EXISTENCE AND UNIQUENESS RESULTS FOR SEMI-LINEAR DIRICHLET PROBLEMS IN ANNULI

by

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Research Report No. 88-14 2

April 1988

$$(1.5) \qquad h(t,v): = (N-2)^{-2}t^{-k}f(t^{1/(2-N)}, v).$$

with $k = (2N-2)/(N-2)$.

The existence of positive radial solutions of (1.1) has been studied in [BCM] and [BP] for functions f which are indepndent of r and satisfy the following conditions,

$$(1.6) \qquad \lim_{u\to\infty} f(u)/u = \infty, \quad \lim_{u\to 0} f(u)/u = 0.$$

In both works, additional assumptions were used in establishing the existence result. (In [BCM] it was assumed that f is monotone increasing. This was replaced by a somewhat weaker assumption in [BP].) In the present note we shall show that conditions (1.6) are sufficient to guarantee the existence of a p.r. solution of (1.1). For a more precise statement of the result we must introduce some additional notation. Let $t_1 = r_0^{2-N}$ and $t_0 = r_1^{2-N}$ and set,

$$(1.7) \qquad \bar{h}(s) = \max_{t_0\le t\le t_1} h(t,s) \ , \ \underline{h}(s) = \min_{t_0\le t\le t_1} h(t,s) \ .$$

Denote by Λ_1 the first eigenvalue of the problem,

$$(1.8) \qquad \begin{cases} \varphi'' + \lambda\varphi = 0 & \text{in} \quad (t_0, t_1) \\ \varphi(t_0) = \varphi(t_1) \end{cases} .$$

Assume that

$$(1.9) \qquad \limsup_{s\to 0} \bar{h}(s)/s < \Lambda_1 ,$$

$$(1.10) \qquad \liminf_{s\to\infty} \underline{h}(s)/s > \Lambda_1 .$$

Then the following result holds.

Th. I Assume (1.2), (1.9), (1.10). Then problem (1.1) possesses a positive radial solution in $\Omega(r_0,r_1)$.

If f is independent of r, the result remains valid even if the continuity assumption on f is replaced by the assumption that f is a locally bounded Borel function.

In general the solution is not unique. However uniqueness holds in the following special case.

Suppose that

$$(1.11) \qquad f(r,u) = r^{-2-\rho} f_0(r^{\rho}u) \qquad \text{for some real } \rho$$

and that f_0 satisfies the following conditions:

$$(1.12)_1 \quad \begin{cases} f_0 \in C(\mathbb{R}) , \quad f_0(0) = 0 , \quad f_0(s) > 0 \quad \text{for} \quad s > 0 \\ \qquad f_0 \ \text{ is locally Lipschitz and } \ f_0 \ \text{ is odd,} \end{cases}$$

$(1.12)_2$ $\quad f_0(s)/s \to 0 \quad$ as $\quad s \to 0+$,

$(1.12)_3$ $\quad f_0(s)/s \to \infty \quad$ as $\quad s \to \infty$,

$(1.12)_4$ $\quad$ the function $s \to f_0(s)/s$ is strictly monotone increasing.

Th. II Assume (1.11) and (1.12). Then problem (1.1) possesses a unique positive radial solution. Furthermore given an integer $k \geq 1$ there exists exactly one solution v of (1.4) in $[t_0, t_1]$ such that $v(t_0) = v(t_1) = 0$, $v'(t_0) > 0$ and v vanishes exactly $k-1$ times in (t_0, t_1).

Given $p > 1$ and an arbitrary real number ν, consider the case $f_0(s) = s^p$ and $\rho = (\nu + 2)/(p - 1)$. In this case the function f in (1.11) is given by $f(r,u) = r^{\nu}u^p$ $(0 < r$ and $0 \leq u)$ so that the equation (1.1) is the Emden-Fowler equation. Clearly f satisfies assumptions (1.12). In this special case Th. II was established in [C] and [Ni - N]. The method presented here is different from the methods used in those papers.

§2. A-priori estimates.

Let $\mathcal{H}$ denote the family of functions h such that,

$$(2.1)\quad \begin{cases} h \in C([t_0,\infty) \times \mathbb{R}),\ h(t,0) = 0,\ h(t,s) > 0 \quad \forall\, t \geq t_0,\ \forall\, s > 0; \\ h(t,s) = h(t_1,s) \qquad\qquad \forall\, t > t_1\ ,\ \ \forall\, s \in \mathbb{R}. \end{cases}$$

For $h \in \mathcal{H}$ we define the functions $\bar{h}$ and $\underline{h}$ as in (1.7)

2.1 Lemma Let $h \in \mathcal{H}$ and assume (1.9). Then there exists a positive number $\bar{A} = \bar{A}(h;\ t_1)$ such that the following statement holds.

If v is a positive solution of (1.4) in an interval $(t_0,\bar{t})$ vanishing at the end points then,

$$(2.2)\qquad \bar{t} < t_1 \ \Rightarrow\ \dot{v}(t_0) \geq \bar{A}.$$

Proof Consider the problem

$$(2.3)\qquad \begin{cases} \ddot{\varphi} + \lambda\varphi = 0 \quad \text{in} \quad (t_0,\bar{t})\ , \\ \varphi(t_0) = \varphi(\bar{t}) = 0. \end{cases}$$

Denote its first eigenvalue by $\lambda_1 = \lambda_1(\bar{t})$ and let φ be a corresponding positive eigenfunction. Further set

$$(2.4)\qquad \bar{g}(s) := \bar{h}(s)/s, \qquad \forall\, s \in \mathbb{R}.$$

Then

$$0 = \int_{t_0}^{\bar{t}} \varphi_1(\ddot{v} + h(t,v))\ dt = \int_{t_0}^{\bar{t}} (\ddot{\varphi}_1 v + \varphi_1 h(t,v))dt$$

$$= \int_{t_0}^{\bar{t}} \varphi_1(-\ \lambda_1 v + h(t,v))dt \le \int_{t_0}^{\bar{t}} \varphi_1 v(-\ \lambda_1 + \bar{g}(v))dt.$$

