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On "Multibump" Bound States for Certain Semilinear Elliptic Equations

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1. INTRODUCTION

In this paper, we study multiplicity of solutions to certain semilinear elliptic equations on $\mathbf{R}^N$. The basic idea is as follows: given finitely many solutions (at low energy), to translate their supports far apart and patch the pieces together to create many "multibump" solutions, i.e., solutions with most of their mass lying in a finite disjoint union of balls. Recent work by Coti-Zelati, Ekeland, and Séré [CES]; Séré [S]; and Coti-Zelati and Rabinowitz [CR1], [CR2] has introduced original and powerful ideas which permit the construction of such "multibump" solutions via variational methods. In particular, they are able to find infinitely many homoclinic-type solutions to periodic Hamiltonian systems ([S], [CR1]) and to certain elliptic equations of nonlinear Schrödinger type on $\mathbf{R}^N$ with periodic coefficients ([CR2]). The goal of our paper is to modify the techniques of [S], [CR1], [CR2] in order to fit variational problems which are only "asymptotically periodic".

We present the method through two general examples of semilinear elliptic equations on $\mathbf{R}^N$. One is the "indefinite" nonlinear Schrödinger equation with periodic potential, introduced in [AL]:

$$(-\Delta + V(x) - E)u = \pm W(x)|u|^{p-2}u \tag{1.1}$$

where V is periodic and bounded, $W(x) \geq 0$ is asymptotic (as $|x| \to \infty$) to a function which is periodic with the same period as V, $p > 2$ (and $p < 2N/(N-2)$ if $N \geq 3$), and E lies in a gap of $\sigma(-\Delta + V(x))$. As in [AL], we will apply a dual variational approach to the study of (1.1), in order to circumvent the technical problems caused by the indefiniteness of the linear operator. In [AL], it is shown that (1.1) possesses at least one solution in the case that W is periodic. [BJ] have shown the existence of one solution with asymptotically periodic W, but under the assumption that W approaches its periodic limit function from above. The case where $W(x) \to 0$ as $|x| \to \infty$ has been extensively studied (see [AL], [HS] and the references contained there); this problem is substancially simpler due to the relative compactness of the nonlinear term.

The other example we treat is the scalar field equation studied in [CR2],

$$-\Delta u + u = f(x, u) \tag{1.2}$$

Here $f(x, u) \geq 0$ is continuous, asymptotically periodic in x for each fixed u, and satisfies certain superlinear growth conditions.

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Elliptic equations of nonlinear Schrödinger type arise (in $N = 3$ dimensions) in nonlinear optics in modelling phenomena such as graded lightguides, laser-induced plasmas, and optical channelling of lasers. These models include a "Kerr effect" in which the self-interaction of light beams leads to a nonlinear relationship between the index of refraction and the field intensity $|u|$, and hence the appearence of a nonlinear potential term of the form $f(x,u) = h(x,|u|^2)u$ (see [Be]).

Our methods continue in the direction pioneered in [S], [CR1], and [CR2]. These techniques provide, roughly speaking, ways of gluing "approximate solutions" together to obtain a genuine solution. There have been many works on "gluing approximate solutions" by using the implicit function theorem (see, for example, [Ta], [Sc], [Sm], [Ka], [Oh], [Po] and the references therein) where more precise information on the linearized problem is needed. However it seems that the methods in Sére ([S]), Coti-Zelati and Rabinowitz ([CR1], [CR2]) have provided an elegant way to glue approximate solutions for certain periodic problems where it is difficult to obtain as precise information as needed for applying the implicit function theorem.

