ELEMENTARY ORDER VARIETIES

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Abstract

We show that the smallest order variety containing the two element chain which is also an elementary class is the class of bounded lattices. On the other hand, all countable lattices belonging to the smallest order variety containing the two element chain and closed under ultraproducts, are complete. We also discuss the general relationships among the operations of taking elementary substructures, retracts, products, and ultraproducts in classes of ordered sets.

Key Words: order varieties, lattices, elementary classes

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0. Introduction

The central purpose of this paper is to answer two questions posed by R.W. Quackenbush at Molokai in January 1987, and Oberwolfach in February 1988. Specifically, the questions are:

Question 1 (Molokai): *Is the smallest order variety containing the two element chain* **2**, *and closed under ultraproducts equal to the variety* $\mathcal{L}$ *of bounded lattices*?

Question 2 (Oberwolfach): *Is the smallest order variety containing the two element chain* **2**, *which is also an elementary class equal to the variety* $\mathcal{L}$?

We will also use this as an opportunity to discuss the relationships between the operators $\mathbf{R}$ (retracts), $\mathbf{P}$ (products), $\mathbf{S}_{\preccurlyeq}$ (elementary substructure), and $\mathbf{P}_{\mathbf{u}}$ (ultraproducts) in the class of ordered sets.

1. Definitions and basic results

In this section we collect together the basic definitions and results of both order theory and model theory which we require to answer the two questions. Our definitions are necessarily somewhat short and incomplete. We recommend [1], [3], and [4] as an introduction to model theory and [2] as an introduction to order varieties. First we will define our four basic operations:

Definition: Let $\mathbf{P} = (P,\leq)$ be an ordered set. Then,

i) $\mathbf{Q}$ is a ***retract*** of $\mathbf{P}$ if $Q \subseteq P$ and there is an order preserving map $f:P \to Q$ which restricts to the identity map on Q,

ii) $\mathbf{Q}$ is an ***elementary substructure*** of $\mathbf{P}$ (written $\mathbf{Q} \preceq \mathbf{P}$) if $Q \subseteq P$, and every first order sentence with parameters from Q which is true in $\mathbf{Q}$, is also true in $\mathbf{P}$.

Definition: Suppose I is a set, that $\mathbf{A}_i$ ($i \in I$) are ordered sets, and that $\mathcal{U}$ is an ultrafilter on I. Then,

i) the ***Cartesian product*** $\prod \mathbf{A}_i$ is the ordered set whose underlying set is $\prod A_i$, and which is ordered coordinatewise,

ii) the ***ultraproduct*** $(\prod \mathbf{A}_i)/\mathcal{U}$ is the ordered set whose underlying set is $\prod A_i$ modulo the equivalence relation which identifies any pair of elements whose coordinates agree on $\mathcal{U}$-many indices (that is, on a subset of I which belongs to $\mathcal{U}$), and whose order relation is also the order relation of $\prod \mathbf{A}_i$ modulo this equivalence relation.

Definition: Let $\mathcal{X}$ be a class of ordered sets. Then,

i) $\mathbf{R}\mathcal{X}$ is the class of ordered sets which are isomorphic to retracts of elements of $\mathcal{X}$,

ii) $\mathbf{P}\mathcal{X}$ is the class of ordered sets which are isomorphic to (Cartesian) products of elements of $\mathcal{X}$,

iii) $\mathbf{P_u}\mathcal{X}$ is the class of ordered sets which are isomorphic to ultraproducts of elements of $\mathcal{X}$,

iv) $\mathbf{S}_{\prec}\mathcal{X}$ is the class of ordered sets which are isomorphic to elementary substructures of elements of $\mathcal{X}$.

Definition: 1) An ***order variety*** is a class of ordered sets which is closed under both $\mathbf{R}$ and $\mathbf{P}$.

2) An ***elementary class*** (of ordered sets) is a class of ordered sets closed under both $\mathbf{P_u}$ and $\mathbf{S}_{\preccurlyeq}$.

