ON THE INDEPENDENCE NUMBER OF RANDOM REGULAR GRAPHS

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Abstract

Let G_r denote a random r-regular graph with vertex set $\{1,2,\ldots,n\}$ and let $\alpha(G_r)$ denote its independence number. We show that with probability going to 1 as $n \to \infty$

$$\left|\alpha(G_r) - \frac{2n}{r}(\log r - \log\log r + 1 - \log 2)\right| \le \frac{\epsilon n}{r}$$

provided $r = O(n^{1/5-\theta})$, $0 < \epsilon < 1$ is constant, $r \ge r_\epsilon$, where r_ϵ depends only on ϵ and $0 < \theta \le 1/5$ is constant.

This note is concerned with the independence number of random regular graphs. Thus let REG(n,r) denote the set of r-regular graphs with vertex set $[n] = \{1,2,\ldots,n\}$. Let G_r denote a random graph sampled uniformly from REG(n,r).

The independence number $\alpha(G)$ of a graph G is the size of the largest set of vertices not containing any edge. This has been studied by, inter alia, Matula [7], Grimmett and McDiarmid [6], Bollobás and Erdös [3] and Frieze [4]. The aim of this note is to apply the approach of [4] to G_r and prove

Theorem

Let $0 < \epsilon < 1$ be fixed. There exists a constant r_ϵ such that if $r \geq r_\epsilon$, $r = O(n^{1/5-\theta})$ for some constant $0 < \theta \leq 1/5$ then

$$|\alpha(G_r) - \frac{2n}{r}(\log r - \log\log r + 1 - \log 2)| \leq \frac{\epsilon n}{r}$$

with probability going to 1 as $n \to \infty$.

□

(All logarithms are natural.)

The upper bound of the theorem is already known (at least for r constant) and straightforward to prove by the first moment method (see Bollobás [1], Theorem XI.27). The lower bound is close to what one might expect given the results of [4]. We can extend the theorem to $r = O(n^{1/5-\theta})$ because of the results of Frieze [5]).

Proof of the theorem

We shall use the model of Bollobás [2] to study G_r. We let $W = [rn]$ and $W_i = \{(i-1)r+1,\dots,ir\}$, $i = 1,2,\dots,n$ be a partition of W into n sets of size r. For $w \in W$ we define $\psi(w) = \lceil w/r \rceil$ so that $w \in W_{\psi(w)}$ holds.

A **configuration** is a partition of W into $m = \frac{1}{2} rn$ pairs. Φ denotes the set of configurations. For $F \in \Phi$ we let $\mu(F)$ be the multigraph with vertex set $[n]$ and m edges $\{\{\psi(x),\psi(y)\}: \{x,y\} \in F\}$.

We consider Φ as a probability space in which each $F \in \Phi$ is equally likely. Let Q be a property of the graphs in $REG(n,r)$ and let Q^* be a property of the configurations in Φ. Suppose these properties are such that for $G_r \in REG(n,r)$ and $F \in \phi^{-1}(G_r)$, G_r has Q if and only if F has Q^*. All we shall need from [2] and [5] is

$$(0) \qquad P(G_r \in Q) \le e^{r^2} P(F \in Q^*).$$

In the analysis we only claim that inequalities hold for r and n sufficiently large and ϵ sufficiently small.

Now for $0 < \epsilon < 1$ let $\alpha_\epsilon(F)$ denote the size of the largest independent set in $\mu(F)$ which is (i) contained in $[n_\epsilon]$, $n_\epsilon = \lfloor (1-\epsilon)n \rfloor$, (ii) of size at most $\frac{2 \log r}{r} n$.

For a positive integer s let Z_s be a random variable which counts the number of independent sets of $\mu(F)$ which are of size s and are contained in $[n_\epsilon]$.

For $F \in \Phi$ let $X_i = X_i(F) = \{p_i, q_i\}$, $p_i < q_i$, $i = 1,2,\dots,m$ denote the pairs of F sorted into lexicographically increasing order and let $\underline{X}^{(i)} = X_1, X_2, \dots, X_i$.

