TESTS FOR INJECTIVITY IN FINITELY GENERATED UNIVERSAL HORN CLASSES

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In this paper we will discuss criteria for determining when the universal Horn class generated by finitely many finite algebras has enough injectives. Section 1 exploits the syntactic results of [1] and illustrates how these may be used, both to prove known results, and also to produce some new ones. Section 2 is more algebraic. Superficially it deals with developing effective tests for when a universal Horn class generated by finitely many finite algebras has enough injectives. However, in the process of discovering such tests, we prove results extendihg those of [6] to universal Horn classes.

For work in this area [9] is an invaluable reference, and we enthusiastically refer readers to its extensive bibliography and cogent summaries of previous work on injectivity, the congruence extension property, and residual smallness. Whole hearted thanks are also due to Ross Willard whose comments and questions inspired much of the work in Section 2.

1. Syntactic Criteria for Injectivity.

In this section we consider criteria for when a finite algebra *A* is injective in the universal Horn class, ISP(A), which it generates. Our criteria will be syntactic ones, involving the elimination of existential quantifiers. For conciseness we introduce the following definitions:

Definition: i) A &at-formula is a formula which is the conjunction of atomic formulas.

ii) An open-positive formula (O^+-formula) is a quantifier free formula formed without negations.

iii) An 3&at-formula is a formula of the form 3 y\|/(x,y) where \|/ is an &at-formula.

iv) An 3^+-formula is a formula of the form 3 y\|/(x,y) where *f* is an O^+-formula.

Note that we use boldface lower case letters to denote tuples of unspecified length. To avoid tiresome qualifications we reserve the symbol $\mathbf{y}$ for non-empty tuples. For a class $\mathcal{K}$ of algebras, we denote the class of injectives in $\mathcal{K}$ by $\mathcal{K}^{\text{inj}}$.

We begin by recalling a result from [1].

Theorem: *If $\mathcal{Q}$ is a finite set of finite algebras then $\mathcal{Q} \subseteq \mathbf{ISP}(\mathcal{Q})^{\text{inj}}$ iff every $\exists$&at-formula is equivalent modulo $\mathcal{Q}$ to a &at-formula.*

We write the latter condition as: $\mathcal{Q} \models \exists\&\text{at} \equiv \&\text{at}$.

This section is devoted to exploring how this criterion may be simplified, or exploited if, first of all $\mathcal{Q}$ consists only of a single finite algebra A, and secondly, this algebra satisfies some additional conditions.

Definition [5]: A is ***self-injective*** if A is injective in $\mathbf{S}A$.

This means simply that any homomorphism from a substructure of A to A extends to an endomorphism of A.

Proposition 1: *A is self injective if every $\exists^+$-formula is equivalent modulo A to an O^+-formula.*

Proof: $A \models \exists^+ \equiv \text{O}^+$

$\Rightarrow \mathbf{S}A$ has TP (by Lemma 1 of [1])

$\Rightarrow A \in (\mathbf{S}A)^{\text{inj}}$ (directly)

$\Rightarrow$ (if A satisfies a positive existential formula at $\mathbf{a}$, and $\mathbf{b}$ is the image of $\mathbf{a}$ under a homomorphism of the subalgebra generated by $\mathbf{a}$ to A, then A satisfies the same existential formula at $\mathbf{b}$)

$\Rightarrow$ (that whether A satisfies a positive existential formula at $\mathbf{a}$ depends only on the positive open diagram of $\mathbf{a}$)

$\Rightarrow \text{A} \models \exists^+ \equiv \text{O}^+$. □

Note that if $A \models (x=y) \vee (u=v) \equiv \&\text{at}$ then $A \models \text{O}^+ \equiv \&\text{at}$. From this simple observation and the syntactic criteria above we get:

Proposition 2 [10] : *If A is quasi-primal then A is injective in* **ISP**(A) *iff A is self-injective.*

Proof: By Proposition 1, A is self-injective $\Leftrightarrow A \models \exists^{+} \equiv O^{+}$. Also, as noted above, $(x=y) \vee (u=v) \equiv$ &at $\Leftrightarrow O^{+} \equiv$ &at, and in the presence of a ternary discriminator this latter condition is automatically true. □

In fact we get the slightly more general:

Proposition 3: *If* $A \models (x=y) \vee (u=v) \equiv$ &at , *then A is injective in* **ISP**(A) *iff A is self-injective.* □

Definition: A^{+} is A with each element added as a constant.

