# ON RANDOM MINIMUM LENGTH SPANNING TREES

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Abstract

We extend and strengthen the result that, in the complete graph $K_n$ with independent random edge-lengths uniformly distributed on [0,1], the expected length of the minimum spanning tree tends to $\zeta(3)$ as $n \to \infty$. In particular, if $K_n$ is replaced by the complete bipartite graph $K_{n,n}$ then there is a corresponding limt of $2\zeta(3)$.

## §1 Introduction

Suppose that we are given a complete graph $K_n$ on $n$ vertices together with lengths on the edges which are independent identically distributed non-negative random variables. Suppose that their common distribution function $F$ satisfies $F(0) = 0$, $F$ is differentiable from the right at zero and $D = F'_+(0) > 0$. Let $X$ denote a random variable with this distribution.

Let $L_n$ denote the (random) length of the minimum spanning tree in this graph. Frieze [3] proved the following:

**Theorem 1**

(a) If $E(X) < \infty$ then $\lim_{n\to\infty} E(L_n) = \zeta(3)/D$, where

$$\zeta(3) = \sum_{k=1}^{\infty} k^{-3} = 1.202\ldots.$$

(b) If $E(X^2) < \infty$ then $\lim_{n\to\infty} \mathrm{Var}(L_n) = 0$, and so in particular $L_n \to \zeta(3)/D$ in probability. □

Recently, Steele [5] has shown that the convergence in probability above holds without assumptions on moments.

In this paper we generalise Theorem 1 to graphs other than $K_n$. We shall also simplify the proofs and sharpen the results.

Let $H$ be a fixed connected multigraph, with vertex set $V(H) = \{v_1, v_2, \ldots, v_h\}$. Corresponding to each edge $e$ of $H$ let $F_e$ be a distribution function of a non-negative random variable such that $F_e(0) = 0$ and $F_e$ has a right derivative $D_e$ at 0. We assume that there exists $D > 0$ such that for each vertex $v$ of $H$,

$$\sum_{e \ni v} D_e = D.$$

(Observe that loops contribute once to this sum.)

For each $n = 1,2,\ldots$ let $H_n$ be a (loopless) graph obtained as follows. Replace each vertex $v_i$ of $H$ by a set $V_i$ of $n$ new vertices, so that $|V(H_n)| = nh$. Now join two distinct vertices of $H_n$ by the same number of edges as join the corresponding vertices of $H$. Thus if $H$ has $\lambda$ loops and $\nu$ non-loops then $H_n$ has $\mu = \binom{n}{2}\lambda + n^2\nu$ edges.

Let the edges of $H_n$ have independent lengths, where the length of an edge $e$ is distributed according to the distribution for the edge of $H$ from which $e$ arose. Let us extend our notation so that the length of $e \in E(H_n)$ has distribution function $F_e$ as well.

For any connected graph $G$ with non-negative edge-lengths let $L(G)$ denote the length of a minimum spanning tree in $G$.

Theorem 2

As $n \to \infty$, $L(H_n) \to (h/D)\zeta(3)$ a.s.

□

This result follows (by a Borel-Cantelli lemma) from

Lemma 0

For any $\epsilon > 0$ there exists $c$, $0 < c < 1$ such that

$$P(|L(H_n) - (h/D)\zeta(3)| > \epsilon) < c^{n^{1/4}}.$$

Theorem 1 follows from the case where $H$ has a single vertex and a single loop, so that $H_n = K_n$. Some other interesting cases are the following, where for simplicity we make each edge length uniform on [0,1].

(1) $$L((K_r)_n) \to \frac{r}{r-1}\zeta(3) \quad \text{a.s.}$$

(Here $(K_r)_n$ is the complete multipartite graph with $r$ blocks each of size n.) In particular $L(K_{n,n}) \to 2\,\zeta(3)$ (se [4]).

(2) $$L((C_k)_n) \to \frac{k}{2}\zeta(3) \quad \text{a.s.}$$

(Here $C_k$ is a cycle with $k$ vertices.)