Hence,

(2.5) $$\sup_{t_0 \le t \le \bar{t}} \bar{g}(v(t)) \ge \lambda_1.$$

Denote,

(2.6) $$\bar{\gamma}(\sigma) := \sup_{0 < s < \sigma} \bar{g}(s).$$

Since v is concave,

(2.7) $$v(t) \le \dot{v}(t_0)(t - t_0)\ ,\quad \forall\ t \in [t_0,\bar{t}].$$

Since $\bar{\gamma}$ is monotone increasing, (2.5) and (2.7) imply,

(2.8) $$\bar{\gamma}(\dot{v}(t_0)(\bar{t} - t_0)) \ge \lambda_1.$$

By (1.9), $\bar{\gamma}(\sigma) \searrow \bar{\gamma}_0 < \Lambda_1$ as $\sigma \searrow 0$. Choose $\tau > 0$ such that $\bar{\gamma}(\tau) < \Lambda_1$. If $\bar{t} \le t_1$, then $\lambda_1 \ge \Lambda_1$. Hence by (2.8),

$$\dot{v}(t_0)(\bar{t} - t_0) > \tau. \tag{2.9}$$

Thus (2.2) holds with $\bar{A} = \tau/(t_1 - t_0)$.

2.2 Remark Let X be a subset of $\mathscr{X}$ such that (1.9) is satisfied uniformly for h in X. More precisely, there exist positive constants $\bar{c}, \bar{\tau}$ such that $\bar{c} < \Lambda_1$ and

$$\sup_{\substack{0<s<\bar{\tau} \\ t_0 \le t \le t}} h(t,s) \le \bar{c}s \quad , \quad \forall\, h \in X. \tag{2.10}$$

Then, for every $h \in X$, the lemma holds with $\bar{A} = \bar{\tau}/(t_1 - t_0)$.

2.3 Lemma Let $c_0 \in (0,1/2)$ and denote

$$\alpha := c_0 t_1 + (1 - c_0)t_0 \ , \quad \beta := c_0 t_0 + (1 - c_0)t_1 .$$

If $v \in C[t_0,t_1]$ and v is concave and positive in (t_0,t_1) then

$$v(t) \ge c_0 \sup_{(t_0,t_1)} v \quad , \ \forall\, t \in (\alpha,\beta). \tag{2.11}$$

Proof Let $M = \sup_{(t_0,t_1)} v$. It is easily seen that the triangle whose vertices are $(t_0,0)$, $(t_1,0)$, $((t_0 + t_1)/2, M/2)$ in the (t,v) plane lies under the graph of the function v. This implies the assertion of the lemma.

2.4 Theorem Let $h \in \mathcal{H}$ and assume (1.10). Let S be the set of positive solutions of (1.4) in (t_0, t_1). Then there exists a positive number $c_1 = c_1(h)$ such that

$$\sup_{(t_0,t_1)} v \leq c_1, \quad \forall\, v \in S. \tag{2.12}$$

Proof Let $[\alpha,\beta]$ be a closed interval contained in (t_0, t_1). Denote $g(s) := \underline{h}(s)/s$, let $v \in S$ and consider the eigenvalue problem,

$$\begin{cases} \ddot{w} + \lambda g(v) w = 0 & \text{in } (\alpha,\beta), \\ w(\alpha) = w(\beta) = 0. \end{cases} \tag{2.13}$$

Let λ_1 be the first eigenvalue and w_1 a corresponding positive eigenfunction of this problem. Then, multiplying $(2.13)_1$ by v and integrating by parts one obtains,

$$-\int_\alpha^\beta \ddot{v} w_1 dt + v\dot{w}_1\Big]_\alpha^\beta + \lambda_1 \int_\alpha^\beta g(v) v w_1 dt = 0. \tag{2.14}$$

Similarly, multiplying (1.4) by w_1 integrating by parts and using (1.7)

$$-\int_\alpha^\beta \ddot{v} w_1 dt + \int_\alpha^\beta g(v) v w_1 dt \leq 0. \tag{2.15}$$

Subtracting (2.15) from (2.14),

$$(\lambda_1 - 1) \int_\alpha^\beta g(v) v w_1 dt + v\dot{w}_1]_\alpha^\beta \geq 0.$$

Since $v(\beta)\dot{w}_1(\beta) - v(\alpha)\dot{w}_1(\alpha) < 0$ it follows that

$$\lambda_1 > 1. \tag{2.16}$$

Denote,

$$\underline{\gamma}(s) := \inf_{(s,\infty)} g, \qquad s > 0. \tag{2.17}$$

Thus $\underline{\gamma}$ is monotone increasing and $\underline{\gamma} \leq g$ in $(0,\infty)$. Let α, β, c_0 be as in Lemma 2.3 and let λ_1, w_1 be as in the first part of this proof. Then, by Lemma 2.3,

$$g(v(t)) \geq \underline{\gamma}(v(t)) \geq \underline{\gamma}(c_0 \sup_{(t_0,t_1)} v) =: \delta_v. \tag{2.18}$$

Next let $\mu_1(\alpha,\beta)$ be the first eigenvalue of the problem

$$\begin{cases} \ddot{\varphi} + \mu\varphi = 0 & \text{in} \quad (\alpha,\beta), \\ \varphi(\alpha) = \varphi(\beta) = 0. \end{cases} \tag{2.19}$$

Since, by (2.13) and (2.18),

$$\begin{aligned} &\ddot{w}_1 + \lambda_1 \delta_v w \leq 0 \quad \text{in} \quad (\alpha,\beta), \\ &w_1(\alpha) = w_1(\beta) = 0, \end{aligned}$$

it follows that $\mu_1(\alpha,\beta) \geq \lambda_1\delta_v$. In view of (2.16) we conclude that,

$$\mu_1(\alpha,\beta) > \underline{\gamma}(c_0 \sup_{(t_0,t_1)} v). \tag{2.20}$$

Now, by (1.10), $\underline{\gamma}(s) \nearrow \gamma_\infty > \Lambda_1$, as $s \nearrow \infty$. Let $\underline{\tau}$, $\underline{c}$ be two positive constants such that

$$\underline{\gamma}(s) \geq \underline{c} > \Lambda_1, \quad \text{for} \quad \forall\, s \geq \underline{\tau}. \tag{2.21}$$