3) An ***elementary order variety*** is a class of ordered sets which is both elementary, and an order variety.

This definition of an elementary class is somewhat disingenuous. In fact, an elementary class of structures in general is the class of structures satisfying a certain collection of first order axioms. In this setting though, it is more natural to think of an elementary class as being defined by some closure conditions, which bear at least a superficial similarity to the closure conditions used to define an order variety.

One final definition:

Definition: We say that $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$, and write $\mathfrak{A} \equiv \mathfrak{B}$, if $\mathfrak{A} \in \mathbf{S}_{\preccurlyeq}\mathbf{P_u}(\{\mathfrak{B}\})$.

We will need the following two results from the theory of order varieties and model theory respectively:

Theorem ([5]): *Every countable lattice is in the order variety generated by* $\mathbb{Q}$.

The Löwenheim-Skolem Theorem (see [4]): *If* $\mathfrak{A}$ *is an infinite structure then for every infinite cardinal* κ*, there exists a structure* $\mathfrak{B}$*, with* $\mathfrak{A} \equiv \mathfrak{B}$*, and* $|\mathfrak{B}| = \kappa$.

2. Order varieties containing 2

We will now answer the two questions posed in the introduction. To answer the first question (in the negative) we will define a property (*) of ordered sets which is preserved by **R**, **P**, and $\mathbf{P_u}$ such that any countable lattice satisfying (*) is complete.

Definition: We say that an ordered set **P** has property (*) if for any subset A of P, of order type $\omega \oplus \omega^d$, there exists an element d of P such that $A \cup \{d\}$ has order type $\omega \oplus 1 \oplus \omega^d$. Such an element d will be called an interpolating element (for A).

Lemma 1: *Property* (*) *is preserved by* **R**, **P**, *and* $\mathbf{P_u}$.

Proof: a) This is obvious for retracts since the only retract of $\omega \oplus 1 \oplus \omega^d$ which contains $\omega \oplus \omega^d$ is $\omega \oplus 1 \oplus \omega^d$.

b) Suppose that $\{\mathbf{P}_i : i \in I\}$ is a collection of ordered sets each satisfying (*), and that $A \subseteq \prod \mathbf{P}_i$ is of order type $\omega \oplus \omega^d$. For each i in I, the projection $\pi_i(A)$ is of one of the following five types: i) $\mathbf{n} \oplus \mathbf{m}$ (**n** denoting the chain with n elements), ii) $\mathbf{n} \oplus \omega^d$, iii) $\omega \oplus \mathbf{m}$, iv) $\omega \oplus \omega^d$, or v) **1**. Here the first term in each sum represents the image of the ω component of A, so in particular n>0, and in iii) m>0. Now choose an element d in $\prod \mathbf{P}_i$ such that $\pi_i(d) = \max(\mathbf{n})$ in cases i) and ii), $\pi_i(d) = \min(\mathbf{m})$ in case iii), $\pi_i(d)$ is an interpolating element for $\pi_i(A)$ in case iv), and in the only way possible in case v). Such an element d is an interpolating element for A in $\prod \mathbf{P}_i$ so **P** preserves (*) as required.

c) Suppose that $\{P_i : i \in I\}$ is a collection of ordered sets each satisfying (*) that $\mathcal{U}$ is an ultrafilter on a set I, and that $A \subseteq \prod P_i/\mathcal{U}$ is order isomorphic to $\omega \oplus \omega^d$. Write $A = \{a_j : j \in \omega\} \cup \{b_k : k \in \omega\}$ with $a_0 < a_1 < \dots < a_n < \dots < b_m < \dots < b_0 < b_1$ where the a_j and b_k are chosen representatives of their equivalence classes in $\prod P_i/\mathcal{U}$. Put

$$A_n = \{ i \in I : (b_0)_i > (b_1)_i > \dots > (b_n)_i > (a_n)_i > \dots > (a_0)_i \}.$$

Then $A_n \in \mathcal{U}$ for all $n \in \omega$. The remainder of the proof breaks down into two cases.

