ONE DIMENSIONAL INFINITE-HORIZON VARIATIONAL PROBLEMS ARISING IN VISCOELASTICITY

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Dedicated to Bernard D. Coleman in
celebration of his sixtieth birthday.

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§1. Introduction

In this paper we study a variational problem for real valued functions defined on an infinite semiaxis of the line. To wit, given $\mathbf{x} \in \mathbb{R}^2$ we seek a "minimal solution" to the problem

$$(P_\infty) \qquad \begin{gathered}\text{Minimize the functional given by} \\ I(w(\cdot)) = \int_0^\infty f(w(s),\dot{w}(s),\ddot{w}(s))ds, \\ w \in A_{\mathbf{x}} = \{v \in W^{2,1}_{\ell oc}(0,\infty) \; : \; (v(0),\dot{v}(0)) = \mathbf{x}\}.\end{gathered}$$

Here $W^{2,1}_{\ell oc} \subset C^1$ denotes the Sobolev space of functions possessing a locally integrable second derivative, and $f = f(w,p,r)$ is a smooth function satisfying

$$(1.1) \quad f_{rr} > 0, \; f(w,p,r) \geq a|w|^\alpha - b|p|^\beta + c|r|^\gamma - d \qquad (a,b,c,d > 0)$$
$$\text{where} \;\; \alpha,\gamma \in (1,\infty), \; \beta \in [1,\infty) \text{ satisfy } \; \alpha > \beta, \; \gamma \geq \beta,$$

as well as an upper growth condition to be described in §2.

It can be appreciated that the notion of minimal solution for (P_∞) is a subtle one, since the infimum of I on $A_{\mathbf{x}}$ is typically either $+\infty$ or $-\infty$. The formulation which is best suited to our problem will be described and analyzed in §3. It will also be shown in §3 that the analysis given for (P_∞) applies to similar problems involving a functional identical to I except for the fact that integration is taken along the entire real line.

Our interest in variational problems of the form (P_∞) stems from a one-dimensional model recently proposed by Bernard Coleman to describe the equilibrium behavior of a long slender bar of polymeric material under tension. It involves a fiber of material distributed along an infinite interval and possessing an equilibrium specific Helmholtz free energy function which, formally, is a higher order version of the van der Waals/ Cahn-Hilliard mean free energy for the density of a two phase fluid ([vdW], [C & H], see also [CGS]). This model goes beyond a model previously analyzed by Coleman in which, starting from a dynamical framework and a general nonlocal constitutive assumption for the stress in a slender rod of polymer, he arrived, by the use of quasistatic- and retardation-type approximations in the limit of zero radius, at a lower order constitutive relation for the equilibrium stress in a stressed one-dimensional fiber ([C1], [C2]). This lower order relation includes, as an important special case, constitutive formulas for the equilibrium stress in a finite fiber which arise from the minimization, under a fixed length constraint, of any one of a large class of free energies of the van der Waals/Cahn-Hilliard type.

To describe the new, higher order, model we utilize an unstressed reference configuration R for the material fiber, where $R = [Z_1, Z_2]$ is a long but finite interval and Z denotes the coordinate in R. The location z in the stressed fiber of the material point at Z in R is given in the form

$$z = \hat{z}(Z) \quad , \quad Z \in [Z_1, Z_2].$$

(time does not enter in the present equilibrium model). Then if we denote the equilibrium <u>stretch</u> <u>ratio</u> (or "stretch") of the stressed fiber at the material point at Z in R by

$$\lambda(Z) = \hat{z}'(Z),$$

it is stipulated that, when the material is held under a fixed tension, the stress at the point Z will be that combination of the values of $\lambda(\cdot)$ and its derivatives at Z which is obtained by minimization of the free energy functional

$$(1.2) \qquad I_{Z_1,Z_2}(\lambda(\cdot)) = \int_{Z_1}^{Z_2} f(\lambda(Z),\lambda'(Z),\lambda''(Z))dZ,$$

under the constraint that the fiber have a prescribed length:

$$(1.3) \qquad \int_{Z_1}^{Z_2} \lambda(Z)dZ = \ell.$$

The form of free energy integrand proposed in this model is given by

$$(1.4) \qquad f(w,p,r) = \Psi(w) - \frac{b}{2}p^2 + \frac{c}{2}r^2 \qquad (b,c > 0),$$

where Ψ is any function possessing some of the basic features of the van der Waals/Cahn-Hilliard potential, for instance

$$(1.5)\quad \Psi(w) = a(w - w_1)^2(w - w_2)^2, \quad w \in \mathbb{R}, \quad \text{with } a > 0,\ w_2 > w_1.$$

Note that the function f given by (1.4), (1.5) obviously satisfies (1.1); in fact, much of our analysis permits a, b, c themselves to vary with w and p. We mention that the characterization of equilibrium states by means of (1.2), (1.3) is the one appropriate to a fiber held in a "hard device", one that maintains the fiber at length ℓ.

It will be shown in §2 that the functional I_{Z_1,Z_2} in (1.2) is bounded below. It then follows by a standard argument involving lower semicontinuity that there exists a stretch field $\lambda(\cdot)$ minimizing I_{Z_1,Z_2} subject to (1.3). Moreover, for f as in (1.4), (1.5), $\lambda(\cdot)$ is four times continuously differentiable and satisfies the Euler-Lagrange equation:

$$(1.6)\qquad \frac{d^2}{dZ^2}(c\lambda_{ZZ}) - \frac{d}{dZ}(b\lambda_Z) + \Psi'(\lambda) = T^0 \quad , \quad Z \in (Z_1, Z_2).$$

Moreover, the tension T^0, which arises as a Lagrange multiplier associated with the constraint (1.3), is uniform over the fiber.

Since we are interested in very long physical fibers we are led to examine limiting cases in which $R = [0,\infty)$ or $R = (-\infty,\infty)$. In such cases the fixed length requirement (1.3) is useless, and we are instead led to postulate that the value T^0 of the tension is specified. This corresponds to the replacement of f in (1.4) by

$$(1.4_0)\qquad f_0(w,p,r) = (\Psi(w) - T^0 w) - \frac{b}{2}p^2 + \frac{c}{2}r^2 \qquad (b,c > 0).$$

It is easily verified that f_0 satisfies the conditions (1.1), whatever be the value $T^0 \in \mathbb{R}$. Thus the first limiting case gives rise to problem (P_∞), the second limiting case to an analogous problem on $(-\infty,\infty)$. For convenience we restrict ourselves for the remainder of this section to the integrand f in (1.4), (1.5). It will be shown in §6 that if the parameter b is sufficiently large then the energy integral $I(\lambda(\cdot))$ in (P_∞) will have the value $-\infty$ for some choices $\lambda(\cdot) \in A_x$. Thus one cannot minimize (P_∞) in the usual sense. One way to overcome that difficulty is to consider the expression

$$(1.7) \qquad J(\lambda(\cdot)) = \liminf_{L\to\infty} \frac{1}{L} \int_0^L f(\lambda(Z),\lambda'(Z),\lambda''(Z))dZ$$

and to look for a stretch field which minimizes J. In this paper we employ a more refined criterion to specify what is meant by a minimal solution for (P_∞), one which is a weakened version of that known in the control theory literature as the <u>overtaking optimality criterion</u> ([B & H], [Ca], [A & L]). The modification which we introduce is closely connected with the notion of <u>minimal energy configuration</u> employed by Aubry and le Daeron in the analysis of an infinite discrete model for crystals which undergo phase transitions in their ability to conduct electricity ([A & D]). This model, due to Frenkel & Kontorova, is the object of current research by several investigators ([A & D], [G & C], [C & D], [Ma]).

The paper is organized as follows. In section 2 we specify our notation and analyze the fixed endpoint variational problem, with f as in

(1.1), corresponding to the integral in (P_∞) but taken over a bounded interval. In section 3 we describe our criterion for a solution of (P_∞) to be minimal. In section 4 we demonstrate the existence of a minimal energy solution, and in section 5 we establish our main result: there always exists a periodic minimal solution for (P_∞). Then in section 6 we prove that in the special case (1.4), (1.5) there is a threshold effect; for fixed a,c there is a value $b_0 > 0$, such that for $b \in (0,b_0)$ the periodic minimal solution mentioned above is constant, while for $b > b_0$ the infimum of I is $-\infty$ and the periodic solution whose existence was shown in section 5 is nonconstant. Finally, in an appendix (section 7) we establish an analytic result utilized in section 4 which may be of independent interest.

§2. The bounded interval problem

In pursuing our goal of analyzing the infinite semi-interval problem (P_∞) we begin by considering, for each $T > 0$ and $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$, the following variational problem for real valued functions on $[0,T]$:

Minimize the functional

$$(P_T) \quad I_T(w(\cdot)) = \int_0^T f(w(t),\dot{w}(t),\ddot{w}(t))dt,$$

$$w(\cdot) \in A_{\mathbf{x},\mathbf{y}} = \{v \in W^{2,1}(0,T) : (v(0),\dot{v}(0)) = \mathbf{x},\ (v(T),\dot{v}(T)) = \mathbf{y}\}.$$

The function $f = f(w,p,r)$ is assumed to be smooth and to satisfy

$$(2.1) \quad \begin{array}{lll} \text{(i)} & f_{,rr} \geq 0 & \\ \text{(ii)} & f(w,p,r) \geq a|x|^\alpha - b|p|^\beta + c|r|^\gamma - d, & a,b,c,d > 0, \\ \text{(iii)} & f(w,p,r) \leq \varphi(w,p) + c'|r|^\gamma, & c' > 0, \end{array}$$

where $\alpha,\gamma > 1$, $\beta \geq 1$ satisfy $\beta < \alpha$, $\beta \leq \gamma$, and φ is continuous.

If we utilize the Sobolev spaces $X = L^\beta(0,T)$, $Y = W^{1,\beta}(0,T)$, $Z = W^{2,\beta}(0,T)$, then it is an elementary consequence of the Arzelá-Ascoli theorem that these Banach spaces are compactly imbedded as follows:

$$Z \subset Y \subset X.$$

Hence by a result of Lions-Magenes [L & M, v.1, p. 102] it follows that for each $\eta > 0$, there exists a $C(\eta) > 0$ such that

$$\|v\|_{W^{1,\beta}} \le \eta\|v\|_{W^{2,\beta}} + C(\eta)\|v\|_{L^\beta}, \quad \forall\, v \in W^{2,\beta}(0,T). \tag{2.2}$$

It follows that

$$\|\dot{v}\|_{L^\beta} \le \eta\|\ddot{v}\|_{L^\beta} + \eta\|\dot{v}\|_{L^\beta} + (C(\eta) + \eta)\|v\|_{L^\beta},$$

so that for $\eta < 1$

$$(1-\eta)^\beta\|\dot{v}\|^\beta_{L^\beta} \le (2\eta)^\beta\|\ddot{v}\|^\beta_{L^\beta} + [2(C(\eta)+\eta)]^\beta\|v\|^\beta_{L^\beta}, \quad \forall\, v \in W^{2,\beta}(0,T).$$

Putting $\eta' = [2\eta/(1-\eta)]^\beta bc^{-1}$ we conclude that for each $\eta' > 0$ there exists $D(\eta') > 0$ satisfying

(2.3)
$$\int_0^T b|\dot{v}(t)|^\beta dt \le \eta' \int_0^T c|\ddot{v}(t)|^\beta dt + D(\eta') \int_0^T |v(t)|^\beta dt, \quad \forall\, v \in W^{2,\beta}(0,T).$$

Now for each $\delta > \beta$ and $a > 0$ there exists a $K = K(\eta',\delta,a) \ge 1$ satisfying

$$ar^\delta + D(\eta')(\eta')^{-1}K^\beta \ge D(\eta')(\eta')^{-1}r^\beta, \quad \forall\, r > 0.$$

Moreover, for each $\gamma \geq \beta$ and $c > 0$ one has

$$c\, r^{\gamma} + c \geq c\, r^{\beta} \quad , \quad \forall\, r > 0.$$

Hence (2.3) ensures that for some $P = P(\eta') \geq 0$ one has

(2.4)
$$(\eta')^{-1} \int_0^T b|\dot{v}(t)|^{\beta} dt \leq \int_0^T (c|\ddot{v}(t)|^{\gamma} + a|v(t)|^{\alpha})dt + P, \quad \forall\, v \in W^{2,\beta}(0,T).$$

(e.g., $P(\eta') = D(\eta')(\eta')^{-1}K^{\beta}T + cT$ will do). By taking $\eta' \leq 1/2$, say, we deduce that for some constants $R = R(\eta') > 0$, $Q = Q(\eta') > 0$ one has,

$$(2.5) \quad \int_0^T f(v(t),\dot{v}(t),\ddot{v}(t))dt \geq \int_0^T R|\dot{v}(t)|^{\beta} dt - Q, \quad \forall\, v \in W^{2,\beta}(0,T).$$

Thus $I = I_T$ is bounded below on $A_{\mathbf{x},\mathbf{y}}$ for each $T > 0$, $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$.