Choose c_0 sufficiently small so that $\Lambda_1 < \mu_1(\alpha,\beta) < \underline{c}$. (This is possible by Lemma 2.3.) Then by (2.20) and (2.21)

$$c_0 \sup_{(t_0,t_1)} v \leq \underline{\tau}\ , \quad \forall\, v \in S. \tag{2.22}$$

<u>2.5</u> <u>Remark</u> Let X be a subset of $\mathscr{H}$ such that (1.10) is satisfied uniformly for h in X. More precisely, there exists positive constants $\underline{\tau},\underline{c}$ with $\underline{c} > \Lambda_1$ such that

$$\inf_{\substack{\underline{\tau}\leq s \\ t_0\leq t\leq t_1}} h(t,s) \geq \underline{c}\, s\,, \quad \forall\, h \in X. \tag{2.23}$$

Then the constant c_1 in the theorem can be chosen uniformly with respect to h in X.

<u>2.6</u> <u>Lemma</u> Let $h \in \mathscr{H}$ and assume (1.10). Then there exists a positive number $\underline{A} = \underline{A}(h;\ t_1)$ such that the following assertion holds.

If v is a positive solution of (1.4) in (t_0, t_1) such that $v(t_0) = 0$ then $\dot{v}(t_0) \leq \underline{A}$.

<u>Proof</u> Let c_1 be as in Theorem 2.4 and denote

$$c_2 := \sup\{h(t,v) : t_0 \leq t \leq t_1,\ 0 \leq v \leq c_1\}. \tag{2.24}$$

Observe that if v is a positive solution of (1.4) in (t_0, t_1) then, by Theorem 2.4,

$$\begin{aligned} \dot{v}(t_0) &= \dot{v}(t) + \int_{t_0}^{t} h(\tau, v(\tau)d\tau \\ &\leq \dot{v}(t) + c_2(t - t_0) \qquad \forall\ t \in [t_0, t_1]. \end{aligned} \tag{2.25}$$

Now either (i) $\dot{v}$ vanishes at some point in $(t_0, t_1]$ or (ii) $\dot{v} > 0$ everywhere in $[t_0, t_1]$. In the first case we have (by (2.25))

$$\dot{v}(t_0) \leq c_2(t_1 - t_0). \tag{2.26}$$

In the second case we note that since v is concave

$$0 < \dot{v}(t_1)(t - t_0) \leq v(t) \quad \forall\ t \in [t_0, t_1]. \tag{2.27}$$

Hence, with $\underline{\gamma}$ as in (2.17),

$$0 \geq \ddot{v}(t) + v\,\underline{\gamma}(v) \geq \ddot{v}(t) + v\,\underline{\gamma}(\dot{v}(t_1)(t - t_0)), \quad \forall\ t \in (t_0, t_1). \tag{2.28}$$

Let $t' \in (t_0, t_1)$ and let $\lambda_1(t')$ be the first eigenvalue of the problem,

$$\begin{cases} \varphi'' + \lambda\varphi = 0 & \text{in } (t', t_1), \\ \varphi(t') = \varphi(t_1) = 0. \end{cases}$$

By (2.28),

$$\lambda_1(t') \geq \underline{\tau}(\dot{v}(t_1)(t' - t_0)). \tag{2.29}$$

Let $\underline{c}$, $\underline{\tau}$ be as in the proof of Theorem 2.4 and choose t' sufficiently near to t_0 so that $\Lambda_1 < \lambda_1(t') < \underline{c}$. Then (2.29) and (2.21) imply that

$$\dot{v}(t_1)(t' - t_0) \leq \underline{\tau}.$$

Hence, by (2.25),

$$\dot{v}(t_0) \leq \underline{\tau}/(t' - t_0) + c_2(t_1 - t_0). \tag{2.30}$$

Finally, (2.26) (in case (i)) and (2.30) (in case (ii)) imply the assertion of the lemma with $\underline{A}$ given by the right hand side of (2.30).

<u>2.7 Remark</u> Let X be a subset of $\mathcal{H}$ which satisfies (2.23) and which is uniformly bounded on compact sets. Then the constant $\underline{A}$ in the previous lemma can be chosen uniformly for $h \in X$.

§3. Existence results

The assertion of Theorem I is equivalent to the statement that there exists a positive solution of (1.4) in (t_0,t_1), which vanishes at the end points. This will be proved in two steps.

3.1 Step 1 In addition to the assumptions of the Theorem (i.e. $h \in \mathcal{H}$ and h satisfies (1.9) and (1.10)) we shall assume that

$$h(t,\cdot) \in C^1(\mathbb{R}) \qquad \forall\ t \in [t_0,t_1]. \tag{3.1}$$

Consider the following problem,

$$\begin{cases} \ddot{v} + h(t,v) = 0 & , \\ v(t_0) = 0\ , & \dot{v}(t_0) = a. \end{cases} \tag{3.2}$$

Denote its solution by $v(\cdot,a)$. For $a > 0$ the solution is positive to the right of t_0. Denote,

$$t_1(a) := \sup\{t : v(\cdot,a) > 0 \ \text{ in } (t_0,t)\}.$$

If $t_1(a) < \infty$ then $v(t_1(a),a) = 0$. Let,

$$\mathfrak{D} := \{a > 0 : t_1(a) < \infty\}.$$

In view of (3.1), $\mathfrak{D}$ is an open set and $t_1(\cdot) \in C(\mathfrak{D})$.

By Lemma 2.6,

(3.3) $$a > \underline{A} \Rightarrow t_1(a) < t_1.$$

Thus $\mathfrak{D}$ contains a half line. Let (a_0,∞) be the maximal half line contained in $\mathfrak{D}$. If $a_0 = 0$ then, by Lemma 2.1, there exists $a' \in (a_0,\infty)$ such that $t_1(a') > t_1$. (In fact this holds for every a' in $(0,\bar{A})$.) If $a_0 > 0$ then, by the proof of Lemma 2.1 (see (2.9)),

$$t_1(a) - t_0 > \tau/a_0 \qquad \forall\, a > a_0$$

for some $\tau > 0$. Therefore $\lim_{a\to a_0+} t_1(a) = \infty$.

