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THE GRADIENT THEORY OF PHASE TRANSITIONS FOR SYSTEMS WITH TWO POTENTIAL WELLS

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1. INTRODUCTION.

In this paper we obtain the r(L*(£2))-liinit of a family of perturbations of a nonconvex functional of vector-valued functions.

The variational problem that we study is of the form

$$E_{£}(u) := J_{\Omega} W(u)\, dx + e^2 \int_{\Omega} |Vu|^2 dx \qquad (1.1)$$

where Q is an open bounded strongly Lipschitz domain of R^n, u: *il* -» R^N and W supports two phases, precisely W attains the minimum value of zero at exactly two points a and b (system with two potential wells of equal depth).With no loss of generality, we assume that meas *(Cl)* =1.

It is clear that the' problem:

minimize $E_0(u)$ with u satisfying a volume constraint

$$\int_{n} u(x)\, dx = m, \text{ Iwhere } m = 9a + (l\text{-}G)b \text{ for some } 0 € (0,1), \qquad (1.2)$$

has infinitely many piecewise constant solutions with values a and b and there is no restriction on the interface between the sets {u = a} and {u = b}. Modulo the volume constraint, the set A = {u = a} is completely arbitrary.

As pointed out by GURTIN [7], this lack of uniqueness is a consequence of the fact that interfaces are allowed to form without an increase in energy.

If we search fotfa mechanism that singles out the solutions that are more likely observed, then we should try to examine which are the limiting cases within theories that penalize the formation of interfaces. In this context, the natural conjecture is that physically preferred solutions are those that minimize the area of the interface {x *e* *Q* | x e dA}.

A theory that includes interfacial energy directly penalizing the interfaces is given by GURTIN [5], [6].

Also, studying the behavior of minimizers of the perturbed problem (1.1) as e -» 0^+ gives another selection criterion to resolve the non-uniqueness in the lower order problem (see GURTIN [7]).

Here, we analyze the asymptotic behavior as e -» 0^+ of a sequence u_e of minimizers of

Eg.We show that if $u_{£}$-» u_0 in L^1 then u_0 only takes the values a and b (corresponding to the two phases in equilibrium since W(a) = 0 = W(b)) and the interface has minimal area, i.e. the portion occupied by the phase u_0 = a minimizes the geometric area-like quantity

Per^(A) (Perimeter of A in *Q)*

among all subsets A' of Ω with meas(A') = meas($\{u_0 = a\}$).

This new variational problem arises as Γ-limit of the functionals

$$J_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega W(u)\,dx + \varepsilon\int_\Omega |\nabla u|^2 dx,$$

where we impose the volume constraint (1.2).

In fact, we show that if W is a locally Lipschitz function growing at least linearly at infinity then

(i) any family (v_ε) such that $J_\varepsilon(v_\varepsilon) \le C < \infty$ for all $\varepsilon > 0$ is compact in $L^1(\Omega)$;

(ii) if $v_\varepsilon \to v_0$ in $L^1(\Omega)$ then $\liminf J_\varepsilon(v_\varepsilon) \ge J_0(v_0)$;

(iii) For any $v_0 \in L^1(\Omega)$ there exists a family (v_ε) such that $v_\varepsilon \to v_0$ in $L^1(\Omega)$ and $\lim J_\varepsilon(v_\varepsilon) = J_0(v_0)$, where

$$J_0(u) := \begin{cases} K\,\mathrm{Per}_\Omega(\{u = a\} & \text{if } u(x) \in \{a, b\} \text{ a.e.} \\ \\ \infty & \text{otherwise,} \end{cases}$$

and

$$K = 2\inf\left\{\int_{-1}^{1}\sqrt{W(g(s))}\,|g'(s)|\,ds \mid g \text{ is piecewise } C^1,\ g(-1) = a,\ g(1) = b\right\}$$

is the energy left on the interface as the boundary layer goes to zero.

Properties (ii) and (iii) say that J_0 is the $\Gamma(L^1(\Omega))$-limit of J_ε.

The form of J_0 and the role played by the geodesic curves was independently conjectured by KOHN & STERNBERG [8], who refer also to MAHONEY & NORBURY [9].

This result confirms the selection criterion of the perturbation process. Moreover, we conclude that the method of KOHN & STERNBERG [8] for constructing local minimizers of J_ε for sufficiently small ε can be applied to systems with two potential wells.

The one-dimensional version of this problem (n=1, N=1) is studied in OWEN [11] (see also CARR, GURTIN & SLEMROD [1]). The n -dimensional case (n arbitrary, N=1) was treated by MODICA [10] and STERNBERG [14] and later by OWEN [12] in a more general setting, where he considers a wider class of perturbations. OWEN [12] concludes that there is no loss of generality in studying the behavior of the simplest perturbation $\varepsilon^2|\nabla u|^2$ of W(u) as a selection criterion as opposed to taking a more complicated perturbation. We conjecture that a similar result must hold for the vector-valued case.