Let $m_\epsilon = m - \lceil \frac{r\epsilon^2 n}{10} \rceil$ and $N_\epsilon = \{v \in [n]: v \cap \psi(X_i) = \phi$ for $i > m_\epsilon\}$. Let $\alpha'_\epsilon(F)$ denote the size of the largest indpendent set in $\mu(F)$ which is (i) contained in N_ϵ and (ii) of size at most $\frac{2 \log r}{r} n$.

The theorem follows from the following

Lemma

(a)
$$P(\alpha_\epsilon(F) > \alpha'_\epsilon(F)) \le e^{-\gamma\epsilon^2 n}$$

for some absolute constant $\gamma > 0$.

(b)

Let $\bar{\alpha}'_\epsilon = E(\alpha'_\epsilon(F))$. Then

$$P(|\alpha'_\epsilon(F) - \bar{\alpha}'_\epsilon| \ge t) \le \exp\{-\frac{t^2 r\epsilon^4}{800(\log r)^2 n}\},$$

for $0 \le t \le \frac{\log r}{r} n$.

(c)

Let $k = \lceil \frac{2n}{r} (\log r - \log\log r + 1 - \log 2 - \frac{\epsilon}{4}) \rceil$. Then

$$P(Z_k > 0) \ge \exp\{-\frac{3(\log r)^2}{r^{3/2}} n\}$$

(d) $P(\alpha(F) \ge \frac{2n}{r} (\log r \log\log r + 1 - \log 2 + \epsilon)) \le \exp\{-\frac{\epsilon \log r}{r} n\}$.

□

Proof of the Theorem

Let $t_0 = \frac{\epsilon n}{4r}$. Then $\bar{\alpha}'_\epsilon \geq k - t_0$ for otherwise

$$P(Z_k > 0) \leq P(\alpha_\epsilon(F) > \alpha'_\epsilon(F)) + P(\alpha'_\epsilon(F) - \bar{\alpha}'_\epsilon > \frac{\epsilon n}{2r})$$

$$\leq e^{-\gamma\epsilon^2 n} + \exp\{-\frac{\epsilon^6 n}{12800(\log r)^2 r}\}$$

which contradicts (c).

But then if $\bar{\alpha}'_\epsilon \geq k - t_0$,

$$P(\alpha(F) < k - 2t_0) \leq P(\alpha'_\epsilon(F) < \bar{\alpha}'_\epsilon - t_0) \leq \exp\{-\frac{\epsilon^6 n}{12800\ r(\log r)^2}\}.$$

Using this, (d) and inequality (0) with

$$Q^* = \{|\alpha(F) - \frac{2n}{r}(\log r - \log\log r + 1 - \log 2)| \geq \frac{\epsilon n}{r}\}$$

establishes the theorem.

Proof of the Lemma

(a)

Now $\alpha'_\epsilon(F) \geq \alpha_\epsilon(F)$ whenever $N_\epsilon \supseteq [n_\epsilon]$ and $N_\epsilon \supseteq [n_\epsilon]$ whenever $\mu(F)$ contains at least $\frac{r\epsilon^2 n}{10}$ edges with both vertices in $[n]-[n_\epsilon]$.

Consider constructing F by first pairing off elements of $W' = W_{[n]-[n_\epsilon]}$. The first $\frac{\epsilon rn}{4}$ times we take an element of W' and find its partner, we have a probability of at least $\frac{\epsilon}{2}$ of choosing its partner in W'.

Thus the number of pairs contained in W' is dominated by $B(\lfloor\frac{\epsilon rn}{4}\rfloor, \frac{\epsilon}{2})$ and the result follows from the Chernoff bound for the tails of the binomial distribution.

(b)

We follow the proof of a simple martingale tail inequality and tighten it for our special case. Let

$$\alpha'_i = \alpha'_i(\underline{X}^{(i)}) = E(\alpha'_\epsilon(F)|\underline{X}^{(i)}), \qquad 1 \le i \le m.$$

Thus $\alpha'_\epsilon = \bar{\alpha}'_\epsilon$ and $\alpha'_m = \alpha'_\epsilon$. Since $\underline{X}^{(m_\epsilon)}$, determines the edges of $\mu(F)$ contained in N_ϵ we see in fact that $\alpha'_i = \alpha'_\epsilon$ for $i \ge m_\epsilon$.