(3) $$L((Q_k)_n) \to \frac{2^k}{k}\zeta(3) \quad \text{a.s.}$$

(Here $Q_k$ is the k-cube.)

We shall prove lemma 0 (and thus Theorem 2) in three stages (sections 3,4,5 below), but first we have:

## §2. Notation and Preliminaries

We use two models of random subgraph of $H_n$.

For $1 \leq m \leq \mu$ $H_{n,m}$ has the same vertex set as $H_n$ and for its edge set a random m-edge subset of $E(H_n)$.

For $0 \leq p \leq 1$ $H_{n,p}$ has the same vertex set as $H_n$ and each of the $\mu$ edges of $H_n$ are independently included with probability $p$ and excluded with probability $1 - p$.

We have need of the following simple relation between $H_{n,m}$ and $H_{n,p}$ where $p = \frac{m}{\mu}$ : for any property $\Pi$

(4) $$P(H_{n,m} \in \Pi) \leq 2\sqrt{\mu}\, P(H_{n,p} \in \Pi).$$

This follows from

$$P(H_{n,p} \in \Pi) = \sum_{m'=0}^{\mu} P(H_{n,p} \in \Pi \mid |E(H_{n,p})| = m')P(|E(H_{n,p})| = m')$$

and the fact that (i) $H_{n,p}$ conditional on $|E(H_{n,p})| = m'$ is distributed as $H_{n,m'}$ and (ii) $|E(H_{n,p})|$ has the binomial distribution $B(\mu,p)$.

§3. Expected value for uniform [0,1] case

Our approach to proving theorem 2 is similar to that of [3] but uses martingale inequalities in place of the Chebycheff inequality. We first discuss the case where edge lengths are uniform on [0,1] and $H$ is r-regular (with loops counting once towards the degree of a node).

Suppose that the edges $E(H_n) = \{u_1,u_2,\dots,u_\mu\}$ are numbered so that $\ell(u_i) \leq \ell(u_{i+1})$, $i = 1,2,\dots,\mu-1$ where $\ell(u)$ is the length of edge $u$.

A minimum length tree may be constructed using the Greedy Algorithm of Kruskal [4]. Let $F_0 = \phi$, $F_1 = \{u_1\}$, $F_2,\dots,F_{hn-1}$ be the sequence of edge sets of the successive forests produced. Here $|F_i| = i$ and $F_{hn-1}$ is the set of edges in a minimum spanning tree.

Next define $t_i = \max\{j: u_j \in F_i\}$. Then

(5) $$L(H_n) = \sum_{i=1}^{hn-1} u_{t_i},$$

and thus

(6) $$E(L(H_n)) = \frac{1}{\mu+1} E\left(\sum_{i=1}^{hn-1} t_i\right).$$

The subgraph $\Gamma_m$ of $H_n$ induced by $U_m = \{u_1, u_2, \ldots, u_m\}$ is distributed as $H_{n,m}$. Let $\kappa_m$ denote the number of connected components of $\Gamma_m$.

Lemma 1

$$\sum_{i=1}^{hn-1} t_i = \sum_{m=1}^{\mu} \kappa_m + hn - \mu - 1.$$

Proof

$$\sum_{m=1}^{\mu} \kappa_m = \sum_{r=1}^{hn-1} (hn-r)(t_{r+1}-t_r)$$

where $t_{hn} = \mu+1$. This is because $\Gamma_{t_r}, \Gamma_{t_r+1}, \ldots, \Gamma_{t_{r+1}-1}$ all have $hn - r$ components. Thus

$$\sum_{m=1}^{\mu} \kappa_m = -(hn-1)t_1 + t_2 + t_3 + \ldots + t_{hn-1} + t_{hn},$$

and the result follows on noting that $t_1 = 1$ and $t_{hn} = \mu + 1$.