The second author has given a slight modification to the minmax procedure in [CR1] and [CR2] and has applied it to certain problems where periodicity is not present, for example, the problem of prescribing scalar curvature on S^3 and S^4 (see [Li1] and [Li2]). However the modification there does not seem to apply to the problem we discuss in section 2 of the present paper. In particular, we are unable to push through the argument in [CR1] (section 4, step 3,4) with our indefinite dual functional. Instead we introduce an auxilliary functional ($\tilde{J}_n$) which has been inspired by the works of Sére ([S]), Coti-Zelati and Rabinowitz ([CR1], [CR2]). This auxilliary functional is introduced in section 3 to deal with our indefinite functional in the asymptotically periodic case. In addition, it seems clear that by using our auxilliary functional one can generalize the results in [S] (on homoclinic orbits to Hamiltonian Systems) to the asymptotically periodic case. An almost identical auxillary functional can also be used in H^1 spaces, where we must replace the discontinuous cut-offs used in defining our auxilliary functional in the dual setting. We have, in section 5, given a sketch of such an application and thus given a different proof of a slightly more general result (asymptotically periodic case) than that of Coti-Zelati and Rabinowitz ([CR2]). One can see that in our proof we do not need to consider a minimization problem as in [CR1] (section 4, step 3,4), and we hope that this may be advantageous in dealing with certain situations where the minimization problem can not be handled.

Our results (Theorems 4.1, 5.1) follow the lines of [S], [CR1] and [CR2]: we define a mountain pass value c, and prove that only one of the following two conditions may hold: either the associated periodic functional has infinitely many critical points with critical value bounded by $c + a$ for some $a > 0$; or the original functional has infinitely many critical points near each level kc for each $k = 2, 3, \ldots$. To prove this result, we suppose that the first case does not hold, i.e., that there are only finitely many critical points (of the periodic problem) at level c. In section 2, we prove a Concentration Compactness result (Lemma 2.4), which shows that (because of asymptotic periodicity) the Palais-Smale condition is violated via sums of translates of critical points, which have no strong limit as the distance separating the pieces tends to infinity. However, it is shown (Lemma 2.5, following [S], [CR1]) that, under our assumption of finitely many critical points at level c, the (PS) condition holds along the pseudo-gradient flow for the functional. In section 3 we introduce our auxilliary functional, which permits us to create mountain pass constructions at levels kc, $k = 2, 3, \ldots$ Then, by modifying an argument of Séré (see Lemmas 4.2, 5.2), we may compare the dynamics of the pseudo-gradient flow for the auxilliary functional with the dynamics of the flow for the original functional and therefore construct a deformation at level kc, $k \geq 2$.

Notations. We write $L^p = L^p(\mathbf{R}^N)$ throughout, and denote the standard $L^p(\mathbf{R}^N)$ norm by $\|u\|_p$. We write $q = \dfrac{p}{p-1}$ with p as in equation (1.1). χ_S denotes the characteristic function of the Borel set S. The open ball of radius r about $v \in L^q$ is written as $B(v,r)$. We also represent the action of integer translation on functions by $* : \mathbf{Z}^N \times L^q \to L^q$,

$$\xi * v(x) = v(x - \xi)$$

where $\xi \in \mathbf{Z}^N$ and $v \in L^q$. Finally, we denote by C various constants whose precise value may change from line to line.

2. Concentration-Compactness

In this section, we define our variational functionals and present lemmas which characterize their Palais-Smale sequences.

Suppose that $V \in L^\infty(\mathbf{R}^N)$ is periodic with respect to some N-dimensional lattice which (for simplicity) we take to be $\mathbf{Z}^N$. Then the Schrödinger operator $H \equiv -\Delta + V$ is defined as a self-adjoint operator on its domain $H^2(\mathbf{R}^N) \subset L^2(\mathbf{R}^N)$. By Floquet-Bloch theory, the spectrum $\sigma(H)$ consists of bands. In what follows, we will assume that V is chosen such that its spectrum contains a gap, i.e., that there exists an open (non-empty) interval (a,b), $b < a$, such that

$$b < \inf \sigma(H), \qquad (a,b) \cap \sigma(H) = \emptyset$$

We also fix a value $E \in (a,b)$.