Remark 2.1 By utilizing the existence for each $T' > T > 0$ of a bounded extension operator $E : W^{2,\beta}(0,T) \to W^{2,\beta}(0,T')$ (E can be chosen uniformly bounded for $T,T' \in [c,C]$ whenever $0 < c < C < \infty$) one readily concludes that $C(\eta)$ in (2.2) may be chosen uniformly for T varying in any compact subinterval of $(0,\infty)$. Thus the constants P,Q,R in (2.4) and (2.5) can be chosen uniformly for T varying in any such interval.

Moreover, we have the following result.

<u>Theorem 2.1</u> The function $U_T : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

(2.6) $$U_T(\mathbf{x},\mathbf{y}) = \inf_{w(\cdot)\in A_{\mathbf{x},\mathbf{y}}} I_T(w(\cdot))$$

satisfies

(2.7) $$\lim_{|\mathbf{x}|+|\mathbf{y}|\to\infty} U_T(\mathbf{x},\mathbf{y}) = +\infty.$$

<u>Note</u>: Hereafter we omit the subscript T where no confusion will arise.

<u>Proof</u>: Given $M > 0$, it follows from (2.5) that

$$I(v(\cdot)) \geq M \quad \text{whenever} \quad \int_0^T |\dot{v}(t)|^\beta dt \geq \frac{1}{R}[M + Q].$$

Thus it will suffice to show that even for those $v(\cdot)$ satisfying

(2.8) $$\int_0^T |\dot{v}(t)|^\beta dt < \frac{1}{R}[M + Q],$$

one has $I(v(\cdot)) \geq M$ provided that $|\mathbf{x}| + |\mathbf{y}|$ is sufficiently large. Suppose first that $|x_1| = |v(0)|$ is sufficiently large so that (2.8) implies

(2.9) $$|v(t)| \geq S \quad , \quad 0 \leq t \leq T,$$

where S satisfies

$$(2.10) \qquad S^{\alpha} \geq \frac{1}{aT}[M + dT + \frac{b}{R}(Q + M)].$$

Then by (2.1 ii) and (2.8)-(2.10),

$$(2.11) \qquad I(v(\cdot)) \geq \int_0^T a|v(t)|^{\alpha}dt - \int_0^T b|\dot{v}(t)|^{\beta}dt - dT \geq M.$$

Similarly, if $|y_1| = |v(T)|$ is sufficiently large, (2.8) again implies that (2.9) holds and (2.11) follows.

Finally, suppose that (2.8) holds while $|x_1|, |y_1|$ are sufficiently small so that the preceding argument does not apply. Note that (2.8) ensures that for some $t_0 \in (0,T)$

$$|\dot{v}(t_0)| \leq [\frac{1}{RT}(M + Q)]^{1/\beta} =: \sigma.$$

Thus if $|x_2| = |\dot{v}(0)| \geq S'$, where S' satisfies

$$(2.12) \qquad S' \geq \sigma + c^{-1/\gamma}T^{1/\gamma'}[M + \frac{b}{R}(M + Q) + dT]^{1/\gamma},$$

then Hölder's inequality gives

$$t_0^{1/\gamma'} \left[\int_0^T |\ddot{v}(t)|^{\gamma}dt\right]^{1/\gamma} \geq \int_0^{t_0} |\ddot{v}(t)|dt \geq S' - \sigma \quad , \quad (\frac{1}{\gamma} + \frac{1}{\gamma'} = 1).$$

Hence (2.1 ii), (2.8), and (2.12) imply

$$(2.13) \qquad I(v(\cdot)) \geq M.$$

A similar argument leads to (2.13) if $|y_2| = |\dot{v}(T)| \geq S'$. This concludes the proof. □

<u>Remark 2.2</u>. A simple modification of this argument (basically by replacing T by $T/2$) reveals that (putting $m(w) := \max\{|w(t)| + |\dot{w}(t)|,\ t \in [0,T]\}$)

$$\lim_{m(w)\to\infty} I_T(w(\cdot)) = +\infty.$$

That is, for each $M > 0$ there exists a rectangle

$$Q_M = \{\mathbf{x} : |x_1| \leq S, |x_2| \leq S'\}$$

such that $I_T(w(\cdot)) \geq M$ for any $w(\cdot)$ such that the corresponding trajectory $t \mapsto \mathbf{x}(t)$, $0 \leq t \leq T$, is not entirely contained in Q_M.

It is an elementary exercise to show that $U_T(\cdot,\cdot)$ is bounded on bounded sets, for instance by constructing polynomials belonging to $A_{\mathbf{x},\mathbf{y}}$, for each $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$. We proceed to show that $U_T(\cdot,\cdot)$ is actually continuous.

<u>Theorem 2.2</u> For each $T > 0$, the function

$$U_T : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

defined in (2.6) is continuous.

<u>Proof</u>: 1. <u>Lower semicontinuity</u>. Given $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$, it follows from the convexity and growth conditions (2.1 i), (2.1 ii) [cf. (2.5)] (recall that $\gamma > 1$) that there exists a minimizer $w(\cdot) \in A_{\mathbf{x},\mathbf{y}}$ for the functional I_T (cf. Morrey [M, Theorem 1.91] or Giaquinta [G, Theorem 3.1]). Moreover by (2.4) one obtains the estimate

(2.14)

$$U_T(\mathbf{x},\mathbf{y}) = \int_0^T f(w(t),\dot{w}(t),\ddot{w}(t))dt \geq (1-\eta') \int_0^T (c|\ddot{w}(t)|^\gamma + a|w(t|^\alpha)dt - (1-\eta')P.$$

Given any sequence $(\mathbf{x}_k,\mathbf{y}_k) \to (\mathbf{x},\mathbf{y})$, let us denote by $w_k(\cdot)$ a minimizer belonging to $A_{\mathbf{x}_k,\mathbf{y}_k}$, $k \geq 1$. It follows from (2.14) and the local boundedness of U_T that the functions $\{w_k(\cdot)\}$ form a bounded subset of $W^{2,\gamma}(0,T)$. Since $\gamma > 1$ we can suppose, by extracting a subsequence and re-indexing, that for some $v(\cdot) \in W^{2,\gamma}$

$$w_k(\cdot) \to v(\cdot) \text{ weakly in } W^{2,\gamma}.$$

Thus

(2.15) $$w_k(t) \to v(t),\ \dot{w}_k(t) \to \dot{v}(t), \quad \text{uniformly in } [0,T],$$
$$\ddot{w}_k(\cdot) \to \ddot{v}(\cdot) \text{ weakly in } L^\gamma.$$

These relations ensure by a lower semicontinuity result of Tonelli's (e.g. cf. Giaquinta [G, Ch. 1, Theorem 2.3]) that

$$\text{(2.16)} \qquad \int_0^T f(v(t),\dot{v}(t),\ddot{v}(t))dt \le \liminf_{k\to\infty} U_T(\mathbf{x}_k,\mathbf{y}_k).$$

Moreover, by (2.15)

$$(v(0),\dot{v}(0)) = \mathbf{x},\ (v(T),\ \dot{v}(T)) = \mathbf{y},$$

so that (2.16) implies

$$U_T(\mathbf{x},\mathbf{y}) \le \liminf_{k\to\infty} U_T(\mathbf{x}_k,\mathbf{y}_k)$$

This completes the proof of lower semicontinuity.

2. Upper semicontinuity. Let $w(\cdot) \in A_{\mathbf{x},\mathbf{y}}$ denote as above a minimizer for I_T, and suppose $(\mathbf{x}_k,\mathbf{y}_k) \to (\mathbf{x},\mathbf{y})$ as $k \to \infty$. Put

$$\mathbf{u}^k = \mathbf{x}_k - \mathbf{x},\ \mathbf{v}^k = \mathbf{y}_k - \mathbf{y},$$

so that $\mathbf{u}^k,\mathbf{v}^k \to \mathbf{0}$ as $k \to \infty$, and define

$$\delta_k(t) = a_k + b_k t + c_k t^2 + d_k t^3,$$

where the coefficients are chosen so that

$$(2.17) \qquad (\delta_k(0),\dot{\delta}_k(0)) = \mathbf{u}^k, \ (\delta_k(T),\dot{\delta}_k(T)) = \mathbf{v}^k, \ k \geq 1.$$

(Explicitly

$$a_k = u_1^k, \ b_k = u_2^k, \ c_k = -\frac{1}{T^2}[3(u_1^k - v_1^k) + T(v_2^k + 2u_2^k)]$$

$$d_k = \frac{1}{T^3}[2(u_1^k - v_1^k) + T(u_2^k + v_2^k)].)$$

It is easy to see that

$$(2.18) \quad \delta_k(t),\dot{\delta}_k(t),\ddot{\delta}_k(t) \to 0 \ \text{ uniformly for } \ t \in [0,T], \text{ as } \ k \to \infty.$$

We define

$$z_k(t) = w(t) + \delta_k(t) \quad , \quad k \geq 1.$$

Then $z_k(\cdot) \in A_{\mathbf{x}_k,\mathbf{y}_k}$ by (2.17), so that (2.18) implies

$$|z_k(t)| \leq |w(t)| + 1, \ |\dot{z}_k(t)| \leq |\dot{w}(t)| + 1, \ |\ddot{z}_k(t)| \leq |\ddot{w}(t)| + 1$$

for all $t \in [0,T]$ and all sufficiently large k. Consequently since $w(\cdot)$ and $\dot{w}(\cdot)$ take values in bounded sets (2.1 iii) implies that

$$|f(z_k(t),\dot{z}_k(t),\ddot{z}_k(t))| \leq \text{const} + c(|\ddot{w}(t)| + 1)^\gamma$$

$$\leq A + B|\ddot{w}(t)|^\gamma \quad , \quad t \in [0,T]$$

for some constants A, B and all sufficiently large k. It now follows from the definition of $z_k(\cdot)$ and the dominated convergence theorem that

$$U_T(\mathbf{x},\mathbf{y}) = I_T(w(\cdot)) = \lim_{k\to\infty} \int_0^T f(z_k(t),\dot{z}_k(t),\ddot{z}_k(t))dt,$$

and since

$$I_T(z_k) \geq U_T(\mathbf{x}_k,\mathbf{y}_k)$$

this implies that

$$U_T(\mathbf{x},\mathbf{y}) \geq \limsup_{k\to\infty} U_T(\mathbf{x}_k,\mathbf{y}_k). \qquad \square$$

By a simple modification of the above proof, utilizing the extension operators $E_{T,T'}$, from $W^{2,\beta}([0,T])$ to $W^{2,\beta}([0,T'])$, we can easily deduce

Corollary 2.3. The mapping $(T,\mathbf{x},\mathbf{y}) \mapsto U_T(\mathbf{x},\mathbf{y})$ is continuous for $T > 0$, $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$.

§3. The optimality criterion

We will treat problem (P_∞) as a minimization, in the limit as $T \to \infty$, of the following functionals

$$(3.1) \qquad \begin{aligned} &I_T(w(\cdot))=\int_0^T f(w(s),\dot{w}(s),\ddot{w}(s))ds, \\ &w(\cdot) \in A_{\mathbf{x}}:= \{v \in W^{2,1}_{\ell oc} : (v(0),\dot{v}(0)) = \mathbf{x}\}. \end{aligned}$$

(Since this is an equilibrium problem, the use of T as parameter should cause no confusion.) However, in many cases it turns out that for every $w(\cdot) \in A_{\mathbf{x}}$ one has $I_T(w(\cdot)) \to \infty$ as $T \to \infty$, in which case the minimization of $I_\infty(w(\cdot))$ has no meaning. Alternatively, it may turn out that there are functions $w(\cdot) \in A_{\mathbf{x}}$ for which $I_T(w(\cdot)) \to -\infty$ as $T \to \infty$, in which case a straightforward minimization of $I_\infty(w(\cdot))$ again has no meaning.

As pointed out in the introduction, one way to reduce this difficulty is to minimize the 'average energy over large intervals', that is to minimize the functional J defined by

$$(3.2) \qquad J(w(\cdot)) = \liminf_{T\to\infty} \frac{1}{T} I_T(w(\cdot)), \; w(\cdot) \in A_{\mathbf{x}}.$$

The infimum of the values assumed by J, namely

$$(3.3) \qquad \mu = \inf J(w(\cdot)) \quad , \quad w(\cdot) \in A_{\mathbf{x}},$$

is then called the minimal growth rate of the energy (it is easily seen that μ is independent of the initial vector $\mathbf{x}$). Unfortunately, this approach also suffers from a serious drawback, which we describe for the case in which the infimum in (3.3) is actually attained. Given an interval $[0,T_0]$, where T_0 may be arbitrarily large, let $w_0(\cdot)$ be <u>any</u> given element of $A_{\mathbf{x}}$ subject only to the condition

$$(w(T_0),\dot{w}(T_0)) = \mathbf{x}.$$

Now define $w_0(\cdot)$ on the half axis $[T_0,\infty)$ in such a way that $s \mapsto w_0(s + T_0)$ is in $A_{\mathbf{x}}$, with

$$\ell im \inf_{T\to\infty} \frac{1}{T - T_0} \int_{T_0}^{T} f(w_0(s),\dot{w}_0(s),\ddot{w}_0(s))ds = \mu.$$

Clearly the extension of $w_0(\cdot)$ to $[0,\infty)$ obtained in this way satisfies

$$J(w_0(\cdot)) = \mu \quad , \quad w_0(\cdot) \in A_{\mathbf{x}}.$$

This is an unsatisfactory situation since for us the infinite horizon problem (P_∞) is merely a mathematical idealization for modelling problems on large intervals, while the above function $w_0(\cdot)$ is a very poor approximation on an interval of length T_0, where T_0 may be very large. This arbitrariness in the definition of $w(\cdot)$ on an initial interval can be removed by imposing some condition of stationarity. However, most conditions of this sort, such as periodicity say, are rather artificial.