Now $\alpha'_\epsilon - \bar{\alpha}'_\epsilon = \sum_{i=1}^{m_\epsilon} (\alpha'_i - \alpha'_{i-1})$ and so

$$P(|\alpha'_\epsilon - \bar{\alpha}'_\epsilon| \ge t) = P(\alpha'_{m_\epsilon} - \bar{\alpha}'_\epsilon \ge t) + P(\alpha'_{m_\epsilon} - \bar{\alpha}'_\epsilon \le - t)$$

$$= P(\exp\{\lambda(\sum_{i=1}^{m_\epsilon} (\alpha'_i - \alpha'_{i-1}) - t)\} \ge 1) + P(\exp\{\lambda(\sum_{i=1}^{m_\epsilon} (\alpha'_i - \alpha'_{i-1}) + t) \ge 1)$$

for all $\lambda > 0$, so from the Markov inequality

$$(1) \qquad P(|\alpha'_\epsilon - \bar{\alpha}'_\epsilon| \ge t) \le e^{-\lambda t}(E(\prod_{i=1}^{m_\epsilon} \exp\{\lambda(\alpha'_i-\alpha'_{i-1})\}) + E(\prod_{i=1}^{m_\epsilon} \exp\{-\lambda(\alpha'_i-\alpha'_{i-1})\})).$$

Now for a given $i \ge 1$

(2) $E_{\underline{X}^{(i)}}(\exp\{\lambda \sum_{t=1}^{i} (\alpha'_t-\alpha'_{t-1})\}) =$

$$E_{\underline{X}^{(i-1)}}(\exp\{\lambda \sum_{t=1}^{i-1} (\alpha'_t-\alpha'_{t-1})\}E_{X_i}(\exp\{\lambda(\alpha'_i-\alpha'_{i-1})\} \mid \underline{X}^{(i-1)}))$$

since $\alpha'_0,\alpha'_1,\ldots,\alpha'_{i-1}$ are determined by $\underline{X}^{(i-1)}$.

Now $e^x \leq x + e^{x^2}$ for all x and so

(3) $E_{X_i}(\exp\{\lambda(\alpha'_i-\alpha'_{i-1})\} \mid \underline{X}^{(i-1)}) \leq E_{X_i}(\lambda(\alpha'_i-\alpha'_{i-1}) \mid \underline{X}^{(i-1)})+E_{X_i}(\exp\{\lambda^2(\alpha'_i-\alpha'_{i-1})^2\} \mid \underline{X}^{(i-1)})$.

$$= E_{X_i}(\exp\{\lambda^2(\alpha'_i-\alpha'_{i-1})^2\} \mid \underline{X}^{(i-1)}).$$

Here we use the fact that $\alpha'_0,\alpha'_1,\ldots,\alpha'_m$ form a martingale to imply that

(4) $$E(Z) = 0 \qquad \text{where} \quad Z = (\alpha'_i - \alpha'_{i-1} \mid \underline{X}^{(i-1)}).$$

We will show that Z satisfies

(5) $$-1 \leq Z \leq \delta = \frac{20 \log r}{\epsilon^2 r},$$

which combined with (4) yields

(6) $$E_{X_i}(\exp\{\lambda^2(\alpha'_i-\alpha'_{i-1})^2\} \mid \underline{X}^{(i-1)}) \leq \frac{\delta}{1+\delta} e^{\lambda^2} + \frac{1}{1+\delta} e^{\delta^2\lambda^2}$$

(Knowing (4) and (5) we use the fact that the function $f(x) = e^{\lambda^2 x^2}$ is convex and maximise $E(e^{\lambda^2 Z^2})$ by putting $Z = -1$ with probability $\frac{\delta}{1+\delta}$ and $Z = \delta$ with probability $\frac{1}{1+\delta}$).

Proof of (5)

Fix $\underline{X}^{(i-1)} = \hat{\underline{X}}^{(i-1)} = \hat{X}_1, \hat{X}_2, \ldots, \hat{X}_{i-1}$ and let $\hat{\Phi} = \{F \in \Phi : \underline{X}^{(i-1)}(F) = \hat{\underline{X}}^{(i-1)}\}$. Let $Y = W - \bigcup_{j=1}^{i-1} \hat{X}_j$ and $\hat{x} = \min Y$, so that if $F \in \hat{\Phi}$ then $X_i(F) = \{\hat{x}, y\}$ for some $y \in Y - \{\hat{x}\}$.

