (1 - \alpha)s_2$ for some $0 < \alpha < 1$. It then follows (as we prove below) that

$$\text{(2.9)} \qquad E \int_0^\infty e^{-st} |x(t;x_0)|^p dt < \infty$$

for almost every $x_0 \in S$, hence by Proposition 2.8, $s \geq g(p)$. This being true for every $s_1 > g(p_1)$ and $s_2 > g(p_2)$ in Q proves the convexity of Q.

We prove now (2.9). With the above notations we have

$$Y_p(s) = \int_0^\infty e^{-[\alpha s_1+(1-\alpha)s_2]t} |x(t;x_0)|^{\alpha p_1+(1-\alpha)p_2} dt =$$

$$= \int_0^\infty [e^{-s_1 t} |x(t;x_0)|^{p_1}]^\alpha [e^{-s_2 t} |x(t;x_0)^{p_2}]^{1-\alpha} dt \leq$$

$$\leq \left[\int_0^\infty e^{-s_1 t} |x(t;x_0)|^{p_1} dt\right]^\alpha \left[\int_0^\infty e^{-s_2 t} |x(t;x_0)|^{p_2} dt\right]^{1-\alpha} =$$

$$= [Y_{p_1}(s_1)]^\alpha [Y_{p_2}(s_2)]^{1-\alpha}$$

where we used the Hölder inequality. Another application of Hölder inequality yields

$$E\, Y_p(s) \leq E[Y_{p_1}(s_1)]^\alpha [Y_{p_2}(s_2)]^{1-\alpha} \leq [E\, Y_{p_1}(s_1)]^\alpha [E\, Y_{p_2}(s_2)]^{1-\alpha} < \infty$$

which prove (2.9). Computing the logarithm of both sides of the inequality

$$E\, Y_p(s) \le [E\, Y_{p_1}(s_1)]^{\alpha}[E\, Y_{p_2}(s_2)]^{1-\alpha}$$

we conclude that $(p,s) \to \log E\, Y_p(s)$ is a convex function, which concludes the proof of the Theorem.

□

3. On the realtion between λ and $g(p)$.

For a homogeneous irreducible process $t \to x(t)$ we define the Lyapunov exponents $\lambda_{\pm}(x_0)$ by

$$(3.1)\quad \lambda_-(x_0) = \liminf_{t\to\infty} \frac{1}{t} \log|x(t;x_0)|, \quad \lambda_+(x_0) = \limsup_{t\to\infty} \frac{1}{t}\log|x(t;x_0)|.$$

Remark 3.1. If $t \to x(t;x_0)$ is the solution of a linear system of differential equations then one may apply Oseledec's Theorem (see Raghunathan [17]) or the Furstenberg and Kesten Theory concerning products of random matrices ([9]) to deduce additional information on $\lambda_{\pm}(x_0)$. E.g. the $\liminf_{t\to\infty}$ in (3.1) might be replaced by $\lim_{t\to\infty}$, λ might be nonrandom and of the same value for all nonzero x_0. In our context of homogeneous irreducible processes we will establish upper and lower bounds which are essentially nonrandom and independent of x_0. In case that $g'(0)$ exists these bounds coincide and we have $\lambda_-(x_0) \le g'(0) \le \lambda_+(x_0)$. Other tools, e.g. the ergodic theorem or the above mentioned theorems of Oseledec and

Furstenberg and Kesten may be used to show that in (3.1) the ℓim inf and the ℓim sup are in fact limits and then the value of the Lyapunov exponent is $g'(0)$. We will do this for a class of two dimensional homogeneous equations by using the Ergodic Theorem.

It follows from Theorem 2.9 that the left and right derivatives of $p \to g(p)$ exist at $p = 0$ and we denote

$$(3.2) \qquad k_+ = \lim_{\substack{p\to 0\\ p>0}} \frac{g(p)}{p}, \qquad k_- = \lim_{\substack{p\to 0\\ p<0}} \frac{g(p)}{p}.$$

Theorem 3.2. Let $t \to x(t;x_0)$ be a homogeneous irreducible process with the initial value x_0. Let $\lambda_\pm(x_0)$ be defined by (3.1) and $k_\pm$ be as in (3.2). We then have that a.s.

$$(3.3) \qquad \lambda_-(x_0) \le k_+ \quad , \quad k_- \le \lambda_+(x_0)$$

for almost every $x_0 \in X$. In particular, if $p \to g(p)$ is differentiable at $p = 0$ then a.s.

$$(3.4) \qquad \lambda_-(x_0) \le g'(0) \le \lambda_+(x_0)$$

for almost every $x_0 \in X$. If it is known that the limit in (3.1) exists then a.s.

$$\lambda(x_0) = g'(0) \text{ for almost every } x_0.$$

<u>Remark 3.3</u>. In the situation considered in section 5 the equality $\Lambda_-(x_0) = \Lambda_+(x_0)$ a.s. is an immediate consequence of the Ergodic Theorem.

<u>Proof</u>: A real number k satisfies

(3.5) $$g(p) \geq kp \quad \text{for every } p > 0 \ (p < 0)$$

if and only if it satisfies

(3.6) $$k \leq k_+ \ (k \geq k_-) .$$

For a given x_0 let $\mu(x_0)$ be a random variable which satisfies

(3.7) $$\mu(x_0) < \lambda(x_0) \text{ a.s.}$$

Then by the definition in (3.1) there is a random variable $\omega \to c(\omega)$, $c(\omega) > 0$, such that a.s.

$$|x(t;x_0)| \geq c(\omega)e^{\mu(x_0)t} \quad \text{for all } t \geq 0.$$

For a positive p and an $s > g(p)$ we thus have

(3.8) $$e^{-st}|x(t;x_0)|^p \geq c(\omega)^p e^{(p\mu(x_0)-s)t} .$$

From the fact that $E\int_0^\infty e^{-st}|x(t;x_0)|^p dt < \infty$ for almost every x_0 (for this value of p) it follows that a.s. $\int_0^\infty e^{-st}|x(t;x_0)|^p dt < \infty$ for almost every x_0. We consider only rational positive values for p and conclude that a.s. $\int_0^\infty e^{-st}|x(t;x_0)|^p dt < \infty$ for almost every x_0 and every rational $p > 0$, which, in view of (3.8) yields that a.s.

$$p\mu(x_0) - s < 0$$

for almost every x_0 and every rational $p > 0$. As the last inequality holds for every $s > g(p)$ it follows that

$$\text{a.s.} \quad g(p) \geq \mu(x_0)p$$

for every rational $p > 0$, for almost every x_0. But then it holds for every $p > 0$, for almost every x_0. Since this is true for an arbitrary random variable $\mu(x_0)$ which satisfies $\mu(x_0) < \lambda_-(x_0)$ a.s. we conclude that

$$\text{a.s.} \; g(p) \geq \lambda_-(x_0)p \tag{3.9}$$

for every $p > 0$, for almost every x_0. A similar argument for negative values of p implies that

(3.10) $$\text{a.s.}\quad g(p) \geq \lambda_+(x_0)p$$

for every $p < 0$, for almost every x_0. By the equivalence of (3.5) and (3.6) this implies (3.3). Then (3.4) follows in case that $p \to g(p)$ is differentiable at $p = 0$.