(If for some sequence $a_n \to a_0+$, $\{t_1(a_n)\}$ converges to a number larger than t_0 then $a_0 \in \mathfrak{D}$ contrary to assumption.) Consequently, in this case too, there exists $a' \in (a_0,\infty)$ such that $t_1(a') > t_1$. Since $t(\cdot)$ is continuous in (a_0,∞) this fact and (3.3) imply that there exists $a \in (a_0,\infty)$ such that $t_1(a) = t_1$.

<u>3.2</u> <u>Step 2</u> Let $h \in \mathfrak{X}$ satisfy (1.9), (1.10). (We do not assume (3.1).) Since we are interested in positive solutions, there is no loss of generality in assuming that $h(t,\cdot)$ is odd. Let J_ϵ denote the ϵ-mollifier on $\mathbb{R}$ and denote $h_n(t,\cdot) = J_{1/n}h(t,\cdot)$. Then, $h_n \in \mathfrak{X}$ and it is easily seen that the set $X = \{h_n\}_1^\infty$ satisfies the assumptions of remarks <u>2.2</u>, <u>2.5</u> and <u>2.7</u>. By the first part of the proof, for each $n \in \mathbb{N}$, there exists a positive solution, say v_n, of the problem

$$(3.4) \qquad \begin{cases} \ddot{v} + h_n(t_1 v) = 0 \, , & t_0 < t < t_1 \\ v(t_0) = v(t_1) = 0. & \end{cases}$$

By Theorem 2.4 and remark 2.5, the sequence $\{v_n\}$ is uniformly bounded in $[t_0, t_1]$. By Lemma 2.6 and remark 2.7 $\{\dot{v}_n(t_0)\}$ is bounded. Consequently, by (3.4), $\{\ddot{v}_n\}$ and $\{\dot{v}_n\}$ are uniformly bounded. By the Theorem of Arzela-Ascoli, there exists a subsequence $\{v_{n_k}\}$ which converges in $C^1[t_0, t_1]$ to a function v. By (3.4),

$$(3.5) \qquad \dot{v}_n(t_0) = \dot{v}_n(t) + \int_{t_0}^{t} h_n(\tau, v_n(\tau)) d\tau \, , \quad \forall \; t \in (t_0, t_1).$$

Since $h_n \to h$ uniformly on compact sets and $v_{n_k} \to v$ in $C^1[t_0, t_1]$ we conclude that

$$(3.6) \qquad \dot{v}(t_0) = \dot{v}(t) + \int_{t_0}^{t} h(\tau, \; v(\tau)) d\tau.$$

Thus v satisfies the equation in (3.4) and clearly $v \geq 0$ in (t_0, t_1) and $v(t_0) = v(t_1) = 0$. It remains to show that $v > 0$ in (t_0, t_1).
We note that by Lemma 2.1 and remark 2.2, the sequence $\{\dot{v}_n(t_0)\}$ is bounded below by a positive number $\bar{A}$. Consequently $\dot{v}(t_0) > 0$ so that $v \equiv 0$. Since $v \geq 0$ and v is concave in (t_0, t_1) it follows that $v > 0$ in this interval.

The argument used in the second part of this proof actually yields the following result.

3.3 Theorem Let $\{h_n\}$ be a sequence in $\mathcal{H}$ which converges uniformly on compact sets to an element $h \in \mathcal{H}$. Suppose that (2.10) and (2.23) hold for $X = \{h_n\}$. Let v_n be a positive solution of (3.4). Then there exists a subsequence $\{v_{n_k}\}$ which converges uniformly in $[t_0,t_1]$ to a positive solution v of (3.4).

Next we consider problem (3.4) for h of the form,

$$h(t,s) = b(t)k(s). \tag{3.7}$$

In this case we can relax the continuity assumption on h, requiring only

$$b \in C\,[t_0,t_1]\ ;\ k \text{ is a Borel function in } L_\infty^{loc}(\mathbb{R}). \tag{3.8}$$

Thus we have,

3.4 Theorem Let h be as in (3.7). Assume (3.8), (1.9), (1.10) and

(3.9) $b(t) > 0$ in $[t_0,t_1]$, $k(s) > 0$ a.e. in $(0,\infty)$ and k is essentially bounded away from zero in every interval $[s_1,s_2] \subset (0,\infty)$.

Then there exists a positive solution v of (1.4) in (t_0,t_1) vanishing at the end points. v is a solution of (1.4) in the following sense: $v \in C^1[t_0,t_1]$ $\dot{v}$ is Lipschitz in $[t_0,t_1]$ and (1.4) holds a.e. in (t_0,t_1).

Proof Since we are interested in positive solutions we may assume that k is odd. Let $k_n = J_{1/n}k$. Then $\{k_n\}$ is bounded in $L_\infty^{loc}(\mathbb{R})$ and $k_n \to k$ a.e. Denote $h_n := b\,k_n$ and let v_n be a positive solution of (3.4). Clearly

$X = \{h_n\}$ satisfies the assumptions of Remarks <u>2.2</u>, <u>2.5</u>, <u>2.7</u>. Therefore (as in the proof of Theorem I) we conclude that there exists a subsequence of $\{v_n\}$ (likewise denoted by $\{v_n\}$) which converges in $C^1[t_0,t_1]$ to a function v and that $\{\dot{v}_n(t_0)\}$ is bounded away from zero so that $\dot{v}(t_0) > 0$. We denote $M: = \sup \|v_n\|_{C[t_0,t_1]}$. We claim that (for this subsequence)

$$(3.10) \qquad k_n(v_n)\dot{v}_n \to k(v)\dot{v} \quad \text{in } L_p(t_0,t_1), \quad \forall\, p \in [1,\infty).$$

First we show that,

$$(3.11) \qquad k_n(v_n)\dot{v}_n - k(v_n)\dot{v}_n \to 0 \quad \text{in } L_1(t_0,t_1).$$