Case 1: $\cap\{A_n : n \in \omega\} \in \mathcal{U}$. In this case $\mathcal{U}$-many of the representatives actually form chains isomorphic to $\omega \oplus \omega^d$. We can find interpolating elements on these coordinates, and choose arbitrary elements on all other coordinates to produce an interpolating element for A.

Case 2: $\cap\{A_n : n \in \omega\} \notin \mathcal{U}$. In this case assume, without loss of generality that $\cap\{A_n : n \in \omega\} = \varnothing$. Define d by setting $(d)_i = (b_j)_i$ for $i \in A_j \setminus A_{j+1}$ and defining d arbitrarily on other coordinates. Then for all $i \in A_n$, $(d)_i \leq (b_n)_i$ so $d \leq b_n$. Similarly $a_m \leq d$ as required. □

Let $\mathcal{C}$ be the smallest order variety which contains the two element chain and is closed under ultraproducts. Then since **2** satisfies (*), so does each ordered set in $\mathcal{C}$. Moreover it is also clear that $\mathcal{C} \subseteq \mathcal{L}$, where $\mathcal{L}$ is the variety of bounded lattices. However, we will now show that the inclusion is proper.

Theorem 2: *Any countable ordered set in $\mathfrak{C}$ is a complete lattice. In particular, $\mathfrak{C} \neq \mathfrak{L}$.*

Proof: Let **P** be countable, and $S \subseteq P$. If S is finite or empty then supS exists since **P** is a bounded lattice. Suppose $S = \{s_n : n \in \omega\}$ is infinite, and let T be the set of upper bounds for S. Again, if T is finite or empty then supS exists so suppose that $T = \{t_n : n \in \omega\}$. For $j \in \omega$ let:

$$a_j = \vee\{s_n : 0 \leq n \leq j\}, \quad b_j = \wedge\{t_n : 0 \leq n \leq j\}.$$

If $\{a_j : j \in \omega\}$ or $\{b_j : j \in \omega\}$ is finite then supS exists. Otherwise,

$$A = \{a_j : j \in \omega\} \cup \{b_j : j \in \omega\} \cong \omega \oplus \omega^d.$$

Since **P** satisfies (*) an interpolating element *d* for A exists. But such a *d* is an upper bound for S, hence an element of T. This contradiction completes the proof that **P** is a complete lattice. □

Now we shall see that if we are also allowed to take elementary substructures then we do in fact obtain the entire order variety $\mathfrak{L}$.

Theorem 3: *The smallest elementary order variety containing* **2** *is $\mathfrak{L}$. In fact, $\mathfrak{L}$ is the smallest order variety containing* **2** *and closed under elementary equivalence.*

Proof: Let **I** denote the unit interval $[0, 1] \cap \mathbb{R}$, and let $\mathbf{Q} = \mathbf{I} \cap \mathbb{Q}$. Then $\mathbf{I} \in \mathbf{RP}(2)$, since **I** is a complete lattice. Hence $\mathbf{Q} \in \mathbf{S}_{\prec}\mathbf{RP}(2)$ since the theory of dense linear orders is model complete. But any bounded countable chain is a retract of **Q**, and so by the result of [5] quoted above, any countable bounded lattice is in $\mathbf{RPS}_{\prec}\mathbf{RP}(2)$. Finally, by the upward Löwenheim-Skolem Theorem (see [4]), any infinite bounded lattice is elementarily equivalent to a countable one and so we obtain:

$$\mathfrak{L} = \mathbf{S}_{\prec}\mathbf{P}_u\mathbf{RPS}_{\prec}\,\mathbf{RP}(2).$$ □

3. Relationships among the operators

In this section we wish to consider the relationships between the operators $\mathbf{R}$, $\mathbf{P}$, $\mathbf{S}_{\prec}$, and $\mathbf{P}_{u}$ as applied to classes of ordered sets.