□

It follows from (6) and the above lemma that

(7) $$E(L(H_n)) = \frac{1}{\mu+1}\left(E\left(\sum_{m=1}^{\mu} \kappa_m\right) + hn\right) - 1.$$

We must therefore estimate $E(\sum_{m=1}^{\mu} \kappa_m)$. It will be easier to work with $H_{n,p}$ and so let $\kappa_p$ denote the (random) number of components in $H_{n,p}$. The following simplification is from Bollobás and Simon [1].

Lemma 2

$$\frac{1}{\mu+1} E(\sum_{m=1}^{\mu} \kappa_m) = \int_0^1 E(\kappa_p)dp.$$

Proof

$$\int_0^1 E(\kappa_p)dp = \int_0^1 \sum_{m=0}^{\mu} \binom{\mu}{m} p^m (1-p)^{\mu-m} E(\kappa_m)dp$$

$$= \sum_{m=0}^{\mu} E(\kappa_m)\binom{\mu}{m} \frac{m!(\mu-m)!}{(\mu+1)!}$$

□

Thus to compute $E(L(H_n))$ we need an accurate estimate of $E(\kappa_p)$.

Lemma 3

If $p \leq 4 \log n/n$ then

(8) $$E(\kappa_p) = hn\,\phi(rnp) + o(n^{3/4})$$

where

$$\phi(a) = \sum_{s=1}^{\infty} \frac{s^{s-2}}{s!} a^{s-1} e^{-as}.$$

(The 'little o' notation in (8) is intended to imply uniformity over relevant p.)

Proof

As we shall see, the most important components from our point of view are small isolated trees. Let therefore $\tau_p$ denote the number of components in $H_{n,p}$ which are trees of order $n^{1/3}$ or less. Let $\mathcal{T}_s(G)$ denote the set of s-vertex subtrees of a graph G. For $T \in \mathcal{T}_s(H_n)$ we find

$$P(T \text{ is a component of } H_{n,p}) = p^{s-1}(1-p)^{rns-\alpha(T)}$$

where, rather crudely,

$$0 \le \alpha(T) \le r\binom{s}{2} + r.$$

Hence

$$E(\tau_p) = \sum_{s=1}^{n^{1/3}} \sum_{T\in\mathcal{T}_s(H_n)} p^{s-1}(1-p)^{rns-\alpha(T)}$$

(9) $$= (1 + o(n^{-1/4})) \sum_{s=1}^{n^{1/3}} |\mathcal{T}_s(H_n)| p^{s-1}e^{-rnsp}.$$

We must now estimate $|\mathcal{T}_s(H_n)|$.

For each tree T in $\mathcal{T}_s(K_s)$ and each tree T′ in $\mathcal{T}_s(H_n)$ let $\mathcal{F}(T,T')$ be the set of bijections f between E(T) and E(T′) that correspond to bijections between V(T) and V(T′).

Now if $T' \in \mathcal{T}(H_n)$ then

$$\sum_{T \in \mathcal{T}_s(K_s)} |\mathcal{F}(T,T')| = s!$$

since each bijection between $\{1,\dots,s\}$ and $V(T')$ contributes exactly one to the sum on the left hand side. Hence

$$(10) \qquad |\mathcal{T}_s(H_n)| = \frac{1}{s!} \sum_{T \in \mathcal{T}_s(K_s)} \sum_{T' \in \mathcal{T}_s(H_n)} |\mathcal{F}(T,T')|.$$

We shall show that for each $T \in \mathcal{T}_s(K_s)$

$$(11) \qquad hn \prod_{k=1}^{s-1} r(n-k) \le \sum_{T' \in \mathcal{T}_s(H_n)} |\mathcal{F}(T,T')| \le hn \prod_{k=1}^{s-1} rn.$$

Using (11) in (10) and $|\mathcal{T}_s(K_s)| = s^{s-2}$ yields

$$|\mathcal{T}_s(H_n)| = (1 + o(n^{-1/4})) \frac{s^{s-2}}{s!} hr^{s-1}n^s ,$$

and then from (9)

$$(12) \qquad E(\tau_p) = (1 + o(n^{-1/4}))hn \sum_{s=1}^{n^{1/3}} \frac{s^{s-2}}{s!} (nrp)^{s-1} e^{-mp}.$$