□

4. Random evolution homogeneous differential equations.

We consider the situation in Example 2.3. The initial value is of the form $x_0 = (y_0, i)$ where $y_0 \in R^n$ is nonzero and i is an integer, $1 \leq i \leq \nu$. Let $p \to g(p)$ be the convex function discussed in Theorem 2.9. Then for almost every $y_0 \in R^n$ the Laplace transform

$$\int_0^\infty e^{-st}\, E|x(t;x_0)|^p dt$$

of the function

(4.1) $$\varphi_i(t,y_0) = E[|x(t;x_0)|^p | x_0 = (y_0,i)]$$

is finite (infinite) for almost every y_0 if $s > g(p)$ $(s < g(p))$. We will derive a system of differential equations which is satisfied by $\varphi(t,y) = \{\varphi_i(t,y)\}_{i=1}^{\nu}$. From this system we will obtain a system of equations satisfied by the Laplace transforms of the functions $\{\varphi_i\}_{i=1}$, namely by

$$(4.2) \qquad \psi_i(s,y) = \int_0^\infty e^{-st}\varphi_i(t,y)dt.$$

This last system of equations will provide information about $g(p)$, since $g(p)$ is characterized by the property that $\psi(s,y)$ should be finite for $s > g(p)$ and infinite for $s < g(p)$.

Let the infinitesimal generator of the Markov chain $\{j_t\}$ be given by $G = (g_{ij})_{i,j=1}^{\nu}$, which is such that $g_{ij} > 0$ for $i \neq j$ and $\sum_{j=1}^{\nu} g_{ij} = 0$. It is a standard result that the expected values in (4.1) define functions $\{\varphi_i\}_{i=1}^{\nu}$ which satisfy the system of equations

$$(4.3) \qquad \begin{cases} \dfrac{\partial\varphi_i}{\partial t} = \dfrac{\partial\varphi_i}{\partial y} \cdot f_i(y) + \displaystyle\sum_{j=1}^{\nu} g_{ij}\varphi_j(t,y) \\ \\ \varphi_i(0,y) = |y|^p \quad , \quad i=1,\dots,\nu \end{cases} .$$

<u>Proposition 4.1</u>. The functions $y \to \varphi_i(t,y)$ are homogeneous of order p for every $1 \leq i \leq \nu$ and $t \geq 0$.

<u>Proof</u>: The assertion follows from the homogeneity of $t \to x(t)$ and the definition (4.1). It is enough to observe that

$$x(t,\alpha x_0) = \alpha x(t,x_0)$$

for every $\alpha > 0$ and $x_0 \in X$ (where $\alpha(y_0,i) = (\alpha y_0,i)$).

□

For sufficiently large s the function ψ_i in (4.2) is finite and well defined. We obtain a system of equations for $\{\psi_i\}_{i=1}^{\nu}$ by computing the Laplace transform of the system (4.3) and then obtain

$$(4.4)\qquad \begin{cases} s\psi_i(s,y) - |y|^p = \dfrac{\partial\psi_i}{\partial y} \cdot f_i(y) + \sum\limits_{j=1}^{\nu} g_{ij}\psi_j(s,y) \\ i = 1,\dots,\nu \end{cases} .$$

It follows from Proposition 4.1 and (4.2) that $y \to \psi_i(s,y)$ is a homogeneous function of order p. We denote a generic element in S^{n-1}, the unit sphere in R^n, by θ and it follows that

$$(4.5)\qquad \psi_i(s,y) = u_i(s,\theta)|y|^p, \text{ for } y = |y|\theta$$

for some functions $\theta \to u_i(s,\theta)$ on S^{n-1}, $1 \le i \le \nu$.

Let the point $\theta \in S^{n-1}$ be represented by an $n - 1$ coordinates system $\theta_1,\dots,\theta_{n-1}$. Thus every point $y \in R^n\setminus\{0\}$ is represented by the n-tupple $(r,\theta_1,\dots,\theta_{n-1})$. There are then n functions $\gamma_i : S^{n-1} \to R^1$ and first order linear differential operators $\{\tilde{\mathscr{L}}_i(\theta)\}_{j=1}^{n}$ in the variables $\theta_1,\dots,\theta_{n-1}$ such that

$$\frac{\partial\psi}{\partial y_j} = \gamma_j(\theta)\frac{\partial\psi}{\partial r} + \frac{1}{r}\tilde{\mathscr{L}}_j(\theta)\psi$$

for C^1 functions ψ on R^n. For ψ_i as in (4.5) we thus have

$$\frac{\partial \psi_i}{\partial y_j} = p\gamma_j(\theta)u_i(s,\theta)r^{p-1} + [\tilde{\mathcal{L}}_j(\theta)u(s,\theta)]r^{p-1}.$$

Using the last equality and denoting

$$f_i(y) = (f_{i1}(y),\dots, f_{in}(y))$$

where each $f_{ij} : R^n \to R^1$ is a homogeneous function, we obtain

$$\frac{\partial \psi_i}{\partial y} \cdot f_i(y) = [pG_i(\theta)u_i(s,\theta) + \mathcal{L}_i(\theta)u(s,\theta)]r^p \tag{4.6}$$

where

$$G_i(\theta) = \sum_{j=1}^{n} \gamma_j(\theta)f_{ij}(\theta) \tag{4.7}$$

$$\mathcal{L}_i(\theta) = \sum_{j=1}^{\nu} f_{ij}(\theta)\tilde{\mathcal{L}}_j(\theta). \tag{4.8}$$

Substituting (4.5)-(4.8) in (4.4) we obtain

$$\begin{cases} \mathcal{L}_i(\theta)u_i(s,\theta) + [pG_i(\theta) - s]u_i + \sum_{j=1}^{\nu} g_{ij}u_j + 1 = 0 \\ i = 1,\dots, \nu \end{cases} \tag{4.9}$$

This is a system of ν first order linear differential equations on the unit sphere S^{n-1}.