For $y \in Y$ let $\Phi_y = \{F \in \hat{\Phi} : X_i(F) = \{\hat{x}, y\}\}$. If $\hat{y}, y \in Y - \{\hat{x}\}$ define $f_{\hat{y},y} : \Phi_{\hat{y}} \to \Phi_y$ as follows:

Suppose $F \in \Phi_{\hat{y}}$ and $\{x, y\} \in F$, then

$$f_{\hat{y},y}(F) = (F \cup \{\{\hat{x}, y\}, \{x, \hat{y}\}\}) - \{\{x, y\}, \{\hat{x}, \hat{y}\}\} \in \Phi_y .$$

Observe that $f_{y,\hat{y}} \circ f_{\hat{y},y}$ is the identity on $\Phi_{\hat{y}}$. Suppose now that we fix $X_i = \{\hat{x}, \hat{y}\}$ and then

$$\alpha_i(\underline{X}^{(i)}) - \alpha_{i-1}(\underline{X}^{(i-1)}) = \frac{1}{|\Phi_{\hat{y}}|} \sum_{\hat{F} \in \Phi_{\hat{y}}} \alpha'(F) - \frac{1}{|Y|} \sum_{y \in Y} \frac{1}{|\phi_y|} \sum_{F \in \Phi_y} \alpha'(F)$$

$$= \frac{1}{|Y|} \sum_{y \in Y} \frac{1}{|\Phi_{\hat{y}}|} \sum_{F \in \Phi_{\hat{y}}} (\alpha'(F) - \alpha'(f_{\hat{y},y}(F))). \tag{7}$$

Fix $\hat{F} \in \Phi_{\hat{y}}$ and an independent set $S \subseteq N_\epsilon$ of size $\alpha'(F)$. Now

(8) $$|\alpha'(f_{\hat{y},y}(\hat{F})) - \alpha'(\hat{F})| \leq 1$$

since (i) by deleting at most one member of $S \cap \psi(\{\hat{x},\hat{y},x,y\})$ we obtain an independent set in $\phi(f_{\hat{y},y}(\hat{F}))$, (ii) we can, symmetrically, compare $\alpha'(F)$, $\alpha'(f_{y,\hat{y}}(F))$ for $F \in \Phi_y$.

(9) $$\alpha'(f_{\hat{y},y}(\hat{F})) \geq \alpha'(\hat{F}) \quad \text{if} \quad S \cap \psi(\{x,y\}) = 0$$

since in this case the added edges cannot join two vertices in S.

Hence (7), (8) and (9) imply

$$-1 \leq \alpha_i(\underline{X}^{(i)}) - \alpha_{i-1}(\underline{X}^{(i+1)}) \leq \frac{1}{|Y|} |\{y \in Y-\{\hat{x}\}: \psi(\{x(y),y\}) \cap S \neq \phi\}|$$

(where $x(y)$ is defined by $\{x(y),y\} \in \hat{F}$)

$$\leq \frac{4(\log r)n}{rn-2i+1}$$

$$\leq \delta \qquad\qquad \text{as} \quad i \leq m_\epsilon$$

and we have proved (5).

Using (3) and (6) inductively in (2) yields

(10) $$E(\exp\{\lambda \sum_{i=1}^{m_\epsilon} (\alpha'_i - \alpha'_{i-1})\}) \leq \left(\frac{\delta}{1+\delta} e^{\lambda^2} + \frac{1}{1+\delta} e^{\delta^2\lambda^2}\right)^{m_\epsilon}$$

and a similar argument yields

(11) $$E(\exp\{-\lambda \sum_{i=1}^{m_\epsilon} (\alpha'_i - \alpha'_{i-1})\}) \le (\frac{\delta}{1+\delta} e^{\lambda^2} + \frac{1}{1+\delta} e^{\delta^2\lambda^2})^{m_\epsilon}.$$

It follows from (1), (7) and (8) that

$$P(|\alpha' - \bar{\alpha}'| \ge t) \le 2e^{-\lambda t}(\frac{\delta}{1+\delta} e^{\lambda^2} + \frac{1}{1+\delta} e^{\delta^2\lambda^2})^{m_\epsilon}$$

$$\le 2e^{-\lambda t + m_\epsilon \delta^2 \lambda^2}$$

provided $\lambda,\ 2\delta\lambda^2 \le 1$.