To prove (11) note that when $s = 1$ it is correct (if we interpret $\prod_{k=1}^{0}$ as 1). Assume that it is true for some $s \ge 1$ : we shall show that it is true for $s+1$. Consider a tree $T$ in $\mathcal{T}_{s+1}(K_{s+1})$ and assume without loss of generality that $s+1$ is a leaf of $T$, with incident edge $e$. Then having fixed a bijection $f$ on the tree $T-(s+1)$ in $\mathcal{T}_s(K_s)$ there are between $r(n-s)$ and $rn$ choices for the image of $e$. This completes our proof of (11) and thus of (12).

We observe that since $s! \geq (s/e)^s$

$$\frac{s^{s-2}}{s!} p^{s-1} e^{-rnsp} \leq \frac{e}{s^2} (rn\, e^{1-rnp})^{s-1}$$

$$\leq \frac{e}{s^2} .$$

This implies, from (12), that

$$E(\tau_p) = hn\phi(rnp) + o(n^{3/4}). \tag{13}$$

We now look at $\sigma_p$ = the number of non-tree components of $H_{n,p}$ of order at most $n^{1/3}$. As each such component consists of a tree $T \in \mathcal{T}_s(H_n)$ plus some $k$ extra edges, we deduce that

$$\text{(14)}\quad E(\sigma_p) \leq \sum_{s=1}^{n^{1/3}} \sum_{T\in\mathcal{T}_s(H_n)} p^{s-1} (1-p)^{rns-\alpha(T)} \sum_{k=1}^{r\binom{s}{2}-s+1} \binom{r\binom{s}{2}}{k} p^k (1-p)^{-k}$$

$$= E(\tau_p) \times o(n^{-1/4}).$$

As $H_{n,p}$ contains at most $n^{2/3}$ components of size exceeding $n^{1/3}$, the lemma follows from (13) and (14).

□

For $p \geq 4 \log n/n$ we use the following.

Lemma 4

(a) If $p = 4 \log n/n$ then

$$P(H_{n,p} \text{ is not connected}) = O(n^{-3}).$$

(b) If $p = n^{-3/4}$ then

$$P(H_{n,p} \text{ is not connected}) = O(ne^{-n^{1/4}}).$$

Proof

(a) If $H_{n,p}$ is not connected then either

(i) $h = 1$

or

(ii) there is a pair of distinct adjacent vertices $v_i, v_j$ in $H$ such that the subgraph of $H_{n,p}$ induced by $V_i \cup V_j$ is not connected.

In case (i) $H_{n,p}$ is the standard model $G_{n,p}$ and in case (ii) the subgraph $K$ induced by $V_i \cup V_j$ contains a random bipartite graph. For brevity we deal with case (ii) and leave case (i) to the reader. Both cases are straightforward.

If $K$ is not connected then there exist $S \subseteq V_i$, $T \subseteq V_j$ such that $1 \leq |S| + |T| \leq n$ and no edge of $H_{n,p}$ joins $S \cup T$ to $V_i \cup V_j - S \cup T$. Hence

$$P(ii) \leq \binom{h}{2} \sum_{\substack{k,\ell=0 \\ 1\leq k+\ell\leq n}}^{n} u(k,\ell)$$

where

$$u(k,\ell) = \binom{n}{k}\binom{n}{\ell}(1-p)^{k(n-\ell)+\ell(n-k)}$$

$$\leq n^{k+\ell - 4(k+\ell) + \frac{8k\ell}{n}}$$

$$\leq n^{-(3 - 2(k+\ell)/n)(k+\ell)}.$$

Part (a) now follows easily, and part (b) may be proved in a similar manner.

□

We can now obtain the limiting value for $E(L(H_n))$ in the special case under consideration.