We will assume now that the solution $t \to x(t;x_0)$ of equation (2.3) is an irreducible process. We want to characterize the values $g(p)$ by properties of the corresponding system (4.9). The functions $\varphi = \{\varphi_i\}_{i=1}^{\nu}$ and $\psi = \{\psi_i\}_{i=1}^{\nu}$ in (4.1) and (4.2) can be used to construct solutions of (4.9) for every $s > g(p)$. By the expected value meaning of φ and ψ it follows that as $s \downarrow g(p)$ the solution ψ of (4.9) must blow up so that

$$\sup_{\theta \in S^{n-1}} |\psi_i(s,\theta)| \to \infty \quad \text{as} \quad s \downarrow g(p). \tag{4.10}$$

We summarize this in the following.

Theorem 4.2. Assume that the solution $t \to x(t)$ of (2.3) is an irreducible process. Then for every p the number $g(p)$ is such that there exists a solution of (4.9) for every $s > g(p)$ and these solutions satisfy (4.10).

Example 4.3. We consider the situation described in Example 2.2 with $n = 2$. Thus the matrix A_i is explicitly written as

$$A_i = \begin{bmatrix} A_i^{11} & A_i^{12} \\ A_i^{21} & A_i^{22} \end{bmatrix}.$$

Introducting the polar coordinates (r,θ)

$$y_1 = r\cos\theta \quad , \quad y_2 = r\sin\theta$$

we have for the gradient of a C^1 function ψ

$$\left(\frac{\partial\psi}{\partial y}\right)^T = \left\{\cos\theta\,\frac{\partial\psi}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial\psi}{\partial\theta}\ ,\ \sin\theta\,\frac{\partial\psi}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial\psi}{\partial\theta}\right\}$$

and

$$\left(\frac{\partial\psi}{\partial y}\right)^T A_i y = \frac{\partial\psi}{\partial\theta}\left[A_i^{21}\cos^2\theta - A_i^{12}\sin^2\theta + (A_i^{22} - A_i^{11})\sin\theta\cos\theta\right] +$$

$$+ r\,\frac{\partial\psi}{\partial r}\left[A_i^{11}\cos^2\theta + A_i^{22}\sin^2\theta + (A_i^{12} + A_i^{21})\sin\theta\cos\theta\right].$$

Thus in this case the system of equation (4.9) becomes

$$\text{(4.11)}\qquad \begin{cases} F_i(\theta)u_i'(\theta) + [pG_i(\theta) - s]u_i(\theta) + \sum_{j=1}^{\nu} g_{ij}u_j(\theta) + 1 = 0 \\ i = 1, \ldots, \nu \end{cases}$$

where

$$\text{(4.12)}\qquad \begin{cases} F_i(\theta) = A_i^{21}\cos^2\theta - A_i^{12}\sin^2\theta + (A_i^{22} - A_i^{11})\sin\theta\cos\theta \\ G_i(\theta) = A_i^{11}\cos^2\theta + A_i^{22}\sin^2\theta + (A_i^{12} + A_i^{21})\sin\theta\cos\theta\ . \end{cases}$$

Here we are looking for periodic solutions of (4.11) for various values of p and s.

5. Two dimensional random evolution differential equations.

We consider now two-dimensional homogeneous systems as in Example 2.3. We write it explicitly as

$$\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = r\begin{bmatrix} \alpha_{j_t}(\theta) \\ \beta_{j_t}(\theta) \end{bmatrix} \tag{5.1}$$

where $\alpha_i(\cdot)$ and $\beta_i(\cdot)$ are Lipschitz continuous scalar functions, periodic of period 2π. To obtain the equations for $t \to r(t)$ and $t \to \theta(t)$ we observe that (5.1) may be written in the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \dot{r}/r \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \alpha_{j_t}(\theta) \\ \beta_{j_t}(\theta) \end{bmatrix}$$

which yields

$$\begin{cases} \dot{r}/r = G_{j_t}(\theta(t)) \\ \dot{\theta} = F_{j_t}(\theta(t)) \end{cases} \tag{5.2}$$

with the notation

$$\begin{cases} F_i(\theta) = \beta_i(\theta)\cos\theta - \alpha_i(\theta)\sin\theta \\ G_i(\theta) = \alpha_i(\theta)\cos\theta + \beta_i(\theta)\sin\theta \end{cases} \tag{5.3}$$

(this notation is consistent with (4.12) as we will see below). In this two dimensional framework it is easy to give conditions which imply the validity of Assumption B. Clearly such is the simple condition stated in the following.

Proposition 5.1. Assume that there is an integer $1 \leq i \leq \nu$ such that $F_i(\theta) \neq 0$ for every $0 \leq \theta \leq 2\pi$. Then Assumption B holds.

We consider now the corresponding system (4.9). Clearly it is of the form (4.11) and we just have to determine how the functions $G_i(\cdot)$ and $F_i(\cdot)$ depend on the functions $\alpha_i(\cdot)$ and $\beta_i(\cdot)$.

Proposition 5.2. The corresponding system (4.9) is given by (4.11) with $F_i(\cdot)$ and $G_i(\cdot)$ defined by (5.3).

Proof: We compute the product $\frac{\partial\psi}{\partial y} \cdot f_j(y) = \frac{\partial\psi}{\partial y} \cdot \begin{bmatrix} r\alpha(\theta) \\ r\beta(\theta) \end{bmatrix}$ which is equal to $(\cos\theta\,\psi_r - \frac{\sin\theta}{r}\psi_\theta,\ \sin\theta\,\psi_r + \frac{\cos\theta}{r}\psi_\theta)\begin{bmatrix} r\alpha(\theta) \\ r\gamma(\theta) \end{bmatrix}$, namely to $r[\alpha(\theta)\cos\theta + \beta(\theta)\sin\theta]\psi_r + [\beta(\theta)\cos\theta - \alpha(\theta)\sin\theta]\psi_\theta$. Thus substituting $\psi(y) = u(\theta)r^p$ gives the expression

$$[pG(\theta)u(\theta) + F(\theta)u'(\theta)]r^p$$

with $G(\theta) = \alpha\cos\theta + \beta\sin\theta$, $F(\theta) = -\alpha\sin\theta + \beta\cos\theta$. This concludes the proof.