Proposition 4: [5] *If* $A \in$ **ISP**$(A)^{\text{inj}}$ *then* $A^{+} \in$ **ISP**$(A^{+})^{\text{inj}}$.

Proof: A direct proof is easy, but a proof from the syntactic criterion of [1] is trivial, just by the usual trick of replacing constants by variables. □

Proposition 5: *If* $A \models (x=y) \vee (u=v) \equiv$ &at *then* $A^{+} \in$ **ISP**$(A^{+})^{\text{inj}}$.

Proof: Suppose $A \models \exists \mathbf{y}\, \varphi(\mathbf{x},\mathbf{y},\mathbf{a}) \leftrightarrow \vee(\mathbf{x}=\mathbf{b})$ (where the disjunction on the right simply enumerates the tuples at which the existential formula holds). Then we can replace the disjunction on the right by a conjunction of atomic formulas with parameters from A, i.e. by an A^{+}-&at formula. □

Proposition 6: *If A is hereditarily subdirectly irreducible and* **ISP**(A) *is congruence distributive, then* $A \in$ **ISP**$(A)^{\text{inj}}$ *iff* $A \in (\mathbf{S}A)^{\text{inj}}$ *and* $A \models (x=y) \vee (u=v) \equiv$ &at.

Proof: Proposition 2.4 of [12] says that the hypotheses on the algebra A imply that $A \models (x=y) \vee (u=v) \equiv \exists$&at. This gives us the left to right implication, and we already have that from right to left. □

Definition: For an algebra A, let $\mathcal{E}_n(A)$ be the meet subsemilattice of $\mathcal{P}(A^n)$ generated by those subsets which are equalizers of terms.

Proposition 7: *A is injective in* **ISP**(*A*) *iff every subset of* A^n *definable by an* ∃&at-*formula is in* $\mathcal{E}_n(A)$.

Proof: This is just a rephrasing of the first Theorem stated above. □

Another way to interpret this proposition is to observe that it is equivalent (by induction) to the condition that the image of the projection map from $\mathcal{E}_{n+1}(A)$ to $\mathcal{P}(A)$ is contained in $\mathcal{E}_n(A)$.

Definition: An ***order primal algebra*** $\mathcal{Q}$, is obtained from an ordered set **Q**, by taking as fundamental operations all order preserving maps from $\mathbf{Q}^n$ to **Q** for all n. The following result of Davey and Quackenbush was announced at Oberwolfach in February 1988, although their proof was by different means.

Proposition 8: *Let* $\mathcal{Q}$ *be an order primal algebra. Then* $\mathcal{Q}$ *is injective in* **ISP**($\mathcal{Q}$).

Proof: If **Q** is an antichain then $\mathcal{Q}$ is primal so we're done. Otherwise choose a<b in **Q**. Then for $c \in Q^n$ define $f_c: Q^n \rightarrow Q$ by:

$$f_c(x) = \begin{cases} a & \text{if } x \leq c \\ b & \text{otherwise.} \end{cases}$$

$$g_c(x) = \begin{cases} a & \text{if } x < c \\ b & \text{otherwise.} \end{cases}$$

Then $\{x: f_c(x) = g_c(x)\} = Q^n \setminus \{c\}$, so $\mathcal{E}_n(\mathcal{Q})$ is $\mathcal{P}(\mathcal{Q}^n)$ and we're done. □

Note that $\mathcal{Q}$ may have one non-trivial quotient, namely the primal algebra $\mathcal{P}$ obtained by collapsing each component of $\mathcal{Q}$ to a single point. It is just as easy to check that both $\mathcal{P}$ and $\mathcal{Q}$ are injective in the universal Horn class which they generate.