Another type of optimality criterion for infinite horizon problems was introduced in the economics literature by Gale [Ga] and von Weizsacker [vW] and has been used in control theory by e.g. Brock and Haurie [B & H], Carlson [Ca] and Artstein and Leizarowitz [A & L]. It is referred to as the "overtaking optimality criterion".

<u>Definition</u> 3.1. A function $w^*(\cdot) \in A_{\mathbf{x}}$ will be called <u>overtaking minimal</u> relative to $\mathbf{x}$ if

$$\limsup_{T\to\infty}[I_T(w^*(\cdot)) - I_T(w(\cdot))] \leq 0, \text{ for all } w(\cdot) \in A_{\mathbf{x}}.$$

Thus if $w^*(\cdot)$ is overtaking minimal then for each $\epsilon > 0$ and $w(\cdot) \in W^{2,1}_{\ell oc}$ with $\mathbf{x}(0) = \mathbf{x}^*(0)$, there is a T_0 such that $I_T(w^*(\cdot)) < I_T(w(\cdot)) + \epsilon$, for every $T > T_0$. This implies, in particular, that $w^*(\cdot)$ is a minimizer of J in (3.2).

Just as the minimal growth rate criterion of minimizing $J(\cdot)$ is too loose, since there are infinitely many functions with a minimal 'average energy over large intervals', the overtaking optimality condition is too strict, and in general there will be no overtaking minimal functions. However, a closely related notion was considered by Aubry and le Daeron [A & D] in their study of the discrete Frankel-Kontorova model describing one-dimensional crystals with phase transitions (cf [N]). There they minimized an energy expression of the form

$$\sum_{k=-N}^{M} u(x_k,x_{k+1}) \text{ , with } -\infty < \dots < x_k < x_{k+1} < \dots < \infty,$$

as $N,M \to \infty$ using the following criterion. A sequence $\{x_k^*\}_{k=-\infty}^{\infty}$ is called a <u>minimal energy configuration</u> if for each $M,N > 0$ the inequality

$$\sum_{k=-N}^{M} u(x_k^*,x_{k+1}^*) \le \sum_{k=-N}^{M} u(x_k,x_{k+1})$$

holds for every increasing sequence $\{x_k\}_{k=-N}^{M}$ satisfying

$$x_{-N} = x_{-N}^*,\ x_M = x_M^*$$

An analogous criterion can be adapted to our framework, as follows.

<u>Definition 3.2</u>. A function $w^* \in W_{\ell oc}^{2,1}$ is called a <u>locally-minimal energy configuration</u> if

$$\int_{T_1}^{T_2} f(w^*(s),\dot{w}^*(s),\ddot{w}^*(s))ds \le \int_{T_1}^{T_2} f(w(s),\dot{w}(s),\ddot{w}(s))ds$$

for each T_1,T_2 such that $0 \le T_1 < T_2$ and each $w \in W^{2,1}([T_1,T_2])$ satisfying:

$$(w(T_1),\dot{w}(T_1)) = (w^*(T_1),\dot{w}^*(T_1)),\ (w(T_2),\dot{w}(T_2)) = (w^*(T_2),\dot{w}^*(T_2)).$$

If in addition to the above property $w^*(\cdot)$ also provides the minimal growth rate of energy, then $w^*(\cdot)$ is called a minimal energy configuration.

It is clear that if $w^*(\cdot)$ is overtaking minimal then it is also a minimal energy configuration. In the next section we will construct for each $\mathbf{x} \in \mathbb{R}^2$ a $w^*(\cdot) \in A_{\mathbf{x}}$ which is a minimal energy configuration. The analysis given there will involve a reformulation of (P_∞) in discrete terms, but as will be seen, the reformulation is not an approximation to (P_∞).

The discrete problem to be analyzed in §4, is of the following type. Consider expressions of the form

$$C_N(X) = \sum_{k=0}^{N-1} v(x_k, x_{k+1}), \tag{3.4}$$

for a given $\mathbf{x} \in \mathbb{R}^2$, where $X = \{x_k\}_{k=0}^{\infty}$ is a sequence in $\mathbb{R}^2$ such that $x_0 = \mathbf{x}$ and $v : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying

$$v(\mathbf{x},\mathbf{y}) \to \infty \quad \text{as} \quad |\mathbf{x}| + |\mathbf{y}| \to \infty. \tag{3.5}$$

Remark: It will be seen that if $\mathbf{x}(\cdot) = \begin{bmatrix} w(\cdot) \\ \dot{w}(\cdot) \end{bmatrix}$ is (globally) bounded for the locally-minimal energy configuration $w(\cdot)$ then $w(\cdot)$ is automatically a minimal energy configuration.

It is desired to minimize $C_N(X)$ as $N \to \infty$, either in the overtaking sense or in the weaker sense of minimal energy configuration. A study of this problem was presented in [L]. There it was shown that when (3.5) holds one can restrict attention, insofar as optimality considerations are concerned, to sequences X lying inside some fixed ball:

$$|x_k| \leq L \qquad \forall\ k \geq 0,$$

where $L > 0$ is a constant which does not depend on X (see [L], Theorem 8.1). Moreover the following result was proved ([L], Theorem 3.1).

<u>Theorem 3.3</u>. Let $v : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying (3.5). Given an $x \in \mathbb{R}^2$ consider the expressions $C_N(X)$ where $x_0 = x$. Then there exist constants μ and M such that

1. For every $X = \{x_k\}_{k=0}^{\infty}$ the inequality

$$\sum_{k=0}^{N} [v(x_k, x_{k+1}) - \mu] \geq -M$$

holds for all $N \geq 1$,

2. There is a sequence X^* satisfying

$$\left|\sum_{k=0}^{N} [v(x_k^*, x_{k+1}^*) - \mu]\right| \leq M \quad , \qquad \forall\ N \geq 1.$$

The scalar μ describes the minimal growth rate for average energy of the energy expressions $C_N(X)$ in (3.4). By Theorem 3.3 every such expression is bounded below by a linear function of N whose slope is μ, while there is a sequence X^* for which $C_N(X^*)$ is bounded both from above and below by such functions.

Definition 3.4. A bounded sequence X^* will be referred to as a minimal energy sequence if for each $N_2 > N_1 > 0$ the inequality

$$\sum_{k=N_1}^{N_1-1} v(x_k^*, x_{k+1}^*) \leq \sum_{k=N_1}^{N_2-1} v(x_k, x_{k+1})$$

holds for every sequence $\{x_k\}_{k=N_1}^{N_2}$ satisfying

$$x_{N_1} = x_{N_1}^* \quad , \quad x_{N_2} = x_{N_2}^* .$$

Now it has also been shown [L, Prop. 5.1] that Theorem 3.3 is equivalent to the following result.

Theorem 3.5. Let $v : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy (3.5). Then $v(\cdot,\cdot)$ can be decomposed in the form

$$v(x,y) = \mu + \pi(x) - \pi(y) + \theta(x,y), \tag{3.6}$$

where μ is a constant, $\pi : \mathbb{R}^2 \to \mathbb{R}$ is continuous, and $\theta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying

$$\min_{\mathbf{y} \in \mathbb{R}^2} \theta(\mathbf{x},\mathbf{y}) = 0 \quad , \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^2. \tag{3.7}$$

§4. Existence of a minimal energy configuration

In this section we will prove the existence, for each $\mathbf{x} \in \mathbb{R}^2$, of a minimal energy configuration in $A_{\mathbf{x}}$. The construction will be given in two stages. First we consider a discrete reformulation of our problem and construct a minimal energy sequence $\mathbf{X}^* = \{x_k^*\}_{k=0}^{\infty}$ for it (recall Definition 3.4). This sequence will determine the values of $(w^*(\cdot),\dot{w}^*(\cdot))$ at the points $\{kT\}_{k=0}^{\infty}$, for some fixed $T > 0$, of a minimal energy configuration $w^*(\cdot)$ via,

$$(w^*(kT),\dot{w}^*(kT)) = x_k^* \quad , \quad k \geq 0.$$

Then $w^*(\cdot)$ will be determined in each interval $[kT,(k + 1)T]$ as a minimizer over $W^{2,1}([kT,(k + 1)T])$ for

$$\int_{kT}^{(k+1)T} f(w(s),\dot{w}(s),\ddot{w}(s))ds, \quad \text{subject to}$$

$$(w(kT),\dot{w}(kT)) = x_k^*, \ (w((k + 1)T),\dot{w}((k + 1)T)) = x_{k+1}^*.$$

For a fixed $T > 0$ consider the function $U_T : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined in (2.6). Examine the energy expressions associated with U_T, namely the quantities $C_N(\mathbf{X})$ defined for each sequence $\mathbf{X} = \{x_k\}_{k=0}^{\infty} \subset \mathbb{R}^2$ by

$$C_N(\mathbf{X}) = \sum_{k=0}^{N-1} U_T(x_k,x_{k+1}). \tag{4.1}$$

The following result will be proved.

<u>Theorem 4.1</u>. For each fixed initial value $x_0 = x \in \mathbb{R}^2$ there is a bounded minimal energy sequence X^*.

<u>Proof</u>: By Theorem 2.1 the function $U_T(\cdot,\cdot)$ satisfies

$$U_T(x,y) \to \infty \quad \text{as} \quad |x| + |y| \to \infty.$$

Then by Theorem 3.5 one can decompose U_T in the form

$$U_T(x,y) = T\mu_T + \pi_T(x) - \pi_T(y) + \theta_T(x,y), \tag{4.2}$$

with μ_T a scalar, $\pi_T : \mathbb{R}^2 \to \mathbb{R}$ continuous, and $\theta_T : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ a continuous function which satisfies,

$$\min_{y \in \mathbb{R}^2} \theta_T(x,y) = 0 \quad , \quad \text{for each} \quad x \in \mathbb{R}^2. \tag{4.3}$$

Now define $\{x_k^*\}_{k=0}^{\infty}$ recursively as a sequence which satisfies

$$x_0^* = x_0 \quad , \quad \theta_T(x_k^*, x_{k+1}^*) = 0 \quad , \quad k = 0,1,2,\ldots \,. \tag{4.4}$$

This scheme is applicable for each $x_0 \in \mathbb{R}^2$ by (4.3), and it results in a bounded sequence as follows easily from Theorem 3.3(1) and the fact that $\pi(\mathbf{y}) \to \infty$ as $|\mathbf{y}| \to \infty$. We claim that $X^* = \{x_k^*\}_{k=0}^{\infty}$ is a minimal energy sequence. For suppose that $1 \le M < N$ and that $\mathbf{X}$ is any sequence in $\mathbb{R}^2$ satisfying

$$x_M = x_M^* \quad , \quad x_N = x_N^*. \tag{4.5}$$

Then (4.2) implies that

$$\sum_{k=M}^{N-1} U_T(x_k, x_{K+1}) = (N - M)\mu_T T + \pi_T(x_M) - \pi_T(x_N) + \sum_{k=M}^{N-1} \theta_T(x_k, x_{k+1}), \tag{4.6}$$

while by (4.4) one has

$$\sum_{k=M}^{N-1} U_T(x_k^*, x_{k+1}^*) = (N - M)\mu_T T + \pi_T(x_M^*) - \pi_T(x_N^*). \tag{4.7}$$

Comparing (4.6), (4.7) in the light of condition (4.5) and the nonnegativity of $\theta_T(\cdot,\cdot)$ yields

$$\sum_{k=M}^{N-1} U_T(x_k, x_{k+1}) \ge \sum_{k=M}^{N-1} U_T(x_k^*, x_{k+1}^*),$$

which concludes the proof. □

We will now use the minimal energy sequence X^* to define a minimal energy configuration $w^*(\cdot)$.