In Section 2 we state some results on functions of bounded variation and sets of finite perimeter. A general discussion of these subjects can be found in De GIORGI [2] and GIUSTI [4].

In Section 3 we prove the main theorem of the $\Gamma(L^1(\Omega))$-limit of the functionals J_ε and we use this result to analyze the behaviour of a $L^1(\Omega)$ limit of a minimizing sequence for E_ε.

In Section 4 we prove a compactness result that allows us to extract a $L^1(\Omega)$ convergent subsequence of any sequence (v_ε) such that $J_\varepsilon(v_\varepsilon) \leq C < \infty$ for all $\varepsilon > 0$. In particular, we conclude that any sequence of minimizers of E_ε admits a subsequence converging in $L^1(\Omega)$ to a minimizer of E_0 with a minimal interfacial area.

2. FUNCTIONS OF BOUNDED VARIATION AND SETS WITH FINITE PERIMETER.

In this section we discuss very briefly the concepts of functions of bounded variation and perimeter of a set. We will restrict ourselves to the properties that will be of later use in this paper.

Let Ω be an open bounded strongly Lipschitz domain of $\mathbb{R}^n$. A function $u \in L^1(\Omega)$ is said to be a *function of bounded variation* ($u \in BV(\Omega)$) if

$$\int_\Omega |\nabla u(x)|\, dx := \sup\left\{ \int_\Omega u(x).\mathrm{div}\, \varphi(x)\, dx \mid \varphi \in C_0^1(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\} < \infty.$$

It follows immediately from this definition that if v_ε converges to v_0 in $L^1(\Omega)$ then

$$\int_\Omega |\nabla u(x)|\, dx \leq \liminf_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon(x)|\, dx. \tag{2.1}$$

Moreover it can be shown that the sets

$$\left\{ u \in L^1(\Omega) \mid \int_\Omega |u(x)|\, dx + \int_\Omega |\nabla u(x)|\, dx \leq C < \infty \right\} \tag{2.2}$$

are compact in $L^1(\Omega)$..

If A is a subset of $\mathbb{R}^n$ then the *perimeter of A in Ω* is defined by

$$\mathrm{Per}_\Omega(A) := \int_\Omega |\nabla \chi_A(x)|\, dx = \sup\left\{ \int_A \mathrm{div}\, \varphi(x)\, dx \mid \varphi \in C_0^1(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\},$$

where χ_A denotes the characteristic function of A. Clearly, if A is a subset of Ω and if

$$u(x) = \begin{cases} a & \text{if } x \in A \\ b & \text{if } x \in \Omega \setminus A \end{cases}$$

then $u \in BV(\Omega)$ if and only $\mathrm{Per}_\Omega(A) < \infty$. Also, if ∂A is smooth then the divergence theorem implies that

$$\mathrm{Per}_\Omega(A) = H_{n-1}(\partial A \cap \Omega),$$

where H_{n-1} is the n-1 dimensional Hausdorff measure.

The next two results are taken from STERNBERG [14]. The first lemma states that every set with finite perimeter can be approximated by sets with smooth boundaries and Lemma 2.4 asserts the smoothness of the signed distance function to the boundary of a sufficiently regular set.

Lemma 2.3.

Let A be a subset of Ω such that $\mathrm{Per}_\Omega(A) < \infty$ and $0 < \mathrm{meas}\,(A) < \mathrm{meas}\,(\Omega)$. There exists a sequence of open sets $\{A_k\}$ satisfying the following properties:

(i) $\partial A_k \cap \Omega \in C^2$;

(ii) $\mathrm{meas}\,(\,((A_k \cap \Omega) \setminus A) \cup (A \setminus (A_k \cap \Omega))\,) \to 0$ as $k \to \infty$;

(iii) $\mathrm{Per}_\Omega(A_k) \to \mathrm{Per}_\Omega(A)$ as $k \to \infty$;

(iv) $H_{n-1}(\partial A_k \cap \partial\Omega) = 0$;

(v) $\mathrm{meas}\,(A_k \cap \Omega) = \mathrm{meas}\,(A)$ for sufficiently large k.

Lemma 2.4.

Let A be an open subset of $\mathbb{R}^n$ with a C^2 compact and nonempty boundary intersecting Ω and such that $H_{n-1}(\partial A \cap \partial\Omega) = 0$. Define the signed distance function to ∂A by

$$d(x) := \begin{cases} \mathrm{dist}(x,\partial A) & \text{if } x \in \Omega \setminus A \\ \\ -\mathrm{dist}(x,\partial A) & \text{if } x \in A \cap \Omega. \end{cases}$$

Then for sufficiently small $\varepsilon > 0$ the restriction of d to the set $\{x \in \Omega \mid |d(x)| < \varepsilon\}$ is a C^2 function with $|\nabla d| = 1$. Furthermore,

$$\lim_{\varepsilon \to 0} H_{n-1}(\{x \in \Omega \mid d(x) = \varepsilon\}) = H_{n-1}(\partial A \cap \Omega).$$

We will also use the *coarea formula* (see FEDERER [3])

$$\int_\Omega f(h(x))\, |\nabla h(x)|\, dx = \int_{-\infty}^{+\infty} f(t)\, H_{n-1}(\{x \in \Omega \mid h(x) = t\})\, dt \tag{2.5}$$

for all measurable functions f and Lipschitz h.