More generally one could define, for a relation R on a finite set A, the **R-primal algebra** on A which has as fundamental operations all those functions which preserve R. In the case where R is a binary relation Ross Willard and I have an almost complete classification of those R for which the R-primal algebras are injective in the universal Horn class which they generate.

2. Algebraic tests for injectivity.

In this section we address the question of whether there exist effective tests for determining when the universal Horn class generated by a finite set, Q, of finite algebras has enough injectives, or is a variety with enough injectives. In providing such tests we also discover internal structural properties of these classes which demonstrate that injectivity in all of ISP(Q) depends only on a finite subset of this class. The latter test is interesting in the case where for some reason, we have an effective method for finding the subdirectly irreducibles in a variety, and overlaps some of the results in [8]. However, our results are purely existential, and do not provide a method for constructing any injectives other than the obvious ones in such classes. Much more structural information is revealed for particular classes satisfying additional conditions in [5]. The techniques utilized here are algebraic, as opposed to the logical ones of Section 1. However, the frequent use of ultraproducts strongly suggests some underlying syntactic results, and is reminiscent of (he Jónsson diagram techniques developed in [7].

For completeness, let us recall the following result (see [4]):

Lemma 9: *A universal Horn class generated by a set* Q *of finitely many finite algebras is an elementary class; in particular it is closed under ultraproducts.* •

The standard proof of this result is to show that any reduced product is a subalgebra of a product of ultraproducts. However, a more "model-theoretic[11] proof is also possible, by adjoining a binary relation symbol to the language, which is intended to denote a congruence with quotient algebra in Q , and using compactness.

In order to effectively test when a finitely generated universal Horn class has enough injectives, we will need to understand an appropriate version of the congruence extension property in such classes. To this end, we introduce the following definitions:

Definition: a) *Let %bea class of algebras, and A* **e** 3C. *We say that* 0 **e** Con(A) *is a* **OC-congruence** *if A/Q* ***e*** 3C.

b) *Let %bea class of algebras. We say that* 3C *has the* **congruence extension property (CEP),** *if for each A* **e** 3C, *each extension* B G X *of A, and every %-congruence* 0 *on A, there exists a %-congruence* $\overline{0}$ *on B with* $\overline{Q}|_A$ = 0.

If $\mathcal{K}$ is closed under **I**, **S**, and **P**, then the $\mathcal{K}$-congruences on A are closed under arbitrary intersections of non-empty families of $\mathcal{K}$-congruences. If in addition, $\mathcal{K}$ contains a one element algebra then the $\mathcal{K}$-congruences are closed under arbitrary intersections and hence form a complete lattice.

We define the **principal $\mathcal{K}$-congruence** $\kappa(a,b)$ on A generated by a pair of elements $a,b \in A$ to be the smallest $\mathcal{K}$-congruence on A containg (a,b), if such a congruence exists, and leave it undefined otherwise. Then the **principal congruence extension property (PCEP)** is just the CEP restricted to defined principal $\mathcal{K}$-congruences.

In [6] it is shown that for varieties of algebras the CEP and PCEP are equivalent. We extend this result below to universal Horn classes generated by finitely many finite algebras, and then show how one may effectively test for the PCEP in such a class.

Lemma 10: *If $\mathcal{K}$ =* **ISP**($\mathcal{Q}$) *where $\mathcal{Q}$ is a finite set of finite algebras, then $\mathcal{K}$ has the CEP iff $\mathcal{K}$ has the PCEP.*

Proof: The implication from left to right is of course trivial. So suppose that $\mathcal{K}$ has the PCEP. Then clearly any fintely generated $\mathcal{K}$-congruence on an element $A \in \mathcal{K}$ can be extended to any $B \in \mathcal{K}$ containing A. Suppose $C, D \in \mathcal{K}$ and $C \subseteq D$, and suppose that θ is any $\mathcal{K}$-congruence on C. For each finitely generated (and hence finite) subalgebra F of D, let $E = C \cap F$, and let $\theta_E = \theta|_E$. Then θ_E is certainly a finitely generated $\mathcal{K}$–congruence, and hence has an extension θ_F on F. There is an ultraproduct U of the finitely generated substructures of D which admits a natural embedding from D to U. Then $U \in \mathcal{K}$ since $\mathcal{K}$ is elementary; and furthermore, the congruence on U induced by the θ_F's is an extension of θ. Restricting this congruence to D provides the required extension of θ. □