Since $v_n > 0$ and $\ddot{v}_n < 0$ in (t_0,t_1) there exists a point $\bar{t}_n$ in this interval such that $\dot{v}_n > 0$ in $(t_0,\bar{t}_n)$ and $\dot{v}_n < 0$ in $(\bar{t}_n,t_1)$. Consequently, if $M_n: = v_n(\bar{t}_n) = \sup\limits_{(t_0,t_1)} v_n$ then

$$\int_{t_0}^{t_1} |k_n(v_n) - k(v_n)|\; |\dot{v}_n| dt$$

$$= \left(\int_{t_0}^{\bar{t}_n} - \int_{\bar{t}_n}^{t_1}\right) |k_n(v_n) - k(v_n)|\; \dot{v}_n\, dt$$

$$= 2\int_0^{M_n} |k_n(s) - k(s)| ds \le 2\int_0^{M} |k_n(s) - k(s)| ds \to 0.$$

This proves (3.11). Next let,

$$\tilde{k}(\sigma) = \begin{cases} k(\sigma), & |\sigma| \le M \\ 0, & \text{otherwise} \end{cases}$$

and denote

$$\tilde{K}(s) = \int_0^s \tilde{k}(\sigma)d\sigma.$$

Then $\tilde{K}$ is uniformly Lipschitz on $\mathbb{R}$. By a result of [MM], if $u_n \to u$ in $W_{1,p}(t_0,t_1)$ (where $p \in [1,\infty)$) then $\tilde{K}(u_n) \to \tilde{K}(u)$ in $W_{1,p}(t_0,t_1)$. Furthermore,

$$\frac{d}{dt}\tilde{K}(u) = \tilde{k}(u)\,\dot{u}.$$

Thus,

$$(3.12) \qquad k(v_n)\dot{v}_n \to k(v)\dot{v} \quad \text{in } L_1(t_0,t_1).$$

Statement (3.10) for $p = 1$ follows from (3.11), (3.12). Since $\{k_n(v_n)\dot{v}_n\}$ is uniformly bounded, it is clear that this implies convergence in $L_p(t_0,t_1)$ for every $p \in [1,\infty)$. Note that $\{v_n\}$ is a bounded sequence in $W_{2,\infty}(t_0,t_1)$ and hence in particular in $H_2(t_0,t_1)$. Therefore one can extract a subsequence (which we continue to denote $\{v_n\}$) which converges weakly in H_2. Since $v_n \to v$ in $C[t_0,t_1]$ it follows that the weak limit of $\{v_n\}$ in H_2 is v. In particular $\ddot{v}_n \to v$ weakly in $L_2(t_0,t_1)$. Since $\dot{v}_n \to \dot{v}$ in $C[t_0,t_1]$ it

follows that $\ddot{v}_n\dot{v}_n \to \dddot{v}v$ weakly in $L_2(t_0,t_1)$. This fact, together with (3.10) and (3.4) imply that

$$\dddot{v}v + bk(v)\dot{v} = 0 \quad \text{a.e. in } (t_0,t_1). \tag{3.13}$$

Clearly $v \geq 0$ in (t_0,t_1), $v(t_0) = v(t_1) = 0$. Recall that $\dot{v}(t_0) > 0$. Let $\bar{t} = \sup\{t > t_0 \colon \dot{v} > 0 \text{ in } (t_0,t)\}$. By (3.13)

$$\ddot{v} + bk(v) = 0 \quad \text{a.e. in } (t_0,\bar{t}). \tag{3.14}$$

since $v(\bar{t}) > 0$ it is clear that there exists $\delta > 0$ and $c > 0$ s.t. $v_n(t) > c$ for $|t - \bar{t}| < \delta$ and $n = 1,2,\ldots.$ Hence there exists $c' > 0$ such that

$$\ddot{v}_n(t) \leq - c' \quad \text{for} \quad |t - \bar{t}| < \delta \quad \text{and} \quad n = 1,2,\ldots.$$

Consequently

$$\dot{v}_n(t) - \dot{v}_n(\bar{t}) \leq - c'(t - \bar{t}),$$

and hence (recall $\dot{v}(\bar{t}) = 0$)

$$\dot{v}(t) \leq - c' \ (t - \bar{t}) \ , \quad |t - \bar{t}| < \delta.$$

Thus $\dot{v} < 0$ in $(\bar{t}, \bar{t} + \delta)$. Furthermore $\dot{v}$ is monotone decreasing (since $\dot{v}_n$ has this property). Thus $\dot{v} < 0$ everywhere in $(\bar{t},t_1)$. Consequently (using (3.13), (3.14))

$$\ddot{v} + bk(v) = 0 \quad \text{a.e. in } (t_0, t_1). \tag{3.15}$$

It is also clear that $v > 0$ in (t_0, t_1). This completes the proof of the theorem.

4. A uniqueness result.

In this section we prove Theorem II. We start with some preliminary observations.

4.1 We consider equation (1.3) with f as in (1.11). In this case the function h in (1.4) is given by

$$h(t,v) = (N - 2)^{-2} t^{-2+\sigma} f_0(vt^{-\sigma}) \quad , \ \sigma := \rho/(N - 2). \tag{4.1}$$

Further applying the Fowler transformation

$$x = \log t \ , \quad z = vt^{-\sigma} \tag{4.2}$$

equation (1.4) obtains the form,

$$z''(x) = (1 - z\sigma)z'(x) + f_1(z(x)) \tag{4.3}$$

where,

$$f_1(z) = \sigma(1 - \sigma)z - f_0(z)/(N - 2)^2. \tag{4.4}$$

If $0 < \sigma < 1$ then, by (1.12), f_1 has exactly three zeros, $z = 0$, $z = z_\sigma > 0$ and $z = - z_\sigma$. If $\sigma \geq 1$ or $\sigma \leq 0$, $f_1(z) \neq 0$ for every $z \neq 0$. Note that f_1 is odd. Accordingly, if $z = z(x)$ is a solution of (4.3) in (α,β) then so is $z = - z(x)$. In addition $z = z(-x)$ is a solution of (4.3), with σ replaced by $\bar{\sigma} := 1 - \sigma$, in $(-\beta,-\alpha)$.