Lemma 5

If $H$ is $r$-regular and edge-lengths are independent and all uniform on [0,1] then

$$\lim_{n\to\infty} E(L(H_n)) = (h/r)\zeta(3).$$

Proof

It follows from (7) and Lemma 2 that

$$E(L(H_n)) = \int_0^1 (E(\kappa_p) - 1)dp + \frac{hn}{\mu+1}.$$

Now if $p_0 = 4\ \log n/n$ then by Lemma 3,

$$\int_0^{p_0} E(\kappa_p)dp = hn \sum_{s=1}^{\infty} \frac{s^{s-2}}{s!} \int_0^{p_0} (rnp)^{s-1} e^{-rnps} dp + o(n^{3/4} p_0)$$

$$= (h/r) \sum_{s=1}^{\infty} \frac{s^{s-2}}{s!} \int_0^{4r\log n} x^{s-1} e^{-sx} dx + o(\log n/n^{1/4})$$

$$= (h/r)\zeta(3) + o(\log n/n^{1/4}).$$

To see the last equation above note that

$$\int_{\omega}^{\infty} x^{s-1} e^{-sx} dx = O(e^{-\omega/2}) \qquad \text{if} \quad \omega = \omega(n) \to \infty$$

and

$$\int_0^{\infty} x^{s-1} e^{-sx} dx = (s-1)!/s^s.$$

It follows from Lemma 4(a) that for $p \geq p_0$, $E(\kappa_p) = 1 + O(n^{-2})$ and so $\int_{p_0}^{1} (E(\kappa_p) - 1)dp = O(n^{-2})$. Hence

(15) $$E(L(H_n)) = (h/r)\zeta(3) + o(\log n/n^{1/4}).$$

□

§4. Probability inequality for uniform [0,1] case

Our aim next is to prove that there is a constant $A = A(r,h) > 0$ such that for any $0 < \epsilon < 2h/r$

(16) $$P(|L(H_n) - (h/r)\zeta(3)| \geq \epsilon) \leq e^{-A\epsilon^2 n^{1/4}}$$

for n sufficiently large. We do this in two stages.

<u>Lemma 6</u>

Let $t_1, t_2, \ldots, t_{hn-1}$ be as in (5) and $0 < \epsilon < 1$ be fixed. Then for n sufficiently large

$$P(|\sum_{i=1}^{hn-1} t_i - (h/r)(\mu+1)\zeta(3)| \geq \epsilon n^2) \leq e^{-\epsilon^2 n^{1/4}/r^3 h^3}.$$

<u>Proof</u>

We prove this using a martingale inequality. Let $X_1, X_2, \ldots, X_N$ be random variables, and for each $i = 1, \ldots, N$ let $\underline{X}^{(i)}$ denote $(X_1, X_2, \ldots, X_i)$. Suppose that the random variable Z is determined by $\underline{X}^{(N)}$. For each $i = 1, 2, \ldots, N$ let

(17) $$\delta_i = \sup |E(Z|\underline{X}^{(i-1)}) - E(Z|\underline{X}^{(i)})|.$$

Here $E(Z|\underline{X}^{(0)})$ means just $E(Z)$. The following inequality is a special case of a martingale inequality due to Azuma. For any $u \geq 0$

(18) $$\Pr(|Z - E(Z)| \geq u) \leq 2 \exp\{-u^2/2 \sum_{i=1}^{m} \delta_i^2\}.$$

To apply (18) we take $N = \lceil \mu/n^{3/4} \rceil$ and let $X_i = u_i$, the $i^{th}$ shortest edge of $H_n$. Let $Z = \sum_{m=1}^{N} \kappa_m$. It is not difficult to see that for $\delta_i$ as defined by (17) we have $\delta_i \le N - i+1$. This follows from the fact (in an obvious notation) that $|\kappa_m(\underline{X}^{(N)}) - \kappa_m(\underline{Y}^{(N)})| \le 1$ if there exists $k$ such that $X_i = Y_i$ for $i \ne k$ or there exist $k,\ell$ such that $X_k = Y_\ell$, $X_\ell = Y_k$ and $X_i = Y_i$ otherwise.