We will use the following estimates concerning the operator $(H-E)^{-1}$:

Lemma 2.1. *$(H-E)^{-1}$ is an integral operator satisfying:*

(a) *$(H-E)^{-1} : \; L^s(\mathbf{R}^N) \to L^t(\mathbf{R}^N)$ is a bounded operator for $1 < s < +\infty$, $0 \leq \frac{1}{s} - \frac{1}{t} < \frac{2}{N}$. In particular, there is a constant C such that*

$$\left| \int w\,(H-E)^{-1} v \right| \leq C \|w\|_q \|v\|_q \quad \forall\, 1 < q \leq 2 \tag{2.1}$$

(b) *The operator $\chi_B(x)(H-E)^{-1} : \; L^s(\mathbf{R}^N) \to L^t(\mathbf{R}^N)$ (s,t as in (a)) is compact for B a bounded set in $\mathbf{R}^N$.*

(c) *There exists constants $C > 0$ and $\kappa > 0$ such that if $u, v \in L^q(\mathbf{R}^N)$ $(1 < q \leq 2)$ are such that $\operatorname{supp} u$ and $\operatorname{supp} v$ are disjoint, then*

$$\left| \int v \cdot (H-E)^{-1} u \, dx \right| \leq C \|u\|_q \, \|v\|_q \, e^{-\kappa d(u,v)} \tag{2.2}$$

where $d(u,v) \equiv \mathrm{dist}(\mathrm{supp}\, u, \mathrm{supp}\, v)$

(d) $(H-E)^{-1}$ *commutes with integer translations in* x*: i.e., if* $\xi \in \mathbf{Z}^N$ *and* $\tau_\xi v(x) \equiv v(x-\xi)$*, then* $\tau_\xi (H-E)^{-1} = (H-E)^{-1}\tau_\xi$.

Properties (a)–(c) follow from estimates of Simon [Si] which show that $(H-E)^{-1}$ is an integral operator whose kernel $k(x,y)$ decays exponentially as $|x-y| \to \infty$.

Now, consider a function $1 \le W_\infty \in L^\infty$ which is periodic with the same period as V, and suppose that $0 \le W \in L^\infty$ satisfies:

$$\lim_{|x|\to\infty} (W(x) - W_\infty(x)) = 0 \tag{2.3}$$

We consider the following problems in $H^2(\mathbf{R}^N)$:

$$(H-E)u = \pm W(x)|u|^{p-2}u \tag{2.4}$$

$$(H-E)u = \pm W_\infty(x)|u|^{p-2}u \tag{2.5}$$

where H, E are as above, and $2 < p < +\infty$ when $N = 1,2$, $2 < p < \dfrac{2N}{N-2}$ when $N \ge 3$.

We next define the operators (Birman-Schwinger kernel– see [RS]),

$$L \equiv W^{1/p}(H-E)^{-1}W^{1/p}$$

$$L_\infty \equiv W_\infty^{1/p}(H-E)^{-1}W_\infty^{1/p}$$

Clearly, L and L_∞ satisfy (a)–(c), and L_∞ satisfies (d). Define the functionals

$$J(v) = \frac{1}{q}\int |v|^q - \frac{1}{2}\int v\, Lv$$

$$J_\infty(v) = \frac{1}{q}\int |v|^q - \frac{1}{2}\int v\, L_\infty v$$

for $v \in L^q(\mathbf{R}^N)$, where $q = \dfrac{p}{p-1}$ is the dual exponent to p. The motivation behind these functionals is the following nonlinear version of the classical Birman-Schwinger principle (see [RS]) which was introduced in [AL]: for each distinct critical point of J (resp. distinct mod $\mathbf{Z}^N$ critical point of J_∞), there exists a distinct solution of (2.4+) (resp. distinct mod $\mathbf{Z}^N$ solution of (2.5+)). The same holds for (2.4–) or (2.5–), as we merely switch the sign of the quadratic term in each functional, and all of the following analysis goes through without change. Note that the condition $2 < p < +\infty$ (when $N = 1,2$), $2 < p < \dfrac{2N}{N-2}$ (when $N \ge 3$) implies that

$$1 < q < 2 (N = 1,2), \quad \frac{2N}{N+2} < q < 2 (N \ge 3)$$

Note also that by Lemma 2.1 (d) above, $J_\infty(\tau_\xi v) = J_\infty(v)$ for all $\xi \in \mathbf{Z}^N$.