For more details on these subjects we refer the reader to De GIORGI [2] and GIUSTI [4].

3. THE r-LIMIT OF A FAMILY OF FUNCTIONALS OF VECTOR VALUED FUNCTIONS.

In what follows, *Q* is an open bounded strongly Lipschitz domain of R^n and W satisfies the following properties:

(HI) W **e wj£(R^N; R)** is a nonnegative function such that

W(u) = 0 if and only if u € {a, b}, where a * b.

(H2) There exist a, 8 > 0 such that

if |u - a| < 5 then cc$|u-a|^2$ < W(u) <^-$|u-a|^2$

and

a$|u - b|^2$ £ W(u) & $\frac{1}{\ }$$|u - b|^2$ whenever |u - b| < 5.

(H3) There exist C, R > 0 such that

if|u|>RthenW(u)£C|u|.

For e > 0 consider the functional

$$J_£(u) := \frac{1}{-} f\ W(u)\, dx + e\, f\ |Vu|^2 dx$$

and let

$$J_0(u) := \begin{cases} K\ Per_Q(\{u = a\}) & \text{if } u(x)\ e\ \{a, b\} \text{ a.e.} \\ oo & \text{otherwise,} \end{cases}$$

where

$$K = 2 \inf\left| J_{-1}^{1} \sqrt{W(g(s))}\ |g'(s)|\, ds \;\middle|\; g \text{ is piecewise } C^1,\ g(-1) = a,\ g(1) = b \right\}.$$

By (H3) we have that $J_e(u)$ < ~ only if u **e** $H^l(Q;\ R^N)$.

Our main theorem states that the J_0 is the r**C$L^1(\Omega)$)-limit of J_ε.**

Theorem 3.1.

Under the hypotheses (HI) - (H3) the following hold:

(i) if v_e -» v_0 in $L^1(Q)$ then lim inf $J_£(v_e) \geq J_0(v_0)$;

(ii) for any v_0 **e** $L^1(Q)$ there exists a family (v_e) such that $v_£$ -> v_0 in $L^1(Q)$ and lim $J_£(v_£)$ =

$J_0(v_0)$.

Before proving Theorem 3.1 we present three technical lemmas concerning the constant K.

Lemma 3.2.

Let K_1 and K_2 be given by

$$K_1 := \inf\left\{\int_{-\infty}^{+\infty} W(g(s)) + |g'(s)|^2\, ds \mid g \text{ is a piecewise } C^1 \text{ curve with } g(-\infty) = a \text{ and } g(+\infty) = b\right\}$$

and

$$K_2 := \inf\left\{\int_{-L}^{L} W(g(s)) + |g'(s)|^2\, ds \mid L>0,\ g \text{ is a piecewise } C^1 \text{ curve with } g(-L) = a \text{ and } g(L) = b\right\}.$$

Then $K_1 = K_2$.

Proof. It is easy to check that $K_1 \le K_2$. In fact, if $g : [-L, L] \to \mathbb{R}^N$ is such that $g(-L) = a$ and $g(L) = b$ then by (H1) the extension

$$g^*(s) := \begin{cases} a & \text{if } s<-L \\ g(s) & \text{if } -L \le s \le L \\ b & \text{if } s>L \end{cases}$$

verifies

$$\int_{-L}^{L} W(g(s)) + |g'(s)|^2\, ds = \int_{-\infty}^{+\infty} W(g^*(s)) + |g^{*\prime}(s)|^2\, ds.$$

Conversely, let $g : \mathbb{R} \to \mathbb{R}^N$ be a piecewise C^1 curve with

$\lim_{s \to -\infty} g(s) = a$ and $\lim_{s \to +\infty} g(s) = b$

and

$$\int_{-\infty}^{+\infty} W(g(s)) + |g'(s)|^2\, ds < \infty.$$

By (H2) we have that

$$g-a \in H^1((-\infty, L);\mathbb{R}^N) \text{ and } g-b \in H^1((-L, +\infty); \mathbb{R}^N) \text{ for all } L > 0. \tag{3.3}$$