Thus

$$(19) \qquad P(|Z - E(Z)| \ge u) \le 2e^{-3u^2/N(N+1)(2N+1)} \quad \text{for} \quad u \ge 0.$$

Now let $Z' = \sum_{m=N+1}^{\mu} \kappa_m$. It follows from (4) and Lemma 4(b) that

$$(20) \qquad P(Z' \ne \mu - N) = O(n^2 e^{-n^{1/4}})$$

and so

$$(21) \qquad E(Z') = \mu - N + o(1).$$

Now (7), (15) and (21) imply that

$$E(Z) = (h/r)(\mu+1)\zeta(3) + O(n^{7/4} \log n).$$

We can then use (19) with $u = \frac{1}{2}\epsilon n^2$ together with Lemma 1, (20) and $\mu \le \frac{1}{2} rhn^2$ to obtain the Lemma.

□

We must now show that sums of order statistics of a large number of independent uniform [0,1] random variables usually behave as expected.

Lemma 7

Let $u_i$, $i = 1,2,\dots,\mu$ denote the order statistics of $\mu$ independent uniform [0,1] random variables. Let $1 \le t_1 < t_2 < \dots < t_{hn-1} \le \mu$ and $T = \sum_{k=1}^{hn-1} t_k$. Then for any fixed $0 < \epsilon < 1$

$$P\left(\left|\sum_{k=1}^{hn-1} u_{t_k} - \frac{T}{\mu+1}\right| > \frac{\epsilon T}{\mu+1}\right) \le e^{-\frac{\epsilon^2 T}{16hn}} . \tag{22}$$

Proof

It is well known (see for example Feller [2]) that if $X_1, X_2, \dots, X_{\mu+1}$ are independent exponential random variables with mean 1 than the variables $Z_i = \frac{Y_i}{Y_{\mu+1}}$, $i = 1,2,\dots,\mu$ are distributed as $u_i$, $i = 1,2,\dots,\mu$ where $Y_i = X_1+X_2+\dots+X_i$. It suffices therefore to prove (22) with $u_{t_k}$ replaced by $Z_{t_k}$. Note now that

$$S = \sum_{k=1}^{hn-1} Y_{t_k} = \sum_{j=1}^{\mu+1} a_j X_j$$

where $a_j = |\{k\colon t_k \ge j\}|$, and that $T = \sum_{j=1}^{\mu+1} a_j$. Now for $\lambda > 0$

$$P(S \ge (1+\epsilon)T) = P(e^{\lambda S - \lambda(1+\epsilon)T} \ge 1)$$

$$\le E(e^{\lambda S - \lambda(1+\epsilon)T})$$

$$= \prod_{j=1}^{\mu+1} \frac{e^{-\lambda(1+\epsilon)a_j}}{1 - \lambda a_j} \qquad \text{if} \quad 0 < \lambda < \min\{1/a_j\}$$

$$\leq \prod_{j=1}^{\mu+1} e^{-\epsilon\lambda a_j + \frac{2}{3}(\lambda a_j)^2} \qquad \text{if} \quad 0 < \lambda < \frac{1}{3}\min\{1/a_j\}$$

and on taking $\lambda = \frac{\epsilon}{3hn}$

$$\leq \prod_{j=1}^{\mu+1} e^{-\frac{\epsilon^2 a_j}{3hn}\left(1 - \frac{2}{9}\frac{a_j}{hn}\right)}$$

(23) $$\leq e^{-\frac{7\epsilon^2}{27}\frac{T}{hn}} \qquad \text{as } a_j \leq hn.$$

Similarly, for any $\lambda > 0$,

$$P(S \leq (1-\epsilon)T) = P(e^{-\lambda S+\lambda(1-\epsilon)T} \geq 1)$$

(24) $$\leq e^{-\frac{\epsilon^2 T}{2hn}}$$

on taking $\lambda = \frac{\epsilon}{hn}$.