Definition: *A set $\mathcal{Q}$ of algebras will be called* **Horn-basic** *if for every embedding of any element of $\mathcal{Q}$ into an element of* **ISP**($\mathcal{Q}$)*, the composite of this embedding with one of the projections is an isomorphism.*

For a variety $\mathcal{V}$, if every subdirectly irreducible is contained in a maximal subdirectly irreducible then the class of maximal subdirectly irreducibles form a Horn-basic generating set for $\mathcal{V}$. Otherwise, no Horn-basic generating set for $\mathcal{V}$ exists.

Lemma 11: *A universal Horn class $\mathcal{K}$ generated by finitely many finite algebras has a Horn-basic generating set, and given an arbitrary generating set $\mathcal{A} = \{A_i : 1 \leq i \leq n\}$ for $\mathcal{K}$ consisting of finite algebras, there is an effective procedure for constructing a Horn-basic generating set.*

Proof: Given a finite generating set $\mathcal{A}$, if it is not Horn-basic then it must be possible to replace an element of $\mathcal{A}$ by a finite or empty set of proper substructures of that element, and produce a generating set $\mathcal{B}$. Clearly this procedure can only be applied finitely often. For effectiveness it suffices to observe that the set of homomorphisms with domain and range in $\mathcal{A}$ is finite. □

Lemma 12: *If $\mathcal{A}$ is Horn-basic then* **ISP**($\mathcal{A}$) *has enough injectives iff* $\mathcal{A} \subseteq \mathrm{ISP}(\mathcal{A})^{\mathrm{inj}}$.

Proof: If $\mathcal{A} \subseteq \mathrm{ISP}(\mathcal{A})^{\mathrm{inj}}$ then clearly **ISP**($\mathcal{A}$) has enough injectives. Conversely, suppose that **ISP**($\mathcal{A}$) has enough injectives. Since $\mathcal{A}$ is Horn-basic, each $A \in \mathcal{A}$ is a retract of any element of **ISP**($\mathcal{A}$) in which it embeds, thus each $A \in \mathcal{A}$ must be injective. □

Definition: *For a class $\mathcal{K}$ of algebras, with $A,B \in \mathcal{K}$, we say that a homomorphism* $f{:}B \to A$ *is an* **injectivity base** *for $\mathcal{K}$ if for every embedding* $g{:}B \to C \in \mathcal{K}$, *there exists a homomorphism* $h{:}C \to A$ *with* $hg=f$.

For a homomorphism $g{:}B \to \prod\{A_i : i \in I\}$ let $\prod(g) = \{\pi_i g : i \in I\}$. We say that a set Σ of embeddings is **representative** (for B in **ISP**($\mathcal{A}$)), if for any embedding h with domain B, and range in **P**($\mathcal{A}$), there exists an embedding $g \in \Sigma$ such that $\prod(g) = \prod(h)$.

Observation: Suppose $\mathcal{A}$ is a finite set of finite algebras, and let $\mathcal{K}$ = **ISP**($\mathcal{A}$). Then:

i) each finite $B \in \mathcal{K}$ has a finite representative set of embeddings;

ii) if $f{:}B \to A$ is a homomorphism, then to determine whether f is an injectivity base for $\mathcal{K}$, it suffices to determine if this holds of a representative set of embeddings for B; and

iii) to check whether $\mathcal{K}$ has the CEP it suffices to check that each congruence on each $A \in \mathcal{K}$ extends over a representative set of embeddings.

The first part of the observation is trivial in as much as there are only finitely many functions from B to the elements of $\mathcal{A}$, while the second and third parts depend only on basic properties of the Cartesian product. The notion of a representative set of embeddings, and part (i) of the observation above are due to G. Bürger [2].