If γ is a smooth cut-off function such that $0 \le \gamma \le 1$ and

$$\gamma(s) = \begin{cases} 1 & \text{if } |s| \le 1 \\ 0 & \text{if } |s| \ge 2, \end{cases}$$

we define

$$g_k(s) := \begin{cases} \gamma\left(\frac{s}{k}\right) g(s) + \left(1 - \gamma\left(\frac{s}{k}\right)\right)a & \text{if } s < 0 \\ \\ \gamma\left(\frac{s}{k}\right) g(s) + \left(1 - \gamma\left(\frac{s}{k}\right)\right)b & \text{if } s > 0. \end{cases}$$

Clearly we have $g_k(-2k) = a$, $g_k(2k) = b$, $g_k(s) = g(s)$ if $|s| \le k$ and (3.3) together with (H2) yield

$$\lim_{k \to \infty} \int_{-2k}^{2k} W(g_k(s)) + |g_k'(s)|^2 \, ds = \int_{-\infty}^{+\infty} W(g(s)) + |g'(s)|^2 \, ds.$$

Next we show that Lemma 3.2 and the hypothesis (H1) yield the following result.

Lemma 3.4.
$K_1 = K_2 = K$.

Proof. (i) First we prove that $K_1 \ge K$. If $g : [-L, L] \to \mathbb{R}^N$ is such that $g(-L) = a$ and $g(L) = b$ and if

$$\xi(s) := g(Ls) \quad \text{for } -1 \le s \le 1$$

then

$$2\int_{-1}^{1} \sqrt{W(\xi(s))}\, |\xi'(s)| \, ds = 2\int_{-L}^{L} \sqrt{W(g(s))}\, |g'(s)| \, ds \le \int_{-L}^{L} W(g(s)) + |g'(s)|^2 \, ds.$$

(ii) In order to show that $K_1 \le K$ it suffices to consider curves $g : [-1, 1] \to \mathbb{R}^N$ with $g(-1) = a$ and $g(1) = b$ for which the arc length

$$\tau(s) := \int_{-1}^{s} |g'(t)| \, dt$$

is a strictly increasing function on $[-1, 1]$. Reparametrizing g with τ^{-1}, we obtain a curve $g^*: [0, L] \to \mathbb{R}^N$ with $g^*(0) = a$, $g^*(L) = b$, $|g^{*\prime}(s)| = 1$ a.e. and

$$\int_{0}^{L} \sqrt{W(g^*(s))}\, |g^{*\prime}(s)| \, ds = \int_{-1}^{1} \sqrt{W(g(s))}\, |g'(s)| \, ds \tag{3.5}$$

where

$$L := \int_{-1}^{1} |g'(t)| \, dt.$$

Define the function
$F(s) := \sqrt{W(g^*(s))}$
which, by (H1), is locally Lipschitz, and consider the initial value problem

$$\begin{cases} h'(s) = F(h(s)) \\ h(0) = \frac{L}{2}. \end{cases}$$

By (H1) there exists an interval (possibly unbounded) (T_0, T_1) such that $h(T_0) = 0$, $h(T_1) = L$ and $h'(s) > 0$ in (T_0, T_1). We extend h by 0 in $(-\infty, T_0)$ and by L in $(T_1, +\infty)$. Setting

$$g^{**}(s) := g^*(h(s))$$

we have that g^{**} is Lipschitz and satisfies $g^{**}(-\infty) = a$, $g^{**}(+\infty) = b$ and

$$|g^{**\prime}(s)|^2 = |g^{*\prime}(h(s))\, h'(s)|^2 = W(g^{**}(s)).$$

Therefore, we obtain

$$\int_{-\infty}^{+\infty} W(g^{**}(s)) + |g^{**\prime}(s)|^2\, ds = 2\int_{T_0}^{T_1} \sqrt{W(g^{**}(s))}\, |g^{**\prime}(s)|\, ds$$

$$= 2\int_0^L \sqrt{W(g^*(s))}\, |g^{*\prime}(s)|\, ds,$$

which, by (3.5) and given the arbitrariness of g, permits us to conclude that $K_1 \leq K$.

Now we define a "geodesic distance" ϕ as follows. Let

$$f(r) := \inf_{|u-c|=r} \sqrt{W(u)}$$

where

$$c := \frac{a+b}{2}.$$

If we set

$$r_0 := \left|\frac{a-b}{2}\right|,$$

then by (H3) there exists $r_1 > r_0$ such that

$$\int_{r_0}^{r_1} f(r)\, dr > \frac{K}{2}. \tag{3.6}$$

Let

$$M := \max_{|u-c| \leq r_1} \sqrt{W(u)}.$$

and define

$$\phi(x) := \inf\left\{ \int_{-1}^{1} T(g(s))\, |g'(s)|\, ds \mid g \text{ is piecewise } C^1,\ g(-1) = a \text{ and } g(1) = x \right\}, \tag{3.7}$$

where

$$T(u) := \min\{\sqrt{W(u)}, M\}.$$

Lemma 3*8.

(i) ϕ is a lipschitz function;

(ii) If u **e** $H^l(\&,$ $R^N)$ then 4>»u € H^1 ^; $\mathbf{R^N}$**)** and

|∇(ϕ•u)(x)| ≤ VW(u(x)) |Vu(x)| a.e. x € *Q*.