□

Since $F(\cdot)$, $G(\cdot)$, $\alpha(\cdot)$ and $\beta(\cdot)$ are related by

$$\begin{bmatrix} G(\theta) \\ F(\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha(\theta) \\ \beta(\theta) \end{bmatrix}$$

which may be inverted to

$$\begin{bmatrix} \alpha(\theta) \\ \beta(\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} G(\theta) \\ F(\theta) \end{bmatrix}$$

there is a 1-1 correspondence between Lipschitz continuous and periodic pairs $(\alpha(\cdot),\beta(\cdot))$ on one hand and such pairs $(G(\cdot),F(\cdot))$ on the other hand. Thus every system (4.12) corresponds to some homogeneous random evolution system.

For the solution $\theta(\cdot)$ of (5.2) we consider the Markov process $t \to (\theta(t),j_t)$ on $\bigcup_{i=1}^{\nu} S_i$, where each S_i is a copy of the unit circle. If Assumption B holds then it has an equilibrium measure whose support is the whole of S, denote it by $\sigma(d\theta) = \{\sigma_i(d\theta)\}_{i=1}^{\nu}$. We have from (5.2) that

$$\text{(5.4)} \qquad \frac{1}{t} \log r(t) = \frac{1}{t} \int_0^t G_{j_s}(\theta(s))ds.$$

The process $t \to (\theta(t),j_t)$ with the initial distribution $\sigma(d\theta)$ is a stationary Markov process on S. It follows from (5.4) by applying the Ergodic Theorem (see Rosenblatt [20], page 105) that the limit $\lim_{t\to\infty} \frac{1}{t} \log r(t)$ exists a.s. for this process. If P_{x_0} is the probability P conditioned on $\{\theta(0) = x_0\}$, and $E(x_0)$ is the event

$$\{\omega : \lim_{t\to\infty} \frac{1}{t} \log r(t) \text{ exists}, \theta(0) = x_0\}$$

then we have the relation

$$(5.5) \qquad P(E) = \int_S P_{x_0}(E(x_0))\, \sigma(dx_0).$$

In (5.5) E is the event $\{\omega : \lim_{t\to\infty} \frac{1}{t} \log r(t) \text{ exists}\}$ for the above stationary process $t \to (\theta(t), j_t)$. Since $P(E) = 1$ it follows from (5.5) that

$$(5.6) \qquad \lim_{t\to\infty} \frac{1}{t} \log r(t) \text{ exists a.s. for } \sigma\text{-almost every } x_0 \text{ in } S.$$

We want to deduce that the limit in (5.6) holds for Lebesgue almost every x_0 in S. This will follow easily for a special class of homogeneous random evolution systems which we introduce now.

Definition 5.3. The random evolution system which is defined in terms of $(\alpha_i(\cdot), \beta_i(\cdot))$, $i = 1,\dots,\nu$, is called nonsingular if the corresponding pairs $(G_i(\cdot), F_i(\cdot))$, $1 \le i \le \nu$, are such that $F_i(\cdot)$ never vanishes for every i. The random evolution system is said to be singular if at least one $F_i(\cdot)$ vanishes at some point.

In the rest of the paper we will consider nonsingular systems. In view of Proposition 5.1 the Assumption B holds in this case.

The assumption in (5.7) below is easily verified and holds in many cases.

Lemma 5.4. Let the system (5.1) be nonsingular and assume that there are integers $i \neq j$, $1 \leq i, j \leq \nu$, such that

$$\int_0^{2\pi} \frac{d\theta}{F_i(\theta)} \neq \int_0^{2\pi} \frac{d\theta}{F_j(\theta)}. \tag{5.7}$$

Then the Lebesgue measure of S is absolutely continuous with respect to the measure $\sigma(d\theta)$.

Remark 5.5. The quantity $\int_0^{2\pi} \frac{d\theta}{F_i(\theta)}$ is the time which takes for a solution of $\dot{\theta}(t) = F_i(\theta(t))$ to change by 2π. It thus follows from (5.7) that if $\theta_i(\cdot)$ and $\theta_j(\cdot)$ are solutions of $\dot{\theta} = F_i(\theta)$ and $\dot{\theta} = F_j(\theta)$ respectively then $|\theta_i(T) - \theta_j(T)| \to \infty$ as $T \to \infty$.

Proof: Let $A \subset S$ be of positive Lebesgue measure. For every $t > 0$ we have

$$\sigma(A) = \int_S P(t,x_0,A)\sigma(dx_0) \tag{5.8}$$

where $(t,x_0,A) \to P(t,x_0,A)$ is the transition probability function for the process $t \to (\theta(t),j_t)$.

The assertion of the Lemma will follow from (5.8) once we have proved that

(5.9) $$P(t,x_0,A) > 0 \quad \text{for every} \quad x_0 \in S$$

for every set $A \subset S$ of positive Lebesgue measure and for t large enough.

We consider continuous functions $\theta : [0,T] \to R^1$ such that $\theta(\cdot)$ is a solution of

(5.10) $$\frac{d\theta(t)}{dt} = \begin{cases} F_i(\theta(t)) & \text{if} \quad 0 \leq t < t_0 \\ F_j(\theta(t)) & \text{if} \quad t_0 < t \leq T \end{cases} \quad \text{for some} \quad 0 \leq t_0 \leq T$$

where i,j are as in (5.7). For a fixed initial value θ_0 we consider functions $\theta_i(\cdot)$ and $\theta_j(\cdot)$ as in Remark 5.5 and it follows from this remark that $|\theta_i(T) - \theta_j(T)| > 2\pi$ if T is large enough, and this uniformly for all possible θ_0. If we consider (5.10) as a differential equation on the unit circle, fix θ_0 and let t_0 vary between 0 and T, then we conclude that for every $o \leq \theta_1 < 2\pi$ there is a $0 \leq t_0 \leq T$ such that the corresponding solution of (5.10) satisfies $\theta(T) = \theta_1$. The time T can be chosen uniformly for all θ_0.