Since f_1 is independent of x, equation (4.3) can be studied by phase plane methods, as in the case of the Emden-Fowler equation. A study of the positive solutions of (1.1) with f as in (1.11), by phase plane methods, was carried out in [BM]. We quote below some of the results of [BM] that will be used in our proof. All of these results hold under assumptions (1.11) and (1.12).

<u>4.2</u> Denote $w = z'(x)$ so that the generic point in the phase plane is (z,w). There exists a curve Γ^* which divides the half plane $\{(z,w): z > 0\}$ into two domains D_0 and D_1 such that D_0 is bounded and D_1 is the union of all phase plane curves corresponding to positive solutions of (1.1) (for arbitrary annuli). The curve Γ^* is the phase plane trajectory of the positive solutions of (4.3) defined on an unbounded interval, for which $(0,0)$ is a limit point. (All these solutions have the same trajectory in the phase plane.) If $z = z(x)$ is such a solution and (x_1,x_2) is the maximal unbounded interval on which it is positive then,

$$
\begin{aligned}
x_1 &= -\infty \text{ and } x_2 < \infty \quad \text{when } \sigma < 1/2, \\
x_1 &= -\infty \text{ and } x_2 = \infty \quad \text{when } \sigma = 1/2, \qquad (4.5)\\
x_1 &> -\infty \text{ and } x_2 = \infty \quad \text{when } \sigma > 1/2.
\end{aligned}
$$

The curve Γ^* intersects the z-axis at exactly one point a^* which divides Γ^* into two branches Γ^*_+, Γ^*_- lying in the upper and lower half plane respectively. These branches can be described in the form $w = w_\pm(z)$, $0 < z < a^*$ which have the following properties

$$(4.6)\qquad \begin{aligned} &\lim_{v\to 0} w_+(z)/z = 1 - \sigma && \text{if } \sigma < 1/2 \\ &\lim_{v\to 0} w_-(z)/z = -\sigma && \text{if } \sigma > 1/2 \\ &\lim_{v\to 0} w_\pm(z)/z = \pm\sigma && \text{if } \sigma' = 1/2. \end{aligned}$$

Furthermore, if $\sigma < 1/2$ (resp. $\sigma > 1/2$) then Γ_-^* (resp. Γ_+^*) intersects the w-axis at a point $w_-^* < 0$ (resp. $w_+^* > 0$) with slope $1 - 2\sigma$. The functions w_+, w_- satisfy the equation

$$(4.7)\qquad ww' = (1 - 2\sigma)w + f_1(z).$$

For $a \geq a^*$ denote by $z(\cdot,a)$ the unique positive solution of (4.3) subject to the conditions $z(0) = a$, $z'(0) = 0$. The set of positive solutions of (1.1) corresponds to $\{z(\cdot,a): a > 0\}$. Denote by $(x_-(a), x_+(a))$ the maximal interval (containing zero) on which $z(\cdot,a)$ is positive and let $\Gamma(a)$ be the corresponding curve in the half plane $\{(z,w): z > 0\}$. By (4.5), $x_-(a^*) = -\infty$ if $\sigma \leq 1/2$ and $x_+(a^*) = \infty$ if $\sigma \geq 1/2$ while $x_+(a^*) < \infty$ if $\sigma < 1/2$ and $x_-(a^*) > -\infty$ if $\sigma > 1/2$. For $a > a^*$, $-\infty < x_-(a) < 0 < x_+(a) < \infty$. Moreover, for $a > a^*$, the curve $\Gamma(a)$ intersects the z-axis exactly once, at $(a,0)$, and this point divides $\Gamma(a)$ into two branches $\Gamma_+(a)$, $\Gamma_-(a)$ lying in the upper and lower quarters respectively. $\Gamma_+(a)$ (resp. $\Gamma_-(a)$) intersects the w-axis at a point $w_+(a) > 0$ (resp. $w_-(a) < 0$) with slope $1 - 2\sigma$. Clearly if $a_1 \neq a_2$ then $\Gamma(a_1)$ does not intersect $\Gamma(a_2)$. As a point moves along $\Gamma(a)$ from $(0,w_+(a))$ towards $(0,w_-(a))$ the corresponding point on the curve $z(\cdot,a)$ moves from $x_-(a)$ towards $x_+(a)$. Finally, two solutions of (4.3) are represented by the same curve in the phase plane if and only if one is a translation of the other. Thus two distinct positive annular solutions in the

same annulus correspond to distinct curves $\Gamma(a)$.

Given $a > 0$, let $v(t,a) := t^{\sigma} z(x,a)$ where $t = e^{x}$ and denote $t_{\pm}(a) = \exp x_{\pm}(a)$. In view of (4.5),

$$(4.8) \qquad t_{+}(a)/t_{-}(a) \to +\infty \quad \text{as} \quad a \to a^{*}.$$

It is easily seen that,

$$(4.9) \qquad t\dot{v} = wt^{\sigma} + \sigma v.$$

Hence,

$$(4.10) \qquad \dot{v}(t_{\pm}(a),a) = w_{\mp}(a) t_{\pm}^{\sigma-1}.$$

<u>4.4 Proof of Theorem II</u> Suppose that in a given annulus there exist two distinct positive solutions of (1.1). Then there exist a_1, a_2 such that $a^{+} < a_1 < a_2 < \infty$ and $t_{\pm}(a_1) = t_{\pm}(a_2)$. Denote $t_{+}(a_i) = \bar{b}$, $t_{-}(a_i) = \underline{b}$ and $v_i(\cdot) = v(\cdot, a_i)$. Since $a_1 < a_2$, $w_{+}(a_2) > w_{+}(a_1)$ and $w_{-}(a_2) < w_{-}(a_1)$. Hence, by (4.10),

$$(4.11) \qquad \dot{v}_2(\underline{b}) > \dot{v}_1(\underline{b}) > 0\ , \quad \dot{v}_2(\bar{b}) < \dot{v}_1(\bar{b}) < 0.$$