Remark : $\mathcal{L} = \mathbf{S}_{\prec}\mathbf{P}_{u}\mathbf{RPS}_{\prec}\mathbf{RP}(2)$, *but* $\mathcal{L}$ *is not the smallest order variety which contains* **2** *and is closed under* $\mathbf{P}_{u}$, *and* $\mathcal{L} \neq \mathbf{S}_{\prec}\mathbf{P}_{u}\mathbf{RP}(2)$.

The first two parts of this remark are simply Theorems 2 and 3 above. The final part says that there exist bounded lattices which do not satisfy the first order theory of complete lattices. This is true because in any ordered set satisfying the first order theory of complete lattices, all definable subsets must have both a supremum and infimum, and in the bounded lattice $\omega^2 \oplus \omega^d$, the collection of left-dense elements (which is first order definable) has no supremum.

Corollary 4: $\mathbf{RPS}_{\prec} \not\subset \mathbf{S}_{\prec}\mathbf{RP}$, *and* $\mathbf{S}_{\prec}\mathbf{RP} \not\subset \mathbf{RPS}_{\prec}$.

Proof: If the first containment were valid, we would get $\mathcal{L} = \mathbf{S}_{\prec}\mathbf{P}_{u}\mathbf{S}_{\prec}\mathbf{RP}(2) = \mathbf{S}_{\prec}\mathbf{P}_{u}\mathbf{RP}(2)$, which by the remark above is not correct, while if the second were valid we would get $\mathcal{L} = \mathbf{S}_{\prec}\mathbf{P}_{u}\mathbf{RPS}_{\prec}(2) = \mathbf{S}_{\prec}\mathbf{P}_{u}\mathbf{RP}(2)$, since the only elementary substructure of **2** is **2**, and this is also not correct. □

Note that the general effect of the operators $\mathbf{S}_{\prec}$ and $\mathbf{R}$ is to make ordered sets "smaller" while $\mathbf{P}_{u}$ and $\mathbf{P}$ generally make things "larger". The following theorem show that in

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Remark : $\mathcal{L} = \mathbf{S}_{\prec}\mathbf{P}_u\mathbf{R}\mathbf{P}\mathbf{S}_{\prec}\mathbf{R}\mathbf{P}(\mathbf{2})$, *but* $\mathcal{L}$ *is not the smallest order variety which contains* **2** *and is closed under* $\mathbf{P}_u$, *and* $\mathcal{L} \neq \mathbf{S}_{\prec}\mathbf{P}_u\mathbf{R}\mathbf{P}(\mathbf{2})$.

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Corollary 4: $\mathbf{R}\mathbf{P}\mathbf{S}_{\prec} \not\subseteq \mathbf{S}_{\prec}\mathbf{R}\mathbf{P}$, *and* $\mathbf{S}_{\prec}\mathbf{R}\mathbf{P} \not\subseteq \mathbf{R}\mathbf{P}\mathbf{S}_{\prec}$.

Proof: If the first containment were valid, we would get $\mathcal{L} = \mathbf{S}_{\prec}\mathbf{P}_u\mathbf{S}_{\prec}\mathbf{R}\mathbf{P}(\mathbf{2}) = \mathbf{S}_{\prec}\mathbf{P}_u\mathbf{R}\mathbf{P}(\mathbf{2})$, which by the remark above is not correct, while if the second were valid we would get $\mathcal{L} = \mathbf{S}_{\prec}\mathbf{P}_u\mathbf{R}\mathbf{P}\mathbf{S}_{\prec}(\mathbf{2}) = \mathbf{S}_{\prec}\mathbf{P}_u\mathbf{R}\mathbf{P}(\mathbf{2})$, since the only elementary substructure of **2** is **2**, and this is also not correct. □

Note that the general effect of the operators $\mathbf{S}_{\prec}$ and **R** is to make ordered sets "smaller" while $\mathbf{P}_u$ and **P** generally make things "larger". The following theorem show that in constructing elementary order varieties we may first make things larger and then smaller, but that neither the process of making things larger, nor that of making things smaller is so nicely balanced.