(iii) K = 2 <|>(b).

Proof, (i) Let x and y be two points in R^N and let *y* be an arbitrary piecewise C^1 curve joining a to x.Then we have

<|>(y)≤ fTds+ f Tds

where [x, y] denotes the segment in R^N with endpoints x and y. Therefore, it follows that

<|>(y)≤<|>(x) + M|x-y|

and, in a similar way

<|>(x)≤<|>(y) + M|x-y|.

Hence, we conclude that

$$|\phi(x) - \phi(y)| \le M\,|x - y|.$$

(ii) Using an argument identical to that of part (i), it is easy to show that

|V(<|)ou)(x)| ≤ T(u(x)) |Vu(x)| a.e. x e f l i f u e $C^l(\bar{Q})$. (3.9)

Let u **e** $H^l(Cl;$ $R^N)$ and consider a sequence

u_k **e** C $^1(\bar{Q})$ such that u_k —» u in H^1.

Since T € L°°, by (3.9) (V((|>*uj_c)) is bounded in L^2, and hence it converges weakly in L^2.

Moreover, by (i) 4>ou € L^2 and so <|>ou € H^1 ^; R^N). Finally, since $T(u_k)$ $|Vu_k|$ converges weakly

in L^2 to T(u) |Vu| we conclude that (3.9) still holds for u.

(iii) Clearly, K £ 2<|>(b). As T(u) = $\overline{Vw(u)}$ if |u - c| ^ r_{l9} it remains to prove that if g: [-1,1] -> R^N

is such that g(-l) = a, g(l) = b and $|g(s_0) - c| = r_2 +$ **e** for some s_0 € (-1,1) and e > 0, then

$$\int_{-1} T(g(s))\,|g'(s)|\,ds \wedge K.$$

In fact, we have that

$$\int_{-1}^{1} T(g(s))\,|g'(s)|\,ds > 2 \int^{fiy*} f(r)\,dr > K.$$

Proof of Theorem 3.1. (i) Let v_e-> v_0 in $L^1(Q)$ and suppose that lim inf $J_e(v_e)$ < oo. By

Fatou's Lemma we have

$$\int_\Omega W(v_0(x))\,dx \le \liminf_{\varepsilon\to 0}\int_\Omega W(v_\varepsilon(x))\,dx = 0$$

and so, by (H1) $v_0 \in \{a, b\}$ a.e. On the other hand, by Lemma 3.8 (ii) we obtain

$$J_\varepsilon(v_\varepsilon) \ge 2\int_\Omega \sqrt{W(v_\varepsilon(x)}\ |\nabla v_\varepsilon(x)|\,dx \ge 2\int_\Omega |\nabla(\phi\circ v_\varepsilon(x))|\,dx,$$

and therefore, by Lemma 3.8 (i), (iii) and (2.1) we conclude that

$$\phi(v_\varepsilon) \to \phi(v_0) \text{ in } L^1,\ \phi(v_0) = \frac{1}{2}K\,\chi_{\{v_0 = b\}} \text{ and } \int_\Omega |\nabla(\phi\circ v_0)(x)|\,dx \le \liminf_{\varepsilon\to 0}\int_\Omega |\nabla(\phi\circ v_\varepsilon)(x)|\,dx < \infty.$$

Hence, $v_0 \in BV(\Omega)$ and

$$J_0(v_0) = K\,Per_\Omega(\{v_0 = b\}) \le \liminf_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon).$$

(ii) By (i) it suffices to consider $v_0 \in BV(\Omega)$ with $v_0 = \chi_A a + (1 - \chi_A)b$ where $A \subset \Omega$. Moreover, using a diagonalization argument, we can assume with no loss of generality that $\partial A \cap \Omega \in C^2$ and $H_{n-1}(\partial A \cap \partial\Omega) = 0$. Let $g : [-L, L] \to \mathbb{R}^N$ be a piecewise C^1 curve such that $g(-L) = a$ and $g(L) = b$ and define

$$w_\varepsilon(x) := \begin{cases} a & \text{if } d(x) < -\varepsilon L \\ g_\varepsilon(d(x)) & \text{if } |d(x)| < \varepsilon L \\ b & \text{if } d(x) > \varepsilon L, \end{cases}$$

where d is the signed distance function to ∂A and

$$g_\varepsilon(s) := g\left(\frac{s}{\varepsilon}\right).$$

By Lemma 2.4 and the coarea formula (2.5) we have for small ε

$$\begin{aligned}\int_\Omega |v_0(x) - w_\varepsilon(x)|\,dx &= \int_{-\varepsilon L}^0 |a - g_\varepsilon(s)|\,H_{n-1}(\{x \in \Omega \mid d(x) = s\})\,ds \\ &\quad + \int_0^{\varepsilon L} |b - g_\varepsilon(s)|\,H_{n-1}(\{x \in \Omega \mid d(x) = s\})\,ds \\ &= \varepsilon\int_{-L}^0 |a - g(s)|\,H_{n-1}(\{x \in \Omega \mid d(x) = \varepsilon s\})\,ds \\ &\quad + \varepsilon\int_0^L |b - g(s)|\,H_{n-1}(\{x \in \Omega \mid d(x) = \varepsilon s\})\,ds \\ &\le \varepsilon L\,(\|a - g\|_\infty + \|b - g\|_\infty)\,(Per_\Omega(A) + 1). \end{aligned} \tag{3.10}$$