We consider first a set $A \subset S$ which is of the form $A = \{j\} \times K$, where K is a subset of the unit circle which has positive Lebesgue measure. We fix the initial condition θ_0 and consider the set of times

(5.11) $$U = \left\{ 0 < t_0 < T : \theta(T) \in K, \text{ where } \theta(\cdot) \text{ is the corresponding solution of } (5.10) \right\}.$$

The mapping $t_0 \to \theta(T)$ which is defined by (5.10) (for a fixed θ_0) is continuously differentiable, thus it follows from the fact that K has positive Lebesgue measure that U in (5.11) has positive Lebesgue measure. Thus if τ is the first jump time of $\{j_t\}_{t\geq 0}$, and if $x_0 = \{i,\theta_0\}$, then $P_{x_0}(\tau \in U) > 0$, which implies that (5.9) holds for x_0 of the form (i,θ_0) and A of the form $\{j\} \times K$. The general claim (5.9) is implied by this as follows. Let T_0 be such that

(5.12) $P(t_0,x_0,A_0) > 0$ for every $t_0 \geq T_0$, $x_0 = (i,\theta_0)$ and $A_0 = j \times K_0$

with $K_0 \subset S^1$ of positive Lebesgue measure. Now let $A = \ell \times K$ with $K \subset S^1$ of positive Lebesgue measure and $x_0 = (k,\theta_0)$. Let τ be the first jump time of $\{j_t\}_{t\geq 0}$ after $t = 0$ and σ be the first jump time after $t = T_0 + 2$. For an $t_0 + 3 \leq s \leq T_0 + 4$ we define a set $A(s) \subset j \times \{S^1\}$ as follows. Let $t \to \theta_s(t,\varphi)$ be the solution of

$$\begin{cases} \dfrac{d\theta}{dt} = \begin{cases} F_j(\theta) & \text{if } T_0 + 2 \leq t < s \\ F_\ell(\theta) & \text{if } s < t \leq T \end{cases} \\ \\ \theta(T_0 + 2) = \varphi \end{cases}.$$

Then

$$A(s) = \{* : (£, 9_s(T»)) € A\}$$

namely, all the initial values such that if the jump of $\{J_t\}_{t\geq 0}$ yr +2 from j to $£$ occurs at $t = s$, then $(£,0(T) € A$.

It now follows that for $T > T_Q + 4$

$$P(T,x_o,A) \geq \int_{T_Q+3}^{V4} \int_{1}^{2} P(r € dt)P(a € ds)P(T_o + 2 - t,x(t),A(s))$$

which is positive since, by (5.12), $P(T_Q + 2 - t,x(t),A(s)) > 0$ for every $1 \leq t \leq 2$ and $T_Q + 3 \leq s \leq T_Q + 4$. This proves (5.9) and, as explained at the beginning, concludes the proof of the Lemma.

D

A consequence of (5.6) and Lemma 5.4 is the following result.

<u>Theorem 5.6</u>. Let the system (2.3) be nonsingular and (5.7) hold for some $l \leq i, j < i >$. $i \neq j$. Then the limit in (3.1) exists a.s. for Lebesgue almost every initial value $x_o € S$ and has the nonrandom value $g'(0)$ which is the Lyapunov exponent for the system (2.3).

We now consider, for every p, the system of equations

(5.13)
$$\begin{cases} F_i(\theta)u_i'(\theta) + [PG.(G) - s]u.(0) + \sum_{J \neq X}^{V} g. .u.(6) + 1 = 0 \\ u_i(0) = u_i(2ir) . \end{cases}$$

We look for a periodic solution under the assumption that $F_i(\theta) \neq 0$ for every $1 \leq i \leq \nu$ and $0 \leq \theta \leq 2\pi$. The value $g(p)$ is such that for every $s > g(p)$ there is a periodic solution for (5.13), and we can find solutions $u^s = \{u_i^s\}_{i=1}^{\nu}$ corresponding to $s > g(p)$ such that

$$m_s = \max_{i,\theta}\{|u_i^s(\theta)|\}$$

satisfies $m_s \to \infty$ as $s \downarrow g(p)$. Thus if we divide (5.13) by m_s we get

$$(5.14)\qquad \begin{cases} F_i(\theta)\, v_i'(\theta) + [PG_i(\theta) - s]v_i(\theta) + \sum_{j=1}^{\nu} g_{ij}v_j(\theta) + \rho_s = 0 \\ v_i(0) = v_i(2\pi), \quad i = 1,\dots,\nu \end{cases}$$

where we denoted $v_i(\theta) = u_i(\theta)/m_s$ so that

$$(5.15)\qquad \max_{i,\theta}\{|v_i(\theta)|\} = 1 \quad \text{for every} \quad s > g(p)$$

and $\rho_s \to 0$ as $s \downarrow g(p)$.

<u>Proposition 5.7</u>. For every p there is a nontrivial periodic solution of

$$(5.16)\qquad \begin{cases} F_i(\theta)v_i'(\theta) + [pG_i(\theta) - g(p)]v_i(\theta) + \sum_{j=1}^{\nu} g_{ij}v_j(\theta) = 0 \\ v_i(0) = v_i(2\pi) \quad , \quad i = 1,\dots, 0 \quad . \end{cases}$$

<u>Proof</u>: We know that there exists a periodic solution $\{v_i\}_{i=1}^{\nu}$ of (5.14) for every $s > g(p)$ and such that (5.15) holds. It then follows from the nonsingularity of the system that there is a constant K such that

$$\left|\frac{d}{d\theta} v_i^s(\theta)\right| \le K$$

for every $s > g(p)$, $1 \le i \le \nu$ and $0 \le \theta \le 2\pi$. But then we can find a subsequence of periodic solutions denoted $\{v^j\}_{j=1}^{\infty}$ corresponding to values $\{s_j\}_{j=1}^{\infty}$ such that $s_j > g(p)$ and $s_j \to g(p)$ as $j \to \infty$, and such that the solutions $\{v^j\}_{j=1}^{\infty}$ converge uniformly on $[0,2\pi]$ to a continuous function v. We have the relation

$$(5.17)\quad (v^j)_i(\theta) = (v^j)_i(0) - \int_0^\theta \frac{1}{F_i}\Big\{[pG_i - s_j]v_i^j + \sum_k S_{ik} v_k^j\Big\}\, d\varphi$$

for $1 \le i \le \nu$, and letting $j \to \infty$ in (5.17) we conclude that $v(\cdot)$ is a solution of (5.16), which completes the proof.

•

We consider the ordinary differential equation

$$(5.18)\quad F_i(\theta)v_i'(\theta) + [pG_i(\theta) - s]v_i(\theta) + \sum_{k=1}^{\nu} S_{ik}v_k(\theta) = 0,\quad 1 \le i \le \nu$$

with periodic coefficients, and denote its fundamental solution by $S_{p,s}(\theta,\theta_0)$. By Proposition 5.7 there is a nontrivial periodic solution of (5.18) for $s = g(p)$, denote it by $v(\cdot)$. Then we have

$$\left[S_{p,g(p)}(2\pi,0) - I\right]v(0) = 0$$

which implies that unity is an eigenvalue of $S_{p,g(p)}(2\pi,0)$ for every p (since $v(0) \neq 0$).