Theorem 5: *The only relationships* $\mathbf{AB \subseteq BA}$ *which hold for* $\mathbf{A, B} \in \{\mathbf{P_u}, \mathbf{S_{\preccurlyeq}}, \mathbf{R}, \mathbf{P}\}$, *are* $\mathbf{P_u S_{\preccurlyeq} \subseteq S_{\preccurlyeq} P_u}$, $\mathbf{PS_{\preccurlyeq} \subseteq S_{\preccurlyeq} P}$, $\mathbf{P_u R \subseteq RP_u}$, *and* $\mathbf{PR \subseteq RP}$.

Proof: The four stated containments are quite well-known, and the first and last have been discussed above. For the rest, the proof consists of a sequence of examples violating or substantiating each of the possible containments in turn.

i) $\mathbf{2^{\omega}}$ has a countable elementary substructure, while the only elementary substructure of $\mathbf{2}$ is $\mathbf{2}$. Therefore, $\mathbf{S_{\preccurlyeq}P \not\subset PS_{\preccurlyeq}}$.

ii) If I is an index set and $\mathbf{A_i}$ and $\mathbf{B_i}$ are ordered sets with $\mathbf{A_i} \preccurlyeq \mathbf{B_i}$ for all $i \in I$, then the natural embedding of $\prod \mathbf{A_i}$ into $\prod \mathbf{B_i}$ is also elementary. Therefore, $\mathbf{PS_{\preccurlyeq} \subseteq S_{\preccurlyeq}P}$.

iii) $\omega^2 \oplus \omega^d \in \mathbf{RS_{\preccurlyeq}(I)}$, but is not in $\mathbf{S_{\preccurlyeq}R(I)}$ since it is not elementarily equivalent to a complete lattice. Therefore, $\mathbf{RS_{\preccurlyeq} \not\subset S_{\preccurlyeq}R}$.

iv) $\omega_1 \in \mathbf{S_{\preccurlyeq}R}(\omega_1 \oplus (2 \times \omega))$ where 2 is the two element antichain, and ω_1 is the first uncountable ordinal, since $\omega_1 \oplus \omega \in \mathbf{R}(\omega_1 \oplus (2 \times \omega))$ and $\omega_1 \preccurlyeq \omega_1 \oplus \omega$. However, any uncountable elementary substructure of $\omega_1 \oplus (2 \times \omega)$ is isomorphic to $\omega_1 \oplus (2 \times \omega)$ since: a) it must contain a pair of incomparables, and b) every element must have a unique successor. Finally, $\omega_1 \notin \mathbf{R}(\omega_1 \oplus (2 \times \omega))$ since its cofinality is too large. Therefore, $\mathbf{S_{\preccurlyeq}R \not\subset RS_{\preccurlyeq}}$.

v) A non-principal ultrapower of $\mathbf{2^{\omega}}$ over a countable index set has $\mathbf{2^{\omega}}$ elements and also has $\mathbf{2^{\omega}}$ atoms, hence is not isomorphic to $\mathbf{2^{\omega}}$. On the other hand, $\mathbf{2}$ has no non-trivial ultrapowers. Therefore, $\mathbf{P_uP \not\subset PP_u}$.

vi) $\mathbf{PP_u}\{\mathbf{n} : n \in \omega\}$ contains an element that has countably many atoms but is not a complete lattice. On the other hand, any ultraproduct of products of chains has either finitely many, or uncountably many atoms. Therefore, $\mathbf{PP_u \not\subset P_uP}$.

vii) $\mathbf{Q}$ belongs to $\mathbf{RP_u}(\omega\oplus 1)$, but is not in $\mathbf{P_uR}(\omega\oplus 1)$ since any element of an ordered set in this class (except the top) has a unique successor. Therefore, $\mathbf{RP_u} \not\subset \mathbf{P_uR}$.