We may argue as above with each $a_j = 1$ (or otherwise) to obtain

(25) $$P(|Y_{\mu+1} - (\mu+1)| \geq \epsilon(\mu+1)) \leq e^{-\frac{\epsilon^2}{4}\mu}.$$

The result follows from (23), (24) and (25) after replacing $\epsilon$ by $\epsilon/2$ throughout the proof.

□

We can now readily establish (16). Let $T = \sum_{i=1}^{hn-1} t_i$, and let

$$A_n = \{|L(H_n) - (h/r)\zeta(3)| \geq \epsilon\},$$

$$B_n = \{|T/(\mu+1) - (h/r)\zeta(3)| \geq \epsilon/2\}.$$

Then

$$P(A_n) \leq P(B_n) + P(A_n|\bar{B}_n).$$

Now Lemma 6 gives

$$P(B_n) \leq P(|T - (h/r)(\mu+1)\zeta(3)| \geq (\epsilon hr/4)\binom{n}{2})$$

$$\leq \exp(-\epsilon^2 n^{1/4}/65rh).$$

Furthermore,

$$P(A_n|\bar{B}_n) \leq P(|L(H_n) - T/(\mu+1)| \geq \epsilon/2|\ \bar{B}_n)$$

$$\leq \exp(-\tilde{\epsilon}^2\tilde{T}/16hn) \qquad \text{by Lemma 7,}$$

where $\tilde{\epsilon} = (\epsilon/2)/((h/r)\zeta(3) + \epsilon/2)$ and

$$\tilde{T} = ((h/r)\zeta(3) - \epsilon/2)(\mu+1).$$

The inequality (16) now follows.

## §5. General case

We will now use the inequality (16) to complete the proof of lemma 0 and thus of Theorem 2 in the general case. We shall assume that $D_e > 0$ for each edge $e$ in $E(H)$. Any edges $e$ with $D_e = 0$ would cause only minor irritation.

We will first use the approach of Steele [5] to relate a random edge-length $X_e$ with distribution function $F_e$ to one which is uniform in $[0,D_e^{-1}]$. Let $A_e$ denote the set of atoms of $F_e$ and define $Y_e$ by

$$Y_e = \begin{cases} D_e^{-1}F_e(X_e) & X_e \notin A_e \\ D_e^{-1}(F_e(X_e-) + U_e(F_e(X_e) - F_e(X_e-)) & X_e \in A_e. \end{cases}$$

where $U_e$ is a uniform [0,1] random variable (and we make a suitable assumption of independence)..

Observe that $Y_e$ is uniform on $[0,D_e^{-1}]$ and $X_e > X_{e'}$ implies $Y_e \geq Y_{e'}$. It follows that there is always a tree $T$ which is simultaneously of minimum length for edge-lengths $\{X_e\}$ and $\{Y_e\}$.

Our hypotheses for the $F_e$, $e \in E(H)$ show that we may write $F_e(x) = D_e x + xg_e(x)$ and $F_e(x-) = D_e x + xh_e(x)$ where $g_e$ and $h_e$ go to zero as $x \to 0$. We then have

$$(27) \qquad \sum_{e\in T} D_e^{-1} X_e h_e(X_e) \le \sum_{e\in T} Y_e - \sum_{e\in T} X_e \le \sum_{e\in T} D_e^{-1} X_e g_e(X_e).$$

Our immediate task is to bound the probability that either of the outside terms of (27) is significant. Let $g_e^*(x) = \sup\{g_e(y): 0 \le y \le x\}$ for $e \in E(H)$. Now fix $\epsilon > 0$. For $e \in E(H)$ let

$$\lambda_e = \lambda_e(\epsilon) = \sup\{\lambda: g_e^*(\lambda) \le \epsilon D_e\}.$$

Let

$$\mu = \min\{\lambda_e: e \in E(H)\}$$

and

$$\upsilon = \min\{P(X_e < \mu): e \in E(H)\},$$

and note that $\mu > 0$, $\upsilon > 0$.