We thus wish to study the eigenvalues of the fundamental solution of (5.18) and address the question: Considering a small neighborhood of the origin in the (p,s) plane, what should be the relation between p and s so that the corresponding matrix $S_{p,s}(2\pi,0)$ has unity as an eigenvalue, up to the order of magnitude $O(p^2 + s^2)$.

The system (5.18) can be written in the form

$$\frac{du}{d\theta} = [A(\theta) + pB(\theta) + sC(\theta)]u(\theta) \tag{5.19}$$

where $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are periodic $\nu \times \nu$ matrix valued functions, p and s are real parameters considered in a small neighborhood of $(0,0)$ in R^2. We have that $B(\cdot)$ and $C(\cdot)$ are diagonal matrices, and explicitly

$$A_{ij}(\theta) = -\frac{g_{ij}}{F_i(\theta)},\ B_{ii}(\theta) = -\frac{G_i(\theta)}{F_i(\theta)},\ C_{ii}(\theta) = -\frac{1}{F_i(\theta)}. \tag{5.20}$$

Let $S_0(\theta,\theta_0)$ denote the fundamental solution of

$$\frac{du}{d\theta} = A(\theta)u(\theta) \tag{5.21}$$

(namely the system (5.19) with $p = s = 0$). The function

$(p,s,\theta,\theta_0) \to S_{p,s}(\theta,\theta_0)$ is a real analytic function of its (p,s) variables so we can write

$$(5.22) \qquad S_{p,s}(\theta,\theta_0) = S_0(\theta,\theta_0) + pM(\theta,\theta_0) + sL(\theta,\theta_0) + o(p,s)$$

where we denote by $o(p,s)$ a function $\varphi(\cdot,\cdot)$ such that $\varphi(p,s)/[|p| + |s|] \to 0$, $|\frac{\partial\varphi}{\partial\theta}|/[|p| + |s|] \to 0$ and $|\frac{\partial\varphi}{\partial\theta_0}|/[|p| + |s|] \to 0$ as $|p| + |s| \to 0$. Substituting (5.22) in (5.19) we obtain

$$\frac{d}{d\theta}[S_0 + pM + sL] = [A + pB + sC][S_0 + pM + sL] + o(p,s)$$

which implies, in view of $\frac{dS_0}{d\theta} = AS_0$, the relations

$$\frac{dM(\theta,\theta_0)}{d\theta} = A(\theta)M(\theta,\theta_0) + B(\theta)S_0(\theta,\theta_0) \quad , \quad M(\theta_0,\theta_0) = 0$$

$$\frac{dL(\theta,\theta_0)}{d\theta} = A(\theta)L(\theta,\theta_0) + C(\theta)S_0(\theta,\theta_0) \quad , \quad L(\theta_0,\theta_0) = 0$$

for every fixed θ_0. We consider now $\theta_0 = 0$ and simplify the notations so that $S_0(\theta)$ denotes $S_0(\theta,\theta)$, $M(\theta)$ and $L(\theta)$ denote $M(\theta,0)$ and $L(\theta,0)$ respectively. We thus have that

$$M(\theta) = \int_0^\theta S_0(\theta,\varphi)B(\varphi)S_0(\varphi)\varphi$$

$$L(\theta) = \int_0^\theta S_0(\theta,\varphi)C(\varphi)S_0(\varphi)d\varphi$$

for $0 \leq \theta \leq 2\pi$. We are interested in the eigenvalues of

$$S_{p,s}(2\pi,0) = S_0(2\pi) + pM(2\pi) + sL(2\pi) + o(p,s)$$

where by the above equations for $M(\theta)$ and $L(\theta)$

$$(5.23) \qquad \begin{cases} M(2\pi) = S_0(2\pi) \displaystyle\int_0^{2\pi} S_0^{-1}(\varphi)B(\varphi)S_0(\varphi)d\varphi \\ L(2\pi) = S_0(2\pi) \displaystyle\int_0^{2\pi} S_0^{-1}(\varphi)C(\varphi)S_0(\varphi)d\varphi. \end{cases}$$

We know by Proposition 5.7 that for $s = g(p)$ the matrix $S_{p,s}(2\pi)$ has unity as an eigenvalue. Moreover, by Theorem 5.6 the function $p \to g(p)$ is differentiable at $p = 0$ and there is a number k such that

$$g(p) = kp + o(p)$$

(where $o(p)$ denotes a function $p \to \rho(p)$ such that $\rho(p)/p \to 0$ as $p \to 0$). We will expand the characteristic polynomial of $S_{p,s}(2\pi)$, taking $s = kp + o(p)$ for some constant k, and we will derive a condition on k that this polynomial has unity as a root, up to order $o(p)$. This will yield an expression for the Lyapunov exponent, as asserted in Theorem 3.2.

We thus consider the characteristic polynomial

$$\Phi_{p,s}(x) = \det[A + pB + sC - xI]$$

and express it in terms of the characteristic polynomial

$$\Phi_0(x) = \det[A - xI]$$

and an expansion in powers of p and s. It follows that

$$\Phi_{p,s}(x) = \Phi_0(x) + pa_{\nu-1}(x) + sb_{\nu-1}(x) + o(p,s) \tag{5.24}$$

where $a_{\nu-1}(x)$ and $b_{\nu-1}(x)$ are polynomial in x of degree $\nu - 1$. We know that $\Phi_0(1) = 0$ and, moreover, there is an $o(p)$ function such that

$$\Phi_{p,kp+o(p)}(1) = 0 \quad \text{for every } p.$$

We substitute in (5.24) $x = 1$ and $s = kp + o(p)$ and obtain

$$pa_{\nu-1}(1) + kpb_{\nu-1}(1) + o(p) = 0. \tag{5.25}$$

Dividing the equation (5.25) by p and letting $p \to 0$ we get that

$$k = -\frac{a_{\nu-1}(1)}{b_{\nu-1}(1)}$$

provided that $b_{\nu-1}(1) \neq 0$. In view of Theorem 5.6 we have thus proved the following Theorem.