(Consider $f(x) = xe^a - e^{x^2 a}$. $f(1) = 0$ and $f'(x) = e^a - 2axe^{x^2 a} \ge 0$ if $x,\ 2ax \le 1$).

Now take $\lambda = \frac{t}{2m_\epsilon \delta^2} \le \frac{rt\epsilon^4}{300(\log r)^2 n} \le \frac{1}{300 \log r}$ for $t \le \frac{\log r}{r} n$,

so that

$$P(|\alpha_\epsilon - \bar{\alpha}_\epsilon| \ge t) \le 2 \exp\{-\frac{t^2}{4m_\epsilon \delta^2}\}$$

and (b) follows.

(c)

We prove this using the inequality

(12) $$P(Z_k > 0) \ge \frac{E(Z_k)^2}{E(Z_k^2)}.$$

Now

(13) $$E(Z_k) = \binom{n_\epsilon}{k} \prod_{i=1}^{rk-1} (1 - \frac{rk-i}{rn-2i+1})$$

$$= \binom{n_\epsilon}{k} \prod_{i=1}^{rk-1} (1 - \frac{i}{r(n-2k)+2i+1})$$

$$\geq \binom{n_\epsilon}{k} \prod_{i=1}^{rk-1} (1 - \frac{i}{r(n-2k)})$$

$$= \binom{n_\epsilon}{k} \exp\{- \sum_{i=1}^{rk-1} (\frac{i}{r(n-2k)} + \frac{i^2}{2r^2(n-2k)^2} + \ldots)\}$$

$$\geq \binom{n_\epsilon}{k} \exp\{- \frac{rk^2}{2(n-2k)} - \frac{rk^3}{6(n-2k)^2}\}$$

$$\geq \binom{n_\epsilon}{k} \exp\{- \frac{rk^2}{2n} (1 + \frac{3k}{n})\}.$$

Now,

$$E(Z_k^2) = \sum_{\substack{S \subseteq [n_\epsilon] \\ |S|=k}} \sum_{\substack{T \subseteq [n_\epsilon] \\ |T|=k}} P(\mathcal{E}_S \mathcal{E}_T)$$

(where $\mathcal{E}_S$ is the event "S is independent in $\mu(F)$")

$$= \binom{n_\epsilon}{k} \sum_{\substack{T \subseteq [n_\epsilon] \\ |T|=k}} P(\mathcal{E}_{S_0} \mathcal{E}_T) \qquad \text{where } S_0 = \{1,2,\ldots,k\}$$

(14)
$$= \binom{n_\epsilon}{k} \sum_{\ell=0}^{k} \binom{k}{\ell} \sum_{\substack{T_\ell \subseteq T \subseteq [n_\epsilon] \\ |T|=k \\ T \cap S_0 = T_\ell}} P(\mathcal{E}_{S_0} \mathcal{E}_T) \quad \text{where} \quad T_\ell = \{1,2,\ldots,\ell\}.$$

Now

$$P(\mathcal{E}_{S_0} \mathcal{E}_T) = \sum_{X \in \Omega} P(\mathcal{E}_T | X) P(X)$$

where $\Omega = \{X:\ X$ is a choice of pairings of elements of $W_{S_0} = \bigcup_{i \in S_0} W_i$ with elements of W for which $\mathcal{E}_{S_0}$ occurs$\}$. For $X \in \Omega$ suppose that when $k+1 \leq i \leq n$, X has $d'_{i,X}$ pairs $\{u,v\}$, $u \in W_{T_\ell}$, $v \in W_i$ and $d''_{i,X}$ pairs $\{u,v\}$ with $u \in W_{S_0 - T_\ell}$, $v \in W_i$. Thus

$$P(\mathcal{E}_T | X) \neq 0 \text{ iff } X \in \Omega_T = \{X \in \Omega:\ d'_{i,X} = 0 \text{ for } i \in T - T_\ell\}.$$