Theorem 13: *There is an effective test to determine if the universal Horn class $\mathcal{K}$ generated by a finite set $\mathcal{A}$ of finite algebras has enough injectives.*

Proof: By Lemma 11 we may assume that $\mathcal{A}$ is Horn-basic. If $\mathcal{K}$ is to have enough injectives then it must have the CEP, and by Lemma 10 it suffices to check whether $\mathcal{K}$ has the PCEP. Suppose that $B, C \in \mathcal{K}$, $a,b \in B$ and that $\kappa(a,b)$ is defined. If $\kappa(a,b)$ cannot be extended to C then there is a pair $(x,y) \in B^2 \setminus \kappa(a,b)$ such that if $f{:}C \to D \in \mathcal{K}$ is a homomorphism and $f(a)=f(b)$ then $f(x)=f(y)$. Consider the subalgebra $\langle a,b,x,y \rangle$ of B generated by $\{a,b,x,y\}$. The principal $\mathcal{K}$-congruence on this algebra generated by (a,b) is also defined and cannot be extended to C either. Thus we see that if every principal $\mathcal{K}$-congruence on every four-generated algebra in $\mathcal{K}$ can be extended, then $\mathcal{K}$ has the PCEP. But by the first and third parts of the observation above, this condition can be verified effectively.

Now suppose that $A \in \mathcal{A}$, $B,C \in \mathcal{K}$, $f{:}B \to A$ is a homomorphism, and $g{:}B \to C$ is an embedding. Assuming that we have verified that $\mathcal{K}$ has the CEP, the $\mathcal{K}$-congruence ker(f) can be extended to C. This yields the diagram below which shows that to determine whether A is injective it suffices to determine whether every inclusion of a subalgebra of A into A is an injectivity base, which by the first two parts of the observation above can also be carried out effectively.

$$\begin{array}{ccc} B & \hookrightarrow & C \\ \downarrow & & \downarrow \\ B/\ker(f) & \hookrightarrow & D \\ \downarrow & & \\ A & & \end{array}$$

□

We now turn our attention to the case where we hope that the class $\mathcal{A}$ constitutes the subdirectly irreducibles of some variety.

Lemma 14: *A universal Horn class* X *generated by finitely many finite algebras which is closed under quotients by ordinary principal congruences, is a variety.*

Proof: Let fie X and **6** *e Con(B).* We must show that *B/Q* e X. But any finitely generated substructure of *B/Q* is the quotient of a finitely generated (hence finite) substructure of B_9 and hence belongs to X (since X is also closed under quotients by finitely generated congruences). Therefore, as in Lemma 10, *B/Q* e SP_U(X), and as X is elementary and universal, *B/Q* e X. •

Theorem 15: *There is an effective test to determine if the universal Horn class* X *generated by a finite set* Q *of finite algebras is a variety with enough injectives.*

Proof: By Lemma 11, we may assume that Q is Horn-basic. First we can use the procedure outlined in Theorem 13 to determine whether X has enough injectives. If so then we note that if X is not a variety then, by Lemma 14 there exists an algebra *B e* X, and a,befi such that £/G(a,b) £ X (where G(a,b) is the ordinary principal congruence generated by the pair (a,b)). Thus there exists a pair (x,y) G B^2\G(a,b) such that such that if *fJf-+De* Q is a homomorphism and f(a)=f(b) then f(x)=f(y). But no homomorphism from the substructure <a,b,x,y> with range in Q, and f(a)=f(b) but f(x)*f(y) can exist either, lest its range fail to be injective. So we already have that <a,b,x,y>/9(a,b) £ X. To summarize, if X has enough injectives but is not a variety then there exists a four-generated algebra, *C* e X, and a principal congruence G on *C* such that *C/Q* £ X.

This condition may be checked effectively since there are only finitely many isomorphism classes of four-generated algebras in X. •

Unfortunately, because we began the procedure in this proof by checking for injectivity, and such a test was required for the latter part of the proof, we have not produced an effective test to determine whether X is a variety, and this remains, for us at least, an open question.

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