APPLICATION OF AVERAGE EIGENVECTORS

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The paper [1] contains a numerical method using the method of successive approximations with initial guess the inner product

(1) $$\langle Ce, f\rangle$$

where

(2) $$C = A^TA - AA^T,$$

the commutator of a non-normal real matrix A and its transpose. The vectors e and f are arbitrary orthogonal unit vectors. The speed of convergence is enhanced by having (1) be as large as possible. In this note we derive an algorithm for finding the optimum pair e and f.

There is no loss of generality in assuming that f is the unit component of Ce in the orthogonal complement of e. That is, f is the unit vector corresponding to

(3) $$F = Ce - \langle Ce, e\rangle e.$$

Then, since substitution of (3) into (1) gives

(4) $$\langle Ce, f\rangle = \|Ce\|^2 - \langle Ce, e\rangle^2$$

and direct computation gives

(5) $$\|F\|^2 = \|Ce\|^2 - \langle Ce, e\rangle^2$$

the problem is equivalent to maximizing (5) subject to the constraint

(6) $$\|e\|^2 = 1.$$

We may forget about the fact that C has the special form (2) and assume that C is an arbitrary symmetric matrix with eigenvalues

(7) $$d_1, d_2, \ldots, d_n.$$

We may also assume that C is diagonal. Our problem then reduces to maximizing

(8) $$\Sigma_{j=1}^{n} d_j^2 x_j^2 - (\Sigma_{j=1}^{n} d_j x_j^2)^2$$

subject to the constraint

(9) $$\Sigma_{j=1}^{n} x_j^2 = 1.$$

The method of Lagrange multiplier then gives

(10) $$d_i^2 x_i - 2 \ (\Sigma_{j=1}^{n} d_j x_j^2) d_i x_i - \lambda x_i = 0$$

for $i = 1, 2, \ldots, n$. For each i for which $x_i \neq 0$, we then have

(11) $$d_i^2 - 2(\Sigma_{j=1}^n d_j x_j^2) d_i - \lambda = 0.$$

If x has another component $x_k \neq 0$, we have also

(12) $$d_k^2 - 2(\Sigma_{j=1}^n d_j x_i^2) d_k - \lambda = 0.$$

Substituting (12) from (11) gives

(13) $$d_i^2 - d_k^2 - 2(\Sigma_{j=1}^n d_j x_j^2)(d_i - d_k) = 0.$$

If $d_i \neq d_k$, we may deduce from (13) that

(14) $$d_i + d_k - 2\ \Sigma_{j=1}^n d_j x_j^2 = 0.$$

If there is a thrid component of x with $x_\ell \neq 0$, it follows from (14) that

(15) $$d_k = d_\ell.$$

If the eigenvalues are distinct then (14) reduces to

(16) $$d_i + d_k - 2(d_i x_i^2 + d_k x_k^2) = 0$$

so, since $x_k^2 + x_k^2 = 1$,

(17) $$(d_i - d_k)(1 - 2\ x_i^2) = 0.$$

Now, if all of the eigenvalues of C are equal, the form (5) is identically zero. Otherwise, for distinct eigenvalues d_i and d_k we obtain

(18) $$x_i = \pm \frac{1}{\sqrt{2}}, \quad x_k = \pm \frac{1}{\sqrt{2}}.$$

This leads to the extreme vectors

(19) $$e = \pm \frac{(v_i \pm v_k)}{\sqrt{2}}$$

where v_i and v_k are the eigenvectors corresponding to d_i and d_k. The extreme value of (5) corresponding to (19) is

(20) $$\frac{(d_i - d_k)^2}{4}.$$

Since (5) is the square of (1) the maximum value in the case of distinct eigenvalue is

(21) $$\frac{|d_i - d_k|}{2}.$$

The reader will verify that, in the case of multiple eigenvalues, with d_i and d_k corresponding to ν and μ non-zero components of x, the same maximum is attained. Thus the maximum is obtained by choosing d_i and d_k to be the maximum and minimum eigenvalues. It is proved in [1] that a commutator (2) can't be diagonal with equal eigenvalues unless it is zero.

References

[1] Hager, W.W. and Pederson, R.N. Norm-Bounded Tridiagonalizing Similarity Transformations for Matrices. CMU Research Report No. 88-10, April 1988.