Consider, for each integer $k \geq 0$, problem (P_T) posed in the beginning of section 2, with $(\mathbf{x},\mathbf{y}) = (\mathbf{x}_k^*,\mathbf{x}_{k+1}^*)$. As indicated in the proof of lower semicontinuity in Theorem 2.2, there exists a minimizer $w_k(\cdot)$ for this problem. Now define $w^* : (0,\infty) \to \mathbb{R}$ as follows

$$w^*(t) = w_k(t - kT) \quad , \quad t \in [kT,(k + 1)T) \quad , \quad k \geq 0. \tag{4.8}$$

thus $w^*|_{[kT,(k+1)T]}$ minimizes the expression

$$\int_{kT}^{(k+1)T} f(w(s),\dot{w}(s),\ddot{w}(s))ds \quad , \quad w(\cdot) \in W^{2,1}([kT,(k + 1)T]),$$

subject to the conditions $\mathbf{x}(kT) = \mathbf{x}_k^*$, $\mathbf{x}((k + 1)T) = \mathbf{x}_{k+1}^*$, where we denote $\mathbf{x}(s) = \begin{bmatrix} w(s) \\ \dot{w}(s) \end{bmatrix}$ (thus by the proof of Theorem 2.1 $\mathbf{x}(\cdot)$ is a bounded function from $(0,\infty)$ to $\mathbb{R}^2$). We proceed to demonstrate that this construction does provide a minimal energy configuration.

<u>Theorem 4.2</u>. The function $w^*(\cdot)$ defined in (4.8) is a minimal energy configuration for problem (P_∞).

In order to prove Theorem 4.2 we will need to compare, for each $w(\cdot) \in A_{\mathbf{x}}$, the quantities

$$\int_{T_1}^{T_2} f(w(s),\dot{w}(s),\ddot{w}(s))ds \quad \text{and} \quad \int_{T_1}^{T_2} f(w^*(s),\dot{w}^*(s),\ddot{w}^*(s))ds$$

for every pair $T_2 > T_1 > 0$, not just integer multiples of some fixed $T > 0$. This will require the use of two results given below.

Lemma 4.3. Let $\{a_k\}_{k=1}^{\infty}$ be an increasing sequence of positive numbers such that $a_k \to \infty$ as $k \to \infty$. Consider numbers $T > 0$ with the property that for every $m \geq 1$,

$$\inf_{k,n}\{a_k - nT : k \geq m, \ \ n \geq 0, \ a_k \geq nT\} = 0. \tag{4.9}$$

Then there is a set $D \subset [0,\infty)$ with $m(D^c) = 0$ such that every $T \in D$ satisfies (4.9). (Here $m(ds)$ is Lebesgue measure and D^c denotes the complement of D in $[0,\infty)$).

The proof of this lemma will be given in section 7.

Now set $S_{\mathbf{x}} := \{X = \{x_k\}_{k=0}^{\infty} \subset \mathbb{R}^2 : x_0 = \mathbf{x}\}$, let $\tilde{S}_{\mathbf{x}}$ denote the set of periodic sequences in $S_{\mathbf{x}}$, and consider the scalar μ_T and the function $\pi_T(\cdot)$ appearing in the decomposition (4.2) for $U_T(\cdot,\cdot)$. It has been shown (cf. [L] §3, Prop. 5.1) that the following formulas define a μ_T and a $\pi_T(\cdot)$ for which (4.2) holds:

$$\mu_T = \inf_{S_{\mathbf{x}}}[\ell im \inf_{N\to\infty} \frac{1}{NT} C_N(X)] = \inf_{\tilde{S}_{\mathbf{x}}}[\ell im_{N\to\infty} \frac{1}{NT} C_N(X)], \tag{4.10}$$

$$(4.11) \qquad \pi_T = \inf_{S_x} \left[\liminf_{N\to\infty} [C_N(X) - NT\mu_T] \right].$$

Remark 4.3a. Here the lack of dependence of μ_T on x follows immediately from the form of (4.10), while its lack of dependence on T follows readily from Lemma 4.3, so that we obtain a $\mu \in \mathbb{R}$ such that

$$(4.12) \qquad \mu_T = \mu \quad , \quad \text{for all} \quad T > 0.$$

Moreover the function $\pi_T(\cdot)$ is (almost) independent of $T > 0$. In fact by using Lemma 4.3 we are able to prove the following.

Proposition 4.4. There exists a continuous function $\pi(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ and a set $D \subset [0,\infty)$, with $m(D^c) = 0$, such that

$$(4.13) \qquad \pi_T(x) = \pi(x) \text{ for every } x \in \mathbb{R}^2 \text{ and } T \in D.$$

Moreover, the decomposition (4.2) for $U_T(\cdot,\cdot)$ can be replaced by

$$(4.14) \quad U_T(x,y) = T\mu + \pi(x) - \pi(y) + \theta'_T(x,y), \ \forall\, x,y \in \mathbb{R}^2 \text{ and } T > 0,$$

where θ'_T is a continuous function which satisfies the condition

$$(4.15) \qquad \min_{y \in \mathbb{R}^2} \theta'_T(\mathbf{x},\mathbf{y}) = 0 \quad , \quad \text{for each } \mathbf{x} \in \mathbb{R}^2.$$

Proof: By the proof of Theorem 2.1 it can be seen that the set $S^1_{\mathbf{x}} = S^1_{\mathbf{x}}(T)$ of all sequences $\{X\} \in S_{\mathbf{x}}$ for which the $\ell\text{im inf}$ in (4.11) does not exceed $\pi_T(\mathbf{x}_0) + 1$ has the feature that

$$(4.16) \qquad X \in S^1_{\mathbf{x}_0} \Rightarrow X = \{\mathbf{x}_k\}_{k=0}^{\infty} \subset K_0,$$

where K_0 is a compact subset of $\mathbb{R}^2$ which depends on the choice of $\mathbf{x}_0$ and $T > 0$. Moreover by Remark 2.1 it follows that for each interval $0 < \alpha \leq T \leq \beta < \infty$ the compact set K_0 can be chosen sufficiently large so that (4.16) is valid for all $S^1_{\mathbf{x}_0}(T)$, $\alpha \leq T \leq \beta$. Furthermore, the proof of Theorem 2.1 together with Remark 2.1 also implies that there is a compact set $K_1 \supset K_0$ with the following property. Given any $X \subset K_0$ and $T \in [\alpha,\beta]$, let $w(\cdot) : [0,\infty) \to \mathbb{R}$ be defined as in (4.8):

$$(4.17) \qquad w(t) = w_k(t - kT), \; t \in [kT,(k + 1)T) \; , \; k = 0,1,2,\dots,$$

with $w_k(\cdot) \in W^{1,2}([0,T])$ a minimizer for problem (P_T) corresponding to $(\mathbf{x},\mathbf{y}) = (\mathbf{x}_k,\mathbf{x}_{k+1})$. Then the associated function $\mathbf{x} : (0,\infty) \to \mathbb{R}^2$ defined by

$$\mathbf{x}(t) = \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix} , \; t \geq 0$$

satisfies

$$(4.18) \qquad \mathbf{x}(t) \in K_1 \quad , \quad \text{all} \quad t \geq 0.$$

Fix $T \in [\alpha,\beta]$. Given any $\epsilon > 0$ let $\mathbf{X} = \mathbf{X}(\epsilon) \in S^1_{\mathbf{x}}$ be such that for this sequence the ℓim inf on the right of (4.11) is within ϵ of $\pi_T(\mathbf{x})$; let $w_\epsilon(\cdot) \in A_{\mathbf{x}}$ be the function associated with $\mathbf{X}(\epsilon)$ as in (4.17); and let $\mathbf{x}^\epsilon(\cdot)$ be the corresponding $\mathbb{R}^2$-valued function. Thus $\mathbf{x}^\epsilon(t) \in K_1$, for all $t \geq 0$. Next let $N_j \to \infty$ be a sequence of integers satisfying

$$(4.19) \qquad \underset{N\to\infty}{\ell\text{im}} \inf[C_N(X(\epsilon)) - NT\mu) = \underset{j\to\infty}{\ell\text{im}}[C_{N_j}(\mathbf{X}(\epsilon)) - N_jT\mu].$$

Set $a^\epsilon_k = N_kT$, $k \geq 0$. Then by Lemma 4.3 there is a set $D = D(\{a^\epsilon_k\}^\infty_{k=0}) \subset (0,\infty)$ of full measure for which the condition (4.9) holds. Given any $T' \in D$ let the sequence $\mathbf{Y} = \{\mathbf{y}_j\}^\infty_{j=0}$ be defined as

$$\mathbf{y}_j = \mathbf{x}^\epsilon(jT') \quad , \quad j \geq 0,$$

with $\mathbf{x}^\epsilon(\cdot)$ as above. Clearly it follows that

$$(4.20) \quad U_{T'}(\mathbf{y}_j,\mathbf{y}_{j+1}) \leq \int_{jT'}^{(j+1)T'} f(w_\epsilon(s),\dot{w}_\epsilon(s),\ddot{w}_\epsilon(s))ds \quad , \quad j \geq 0.$$

Now by (4.9) there exists a sequence $M_\ell \to \infty$, $\ell \geq 1$, such that

$$\mathrm{dist}(M_\ell T', \{a_k\}) \to 0.$$

If we denote by a_{k_ℓ} that element of $\{a_k\}$ satisfying

$$a_{k_\ell - 1} \le M_\ell T' < a_{k_\ell} \quad , \quad \ell \ge 1$$

then we obtain by use of (4.16)

$$C_{M_\ell}(Y) - \mu M_\ell T' - (C_{N_{K_\ell}}(X_\epsilon) - \mu N_{k_\ell} T)$$

$$\text{(4.21)} \qquad \le - \int_{M_\ell T'}^{a_{k_\ell}} f(w_\epsilon(s)\dot{w}_\epsilon(s), \ddot{w}_\epsilon(s))ds - \mu(a_{k_\ell} - M_\ell T')$$

$$\le - \int_{M_\ell T'}^{a_{k_\ell}} (a|w_\epsilon(s)|^\alpha - b|\dot{w}_\epsilon(s)|^\beta - d)ds - \mu(a_{k_\ell} - M_\ell T'),$$

where we have used (2.1 ii) to obtain the last inequality. Now $\mathbf{x}^\epsilon(t) \in K_1$ for all $t \ge 0$, so the integrand in the last integral in (4.21) is bounded uniformly, while by (4.9) $\liminf_{\ell \to \infty}\{a_{k_\ell} - M_\ell T'\} = 0$. Hence (4.21) implies that

$$\text{(4.22)} \qquad \pi_{T'}(x_0) \le \pi_T(x_0) + \epsilon \quad , \quad \forall\, T' \in D(\{a_k\}_{k=0}^{\infty}).$$

By taking a sequence $\epsilon_m \to 0$, we deduce that the set

$$D^* = \bigcap_{m=0}^{\infty} D(\{a_k^{\epsilon_m}\}_{k=0}^{\infty})$$

has full measure. Moreover it follows by (4.22) that

$$\pi_{T'}(x_0) \leq \pi_T(x_0) \quad , \quad \text{for all} \quad T' \in D^*. \tag{4.22'}$$

Now set

$$\pi(x_0) = \inf\{\pi_T(x_0); T > 0\}. \tag{4.23}$$

It will be seen below that $\pi(x_0) \neq -\infty$. Given $\delta > 0$, select $T := T_\delta > 0$ such that

$$\pi_T(x_0) < \pi(x_0) + \delta.$$

Thus we obtain from (4.22') the existence of a set $D^* = D^*(\delta)$ such that $M((D^*)^c) = 0$ and

$$\pi_{T'}(x_0) \in [\pi(x_0), \pi(x_0) + \delta] \text{ , for all } T' \in D^*(\delta).$$

By taking a sequence $\delta_\ell \to 0$ and setting $D^{**}(x_0) = \bigcap_{\ell=0}^{\infty} D^*(\delta_\ell)$ we deduce that

$$(4.24) \qquad \pi_{T'}(x_0) = \pi(x_0) \quad \text{for all} \quad T' \in D^{**}(x_0).$$

Note that if in (4.23) $\pi(x_0) = -\infty$, the same basic reasoning would show that $\pi_{T'}(x_0) = -\infty$ for all T' in a full set D^{**}, contradicting the fact that $\pi_{T'}(x_0) \in \mathbb{R}$.

The continuity of the function $\pi(\cdot)$ defined as in (4.23) now follows from the observation that for any sequence $y_{0,n} \to x_0$ there are values $T' \in \bigcap_{n=0}^{\infty} D^{**}(y_{0,n})$. Hence by the continuity of $\pi_{T'}(\cdot)$,

$$\pi(y_{0,n}) = \pi_{T'}(y_{0,n}) \Rightarrow \pi_{T'}(x_0) = \pi(x_0).$$

To demonstrate (4.13), it suffices to examine a countable dense set $\{z_i\} \subset \mathbb{R}^2$. Then setting $D = \bigcap_{i=0}^{\infty} D^{**}(z_i)$ we conclude that whenever $T' \in D$

$$\pi_{T'}(z_i) = \pi(z_i) \quad , \quad i \geq 0.$$

It now follows by the continuity of $\pi_{T'}(\cdot)$ and $\pi(\cdot)$ that (4.13) is valid.