Theorem 5.8. Consider the two-dimensional homogeneous nonsingular random evolution system (2.3). Let $\theta \to S_0(\theta)$ be the fundamental solution of (5.21) and let M and L be the matrices in (5.23), where the matrices A, B and C are as in (5.20). The polynomials of $\nu - 1$ degree $x \to a(x)$ and $x \to b(x)$ are defined by the relation

$$\det[S_0(2\pi) + pM + sL - xI] = \det[S_0(2\pi) - xI] + pa(x) + sb(x) + \rho(p,s)$$

where $\rho(p,s)/[|p| + |s|] \to 0$ as $|p| + |s| \to 0$. If $b(1) \neq 0$ then the Lyapunov exponent is given by

$$\lambda(x_0) = -\frac{a(1)}{b(1)} \quad \text{a.s.}$$

for almost every x_0 (with respect to the Lebesgue measure).

6. The random evolution harmonic oscillator.

We consider a random harmonic oscillator which is exposed to random perturbations so that it is modeled by the following equation

$$\ddot{\zeta}(t) + (1 + \epsilon_{j_t})\zeta(t) = 0. \tag{6.1}$$

The real numbers ϵ_i, $1 \leq i \leq \nu$, in (6.1) are such that $|\epsilon_i| < 1$ for every $1 \leq i \leq \nu$ and $\{j_t\}_{t \geq 0}$ is a Markov chain on the state space $\{1, \ldots, \nu\}$. Then the two dimensional process

$$y(t) = \begin{bmatrix} \zeta(t) \\ \dot{\zeta}(t) \end{bmatrix}$$

satisfies the random evolution linear differential equation

(6.2) $$\dot{y}(t) = A_{j_t}\, y(t)$$

where $A_i = \begin{bmatrix} 0 & 1 \\ -(1+e_i) & 0 \end{bmatrix}$ for $1 \le i \le \nu$. As discussed in section 5, there is associated with (6.2) an ordinary differential equation (5.4). In the present situation we have

$$\begin{cases} F_i(\theta) = -1 - e_i \cos^2\theta \\ G_i(\theta) = -e_i \sin\theta \cos\theta \end{cases}$$

for $1 \le i \le \nu$, and the condition $|e_i| < 1$ implies that (6.2) is nonsingular. We consider the fundamental solution of the periodic coefficient linear differential equation

(6.3) $$\frac{du}{d\theta} = A(\theta)u(\theta)$$

where $A(\theta)$ is the $\nu \times \nu$ matrix defined by

$$A_{ij}(\theta) = \frac{g_{ij}}{1 + e_j \cos^2\theta}.$$

In this situation it is well known that the Lyapunov exponent exists as a limit in (3.1), thus we don't have to employ Theorem 5.6 to deduce that. (Theorem 5.6 is applicable if (5.7) holds for some $1 \le i, j \le \nu$ which is the case for "most" choices of values ϵ_i, $1 \le i \le \nu$. Nevertheless, if $\nu = 2$ and $\epsilon_1 + \epsilon_2 = 0$ then (5.7) fails.)

Once the fundamental solution of (6.3) is computed then we find the matrices M and L by using (5.23), and the polynomials $a(\cdot)$ and $b(\cdot)$ are obtained as described in Theorem 5.6. The Lyapunov exponent is given by $\lambda = -\frac{a(1)}{b(1)}$, using Theorem 5.6.

<u>Example 6.1</u>. We assume that $\nu = 2$ and for some $0 < \epsilon < 1$ we have $\epsilon_1 = \epsilon$ and $\epsilon_2 = -\epsilon$. The two dimensional ordinary differential equation (6.3) is

$$\text{(6.4)} \qquad \frac{d}{d\theta}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\dfrac{\alpha}{1 + \epsilon \cos^2\theta} & \dfrac{\alpha}{1 + \epsilon \cos^2\theta} \\ \dfrac{\beta}{1 - \epsilon \cos^2\theta} & -\dfrac{\beta}{1 - \epsilon \cos^2\theta} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

where $G = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$ is the infinitesimal generator of the Markov chain $\{j_t\}_{t\ge 0}$, $\alpha, \beta > 0$. The special form of (6.4) enables a solution by quadratures. In fact this is the case whenever $\nu = 2$. We multiply (6.4) from the left by $(1,-1)$ and define $w(t) = u(t) - v(t)$. Then

$$\text{(6.5)} \qquad w(\theta) = \exp\left\{-\int_0^\theta \left[\frac{\alpha}{1 + \epsilon \cos^2\varphi} + \frac{\beta}{1 - \epsilon \cos^2\theta}\right] d\varphi\right\} \; w(0)$$

and we have for $u(\cdot)$ and $v(\cdot)$

$$\text{(6.6)} \qquad \frac{du}{d\theta} = -\frac{\alpha}{1+\epsilon\cos^2\theta}\,w(\theta) \quad , \quad \frac{dv}{d\theta} = -\frac{\beta}{1-\epsilon\cos^2\theta}\,w(\theta).$$

When (6.6) is integrated with the initial values $u(0) = 1$, $v(0) = 0$ (which yield $w(0) = 1$ in (6.5)) we get the first column of the fundamental matrix $\theta \to S_0(\theta)$, while choosing $u(0) = 0$ and $v(0) = 1$ gives the second column. We thus obtain

$$\text{(6.7)} \qquad S_0(\theta) = \begin{bmatrix} 1 - \alpha \int_0^\theta \frac{w(\varphi)d\varphi}{1+\epsilon\cos^2\varphi} & \int_0^\theta \frac{w(\varphi)d\varphi}{1+\epsilon\cos^2\varphi} \\ \beta \int_0^\theta \frac{w(\varphi)d\varphi}{1-\epsilon\cos^2\varphi} & 1 - \beta \int \frac{w(\varphi)d\varphi}{1-\epsilon\cos^2\varphi} \end{bmatrix}$$

where here we denoted

$$w(\theta) = \exp\left\{-\int_0^\theta \left[\frac{\alpha}{1+\epsilon\cos^2\varphi} + \frac{\beta}{1-\epsilon\cos^2\varphi}\right]d\varphi\right\}$$

(the one which corresponds to $w(0) = 1$ in (6.5)). As described above, the expression (6.7) is used to obtain the matrices M and L in quadratures. We thus denote

$$S_0 = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{bmatrix}, \; M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}, \; L = \begin{bmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{bmatrix}$$

and consider the characteristic polynomial

$$\det[S_0 + pM + sL - xI].$$

It is equal to

$$\det[S_0 - xI] + (\sigma_1 - x)(pm_4 + s\ell_4) + (\sigma_4 - x)(pm_1 + s\ell_1) -$$

$$- \sigma_2(pm_3 + s\ell_3) - \sigma_3(pm_2 + s\ell_2) + o(p,s)$$

from which we deduce that

$$a(x) = -(m_1 + m_4)x + (\sigma_1 m_4 + \sigma_4 m_1 - \sigma_2 m_3 - \sigma_3 m_2)$$

$$b(x) = -(\ell_1 + \ell_4)x + (\sigma_1 \ell_4 + \sigma_4 \ell_1 - \sigma_2 \ell_3 - \sigma_3 \ell_2).$$

Then the Lyapunov exponent is

$$\lambda = - \frac{m_1 + m_4 - \sigma_1 m_4 - \sigma_4 m_1 + \sigma_2 m_3 + \sigma_3 m_2}{\ell_1 + \ell_4 - \sigma_1 \ell_4 - \sigma_4 \ell_1 + \sigma_2 \ell_3 + \sigma_3 \ell_2}$$

provided that the denumerator does not vanish.