Therefore, $w_\varepsilon \to v_0$ in $L^1(\Omega)$. Moreover, (H1), Lemma 2.4 and (2.5) yield

$$J_\varepsilon(w_\varepsilon) = \int_{-\varepsilon L}^{\varepsilon L}\left[\frac{1}{\varepsilon}W(g_\varepsilon(s)) + \varepsilon\,|g'_\varepsilon(s)|^2\right] H_{n-1}(\{x \in \Omega \mid d(x) = s\})\,ds$$

$$= \int_{-L}^{L} [W(g(s)) + |g'(s)|^2]\, H_{n-1}(\{x \in \Omega \mid d(x) = \varepsilon s\})\, ds$$

and so,

$$\lim_{\varepsilon \to 0} J_\varepsilon(w_\varepsilon) = \mathrm{Per}_\Omega(A) \int_{-L}^{L} [W(g(s)) + |g'(s)|^2]\, ds.$$

Finally, Lemma 3.4 and a diagonalization argument allow us to construct a sequence $v_\varepsilon \to v_0$ in $L^1(\Omega)$ such that

$$\limsup_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon) \le K\, \mathrm{Per}_\Omega(A)$$

which, together with part (i), implies that

$$\lim_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon) = K\, \mathrm{Per}_\Omega(A).$$

Assume that $\mathrm{meas}(\Omega) = 1$ and consider the following variational problem:

(P_ε) minimize

$$E_\varepsilon(u) := \int_\Omega W(u)\, dx + \varepsilon^2 \int_\Omega |\nabla u|^2 dx$$

on $\left\{ u \in W^{1,1}(\Omega) \mid \int_\Omega u(x)\, dx = m \right\}$, where $m = \theta a + (1-\theta) b$ for some $\theta \in (0, 1)$.

Clearly, E_0 admits infinitely many piecewise C^1 solutions with values a and b. We show that the regularization E_ε of E_0 selects the solution with minimal interfacial area, namely

Theorem 3.11.

If the hypotheses (H1) - (H3) are verified and if u_ε is a sequence of minimizers of E_ε converging to u_0 in $L^1(\Omega)$ then u_0 is a solution of the geometric variational problem:

minimize $\mathrm{Per}_\Omega(\{u = a\})$

on $\left\{ u \in BV(\Omega) \mid W(u) = 0 \text{ a.e. and } \int_\Omega u(x)\, dx = m \right\}$.

Remark 3.12.(i) The new variational problem is said to be "geometric" because u_0 is a solution if and only if $A := \{u_0 = a\}$ minimizes $\mathrm{Per}_\Omega(A')$ with A' satisfying the volume constraint $\mathrm{meas}(A') = \mathrm{meas}(A)$.

(ii) The existence of a minimizer u_ε of E_ε is obtained easily by means of the direct method of the Calculus of Variations. Moreover there exists a constant $C_2 > 0$ such that Jgdig) $\leq C_x$ for sufficiently small $e > 0$. (3.12)

In fact, let y be a smooth function with compact support and satisfying T(-1) = 0,7(1) = 1 and 0 ≤ y ≤ 1. Given e > 0 choose $T|_e$ such that

$$\text{meas}\{x \in Q \mid x_n > Ti_\varepsilon + e\} + \int M\left(\frac{x_n - \eta_\varepsilon}{}\right) dx = e$$

and define

$$w_\varepsilon(x) := \begin{cases} si & \text{if } x_n > Ti_\varepsilon - fe \\ \gamma\left(\frac{x_n - \eta_\varepsilon}{\varepsilon}\right)a + \left(1 - \gamma\left(\frac{x_n - \eta_\varepsilon}{\varepsilon}\right)\right)b & \text{if } |x_n - \eta_\varepsilon| < \varepsilon \\ b & \text{if } x_n < T|_e - e. \end{cases}$$

Since

$$\int_a w_\varepsilon(x)\, dx = m,$$

we have

$$J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(w_\varepsilon) = \text{If f } W(w_\varepsilon(x))\, dx + |yf^\wedge k)|^2 |a-b|^2 dx]$$

$$\leq \text{; Const!} \max_{v \in [a,b]} W(v) + ||y||i|a-b|^2 l.$$

Proof of Theorem 3.11. By Remark 3.12 (ii) and Theorem 3.1 (i) we deduce that $u_0 \in BV(Q)$, $W(u_0) = 0$ a.e., the average of u_0 is equal to m and

$$\liminf_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) \geq J_0(u_0). \tag{3.13}$$