It remains to prove that (4.14), (4.15) hold. Examine for each $T > 0$ the function $\theta_T'(\cdot,\cdot)$ defined by

$$(4.24) \qquad \theta_T'(x,y) := U_T(x,y) - T\mu + \pi(y) - \pi(x) \quad , \quad x,y \in \mathbb{R}^2.$$

Since $(T,\mathbf{x},\mathbf{y}) \mapsto U_T(\mathbf{x},\mathbf{y})$ is continuous (see Corollary 2.3) it follows that the function

$$(T,\mathbf{x},\mathbf{y}) \mapsto \theta'_T(\mathbf{x},\mathbf{y})$$

is continuous, and we know by (4.13) that

$$\theta'_T(\cdot,\cdot) = \theta_T(\cdot,\cdot) \quad , \quad \text{for all} \quad T \in D. \tag{4.25}$$

It follows from (4.24) that $\theta'_T(\mathbf{x},\mathbf{y}) \to \infty$ as $|\mathbf{y}| \to \infty$, uniformly on compact sets of the form

$$(T,\mathbf{x}) \in [\alpha,\beta] \times S.$$

Thus there is a bounded set S_1 such that $\theta'_T(\mathbf{x},\mathbf{y}) = 0$ with $(T,\mathbf{x}) \in [\alpha,\beta] \times S \Rightarrow \mathbf{y} \in S_1$. This together with the fact that for all $T \in D$

$$\min_{\mathbf{y}\in\mathbb{R}^2} \theta'_T(\mathbf{x},\mathbf{y}) = 0 \quad \text{for} \quad \mathbf{x} \in S, \tag{4.26}$$

implies the validity of (4.26) for <u>all</u> $T > 0$. □ (Prop. 4.4.)

<u>Proof of Theorem 4.2</u>: Given $T_2 > T_1 > 0$, let $w(\cdot) \in W^{2,1}_{\ell oc}$ satisfy

$$x(T_1) = x^*(T_1), \quad x(T_2) = x^*(t_2)$$

$$\text{where} \quad x(\cdot) = \begin{bmatrix} w(\cdot) \\ \dot{w}(\cdot) \end{bmatrix}, \quad x^*(\cdot) = \begin{bmatrix} w^*(\cdot) \\ \dot{w}^*(\cdot) \end{bmatrix},$$

with $w^*(\cdot)$ defined as in (4.8). We shall suppose that neither T_1 nor T_2 is a multiple of the fixed T relative to which $w^*(\cdot)$ was defined (the other cases are simpler). Denote by M, N the nonnegative integers determined by

$$T_1 \in ((M - 1)T, MT) \quad , \quad T_2 \in (NT, (N + 1)T).$$

Then we can estimate

$$\int_{T_1}^{T_2} f(w(s), \dot{w}(s), \ddot{w}(s))ds \geq U_{MT-T_1}(x(T_1), x(MT)) + \sum_{k=M}^{N-1} U_T(x(kT), x(k + 1)T))$$

$$+ U_{T_2-NT}(x(NT), x(T_2))$$

$$\geq (T_2 - T_1)\mu + \pi(x(T_1)) - \pi(x(T_2)),$$

where we have utilized (4.13).

We proceed to show that $w^*(\cdot)$ yields equality:

$$(4.28)\quad \int_{T_1}^{T_2} f(w^*(s),\dot{w}^*(s),\ddot{w}^*(s))ds = (T_2 - T_1)\mu + \pi(x(T_2)) - \pi(x(T_1)).$$

For this we examine the integral over $[(M-1)T, (N+1)T]$. By the definition of $w^*(\cdot)$ we have

$$\int_{(M-1)T}^{(N+1)T} f(w^*(s),\dot{w}^*(s),\ddot{w}^*(s))ds = (N-M+2)T\mu + \pi(x^*((M-1)T)) - \pi(x^*((N+1)T)).$$

On the other hand, using the decomposition into integrals over the intervals $[(N-1)T,T_1]$, $[T_1,T_2]$ and $[T_2,N+1)T]$ we obtain by (4.14)

$$\int_{(M-1)T}^{(N+1)T} f(w^*(s),\dot{w}^*(s),\ddot{w}^*(s))dr \geq U_{T_1-(M-1)T}(x^*((M-1)T),x^*(T_1))$$

$$+ U_{T_2-T_1}(x^*(T_1),x^*(T_2)) + U_{(N+1)T-T_2}(x^*(T_2),x^*((N+1)T))$$

$$\geq (N-M+2)T\mu + \pi(x^*((M-1)T)) - \pi(x^*((N+1)T))$$

$$+ \theta'_{T_1(M-1)T}(x^*((M-1)T),x^*(T_1)) + \theta'_{T_2-T_1}(x^*(T_1),x^*(T_2))$$

$$+ \theta'_{(N+1)T-T_2}(x^*(T_2),x^*((N+1)T))$$

By comparing these two decompositions we conclude that each of the θ'

terms is 0. In particular

$$\theta'_{T_2-T_1}(x^*(T_1),x^*(T_2)) = 0,$$

whence (4.28) follows. □

§5. Periodic minimal energy configurations.

In this section we will demonstrate the existence of periodic minimal energy configurations. In order to state our result concisely we introduce the following notation for use with any integrand f of the sort described in (2.1):

(5.1) $$m_f := \inf\{f(w,0,s) : (w,s) \in \mathbb{R}^2\}.$$

We will prove the following assertion.

Theorem 5.1. Suppose that the integrand f in (P_∞) is such that

(5.2) $$\mu < m_f,$$

where μ is the minimal growth rate for (P_∞). Then there exists a nonconstant periodic minimal energy configuration $w^*(\cdot)$ for (P_∞).

If f has the further property

(5.1') $$\inf\{f(w,0,0) : w \in \mathbb{R}\} = m_f,$$

then whenever (5.2) fails there exists a constant minimal energy configuration

$$w^*(s) = \bar{w}, \; \forall \; s \geq 0,$$

where $\bar{w} \in \mathbb{R}$ is any value for which

$$m_f = f(\bar{w},0,0).$$

Remark: It will be shown in §6 that (5.2) and (5.1′) hold for a large and interesting class of problems.

The following result, which characterizes those periodic configurations which are minimal energy configurations, will be needed.

Lemma 5.2. Let $w(\cdot)$ be a periodic configuration of period $T > 0$:

$$w(t + T) = w(t) \quad , \quad \forall\, t \geq 0.$$

Then $w(\cdot)$ is a minimal energy configuration if and only if

$$\frac{1}{T}\int_0^T f(w(s),\ \dot{w}(s),\ddot{w}(s))ds = \mu. \tag{5.3}$$

Proof: By the definition of μ [cf. (3.3)] one deduces the inequality

$$\mu \leq \liminf_{k\to\infty} \frac{1}{kT}\int_0^{kT} f(w(t),\dot{w}(t),\ddot{w}(t))dt = \frac{1}{T}\int_0^T f(w(t),\dot{w}(t),\ddot{w}(t))dt.$$

Moreover if the inequality is strict then it is easily seen (using the periodicity) that $w(\cdot)$ cannot be a minimal energy configuration. Hence

(5.3) is certainly necessary for $w(\cdot)$ to be a minimal energy configuration.

Now suppose (5.3) holds. Let $v(\cdot)$ be any configuration and let T_1, T_2 be a given pair of points in $[0,\infty)$, with $0 \leq T_1 < T_2$. We proceed to show that if $v(T_1) = w(T_1), \dot{v}(T_1) = \dot{w}(T_1), v(T_2) = w(T_2), \dot{v}(T_2) = \dot{w}(T_2)$ then

$$(5.4) \qquad \int_{T_1}^{T_2} f(w(t),\dot{w}(t),\ddot{w}(t))dt \leq \int_{T_1}^{T_2} f(v(t),\dot{v}(t),\ddot{v}(t))dt.$$

Let the integers m,n be determined by

$$(m - 1)T \leq T_1 < mT \quad , \quad nT \leq T_2 < (n + 1)T.$$

We now compute the integral I over $[(m - 1)T,(n + 1)T]$ in two ways, as follows. By (5.3) and the periodicity of $w(\cdot)$

$$(5.5)\ I = \sum_{k=m-1}^{n} \int_{kT}^{(k+1)T} f(w(t),\dot{w}(t)),\ddot{w}(t))dt = \sum_{k=m-1}^{n} \mu T = (n - m + 1)\mu T.$$

On the other hand, putting $x(t) = \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}$ we obtain

$$I \geq U_{T_1-(m-1)T}(x((m-1)T),x(T_1))+U_{T_2-T_1}(x(T_1),x(T_2))+U_{(n+1)T-T_2}(x(T_2),x((n+1)T)).$$

Using the decomposition (4.15) for $U(\cdot,\cdot)$ we obtain

$$I \geq (T_1-(m-1)T)\mu+\theta_{T_1-(m-1)T}(x((m-1)T),x(T_1))+(T_2-T_1)\mu+\theta_{T_2-T_1}(x(T_1),x(T_2)) +$$

$$+ ((n+1)T-T_2)\mu + \theta_{(n+1)T-T_2}(x(T_2),x((n+1)T)).$$

Thus by the nonnegativity of the functions $\theta(\cdot,\cdot)$,

$$I \geq (n - m + 2)T\mu + \theta_{T_2-T_1}(x(T_1),x(T_2)). \tag{5.6}$$

A comparison with (5.5) shows that equality holds in (5.5) and

$$\theta_{T_2-T_1}(x(T_1),x(t_2)) = 0.$$

Hence

$$\int_{T_1}^{T_2} f(w(t),\dot{w}(t),\ddot{w}(t))dt = \mu(T_2 - T_1) + \pi(x(T_1)) - \pi(x(T_2)). \tag{5.7}$$

On the other hand, since $v(\cdot)$ has the same end data as $w(\cdot)$,

$$\int_{T_1}^{T_2} f(v(t),\dot{v}(t),\ddot{v}(t))dt \geq U_{T_2-T_1}(x(T_1),x(T_2)) \geq \mu(T_1-T_2) + \pi(x(T_1))-\pi(x(T_2)).$$

Comparing this with (5.7) we conclude that (5.3) suffices for $w(\cdot)$ to be a minimal energy configuration. □

Proof of Theorem 5.1. Given $x_0 = x \in \mathbb{R}^2$ we consider the function $w^*(\cdot)$ defined in (4.8). By theorem 4.2 this function is a minimal energy configuration. We will examine the phase-plane orbit

$$t \mapsto x^*(t) = \begin{bmatrix} w^*(t) \\ \dot{w}^*(t) \end{bmatrix}.$$

Recall that $w^*(\cdot)$ was constructed by using the minimal energy sequence (4.4). Such sequences are uniformly bounded if x_0 belongs to a bounded set in $\mathbb{R}^2$, as follows from Theorem 2.1 and Theorem 8.1 in [L]. It then follows that the orbits

$$t \mapsto x^*(t),$$

with $w^*(\cdot)$ as above and x_0 in a bounded set are uniformly bounded.

Suppose, for the sake of definiteness, that x_0 lies in the first quadrant of $\mathbb{R}^2$. Then $t \mapsto x_1^*(t)$, where $x_1^*(\cdot) = w(\cdot)$ is the first coordinate of $x^*(\cdot)$, is an increasing function so long as $x^*(\cdot)$ remains in the first quadrant. Moreover by (2.7), $x_1^*(\cdot)$ is bounded. Consequently $x^*(\cdot)$ either crosses the x_1-axis and enters the fourth quadrant in finite "time" or else $x^*(t)$ converges (essentially) to the point $(m_1,0)$ as $t \to \infty$, where $M_1 = \lim_{t\to\infty} x_1^*(t) = \sup\{x_1^*(t) : t \geq 0\}$. In the latter case, the finiteness of M_1 implies that for each $\epsilon > 0$, the fraction of the time interval $[0,T]$ during which $x_2^*(t) \in (0,\epsilon)$ and $x_1^*(t) \in (M_1-\epsilon, M_1)$ approaches 1 as $T \to \infty$. Hence when (5.2) holds this second possible behavior of $x^*(\cdot)$ contradicts the fact that $w^*(\cdot)$ is a minimal energy configuration in light of (5.2) and the growth condition (2.1 ii) for f as a function of r. On the other hand, when (5.2) is false and (5.1′) holds, then the constant function $w^*(s) = \bar{w}$, $s \geq 0$, is obviously a minimal energy configuration (so that (5.2) is replaced by $\mu = m_f$).

Hereafter we will suppose that (5.2) holds so that $x^*(\cdot)$ crosses the x_1-axis at some time $t_1 > 0$, and we have $x_2^*(t_1) = 0$, $x_2^*(t) < 0$ for $t > t_1$ sufficiently small. The same reasoning as above when applied to the decreasing function $t \to x_1^*(t)$ implies that there is a $t_2 > t_1$ at which another insection of $x^*(\cdot)$ with the x_1-axis occurs. In this manner one obtains a sequence $\{t_k\}_{k\geq 1}$ of successive times at which the orbit $x^*(\cdot)$ crosses the x_1-axis. We distinguish between two different cases.

(5.8) <u>First case</u>: The orbit $x^*(\cdot)$ intersects itself, so that for some $0 \leq T_1 < T_2$ $\quad x^*(T_2) = x^*(T_1)$.

(5.9) <u>Second case</u>: The orbit $x^*(\cdot)$ does not self-intersect.

<u>Proposition 5.3</u>. Assume that (5.8) holds. then there exists a periodic minimal energy configuration.