Then

$$P\Big(\sum_{e\in T} D_e^{-1} X_e g_e(X_e) > \epsilon \sum_{e\in T} X_e\Big)$$

$$\le P(X_e \ge \mu \quad \text{for some} \quad e \in E(H))$$

$$\le P(H_{n,\upsilon} \text{ is not connected}).$$

But this last quantity is at most $e^{-n\nu/3}$ (for $n$ sufficiently large) by an argument similar to that of Lemma 4. An analogous argument yields

$$P\left(\sum_{e\in T} D_e^{-1} X_e h_e(X_e) < -\epsilon \sum_{e\in T} X_e\right) \le e^{-n\nu'/3}$$

for some $\nu' = \nu'(\epsilon) > 0$.

Thus if $L(H_n')$ denotes the length of a minimum spanning tree when the length $X_e'$ of edge $e \in E(H)$ is uniform in $[0,D_e^{-1}]$ then we can write, for small fixed $\epsilon > 0$,

(28a)
$$P(L(H_n) \ge (1+\epsilon)^2(h/D)\zeta(3))$$
$$\le e^{-n\nu/3} + P(L(H_n') \ge (1+\epsilon)(h/D)\zeta(3))$$

and

(28b)
$$P(L(H_n) \le (1-\epsilon)^2(h/D)\zeta(3))$$
$$\le e^{-n\nu'/3} + P(L(H_n') \le (1-\epsilon)(h/D)\zeta(3)).$$

These results reduce the general case of the theorem to the case of uniform edge-lengths. Thus in particular the inequality (16) holds also when all edge lengths have the negative exponential distribution with mean 1.

However, the above argument works also in the other direction; and we have

(29a) $$P(L(H_n') \geq ((1+\epsilon)/(1-\epsilon))(h/D)\zeta(3))$$

$$\leq e^{-n\upsilon'/3} + P(L(H_n) \geq (1+\epsilon)(h/D)\zeta(3))$$

and

(29b) $$P(L(H_n') < ((1-\epsilon)/(1+\epsilon))(h/D)\zeta(3))$$

$$\leq e^{-n\upsilon/3} + P(L(H_n) < (1-\epsilon)(h/D)\zeta(3)).$$

Thus the case of uniform edge-lengths reduces to the case of (negative) exponential edge-lengths.

Now we are almost home. We wish to show that lemma 0 holds when the edge-lengths have exponential distributions.

Let us check first that we may take each $D_e$ rational. Let $D'$ be rational, $0 < D' < D$. We shall show that there exist rational $D_e'$, $0 < D_e' \leq D_e$ for $e \in E(H)$ such that $\sum_{e \ni v} D_e' = D'$ for $v \in V(H)$. A similar approximation from above may be obtained by the reader.

Suppose then that $0 < \epsilon < 1$ and $D' = (1-\epsilon)D$ is rational. Write $D' = M/N$ where $M$ and $N$ are positive integers such that both $\epsilon ND_e \geq 1$ and $(1-\epsilon)ND_e \geq 1$ for each $e \in E(H)$. Observe next that the polyhedron

$$\sum_{e \ni v} x_e = (1-\epsilon)D$$

$$1/N \leq x_e \leq \lceil (1-\epsilon)ND_e \rceil /N$$

is non-empty, since it contains the point $x_e = (1-\epsilon)D_e$, $e \in E(H)$. But the polyhedron is rational, and so it contains a rational point, as required.

Finally then we wish to show that lemma 0 holds when each edge $e$ of $H$ has exponential distribution with rational parameter $\lambda_e = D_e = P_e/Q$. Consider the graph $\tilde{H}$ obtained from $H$ by replacing each edge $e$ by $P_e$ parallel copies, each with edge-length exponentially distributed with parameter $1/Q$ (mean Q). Then $L(H_n)$ and $L(\hat{H}_n)$ have the same distribution, and we have already shown the required result for $L(\hat{H}_n)$.

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