□

We consider now the situation where $\epsilon_i = \epsilon c_i$ where c_i,

$i = 1, \ldots, \nu$, are fixed numbers and we want to study the asymptotic behavior of the Lyapunov expnent as $\epsilon \to 0$. Then the matrix $A(\theta)$ in (6.3) can be written as

$$A(\theta) = G + \epsilon C \cos^2\theta + o(\epsilon)$$

where G is the infinitesimal generator matrix and C is given by $C_{ij} = -g_{ij}c_i$. If $\theta \to S(\theta)$ is the fundamental solution of (6.3) with $S(0) = I$ then it is of the form

$$S(\theta) = S_0(\theta) + \epsilon X(\theta) + o(\epsilon)$$

where $S_0(\theta) = \exp G\theta$ and $X(\cdot)$ satisfies

$$\frac{dX(\theta)}{d\theta} = GX(\theta) + \cos^2\theta \, Ce^{G\theta} \quad , \quad X(0) = 0.$$

Thus $\theta \to X(\theta)$ is given by

$$X(\theta) = \int_0^\theta e^{G(\theta-\varphi)} \, Ce^{G\varphi} \cos^2\varphi d\varphi$$

which gives the first order approximation for $\theta \to S(\theta)$ in quadratures. It is just as easy to obtain higher order approximations of $S(\theta)$, for every fixed prescribed order, by using an iterative method and developing the matrix $A(\theta)$ in (6.3) in powers of ϵ up to that order.

References

[1] Arnold, L. (1984). A formula connecting sample and moment stability of linear stochastic systems. *SIAM Journal on Applied Mathematics*, **44**, pp. 793-802.

[2] Arnold, L. and Kleimann, W. (1983). Qualitative theory of stochastic systems. In *Probabilistic Analysis and related topics*, vol. 3, Ed. A.T. Barucha-Reid, pp. 1-79, Academic Press, New York.

[3] Arnold, L., Kleimann, W. and Oeljeklaus, E. (1985). Lyapunov exponents of linear stochastic systems. In *Lecture notes in Mathematics*, *1186*, *Lyapunov Exponents*, Ed. L. Arnold and V. Wihstutz, pp. 85-125. Springer-Verlag, New York.

[4] Arnold, L., Oeljeklaus, E. and Pardoux, E. (1985). Almost sure and moment stability for linear Ito equations. In *Lecture notes in Mathematics*, *1186*, *Lyapunov exponents*, Ed. L. Arnold and V. Wikstutz, pp. 129-159. Springer-Verlag, New York.

[5] Arnold, L, Papanicolaou, G. and Wihstutz, V. (1986). Asymptotic analysis of the Lyapunov exponent and rotation number of the random oscillator and applications. *SIAM Journal on Applied Mathematics*, **46**, pp, 427-450.

[6] Blankenship, G.L. and Li, C.W. (1986). Almost sure stability of linear stochastic systems with Poisson coefficients. *SIAM Journal on Applied Mathematics*, **46**, pp. 875-911.

[7] Blankenship, G.L. and Loparo, K.A. (1985). Almost sure instability of a class of linear stochastic systems with jump processes coefficients. In *Lecture Notes in Mathematics*, *1186*, *Lyapunov Exponents*, Ed. L. Arnold and V. Wihstutz, pp. 160-190.

[8] Crauel, H. (1984). Lyapunov number of Markov solutions of linear stochastic systems. *Stochastics* **14**, pp. 1-18.

[9] Furstenberg, H. and Kesten, H. (1960). Products of random matrices. *The Annals of Mathematical Statistics*, **31**, pp. 457-469.

[10] Has'minskii, R.Z. (1967). Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems. *Theory of Probability and Applications*, **12**, pp. 144-147.

[11] Has'minskii, R.Z. (1980). Stochastic Stability of Differential Equations. Sijthoff and Noordhoof, Alphan aan den Rijn, the Netherland.

[12] Kleimann, W. (1979). Some exact results on stability and growth of linear parameter excited stochastic systems. In *Lecture Notes on Control and Information Sciences*, **16**, Stochastic Control Theory and Stochastic Differential Equations. Ed. M. Kohlman and W. Vogel.

[13] Leizarowitz, A. (1988). Estimates and exact expressions for Lyapunov exponents of stochastic differential equations. To appear in *Stochastics*.

[14] Oseledec, V.I. (1968). A multiplicative ergodic theorm. Lyapunov characteristic numbers for dynamical systems. *Transactions of the Moscow Mathematical Society*, **19**, pp. 197-221.

[15] Pardoux, E. and Wihstutz, V. (1988). Lyapunov exponents and rotation number of two-dimensional linear stochastic systems with small diffusion. *SIAM Journal on Applied Mathematics*, **44**, pp. 442-457.

[16] Pinsky, M. (1974). Stochastic stability and Dirichlet problems. *Communications in Pure and Applied Mathematics*, **27**, pp. 311-350.

[17] Pinsky, M. (1986). Instability of the harmonic ascillator with small noise. *SIAM Journal on Applied Mathematics*, **46**, pp. 451-463.

[18] Raghunathan, M.S. (1979). A proof of Oseledec's multiplicative ergodic theorem. *Israel Journal of Mathematics*, **32**, pp. 356-362.

[19] Rosenblatt, R. (1973). Random Processes, second edition. Springer-Verlag, New York.

[20] Wihstutz, V. (1985). Parameter dependence of the Lyapunov exponent for linear stochastic systems. A survey. In *Lecture Notes in Mathematics, 1186, Lyapunov Exponents*. Ed. L. Arnold and V. Wihstutz, pp. 200-215. Springer-Verlag, New York.