If $X \in \Omega_T$ then, for $d''_{T,X} = \sum_{i \in T - T_\ell} d''_{i,X}$, we have

$$P(\mathcal{E}_T | X) = \prod_{i=1}^{r(k-\ell)-d''_{T,X}} \left(1 - \frac{r(k-\ell) - d''_{T,X} - i}{r(n-2k) - 2i + 1}\right)$$

$$\leq \exp\left\{ \sum_{i=1}^{r(k-\ell)-d''_{T,X}} \frac{r(k-\ell)-d''_{T,X}-i}{r(n-2k)} \right\}$$

$$\leq 2 \exp\left\{ - \frac{(r(k-\ell) - d''_{T,X})^2}{2r(n-2k)} \right\}.$$

Hence

$$\sum_{\substack{T_\ell \subseteq T \subseteq [n_\epsilon] \\ |T|=k \\ T\cap S_0 = T_\ell}} P(\mathcal{E}_S \mathcal{E}_T) \leq 2 \sum_{\substack{T_\ell \subseteq T \subseteq [n_\epsilon] \\ |T|=k \\ T\cap S_0 = T_\ell}} \sum_{X \in \Omega_T} \exp\{-\frac{(r(k-\ell) - d''_{T,X})^2}{2r(n-2k)}\} P(X)$$

$$= 2 \sum_{\substack{T_\ell \subseteq T \subseteq [n_\epsilon] \\ |T|=k \\ T\cap S_0 = T_\ell}} P(X \in \Omega_T | X \in \Omega) P(X \in \Omega) E_X(\exp\{-\frac{(r(k-\ell) - d''_{T,X})^2}{2r(n-2k)}\} | X \in \Omega_T)$$

$$= 2 \binom{n_\epsilon - k}{k-\ell} \Big(\prod_{i=1}^{r\ell} (1 - \frac{r(k-\ell)}{r(n-k)-i})\Big) P(X \in \Omega) E_X(\exp\{-\frac{(r(k-\ell) - d''_{T_0,X})^2}{2r(n-2k)}\} | X \in \Omega_T)$$

$$(15) \quad \leq 2 \binom{n_\epsilon - k}{k-\ell} \exp\{-\frac{r\ell(k-\ell)}{n-k}\} P(X \in \Omega) E_X(\exp\{-\frac{(r(k-\ell) - d''_{T_0,X})^2}{2r(n-2k)}\} | X \in \Omega_T)$$

where $T_0 = \{1,2,\ldots,\ell,\ k+1,\ k+2,\ \ldots,\ 2k-\ell\}$.

Let now

$$\mu = E_X(d''_{T_0,X} | X \in \Omega_{T_0}) = \frac{r(k-\ell)^2}{n-k-\ell}$$

($r(k-\ell)$ pairings in which the expected individual contribution is $\frac{k-\ell}{n-k-\ell}$.)

and observe that

$$(16)\qquad P(|d''_{T_0,X}-\mu| \geq t \,|\, X \in \Omega_{T_0}) \leq 2\exp\{-\frac{t^2}{4r(k-\ell)}\}.$$

We can prove (16) using the "martingale" approach used to prove part (a). Assume some fixed choice of pairings of elements in T_ℓ with elements in $W_{\overline{S_0 \cup T_0}}$. Let W' denote $W_{\bar{S}_0}$ with these latter elements removed. Then let X_i, $1 \leq i \leq \rho = r(k-\ell)$ denote the (random) choice of "partner" in $W - [rk]$ of the element $i+r\ell$ (in $W_{S_0-T_\ell}$). We replace the random variable α' by $\Delta = d''_{T,X}$ and define $\Delta_i = E(\Delta|\underline{X}^{(i)})$. It is straightforward to show that $|\Delta_i - \Delta_{i-1}| \leq 1$ by an argument similar to that in part (a). Indeed consider a fixed $\hat{\underline{X}}^{(i-1)}$ and let $\Theta_x = \{\underline{X}^{(\rho)} : \underline{X}(i-1) = \hat{\underline{X}}^{(i-1)}$ and $X_i = x\}$ for $x \in W' - \{\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_{i-1}\}$. To prove $|\Delta_i - \Delta_{i-1}| \leq 1$ we need only construct bijections $g_{x,x'}: \Theta_x \to \Theta_{x'}$, for all x,x', so that $|\Delta(\underline{X}^{(\rho)}) - \Delta(g_{x,x'}(\underline{X}^{(\rho)}))| \leq 1$ for $\underline{X}^{(\rho)} \in \Theta_x$ and $x' \neq x$. This is easily done. If $\underline{X}^{(\rho)} \in \Theta_x$ and $x' = X_j$ for some $j > i$ let $g_{x,x'}(\underline{X}^{(\rho)})$ be $\underline{X}^{(\rho)}$ with X_i, X_j interchanged. Otherwise just replace X_i by x'. This yields $|\Delta_i - \Delta_{i-1}| \leq 1$ after arguing as in part (a). Inequality (15) now follows.