<u>Proof</u>: Let $0 \leq T_1 < T_2$ be as in (5.8), set $T = T_2 - T_1$ and define for each t

$$w(t) = w^*(\tau),$$

where τ is the unique point satisfying

$$T_1 \le \tau < T_2 \quad , \quad t - \tau = nT \quad \text{for some integer } n.$$

Then the orbit $t \mapsto \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}$ associated with this function is obviously periodic of period T. In order to prove that $w(\cdot)$ is a minimal energy configuration it is enough by Lemma 5.2 to show that (5.3) holds, or equivalently, that

$$\text{(5.10)} \qquad \int_{T_1}^{T_2} f(w^*(s), \dot{w}^*(s), \ddot{w}^*(s))ds = (T_2 - T_1)\mu.$$

But arguing as in the proof of Lemma 5.1 we conclude that

$$\int_{T_1}^{T_2} f(w^*(s), \dot{w}^*(s), \ddot{w}^*(s))ds = U_{T_2-T_1}(x^*(T_1), x^*(T_2))$$

$$= (T_2-T_1)\mu + \pi(x^*(T_1)) - \pi(x^*(T_2)) + \theta_{T_2-T_1}(x^*(T_1), x^*(T_2))$$

and that the $\theta_{T_2-T_1}(\cdot,\cdot)$ term must vanish. Since $x^*(T_2) = x^*(T_1)$, the above equality reduces to (5.10), which completes the proof. □

Proposition 5.4. Assume that (5.9) holds. Then there exists a periodic minimal energy configuration.

Proof: Recall the increasing sequences $\{t_k\}_{k\geq 1}$ of successive times where the orbit $t \mapsto \mathbf{x}^*(t)$ intersects the x_1-axis. For every odd integer k we define

$$(5.11)\quad \mathbf{y}_k(t) = \mathbf{x}^*(t + t_k) \quad , \quad 0 \leq t \leq t_{k-1} - t_k \quad , \quad k = 1,3,5,\dots .$$

We proceed to show that there is an interval $[0,\tau_1]$ such that $t_{k_i+1} - t_{k_i} \to \tau_1 > 0$ as $i \to \infty$ for some subsequence of odd numbers $k_i \to \infty$, and there is an orbit $\mathbf{y}(\cdot)$ of the form $\mathbf{y}(t) = \begin{bmatrix} v(t) \\ \dot{v}(t) \end{bmatrix}$ such that

$$(5.12)\qquad \max_{0\leq t\leq \tau_1} |\mathbf{y}_{k_i}(t) - \mathbf{y}(t)| \to 0 \quad \text{as} \quad i \to \infty.$$

In (5.12), when $t_{k_i+1} - t_{k_i} < \tau_1$ we extend $\mathbf{y}_{k_i}(\cdot)$ to all of $[0,\tau_1]$ by setting

$$\mathbf{y}_{k_i}(t) = \mathbf{y}_{k_i}(t_{k_i+1} - t_{k_i}) \ , \ \text{for} \quad t \in [t_{k_i+1} - t_{k_i}, \tau_1].$$

For $\mathbf{y}_k(\cdot)$ in (5.11) we denote $\alpha_k = t_{k+1} - t_k$. Thus $\mathbf{y}_k(0)$ and $\mathbf{y}_k(\alpha_k)$ both belong to the x_1-axis and $(\mathbf{y}_k)_2(t) < 0$ for $0 < t < \alpha_k$. Moreover since the orbit $\mathbf{y}_k(\cdot)$ is part of the minimal energy configuration $\mathbf{x}^*(\cdot)$, there is a bound

$$(5.13)\qquad |\mathbf{y}_k(t)| \leq M \ , \ 0 \leq t \leq \alpha_k$$

which is uniform for all k's. We show next that there is a bound of the form

(5.14) $$\alpha_k \le T_0 \quad \text{for all} \quad k = 1,3,5,\dots .$$

To prove (5.14) let $\epsilon > 0$ be small and consider the strip

$$S_\epsilon = \{x \in \mathbb{R}^2 : -\epsilon < x_2 < 0\}.$$

Observe that the boundedness of the first componenet of $\mathbf{y}_k(\cdot)$ ensures that the total time spent outside ϵ cannot exceed $T_\epsilon = \frac{2M'}{\epsilon}$, where M' is a bound for $|y_1(\cdot)|$. But if α_k were very large compared to T_ϵ then the fraction of time spent by the orbit in S_ϵ would be arbitrarily close to one. For ϵ sufficiently small we would have by (5.2) that

(5.15) $$\inf_{\substack{(x,p)\in S_\epsilon \\ s>0}} f(x,p,s) > \mu.$$

Thus by (5.15) we would have for some $\delta > 0$

(5.16) $$\frac{1}{\alpha_k}\int_0^{\alpha_k} f(w_k(t),\dot{w}_k(t),\ddot{w}_k(t))dt > \mu + \delta,$$

(where $w_k(\cdot)$ is the first component of $\mathbf{y}_k(\cdot)$), provided that α_k is large enough. On the other hand since $\mathbf{y}_k(\cdot)$ corresponds to a minimal energy configuration,

$$\frac{1}{\alpha_k}\int_0^{\alpha_k} f(w_k(t),\dot{w}_k(t),\ddot{w}_k(t))dt = \frac{1}{\alpha_k}[\alpha_k\mu + \pi(\mathbf{y}_k(0)) - \pi(\mathbf{y}_k(\alpha_k))],$$

which is below $\mu + \delta$ for α_k sufficiently large. This contradiction implies the validity of (5.14).

We claim that there is a $C > 0$ such that

$$(5.17) \qquad |\mathbf{y}_k(0) - \mathbf{y}_k(\alpha_k)| \geq C \quad , \quad k = 1,3,5,\dots \ .$$

For by the construction of the $\{\mathbf{y}_k(\cdot)\}$ there are points $\mathbf{x}$, $\mathbf{z}$ on the x_1-axis, such that monotonically (this uses (5.9))

$$\mathbf{y}_k(0) \to \mathbf{x} \quad , \quad \mathbf{y}_k(\alpha_k) \to \mathbf{z} \quad \text{as} \quad k \to \infty.$$

Now $\mathbf{x} \neq \mathbf{z}$, since otherwise $\mathbf{x}^*(\cdot)$ would converge to a point on the x_1-axis in contradiction to (5.2), so (5.17) holds. Now (5.17) ensures that for some $t_0 > 0$ one has

$$(5.18) \qquad \alpha_k > t_0 \quad k = 1,3,5,\dots \ .$$

Otherwise the quantities $\dot{w}_k(t)$ would be unbounded. It thus follows from (5.14) and (5.18) that there is a subsequence $\{k_i\}_{i\geq 1}$ of the odd integers with

$$\alpha_{k_i} \to \tau_1 > 0 \quad \text{as} \quad i \to \infty.$$

Since $\{\mathbf{y}_{k_i}(\cdot)\}$ correspond to a minimal energy configuration it follows that the quantitives

$$\|\ddot{w}_{k_i}(\cdot)\|_{L^{\gamma}(0,\alpha_k)} \quad \text{are uniformly bounded.}$$

Hence we may also suppose that

$$\{\dot{w}_{k_i}(\cdot)\}_{i\geq 1} \quad , \quad \{w_{k_i}(\cdot)\}_{i\geq 1}$$

both converge uniformly on $[0,\tau_1]$ to limits $\dot{v}_1(\cdot)$ and $v_1(\cdot)$, respectively. Since each $\mathbf{y}_{k_i}(\cdot)$ satisfies

$$(5.19) \qquad \int_0^{\alpha_{k_i}} f(w_{k_i}(t),\dot{w}_{k_i}(t),\ddot{w}_{k_i}(t))dt = \alpha_{k_i}\mu + \pi(\mathbf{y}_{k_i}(0)) - \pi(\mathbf{y}_{k_i}(\alpha_{k_i}))$$

it follows, by letting $i \to \infty$ and using the lower semicontinuity, that

$$(5.20) \qquad \int_0^{\tau_1} f(v_1(t),\dot{v}_1(t),\ddot{v}_1(t))dt \leq \tau_1\mu + \pi(\mathbf{y}(0)) - \pi(\mathbf{y}(\tau_1)),$$

where $\mathbf{y}(\cdot) = \begin{bmatrix} v_1(\cdot) \\ \dot{v}_1(\cdot) \end{bmatrix}$.

Now consider the functions $\mathbf{y}_k(\cdot)$ defined for every <u>even</u> integer k by

$$\mathbf{y}_k(t) = \mathbf{x}^*(t + t_k) \quad , \quad 0 \le t \le t_{k+1} - t_k \quad , \quad k = 2,4,6,\dots .$$

It follows by the same argument as above that there is an interval $[0,\tau_2], \tau_2 > 0$ and a subsequence $\{k_i\}_{i \ge 1}$ of the even integers such that $t_{k_i+1} - t_{k_i} \to \tau_2$ and an orbit $\mathbf{z}(t) = \begin{bmatrix} v_2(t) \\ \dot{v}_2(t) \end{bmatrix}$ such that

$$\max_{0 \le t \le \tau_k} |\mathbf{y}_{k_i}(t) - \mathbf{z}(t)| \to 0 \quad \text{as} \quad i \to \infty.$$

Then we have

$$\text{(5.21)} \qquad \int_0^{\tau_2} f(v_2(t), \dot{v}_2(t), \ddot{v}_2(t))dt \le \mu\tau_2 + \pi(\mathbf{z}(0)) - \pi(\mathbf{z}(\tau_2)).$$

Clearly

$$\text{(5.22)} \qquad \mathbf{y}(\tau_2) = \mathbf{z}(0) \quad , \quad \mathbf{y}(0) = \mathbf{z}(\tau_1).$$

Now define a periodic configuration $v(\cdot)$ as follows

$$v(t) = \begin{cases} v_1(t) & 0 \le t \le \tau_1 \\ v_2(t - \tau_1) & \tau_1 < t \le \tau_1 + \tau_2. \end{cases}$$

while for $t > \tau_1 + \tau_2$ $v(t)$ is defined by $v(t) = v(t')$, where $t' \in (0,\tau_1 + \tau_2)]$ is the unique point such that $t - t' = n(\tau_1 + \tau_2)$ for some integer n. In order to show that this is a minimal energy configuration is suffices by Lemma 5.1 to prove that

$$\int_0^{\tau_1+\tau_2} f(v(t),\dot{v}(t),\ddot{v}(t))dt = (\tau_1 + \tau_2)\mu. \tag{5.23}$$

But (5.20), (5.21) and (5.22) imply that

$$\int_0^{\tau_1+\tau_2} f(v(t),\dot{v}(t),\ddot{v}(t))dt \leq (\tau_1 + \tau_2)\mu,$$

whence (5.23) holds, which completes the proof. □

§6. A class of examples; single- and double-well potentials.

In this section we study (P_∞) for integrands of the form

$$(6.1)\qquad f(w,p,r) = \psi(w) - bp^2 + cr^2 \quad , \quad b,c > 0,$$

where $\psi(\cdot)$ is a smooth function satisfying

$$(6.2)\qquad \psi(w) \geq a|w|^\alpha - d \quad , \quad w \in \mathbb{R} \quad , \quad \text{for some } \alpha > 2,\ a,d > 0.$$

Thus f satisfies conditions (2.1). Furthermore, condition (5.1)′ clearly holds:

$$(6.3)\ m_f = \inf\{f(w,0,r) : (w,r) \in \mathbb{R}^2\} = \inf\{f(w,0,0) : w \in \mathbb{R}\} = \min \psi(\cdot).$$

We have the following result.

Theorem 6.1. Suppose that there are at most two absolute minimizers of $\psi(\cdot)$: $\psi(w) = m_f \Leftrightarrow w \in M$, where $M = \{w_1\}$ or $M = \{w_1,w_2\}$. Furthermore suppose that

$$(6.4)\qquad \psi''(w) > 0 \quad \forall\, w \in M.$$

Then for each fixed $c,\psi(\cdot)$ as in (6.1) and (6.2) there is a scalar $b_0 = b_0(c;\psi(\cdot)) > 0$ such that the minimal energy growth rate for f, $\mu = \mu(b,c;\psi(\cdot))$, satisfies

$$(6.5)\qquad \begin{cases} \mu < m_f \ , & \text{for } b \in (b_0,\infty) \\ \mu = m_f \ , & \text{for } b \in [0,b_0]. \end{cases}$$

Proof: Consider for all $T > 0$ and all $w(\cdot) \in W^{2,1}_{\ell oc}$ the Rayleigh quotient $R_T(w(\cdot)) = R_T(w(\cdot);\psi_0(\cdot))$ defined by

$$(6.6)\qquad \begin{cases} \dfrac{\int_0^T [\psi_0(w(t)) + c\ddot{w}^2(t)]dt}{\int_0^T \dot{w}^2(t)dt} \ , & \text{if } \int_0^T \dot{w}^2(t)dt > 0 \\ +\infty \ , & \text{otherwise} \ , \end{cases}$$

where $\psi_0(\cdot)$ is defined by

$$(6.7)\qquad \psi_0(w) = \psi(w) - m_f \geq 0 \ , \quad \forall\, w \in \mathbb{R};$$

thus $\psi_0(w) = 0 \Leftrightarrow w \in M$.