Suppose that $u \in BV(\Omega)$, $u \in \{a, b\}$ a.e. and

$$\int_Q u(x)dx = m.$$

We claim that there exists a family (v_e) such that $v_e \to u$ in $L^1(\Omega)$, $\lim J_e(v_\varepsilon) = J_0(u)$ and

$$\int v_\varepsilon(x)\, dx = m.$$

Then, since u_e is a solution of (P_ε), by (3.13) it follows that

$$J_0(u) = \lim J_\varepsilon(v_\varepsilon) \geq \limsup J_\varepsilon(u_\varepsilon) \geq J_Q(u_Q)$$

and so, u_0 is a solution of the geometric variational problem.

We prove our claim by showing that it is possible to modify the sequence w_E constructed in the proof of Theorem 3.1 (ii), obtaining a new sequence w^*_ε such that

$$J_\varepsilon(w^*_\varepsilon) = J_\varepsilon(w_\varepsilon) + o\,(1) \text{ and } \int_\Omega w^*_\varepsilon(x)\,dx = m.$$

In fact, define

$$w^*_\varepsilon := w_\varepsilon + m - \int_\Omega w_\varepsilon\,dx.$$

Clearly $\nabla w^*_\varepsilon = \nabla w_\varepsilon$ and by (H1) we have

$$\begin{aligned}\frac{1}{\varepsilon}\int_\Omega W(w^*_\varepsilon(x))\,dx = \frac{1}{\varepsilon}\int_\Omega W(w_\varepsilon(x))\,dx &+ \frac{1}{\varepsilon}\int_{\{x\in\Omega\,|\,d(x)<-\varepsilon L\}} W\left(a+m-\int_\Omega w_\varepsilon\,dx\right)dx\\ &+\frac{1}{\varepsilon}\int_{\{x\in\Omega\,|\,|d(x)|<\varepsilon L\}}\left[W\left(w_\varepsilon(x)+m-\int_\Omega w_\varepsilon\,dx\right)-W(w_\varepsilon(x))\right]dx\\ &+\frac{1}{\varepsilon}\int_{\{x\in\Omega\,|\,d(x)>\varepsilon L\}} W\left(b+m-\int_\Omega w_\varepsilon\,dx\right)dx.\end{aligned}$$

It suffices to notice that (H2) and (3.10) yield

$$\frac{1}{\varepsilon}\int_{\{x\in\Omega\,|\,d(x)<-\varepsilon L\}} W\left(a+m-\int_\Omega w_\varepsilon\,dx\right)dx \le \frac{1}{\varepsilon\alpha}\left|\int_\Omega u(x)-w_\varepsilon(x)\,dx\right|^2 = O(\varepsilon),$$

$$\frac{1}{\varepsilon}\int_{\{x\in\Omega\,|\,d(x)>\varepsilon L\}} W\left(b+m-\int_\Omega w_\varepsilon\,dx\right)dx \le \frac{1}{\varepsilon\alpha}\left|\int_\Omega u(x)-w_\varepsilon(x)\,dx\right|^2 = O(\varepsilon)$$

and, since $\| w_\varepsilon\|_\infty \le$ Const. and W is locally Lipschitz, we deduce that

$$\left|\frac{1}{\varepsilon}\int_{\{x\in\Omega\,|\,|d(x)|<\varepsilon L\}}\left[W\left(w_\varepsilon(x)+m-\int_\Omega w_\varepsilon\,dx\right)-W(w_\varepsilon(x))\right]dx\right| \le$$
$$\text{Const.}\frac{1}{\varepsilon}\,\text{meas}\,\{x\in\Omega\,|\,|d(x)|<\varepsilon L\}\left|\int_\Omega u(x)-w_\varepsilon(x)\,dx\right| = o\,(1).$$

4. A COMPACTNESS RESULT.

The main result of this section is the following theorem:

Theorem 4.1.

Under the assumptions (H1) and (H3), any family (v_ε) such that $J_\varepsilon(v_\varepsilon) \le \text{Const.} < \infty$ for all $\varepsilon > 0$ is relatively compact in $L^1(\Omega)$.

Proof. Let (v_ε) be such that $J_\varepsilon(v_\varepsilon) \le \text{Const.} < \infty$ for all $\varepsilon > 0$ and let

$$R_1 := \max\left\{R, \frac{W(0)}{C}\right\},$$ where R and C are as in (H3).