Thus $v_2(t) > v_1(t)$ for t near $\underline{b}$ and for t near $\bar{b}$. Now, v_1, v_2 are two positive solutions of (1.4) (with h given by (4.1)) in $(\underline{b}, \bar{b})$ vanishing at the end points. In view of $(1.12)_4$, the comparison theorem implies that the two solutions must intersect in $(\underline{b}, \bar{b})$. Further, (4.11) implies that they must intersect at least twice in $(\underline{b}, \bar{b})$. Let $\underline{c}$ (resp. $\bar{c}$) be the point of

intersection nearest to $\underline{b}$ (resp. $\bar{b}$). Thus (by (4.11))

$$\begin{cases} v_2(t) > v_1(t) \quad \text{in} \quad (\underline{b},\underline{c}) \cup (\bar{c},\bar{b}), \\ v_1(\underline{c}) = v_2(\underline{c}) \ , \quad v_1(\bar{c}) = v_2(\bar{c}). \end{cases} \tag{4.12}$$

This implies,

$$\dot{v}_2(\underline{c}) < \dot{v}_1(\underline{c}) \ , \quad \dot{v}_2(\bar{c}) > \dot{v}_1(\bar{c}) \tag{4.13}$$

Set $\underline{z} := v_i(\underline{c})\underline{c}^{-\sigma}$, $\bar{z} := v_i(\bar{c})\bar{c}^{-\sigma}$ (see (4.2)) and let $(\underline{z},\underline{w}_i)$ and $(\bar{z},\bar{w}_i)$ be the points corresponding to $\underline{c}$ and $\bar{c}$ on $\Gamma(a_i)$, $i = 1,2$.
Then by (4.9),

$$\underline{c}\dot{v}_i(\underline{c}) = \underline{w}_i\underline{c}^{\sigma} + \sigma v_i(\underline{c}) \quad , \quad i = 1,2.$$

Since $v_1(\underline{c}) = v_2(\underline{c})$, (4.13) implies

$$\underline{w}_2 < \underline{w}_1. \tag*{$(4.14)_1$}$$

Similarly we obtain,

$$\bar{w}_2 > \bar{w}_1. \tag*{$(4.14)_2$}$$

Since $a_2 > a_1$, if $(z,w_2) \in \Gamma(a_2)$ and $(z,w_1) \in \Gamma(a_1)$ then $w_2 \neq 0$ and the following statement holds,

$$(4.15)\qquad \begin{aligned} &w_2 > 0 \Rightarrow w_2 > w_1 ,\\ &w_2 < 0 \Rightarrow w_2 < w_1 . \end{aligned}$$

Thus $(4.14)_1$ implies that $\underline{w}_2 < 0$. Since $\underline{c} < \bar{c}$ it follows that $(\bar{z},\bar{w}_2)$ lies between $(\underline{z},\underline{w}_2)$ and $(0,w_-(a_2))$. Hence, $\bar{w}_2 < 0$ and hence, by (4.15), $\bar{w}_2 < \bar{w}_1$ in contradiction to $(4.14)_2$. Thus in any annulus there is at most one positive solution of (1.1). Since the existence is guaranteed by Th. I, the first part of Th. II is proved.

This uniqueness result implies that the function

$$(4.16)\qquad (a^*,\infty) \ni a \to t_+(a)/t_-(a)$$

is strictly monotone. The existence result established before implies that the range of this function is $(1,\infty)$. Finally, in view of (4.8) we conclude that

$$(4.17)\qquad \lim_{a\to\infty} t_+(a)/t_-(a) = 1.$$

If $v = v(t)$ is a solution of (1.4) in (t_0,t_1) then $v = -v(t)$ is a solution of (1.4) in the same interval while $v = t\,v(1/t)$ is a solution of (1.4) in $(1/t_1,\ 1/t_0)$ with σ replaced by $\bar{\sigma} = 1-\sigma$. Therefore it is sufficient to establish the assertion of the theorem for $\sigma \lesssim 1/2$.

Let Γ be a curve in the phase plane (z,w) corresponding to a solution $z = z(x)$ of (4.3). Then $-\Gamma$ corresponds to the solution $z = -z(x)$. (Recall that f_1 is odd.) Thus $-\Gamma^*$ divides the left hand plane $z < 0$ into two domains, $-D_0$ and $-D_1$, such that $-D_1$ is precisely the union of all curves representing negative solutions of (1.1) (for arbitrary annuli).

Note that the function $a \to w_+(a)$ is strictly monotone increasing in

(a^*,∞) because $\Gamma(a_1) \cap \Gamma(a_2) = \phi$ whenever $a_1 \neq a_2$. If $\sigma < 1/2$ then $w_+(a) \to 0$ as $a \to a^*$ and $w_+(a) \to \infty$ as $a \to \infty$, because for every point $(0,w)$ with $w > 0$ there exists a curve $\Gamma(a)$ starting at that point.

Assume $\sigma < 1/2$. Given $b > 0$ let $a > a^*$ be such that $w_+(a) = b$ and denote $\mathfrak{X}(b) := \Gamma(a)$ and $\tilde{w}(b) := w_-(a)$. For $b < 0$ let $\mathfrak{X}(b) = -\mathfrak{X}(-b)$ and denote $\tilde{w}(b) = -\tilde{w}(-b)$. Note that $|\tilde{w}(b)| > |w_-^*|$, for every $b \neq 0$.

Now for $b > 0$ let

$$\mathfrak{X}_\infty(b) := \bigcup_{i=0}^{\infty} \mathfrak{X}(b_i) \,, \tag{4.18}$$

where $b_0 = b$, $b_i = \tilde{w}(b_{i-1})$ for $i = 1,2,\dots$. Then $\mathfrak{X}_\infty(b)$ is a continuous curve in $D_1 \cup (-D_1)$ which corresponds to an oscillating solution of (1.4). Given $t_0 > 0$ there exists exactly one such solution of (1.4) defined in some interval (t_0,T), namely, the solution v with initial data $v(t_0) = 0$, $\dot{v}(t_0) = b\, t_0^{\sigma-1}$. We denote this solution by $\bar{v}(\cdot;b)$.