We will prove that $b_0 > 0$, where b_0 is defined by

$$(6.8)\qquad b_0 := \inf_{\substack{T>0 \\ x(0)=x(T)}} R_T(w(\cdot)) \ , \quad \text{with } x(\cdot) = \begin{bmatrix} w(\cdot) \\ \dot{w}(\cdot) \end{bmatrix}.$$

Relation (6.5a) is a direct consequence of the positivity of b_0; for each $b > b_0$ there exists by (6.8) a periodic function $w(\cdot)$ of period $T_0 > 0$ satisfying

$$\frac{1}{T_0}\int_0^{T_0} [\psi_0(w(t) - b\dot{w}^2(t) + c\ddot{w}^2(t)]dt < 0.$$

Thus (6.5a) follows from the definitions (3.3), (6.7) of μ and $\psi_0(\cdot)$.

Similarly, (6.5b) follows from Lemma 5.2 and the observation that when $b \geq b_0$ then for all periodic $w(\cdot)$ of period $T_0 > 0$

$$\frac{1}{T_0}\int_0^{T_0} [\psi_0(w(t)) - b\dot{w}^2(t) + c\ddot{w}^2(t)]dt \geq 0,$$

whereas for the functions $w(\cdot) \equiv w_i$, $w_i \in M$, equality holds.

In the case where $M = \{w_1\}$ is a singleton, we note that by (6.2), (6.4) there is a constant e such that

$$0 < e \leq \frac{1}{2}\psi''(w_1) \text{ and } \psi(w) \geq e(w - w_1)^2 \quad \forall\, w \in \mathbb{R}.$$

Hence it follows from (6.8) that

$$\text{(6.9)} \qquad b_0 \geq \inf_{\substack{T>0 \\ x(0)=x(T)}} R_T^e(w(\cdot) - w_1) = \inf_{\substack{T>0 \\ x(0)=x(T)}} R_T^e(w(\cdot)) =: b_1,$$

where

$$R_T^e(w(\cdot)) = \frac{\int_0^T [ew^2(t) + c\ddot{w}^2(t)]dt}{\int_0^T \dot{w}^2(t)dt}.$$

It will be seen later that $b_1 > 0$, whence $b_0 > 0$ as claimed.

To establish the positivity of b_0 when $M = \{w_1, w_2\}$ is a doubleton we begin by noting that (6.8) leads to an alternative recipe. We claim that for each fixed $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ we have

$$(6.8') \qquad b_0 = \inf_{\substack{T>0 \\ \mathbf{x}(0)=\mathbf{x}(T)=\mathbf{y}}} R_T(w(\cdot)).$$

This follows from the observation that a function $w(\cdot)$ of period T is also of period kT for all positive integers k. Now for large k we can join the end values assumed by $\mathbf{x}(\cdot)$ on an interval of length kT to the prescribed end values $\mathbf{y}$ (by extending $w(\cdot)$ onto the concentric interval of length $kT + 2$, say) without making much change in the ratio (6.6). Thus (6.8′) holds.

Now take $\mathbf{y} = \begin{bmatrix} w_* \\ 0 \end{bmatrix}$ where w_* is any interior point of the interval $[w_1, w_2]$. Observe that shifting each $w(\cdot) \in W^{2,1}_{\ell oc}$ by an additive constant k corresponds to translating the function $\psi_0(\cdot)$ by amount k. Thus we obtain the formula

$$(6.10) \qquad b_0 = \inf_{\substack{T>0 \\ \mathbf{x}(0)=\mathbf{x}(T)=\mathbf{y}}} R_T(w(\cdot)) = \inf_{\substack{T>0 \\ \mathbf{x}(0)=\mathbf{x}(T)=\mathbf{0}}} R^*_T(w(\cdot)),$$

where $R^*_T(w(\cdot)) = R_T(w(\cdot)\ ;\ \psi_0(\cdot - w_*))$. Note that the zeros of this translate $\bar{\psi}_0(\cdot) = \psi_0(\cdot - w_*)$ are

$$\bar{w}_1 = w_1 - w_* \ , \ \bar{w}_2 = w_2 - w_*.$$

Next, for functions $w(\cdot)$ satisfying $\mathbf{x}(0) = \mathbf{x}(T) = \mathbf{0}$ we shall examine the values assumed by the ratio in R_T^* over subintervals of $[0,T]$ where $w(\cdot)$ has constant sign. Given any such $w(\cdot)$ we decompose $[0,T]$ into three disjoint sets

$$A = \{t : w(t) > 0\}, \quad B = \{t : w(t) < 0\}, \quad C = \{t : w(t) = 0\}.$$

As is well known, $\dot{w}(t) = 0$ a.e. on C, whence $\ddot{w}(t) = 0$ a.e. on C as well. Hence

$$R_T^*(w(\cdot)) = \frac{\int_A[\bar{\psi}_0(w(t))+c\ddot{w}^2(t)]dt+\int_B[\bar{\psi}_0(w(t))+c\ddot{w}^2(t)]dt+\int_C\bar{\psi}_0(0)dt}{\int_A \dot{w}^2(t)dt + \int_B \dot{w}^2(t)dt}$$

(6.11)

$$\geq \min\left\{\frac{\int_A[\bar{\psi}_0(w(t))+c\ddot{w}^2(t)]dt}{\int_A \dot{w}^2(t)dt}, \frac{\int_B[\bar{\psi}_0(w(t))+c\ddot{w}^2(t)]dt}{\int_B \dot{w}^2(t)dt}\right\}.$$

Now denote by $\psi^L(\cdot)$ any smooth nonnegative extension of $\bar{\psi}_0(\cdot)$ from $[0,\infty)$ to $\mathbb{R}$ which possesses no zeros other than $\bar{w}_2$ and which satisfies the growth condition (6.2) on $\mathbb{R}$. Likewise denote by $\psi^R(\cdot)$ any smooth nonnegative extension of $\bar{\psi}_0(\cdot)$ from $(-\infty,0]$ to $\mathbb{R}$ which possesses no zeros other than $\bar{w}_1$ and which satisfies the growth condition (6.2). It then follows from (6.11), using the fact that the open sets A,B are disjoint unions of intervals, that

(6.12)

$$R_T^*(w(\cdot)) \geq \min\left\{ \inf_{\substack{T_0 \in [0,T] \\ w(0)=w(T)=0}} R_{T_0}(w(\cdot);\psi^L(\cdot)),\ \inf_{\substack{T_0 \in [0,T] \\ w(0)=w(T)=0}} R_{T_0}(w(\cdot);\psi^R(\cdot)) \right\}.$$

Thus (6.10) implies

$$(6.13)\quad b_0 \geq \min\left\{ \inf_{\substack{T>0 \\ w(0)=w(T)=0}} R_T(w(\cdot);\psi^L(\cdot)),\ \inf_{\substack{T>0 \\ w(0)=w(T)=0}} R_T(w(\cdot);\psi^R(\cdot)) \right\}.$$

We will proceed to show that both infima in (6.13) are positive. For the sake of brevity, we focus attention in what follows on the quantity

$$b_0^L := \inf_{\substack{T>0 \\ w(0)=w(T)=0}} R_T(w(\cdot)\ ;\ \psi^L(\cdot)),$$

but the treatment of the quantity

$$b_0^R := \inf_{\substack{T>0 \\ w(0)=w(T)=0}} R_T(w(\cdot)\ ;\ \psi^R(\cdot)),$$

is carried out in an identical fashion.

Now by (6.4) and the construction of $\psi^L(\cdot)$ there is a constant $e' = e(\psi^L(\cdot))$, $0 < e \leq \frac{1}{2}\psi''(w_2)$, such that

(6.14) $$\psi^L(w) \geq e'(w - \bar{w}_2)^2 \quad , \quad \forall\, w \in \mathbb{R}.$$

Again shifting each $w(\cdot) \in W^{2,1}_{\ell oc}$ by an additive constant so as to translate $\psi^L(\cdot)$ we obtain

(6.15) $$b_0^L \geq \inf_{\substack{T>0 \\ w(0)=w(T)=0}} R_T^{e'}(w(\cdot) - \bar{w}_2) = \inf_{\substack{T>0 \\ w(0)=w(T)=-\bar{w}_2}} R_T^{e'}(w(\cdot)) =: b_1'$$

where $R_T^{e'}$ denote the Rayleigh quotient associated with $\psi(w) = e'w^2$, i.e.

$$R_T^{e'}(w(\cdot)) = \frac{\int_0^T [e'w^2(t) + c\ddot{w}^2(t)]dt}{\int_0^T \dot{w}^2(t)dt}.$$

Next we show that the infimum giving b_1' is not attained for small values of T. Since the end conditions in (6.15) imply

$$\int_0^T \dot{w}(t)dt = 0$$

it follows that for some $t_0 \in [0,T]$, $\dot{w}(t_0) = 0$. Hence by Schwarz's inequality

$$\int_0^T \dot{w}^2(t)dt \leq \int_0^T |(t - t_0) \int_{t_0}^t \ddot{w}^2(s)ds|dt \leq T^2/2 \int_0^T \ddot{w}^2(t)dt,$$

so that for each $w(\cdot)$ entering (6.15) one has

$$R_T^{e'}(w(\cdot)) \geq \frac{2c}{T^2}.$$

Consequently for $\delta > 0$ sufficiently small we can give the following alternate formula for the right hand side of (6.15):

$$b_1' = \inf_{\substack{T \geq \delta \\ w(0)=w(T)=-w_2}} R_T^{e'}(w(\cdot)). \tag{6.15'}$$

We now relax the conditions on $w(\cdot)$ under which the infimum in (6.15′) is taken; it will only be required that on [0,T] $\dot{w}(\cdot) \not\equiv 0$. Clearly

$$b_1' \geq \inf_{\substack{T \geq \delta \\ \dot{w} \not\equiv 0}} R_T^{e'}(w(\cdot)) := b_2'. \tag{6.16}$$

Furthermore we observe that b_2' is also given by the formula

$$b_2' = \inf_{\substack{T \in [\delta, 2\delta) \\ \dot{w} \not\equiv 0}} R_T^{e'}(w(\cdot)). \tag{6.16'}$$

This version of (6.16) holds because for each $T > \delta$ the interval [0,T] can be decomposed into finitely many disjoint subintervals $I_j = [t_j, t_j + T_0)$ of common length $T_0 \in [\delta, 2\delta)$; hence

$$R_T^{e'}(w(\cdot)) = \frac{\Sigma \int_{I_j} [e'w^2(t) + c\ddot{w}^2(t)]dt}{\Sigma \int_{I_j} \dot{w}^2(t)dt} \geq \min_j \frac{\int_{I_j} [e'w^2(t) + c\ddot{w}^2(t)]dt}{\int_{I_j} \dot{w}^2(t)dt}.$$

Finally, we use (6.16′) to demonstrate the positivity of b_2' (positivity of the analogous quantity associated with $\psi^R(\cdot)$ is proved in the same way), so that the positivity of b_0 will follow from (6.13)-(6.16). Let $\{(w_n(\cdot),T_n)\}_{n\geq 1}$ denote a minimizing sequence for (6.16′); i.e.

$$(6.17) \qquad R_{T_n}^{e'}(w_n(\cdot)) \to b_2', \text{ with } T_n \in [\delta,2\delta), \dot{w}_n \not\equiv 0 \text{ on } [0,T_n].$$

Without loss of generality we can suppose that

$$(6.18) \qquad T_n \to T_0 \in [\delta,2\delta].$$

Moreover by the homogeneity of $R_T^{e'}$ we can suppose that

$$(6.19) \qquad \int_0^{T_n} \dot{w}_n^2(t)dt = 1, \quad \forall\, n \geq 1.$$

For those values of n with $T_n < T_0$ we extend $w_n(\cdot)$ from $[0,T_n]$ onto $[T_n,T_0]$ as that (linear) function corresponding to the identically zero extension of $\ddot{w}_n(\cdot)$ onto $[T_n,T_0]$. Denote the resulting function in $W^{2,1}(0,T_0)$ by $\tilde{w}_n(\cdot)$. On the other hand, for values of n such that

$T_n \geq T_0$ let $\tilde{w}_n(\cdot)$ denote the restriction of $w_n(\cdot)$ to $[0,T_0]$. In general

$$\int_0^{T_0} \dot{\tilde{w}}_n^2(t)dt \neq 1,$$

but it is easy to see that (6.17)-(6.19) imply

$$\int_0^{T_0} \dot{\tilde{w}}_n^2(t)dt \to 1, \tag{6.20}$$

as well as

$$\int_0^{T_0} \tilde{w}_n^2(t)dt \leq M, \int_0^{T_0} \ddot{\tilde{w}}_n^2(t)dt \leq M, \qquad \forall\ n \geq 1, \text{ for some } M < \infty. \tag{6.21}$$