viii) If I is an index set and $\mathbf{A_i}$ and $\mathbf{B_i}$ are ordered sets with $\mathbf{A_i} \in \mathbf{R}(\mathbf{B_i})$ then for any ultrafilter $\mathcal{U}$ on I the ultraproduct of the $\mathbf{A_i}$ modulo $\mathcal{U}$ is a retract of the ultraproduct of the $\mathbf{B_i}$ modulo $\mathcal{U}$ via the obvious map.

Therefore, $\mathbf{P_uR} \subseteq \mathbf{RP_u}$. □

The next proposition and its corollary show that if we begin with an elementary class of orderings, then the retracts of the countable chains in the class already encompass all countable chains (with suitable bounds).

Proposition 6: *If* $\mathcal{E}$ *is an elementary class of bounded linear orderings, containing an infinite chain, then* $\mathbf{1\oplus\eta\oplus 1}$ *is in* $\mathbf{R}\{\mathbf{C} \in \mathcal{E}\text{: }\mathbf{C}\text{ is countable}\}$.

Proof: If $\mathcal{E}$ contains a countable model in which all discrete intervals are finite then $\mathbf{1\oplus\eta\oplus 1}$ is a retract of this model. Otherwise, there is certainly a model $\mathbf{C} \in \mathcal{E}$ containing an interval isomorphic to either ω or ω^d. Then some non-principal ultrapower of $\mathbf{C}$ contains an interval isomorphic to the sum of copies of ζ (the order type of the integers) over a dense chain. This model contains a countable elementary substructure (which must be in $\mathcal{E}$) also having such an interval. By collapsing each copy of ζ to a point, and sending all other elements to 0 or 1 as appropriate, we obtain the desired retraction. □

Corollary 7: *If* $\mathcal{E}$ *is an elementary class of bounded orders which contains an infinite chain, then* $\mathbf{R}\{\mathbf{C} \in \mathcal{E}\text{: }\mathbf{C}\text{ is countable}\}$ *contains all countable bounded chains.*

In fact it is possible to show by a similar, but slightly more delicate argument that the bounds above may be removed unless all the chains in $\mathcal{E}$ have such bounds.

In constructing elementary order varieties it would probably be most appropriate to work with the operators $\mathbf{E} = \mathbf{S}_{\leq}\mathbf{P}_{u}$ (elementary closure) and $\mathbf{V} = \mathbf{RP}$ (formation of order varieties). Each of these operators is idempotent. We have seen (Corollary 4) that $\mathbf{VEV} \neq \mathbf{EVE}$. Also in view of Theorem 5, relationships between the operators $\mathbf{P}$ and $\mathbf{P}_{u}$; and $\mathbf{S}_{\leq}$ and $\mathbf{R}$ could be studied. This suggests the question: are the three semigroups of class operators generated by $\{\mathbf{V},\mathbf{E}\}$, $\{\mathbf{P},\mathbf{P}_{u}\}$, and $\{\mathbf{S}_{\leq},\mathbf{R}\}$ respectively all infinite?

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References

[1] J.L. Bell and A. B. Slomson, ***Models and Ultraproducts***, North Holland, 1969.

[2] D. Duffus and I. Rival, *A structure theory for ordered sets*, Discrete Math., **35** (1979) 53-118

[3] Paul C. Eklof, *Ultraproducts for algebraists*, in, ***A Handbook of Mathematical Logic***, ed. J.Barwise, North Holland, 1977.

[4] H. Jerome Keisler, *Fundamentals of model theory*, in, ***A Handbook of Mathematical Logic***, ed. J.Barwise, North Holland, 1977.

[5] M. Pouzet and I. Rival, *Every countable lattice is a retract of a direct product of chains*, Algebra Universalis **18** (1984), 295-307.