Write $v_\varepsilon = u_\varepsilon + z_\varepsilon$, with

$$z_\varepsilon := v_\varepsilon\, \chi_{\{|v_\varepsilon| > R_1\}}.$$

Since

$$\int_\Omega W(v_\varepsilon(x))\, dx \le \varepsilon \text{Const.},$$

by (H3) we have that

$$z_\varepsilon \to 0 \text{ in } L^1(\Omega) \tag{4.2}$$

and

$$\lim_{\varepsilon \to 0} \int_\Omega W(u_\varepsilon(x))\, dx = \lim_{\varepsilon \to 0} \left[\int_{\{|v_\varepsilon| \le R_1\}} W(v_\varepsilon(x))\, dx + \int_{\{|v_\varepsilon| > R_1\}} W(0)\, dx \right]$$

$$\le \lim_{\varepsilon \to 0} \int_\Omega W(v_\varepsilon(x))\, dx = 0. \tag{4.3}$$

As (u_ε) is bounded in L^∞, there is a subsequence (that we will still denote by u_ε) and a Young's probability measure μ (see TARTAR [15]) such that if f is a continuous function then

$$f(u_\varepsilon) \to \left(x \to \int_{\mathbb{R}^N} f(y)\, d\mu_x(y) \right) \text{ in } L^\infty \text{ weak } *.$$

Hence, by (H1) and (4.3) we conclude that

$$\mu_x = \theta(x)\, \delta_{y=a} + (1 - \theta(x))\, \delta_{y=b} \text{ a.e., where } 0 \le \theta \le 1.$$

Now consider the function ϕ as defined by (3.7). We obtain

$$\int_\Omega \phi(v_\varepsilon(x))\, dx \le \text{meas}(\Omega)\, \|\phi \circ u_\varepsilon\|_\infty + \int_\Omega \phi(z_\varepsilon(x))\, dx$$

which, by Lemma 3.8 (i) and (4.2), implies that

$$(\phi \circ v_\varepsilon) \text{ is bounded in } L^1(\Omega). \tag{4.4}$$

Also, by Lemma 3.8 (ii) we have

$$\int_\Omega |\nabla(\phi \circ v_\varepsilon)(x)|\, dx \le \int_\Omega \sqrt{W(v_\varepsilon(x))}\, |\nabla v_\varepsilon(x)|\, dx \le \frac{1}{2} J_\varepsilon(v_\varepsilon) \le \text{Const.} \tag{4.5}$$

From (4.4) and (4.5), together with (2.2), we conclude that (for some subsequence) there exists a function h such that

$$\phi \circ v_\varepsilon \to h \quad \text{in } L^1(\Omega).$$

Moreover, as ϕ is a Lipschitz function by (4.2) we obtain

$$\phi \circ u_\varepsilon \to h \quad \text{in } L^1(\Omega) \text{ strong}. \tag{4.6}$$

Since the Young's probability measure associated with $\phi \circ u_\varepsilon$ is given by

$$\nu_x = \theta(x)\, \delta_{y=\phi(a)} + (1 - \theta(x))\, \delta_{y=\phi(b)},$$

(4.6) yields

$$\nu_x = \delta_{y=h(x)} \text{ a.e.}$$

and so, we have that

$$\theta(x) = \chi_A(x) \text{ in } \Omega, \text{ for some } A \subset \Omega.$$

Define the function

$$u_0 := \chi_A\, a + (1 - \chi_A)\, b.$$

From the fact that

$$\mu_x = \delta_{y = u_0(x)} \text{ a.e.}$$

it follows that

$$u_\varepsilon \to u_0 \text{ in } L^p \text{ strong, for all } 1 \le p < \infty,$$

which, together with (4.2), permits us to conclude that (for some subsequence)

$$v_\varepsilon \to u_0 \text{ in } L^1(\Omega).$$

Remark 4.7. From the previous theorem and Remark 3.12 we deduce that every sequence of solutions of (P_ε) (i.e. minimizers of E_ε) admits a subsequence converging in L^1 to a solution of (P_0) with minimal interfacial area.

FINAL COMMENTS.

We remark that our hypotheses are considerably weaker then those found in the literature for the case $N = 1$. In fact, in order to establish the L^1 compactness it is often assumed that W grows quadratically at infinity (see KOHN & STERNBERG [8], MODICA [10], OWEN [12], OWEN & STERNBERG [13]), i.e. there exist $C_1, C_2, r > 0$ and $p \ge 2$ such that

$$C_1 |\, t\, |^p \le W(t) \le C_2 |\, t\, |^p \text{ for all } t > r.$$

However, we only needed W to grow at least linearly as in Theorem 4.1.

Also, usually one has $W \in C^2$ and $W''(a) > 0$, $W''(b) > 0$ (note that MODICA [10] assumes only continuity for W). For $N > 1$ KOHN & STERNBERG [8] proposed having the Hessian of W positive definite at a and b and, in this case, it is clear that (H1) and (H2) hold trivially.

Added in proof : After submission of this article we learned of an analogous result obtained by P. STERNBERG for $N = 2$. His analysis requires W to be more regular and grow at least quadratically at infinity.

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