Hence without loss of generality we can suppose, by extracting a subsequence, that there is an element $\ddot{v}(\cdot) \in L^2(0,T_0)$ and continuous functions $v(\cdot), \dot{v}(\cdot)$ for which

$$\begin{aligned} &\ddot{\tilde{w}}_n(\cdot) \to \ddot{v}(\cdot) \text{ weakly in } L^2(0,T_0), \\ &\tilde{w}_n(\cdot) \to v(\cdot),\ \dot{\tilde{w}}_n(\cdot) \to \dot{v}(\cdot) \text{ uniformly in } C[0,T_0]. \end{aligned} \tag{6.22}$$

That is, $w_n(\cdot) \to v(\cdot)$ weakly in $W^{2,2}(0,T_0)$. It follows from (6.22) and the sequential weak lower semicontinuity of the L^2-norm that

$$(6.23)\quad \int_0^T \dot{v}^2(t)dt = 1, \qquad R_{T_0}^{e'}(v(\cdot)) = \int_0^{T_0} (e_1' v^2(t) + c\ddot{v}^2(t)]dt = b_2'.$$

This obviously implies the asserted positivity of b_2', hence the positivity of b_1' in (6.15) (as well as the positivity of b_1 defined in (6.9)). □

Corollary 6.2. Suppose that $\psi(\cdot)$ as in Theorem 6.1 has a single absolute minimizer (i.e., $M = \{w_1\}$) and in addition that $\psi(\cdot)$ satisfies

$$(6.24)\qquad \psi(w) \geq e(w - w_1)^2 \quad, \quad w \in \mathbb{R}, \quad \text{where} \quad e = \frac{1}{2}\psi''(w_1).$$

In this case the threshold value $b_0 = b_0(c\ ;\ \psi(\cdot))$ is given by

$$(6.25)\qquad b_0 = \sqrt{2c\psi''(w_1)}.$$

Proof: According to (6.9)

$$(6.26)\qquad b_0 \geq b_1 := \inf_{\substack{T>0 \\ x(0)=x(T)}} R_T^e(w(\cdot)).$$

We proceed to appraise b_1 by making use of the arithmetic-geometric mean inequality, followed by integration by parts, for functions $w(\cdot)$ in (6.26):

$$\int_0^T [ew^2(t) + c\ddot{w}^2(t)] \geq \int_0^T [-2\sqrt{ec}\, w(t)\ddot{w}(t)]dt$$

(6.27)

$$= 2\sqrt{ec}\left\{\int_0^T \dot{w}^2(t)dt - w(t)\dot{w}(t)\Big]_0^T\right\} = 2\sqrt{ec}\int_0^T \dot{w}^2(t)dt.$$

This yields the inequality

(6.28) $$b_1 \geq 2\sqrt{ec}.$$

Moreover equality holds in (6.27) if and only if

$$\sqrt{e}\, w(t) + \sqrt{c}\, \ddot{w}(t) = 0 \quad \text{a.e.} \quad t \in [0,T].$$

It follows that

$$R_T^e(w(\cdot)) = 2\sqrt{ec}$$

whenever T is a multiple of $T^* = (c/e)^{1/4}\pi$ and $w(\cdot)$ has the form

(6.29) $$w(t) = C\cos((e/c)^{1/4}t - \theta), \quad C,\theta \text{ constants}.$$

Hence (6.28) is actually an equality.

In order to verify (6.25) we now write (6.8) in the form

(6.8′)
$$b_0 = \inf_{\substack{T>0 \\ x(0)=x(T)}} R_T(w(\cdot) + w_1 \; ; \; \psi(\cdot)).$$

Now when $w(\cdot)$ in (6.8′) is replaced by $\lambda w(\cdot)$, $\lambda \in (0,1)$ then (6.24) implies that $R_T(\lambda w(\cdot) + w_1)$ satisfies

$$\frac{\int_0^T (\psi(\lambda w(t) + w_1) + c\lambda^2 \ddot{w}^2(t))dt}{\int_0^T \lambda^2 \dot{w}^2(t)dt} \geq \frac{\int_0^T (ew^2(t) + c\ddot{w}^2(t))dt}{\int_0^T \dot{w}^2(t)dt} = R_T^e(w(\cdot)).$$

On the other hand by the formula $e = \frac{1}{2}\psi''(w_1)$ and the smoothness of $\psi(\cdot)$ we know that for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$ev^2 \leq \psi(v + w_1) \leq (1 + \epsilon)ev^2 \quad , \quad |v| \leq \delta.$$

Consequently for each fixed $w(\cdot)$ as in (6.8′) there exists $0 < \lambda << 1$ such that

(6.30)
$$R_T^e(w(\cdot)) \leq R_T(\lambda w(\cdot) + w_1) \leq \frac{\int_0^T [(1 + \epsilon)ew^2(t) + c\ddot{w}^2(t)]dt}{\int_0^T \dot{w}^2(t)dt} = R_T^{(1+\epsilon)e}(w(\cdot)).$$

It follows from (6.8′) and (6.30) that for each $\epsilon > 0$

$$\inf_{\substack{T>0 \\ x(0)=x(T)}} R_T^e(w(\cdot)) \leq b_0 \leq \inf_{\substack{T>0 \\ x(0)=x(T)}} R_T^{(1+\epsilon)e}(w(\cdot))$$

By (6.28) this is equivalent to

$$2\sqrt{ec} \leq b_0 \leq 2\sqrt{(1 + \epsilon)ec},$$

so that (6.25) follows. □

§7. Proof of Lemma 4.3.

We first prove the assertion of the Lemma for values of T in the interval $(0,1)$. Then for any $T_0 > 1$ the assertion follows for all T in the interval $(0,T_0)$ by considering the sequence $a'_k = a_k/T_0$ and applying the result for T in $(0,1)$ to the sequence $\{a'_k\}$. Thus the assertion follows for all T in $(0,\infty)$.

Let $\epsilon > 0$ be fixed and let $I_k = (a_k - \epsilon, a_k)$. For a fixed k we define

$$A_k = \bigcup_{n \geq a_k} \frac{1}{n} I_k,$$

where for an interval I and a scalar $c \neq 0$

$$\frac{1}{c} I = \{x : cx \in I\}.$$

Then A_k is the set of all points in $(0,1)$ such that

$$a_k - \epsilon < nx < a_k \quad \text{for some integer } n.$$

Thus $\bigcup_{k=m}^{\infty} A_k$ is the set of all points x in $(0,1)$ such that the relation $a_k - \epsilon < nx < a_k$ holds for some $k \geq m$, $n \geq 1$, and

$$B_\epsilon = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k$$

is the set of all $x \in (0,1)$ such that $a_k - \epsilon < nx < a_k$ holds for infinitely many pairs of integers (k,n). Our goal is to prove that

$$(7.1) \qquad m(B_\epsilon) = 1 \quad \text{for each} \quad \epsilon > 0.$$

We then define $B = \bigcap_{n=2}^{\infty} B_{1/n}$, and (7.1) implies that $m(B) = 1$, so that the relation (4.9) holds for every $T \in B$. Since $B_\epsilon = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k$ it suffices, in order to establish (7.1), to show that

$$(7.2) \qquad m\left[\bigcup_{k=m}^{\infty} A_k\right] = 1 \quad \text{for each} \quad m \geq 1.$$

This will conclude the proof of the Lemma, and the argument is given below.

We first examine the structure of the set A_k. It is a union of intervals $\frac{1}{n} I_k$ for $n \geq a_k$. Put

$$(7.3) \qquad n_0 = \left[\frac{a_k}{\epsilon}\right],$$

(where [x] is the largest integer not exceeding x), and consider any $n < n_0$. Then

$$\frac{a_k - \epsilon}{n} \geq \frac{a_k}{n+1}$$

so that the intervals $\frac{1}{n}I$ and $\frac{1}{n+1}I$ are nonoverlapping. That is, the overlapping portions of A_k are composed of

$$\bigcup_{n \geq n_0} \frac{1}{n} I_k$$

(if $\frac{a_k}{\epsilon}$ is not an integer, which can be assumed without loss of generality by suitable choice of ϵ), and thus lie in the interval $(0,\frac{a_k}{n_0})$. This contains the interval $(0,\epsilon)$ and is as close to it as we please for large k.

The measure of the nonoverlapping parts of A_k is $\epsilon \sum\limits_{a_k \leq j < n_0} \frac{1}{j}$ and this can be approximated by $\epsilon \, \ell og \frac{n_0}{a_k}$, which in turn can be approximated by the quantity

$$\delta := \epsilon \, \ell og \, 1/\epsilon.$$

The approximation is valid in the sense that the measure of the nonoverlapping parts of A_k lies between $(1 - \theta)\delta$ and $(1 + \theta)\delta$ for any prescribed small $\theta > 0$, provided that k is sufficiently large.

Now we take an interval (a,b) which is contained in $(\epsilon,1)$ and we estimate the measure of $(a,b) \cap A_k$. Since $(a,b) \subset (\epsilon,1)$, for large enough

k only intervals in the nonoverlapping part of A_k will intersect (a,b), so

$$m[(a,b) \cap A_k] = \epsilon \sum_{j \in S} \frac{1}{j},$$

S being the set of integers $\{j : [a_k/b] < j \leq [a_k/a]\}$, which implies that

$$m[(a,b) \cap A_k] \geq \epsilon \frac{1}{2} \log \frac{b}{a} = \frac{\epsilon}{2}(-\log \frac{a}{b}) \geq \frac{\epsilon}{2}(1 - a/b)$$

if k is large enough. Thus

$$m[(a,b) \cap A_k] \geq \frac{\epsilon}{2}(b - a), \tag{7.4}$$

provided that k is sufficiently large.

We will now select a sequence of integers $\{k_i\}_{i=1}^{\infty}$ increasing to infinity, and we will show that $m\left[\bigcup_{i=i_0}^{\infty} A_{k_i}\right] = 1$ for each i_0. Suppose $A_{k_1}, \ldots, A_{k_\ell}$ to have been already chosen. Then $\bigcup_{i=1}^{\ell} A_{k_i}$ is a finite union of intervals and so is its complement in $(0,1)$, which we denote by Δ_ℓ. We know that $\Delta_\ell \subset (\epsilon,1)$ and we write it as a finite union of intervals $\Delta_\ell = \bigcup_{p=1}^{m} J_p$. For each J_p we select a closed interval K_p satisfying

$$K_p \subset \text{int } J_p \tag{7.5}$$

$$m(K_p) > \frac{1}{2}\, m(J_p). \tag{7.6}$$

It follows from (7.5) that if k is sufficiently large then no interval in A_k will intersect both K_p and and $\bigcup_{i=1}^{\ell} A_{k_i}$. It also follows from (7.4) and (7.6) that if k is sufficiently large then

$$m[K_p \cap A_k] > \frac{1}{2}\,\epsilon\,\frac{1}{2}\, m(J_p) = \frac{\epsilon}{4}\, m(J_p). \tag{7.7}$$

Since we have only a finite number of intervals K_p we can find an integer k large enough so that (7.7) holds for every p, $1 \leq p \leq m$, and we choose $k_{\ell+1}$ to be such an integer k. In fact by the same argument, $k_{\ell+1}$ can be chosen large enough so that (7.7) holds for every K_p which satisfies (7.5) and (7.6) where now $\bigcup_{p=1}^{m} J_p$ denotes the complement of $\bigcup_{i=i_0}^{\ell} A_{k_i}$ for some i_0, $1 \leq i_0 \leq \ell$.

Now let $i_0 \geq 1$ be given and denote the measure of $\bigcup_{i=i_0}^{\ell} A_{k_i}$ by μ_ℓ. We have the following relation

$$\mu_{\ell+1} = \mu_\ell + m\left\{\left[(0,1)\setminus \bigcup_{i=i_0}^{\ell} A_{k_i}\right] \cap A_{k_{\ell+1}}\right\}$$

$$= \mu_\ell + m\left\{\left[(\epsilon,1)\setminus \bigcup_{i=i_0}^{\ell} A_{k_i}\right] \cap A_{k_{\ell+1}}\right\} \geq \mu_\ell + \sum_{p=1}^{m} \frac{\epsilon}{4}\, m(J_p),$$

where the last inequality follows from (7.7) and, as above $\bigcup_{p=1}^{m} J_p$ denotes the complement of $\bigcup_{i=i_0}^{\ell} A_{k_i}$. We thus deduce

(7.8)
$$\mu_{\ell+1} \geq \mu_\ell + \frac{\epsilon}{4} \sum_{p=1}^{m} m(J_p).$$

We know, however, that $\mu_\ell + \sum_1^m m(J_p) = 1$ (by definition) so that (7.8) implies

(7.9)
$$\mu_{\ell+1} \geq \mu_\ell + \frac{\epsilon}{4}(1 - \mu_\ell).$$

The sequence $\{\mu_\ell\}_{\ell=i_0}^{\infty}$ is nondecreasing and bounded by 1, so it tends to a limit μ. Since for $\mu < 1$ (7.9) implies that $\mu_\ell \to \infty$ it follows that $\mu = 1$, so the measure of $\bigcup_{i=i_0}^{\infty} A_{k_i}$ is equal to 1 for each $i_0 \geq 1$. □

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