By the results of [BM] (see in particular Lemma 2.2), if $\sigma \neq 1/2$, a curve Γ in the phase plane corresponding to a solution of (1.4) cannot contain a closed loop (and, by uniqueness, it cannot intersect itself). Furthermore $\mathfrak{X}_\infty(b)$ is unbounded. Indeed, if $\mathfrak{X}_\infty(b)$ were bounded then it follows that the solution $\bar{v}(\cdot;b)$ is defined on (t_0,∞) and consequently, using the Poincaré-Bendixon theory, one concludes (as in [BM, Lemma 2.7]) that $\lim_{x\to\infty} (z,w)$ exists and is a stationary point. However this is impossible because $|b_k| > |w_-^*|$ for $k \geq 1$ so that $\{b_k\}$ cannot converge to zero. (Recall that there are exactly three stationary points and they are all located on the z-axis.)

Next we observe that $\{b_{2j}\}_0^\infty$ and $\{b_{2j-1}\}_1^\infty$ are strictly monotone. Since

$\mathfrak{X}_\infty(b)$ is unbounded, it follows that $b_{2j} \to +\infty$ and $b_{2j-1} \to -\infty$. (If $\{b_{2j}\}$ were bounded then choosing an upper bound $\bar{b}$ we would conclude that $\mathfrak{X}_\infty(b)$ lies inside the domain bounded by $\mathfrak{X}(\bar{b}) \cap \mathfrak{X}(\tilde{w}(\bar{b}))$.) In particular we deduce that $\{b_{2j}\}_0^\infty$ is monotone increasing while $\{b_{2j-1}\}_1^\infty$ is monotone decreasing.

To each point $(0,b_j)$ there corresponds a unique point $t_j(b) \in [t_0,\infty)$ such that $\bar{v}(t_j(b);b) = 0$ and $\dot{\bar{v}}(t_j(b);b) = b_j t_j(b)^{\sigma-1}$. (In particular $t_0 = t_0(b)$.) $\{t_j(b)\}_0^\infty$ is precisely the set of zeros of $\bar{v}(\cdot;b)$ and $t_j(b) < t_{j+1}(b)$, $j = 0,1,\ldots.$ Let $\bar{v}_k(\cdot;b)$ denote the restriction of $\bar{v}(\cdot;b)$ to $[t_0,t_k(b)]$. Then $\bar{v}_k(\cdot;b)$ is a solution of (1.4) in this interval which vanishes at $k+1$ points (including the end points) and which satisfies $\dot{\bar{v}}_k(t_0;b) > 0$. Conversely, if v is a solution of (1.4) in $[t_0,T]$ satisfying the above conditions then there exists $b > 0$ such that $T = t_k(b)$. Therefore, to establish the second assertion of Th. II it is sufficient to show that $b \to t_k(b)/t_0$ is monotone and that its range is $(1,\infty)$.

Let a_j denote the point where $\mathfrak{X}(b_j)$ intersects the z-axis. In view of the monotonicity properties of $\{b_{2j}\}$ and $\{b_{2j-1}\}$ it follows that $\{a_{2j}\}_0^\infty$ is monotone increasing in $(0,\infty)$ and $\{a_{2j-1}\}_1^\infty$ is monotone decreasing in $(-\infty,0)$. Since $\mathfrak{X}(b_{2j}) = \Gamma(a_{2j})$ and $\mathfrak{X}(b_{2j-1}) = -\Gamma(-a_{2j-1})$ we have,

$$t_k(b)/t_0 = \prod_{j=0}^{k-1} t_+(|a_j|)/t_-(|a_j|). \tag{4.19}$$

By the first part of the proof,

$$a \to t_+(a)/t_-(a) \text{ is monotone decreasing in } (a^*,\infty) \text{ and its range is } (1,\infty). \tag{4.20}$$

It is easy to see that for each fixed j $(j = 0,1,\dots)$ $|a_j| \to \infty$ as $b \to \infty$. Therefore, (4.19) and (4.20) imply that

$$t_+(a_0)/t_-(a_0) \le t_k(b)/t_0 \le [\max\,(t_+(a_0)/t_-(a_0),\ t_+(-a_1)/t_-(-a_1))]^k$$

and hence $t_k(b)/t_0$ varies from 1 to ∞ as b varies from ∞ to zero. Furthermore, if $0 < b < b'$ then $|a_j| < |a_j'|$ (where a_j' is in the same relation to b' as a_j to b) and consequently

$$t_+(|a_j|)/t_-(|a_j|) > t_+(|a_j'|)/t_-(|a_j'|) \qquad j = 0,\dots,k.$$

Hence by (4.19),

$$t_k(b)/t_0 > t_k(b')/t_0.$$

This completes the proof of the second part of the theorem in the case $\sigma \ne 1/2$.

Finally, assume $\sigma = 1/2$. In this case if $z = z(x)$ is a solution of (4.3) then $z = z(-x)$ and $z = -z(x)$ are also solutions of (4.3). Hence if Γ is a curve in the phase plane corresponding to a solution of (4.3) then Γ^* (its symmetric image with respect to z-axis), as well as $-\Gamma$, are also curves of this type. In particular $\Gamma(a)$, $a > a^*$, is symmetric with respect to the z-axis and $-\Gamma(a)$ is its continuation as an integral curve (corresponding to solutions of (4.3)) to the left hand plane $z < 0$. Consequently, setting $b := w_+(a)$, the curve $\mathfrak{X}_\infty(b)$ defined in (4.18) reduces to the closed curve $\Gamma(a) \cup (-\Gamma(a))$. Therefore if $\bar{v}(\cdot;b)$ and $\{t_j(b)\}_0^\infty$ are defined as before we have:

$$t_{j+1}(b)/t_j(b) = t_+(a)/t_-(a) , \quad j = 0,1,\ldots$$

so that (with $t_0 = t_0(b)$):

$$t_k(b)/t_0 = (t_+(a)/t_-(a))^k.$$

By the first part of the proof it follows that the function $b \to t_k(b)/t_0$ is monotone decreasing in $(0,\infty)$ and its range is $(1,\infty)$. This implies the second assertion of the theorem for $\sigma = 1/2$ and the proof is complete.

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Existence and Uniqueness Results for Semi-linear Dirichlet Problems in Annuli

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(*) Partially supported by NSF grant #DMS-8704530.