So

$$E_X(\exp\{-\frac{(r(k-\ell) - d''_{T_0,X})^2}{2r(n-2k)}\} \,|\, X \in \Omega_{T_0})$$

$$= \sum_{d''=0}^{r(k-\ell)} \exp\{-\frac{(r(k-\ell)-\mu-(d''-\mu))^2}{2r(n-2k)}\} P(d''_{T_0,X} = d'')$$

$$\leq 2 \sum_{d''=0}^{r(k-\ell)} \exp(-\frac{(r(k-\ell)-\mu-(d''-\mu))^2}{2r(n-2k)} - \frac{(d''-\mu)^2}{4r(k-\ell)}\}$$

$$(17) \qquad = 2 \sum_{d''=0}^{r(k-\ell)} \exp\{-\frac{(r(k-\ell)-\mu)^2}{2r(n-2k)} - (d''-\mu)(\frac{d''-\mu}{2r(n-2k)} + \frac{d''-\mu}{4r(k-\ell)} - \frac{r(k-\ell)-\mu}{r(n-2k)})\}.$$

Now let $\hat{d} = \mu + \frac{4(k-\ell)(r(k-\ell)-\mu)}{n-2k} \leq 6\mu$

so that $\frac{\hat{d}-\mu}{4r(k-\ell)} = \frac{r(k-\ell)-\mu}{r(n-2k)}$.

Thus the sum in (17) is bounded above by

$$2 \sum_{d''=\hat{d}}^{r(k-\ell)} \exp\{-\frac{(r(k-\ell)-\mu)^2}{2r(n-2k)}\} + 2 \sum_{d''=0}^{\hat{d}} \exp\{-\frac{(r(k-\ell)-\mu)^2}{2r(n-2k)} + \frac{5\mu(r(k-\ell)-\mu)}{r(n-2k)}\}$$

$$\leq 2 \sum_{d''=0}^{r(k-\ell)} \exp\{-\frac{(r(k-\ell)-\mu)^2}{2r(n-2k)} (1 - \frac{10\mu}{r(k-\ell)-\mu})\}$$

$$\leq 2rk \exp\{\frac{10\mu(r(k-\ell)-\mu)}{2r(n-2k)}\} \exp\{-\frac{(r(k-\ell)-\mu)^2}{2r(n-2k)}\}$$

$$= 2rk \exp\{\frac{6rk^3}{n^2}\} \exp\{-\frac{r(k-\ell)^2(n-2k)}{2(n-k-\ell)^2}\}$$

$$(18) \qquad \leq 2rk \exp\{\frac{7rk^3}{n^2}\}\exp\{-\frac{r(k-\ell)^2}{2(n-k)}\}.$$

Hence, by (14), (15) and (18)

$$E(Z_k^2) \leq 4rk \exp\{\frac{7rk^3}{n^2}\}E(X_k) \sum_{\ell=0}^{k} \binom{k}{\ell} \begin{bmatrix} n_\epsilon-k \\ k-\ell \end{bmatrix} \exp\{-(\frac{r\ell(k-\ell)}{n-k} + \frac{r(k-\ell)^2}{2(n-k)})\}.$$

Applying (12) and (13) to the above inequality and simplifying yields

$$(19) \qquad P(Z_k > 0)^{-1} \le 4rk \exp\{\frac{17rk^3}{2n^2}\} \sum_{\ell=0}^{k} \frac{\binom{k}{\ell}\begin{bmatrix} n_\epsilon - k \\ k-\ell \end{bmatrix}}{\begin{bmatrix} n_\epsilon \\ k \end{bmatrix}} \exp\{\frac{r\ell^2}{2n}\}.$$

Let $u_\ell = \dfrac{\binom{k}{\ell}\begin{bmatrix} n_\epsilon - k \\ k-\ell \end{bmatrix}}{\begin{bmatrix} n_\epsilon \\ k \end{bmatrix}} \exp\{\frac{r\ell^2}{2n}\}$.

Observe that $(A/\ell)^\ell$ is maximised at $\ell = A/e$ and so

$$(20) \qquad (A/\ell)^\ell \le e^{A/e}$$

and

$$(21) \qquad u_\ell \le (\frac{ke}{\ell} \cdot \frac{k}{n_\epsilon} \cdot \exp\{\frac{r\ell}{2n}\})^\ell$$

$$\le (\frac{k}{\ell} \cdot \frac{6 \log r}{r} \cdot \exp\{\frac{\ell d}{2n}\})^\ell.$$

<u>Case 1</u>: $0 \le \ell \le k/2$

Here $\exp\{\frac{r\ell}{2n}\} \le \sqrt{r}$ and so, by (21)

$$u_\ell \le (\frac{6\ k \log r}{\ell \sqrt{r}})^\ell$$

$$\le \exp\{\frac{6\ k \log r}{e^2 \sqrt{r}}\} \qquad , \text{ by (3)}$$

$$(22) \qquad \le \exp\{\frac{2\ (\log r)^2}{r^{3/2}}\ n\}.$$

Case 2: $k/2 < \ell \le \frac{2n}{r}(\log r - \log\log r - 3)$

By (20),

$$u_\ell \le \left(\frac{12 \log r}{r} \exp\{\frac{\ell r}{2n}\}\right)^\ell$$

$$\le \left(\frac{12 \log r}{r} \cdot \frac{r}{e^3 \log r}\right)^\ell$$

(23) $$\le 1.$$

Case 3: $\frac{2n}{r}(\log r - \log\log r - 3) < \ell \le k.$

Now

$$\frac{u_\ell}{u_{\ell+1}} = \frac{(\ell+1)(n_\epsilon - 2k + \ell + 1)}{(k-\ell)^2} \exp\{-\frac{(2\ell+1)r}{2n}\}$$

$$\le \frac{kn}{(k-\ell)^2} \frac{e^3 \log r}{r^2}.$$

Hence

$$u_\ell \le \frac{1}{((k-\ell)!)^2} \left(\frac{kne^3(\log r)^2}{r^2}\right)_k^{k-\ell} u_k,$$

$$\le \left(\frac{kne^5(\log r)^2}{(k-\ell)^2 r^2}\right)^{k-\ell} u_k.$$

Now observe that $(A/\ell^2)^\ell$ is maximised at $\ell = (A/e)^{1/2}$ and so

$$u_\ell \le \exp\{(\frac{kne^4(\log r)^2}{r^2})^{1/2}\}u_k$$

$$(24) \qquad \le \exp\{\frac{11\ (\log r)^{3/2}}{r^{3/2}}\ n\}u_k\ .$$

Now

$$u_k^{-1} = \begin{bmatrix} n_\epsilon \\ k \end{bmatrix} \exp\{-\frac{k^2 r}{2n}\}$$

$$(25) \qquad \ge (\frac{n_\epsilon e}{k} \exp\{-(\frac{k}{2n_\epsilon} + (\frac{k}{n_\epsilon})^2\} \exp\{-\frac{kr}{2n}\})^k$$

$$\ge e^{\epsilon k/5}.$$

Part (c) follows from (19), (22), (23), (24), (25).

(d)

Let now $\ell = \lceil \frac{2n}{r} (\log r - \log\log r + 1 - \log 2 + \frac{\epsilon}{2}) \rceil$ and Y be the random variable which counts the number of independent sets of $\mu(F)$ of size ℓ. Then

$$P(\alpha(F) \ge \ell) \le E(Y)$$

$$= \binom{n}{\ell} \prod_{i=1}^{r\ell-1} (1 - \frac{r\ell-i}{rn-2i+1})$$

$$\le \binom{n}{\ell} \sum_{i=1}^{2\ell-1} (1 - \frac{r\ell-i}{rn})$$

$$\le 2\binom{n}{\ell}\exp\{-\frac{r\ell^2}{2n}\}$$

$$\le 2\left(\frac{ne}{\ell}\exp\{-\frac{r\ell}{2n}\}\right)^{\ell}$$

and (d) follows. □

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