Denote by $\mathcal{C}$ the critical points of J_∞. In [AL] we show that $\mathcal{C}$ is non-empty. Define also

$$\begin{aligned} J^t &= \{v \in L^q : \quad J(v) \le t\} \\ J_s &= \{v \in L^q : \quad J(v) \ge s\} \\ J_s^t &= J_s \cap J^t \end{aligned}$$

and similarly for $(J_\infty)^t, (J_\infty)_s, (J_\infty)_s^t$. In addition, we will use the notation: $\mathcal{C}^t = \mathcal{C} \cap J^t$, $\mathcal{C}_s^t = \mathcal{C} \cap J_s^t$.

LEMMA 2.2. *0 is an isolated critical point of both J and J_∞, i.e., there exists $\nu > 0$ so that for any v which is a nontrivial critical point of either J or J_∞, $\|v\|_q \ge \nu > 0$*

PROOF: Suppose v is any nontrivial critical point of J (the case for J_∞ is identical). Then

$$0 = J'(v)v = \int |v|^q - \int v\, Lv$$

Using (2.1), we obtain

$$\int |v|^q = \int v\, Lv \le C\|v\|_q^2$$

with C independent of v. The result follows, as $q < 2$. ∎

LEMMA 2.3.

(i) *There exists $\beta > 0$ with $J_\infty(v) \ge \beta$ for any $v \in \mathcal{C}\backslash\{0\}$.*

(ii) *If $v \in \mathcal{C}\backslash\{0\}$ and $J_\infty(v) = b$, then*

$$\|v\|_q = \left(\frac{2qb}{2-q}\right)^{1/q}$$

PROOF: (i) If $v \in \mathcal{C}\backslash\{0\}$, Lemma 2.2 yields $\|v\|_q \ge \nu > 0$. Again, we have

$$\int |v|^q = \int v\, L_\infty v$$

so

$$J_\infty(v) = (\frac{1}{q} - \frac{1}{2}) \int |v|^q \ge \left(\frac{2-q}{2q}\right) \nu^q$$

(ii)

$$\begin{aligned} b = J_\infty(v) &= J_\infty(v) - \frac{1}{2} J'_\infty(v)v \\ &= (\frac{1}{q} - \frac{1}{2}) \int |v|^q \end{aligned}$$

gives the desired equality. ∎

REMARK: The same conclusions hold if we replace J_∞ by J.

We now demonstrate a Concentration-Compactness property for our functional J_∞. Namely, we show that Palais-Smale sequences fail to be compact because mass may split off and translate to infinity (see [Lns] for the Concentration-Compactness method).

LEMMA 2.4. *Suppose $\{v_n\}$ is a sequence in $L^q(\mathbf{R}^N)$ such that (as $n \to \infty$)*

$$J_\infty(v_n) \to d > 0$$
$$J'_\infty(v_n) \to 0 \qquad (\text{in } L^p(\mathbf{R}^N))$$

Then, there is a positive integer $m < \infty$, m critical points $\{v^{(j)}\}_{j=1,\ldots,m} \subset \mathcal{C} \cap (J_\infty)^c$, and m sequences of integer coordinate vectors $\{y_n^j\}_{j=1,\ldots,m} \subset \mathbf{Z}^N$ such that along some subsequence $n \to \infty$:

$$\left.\begin{aligned} \|v_n - \sum_{j=1}^{m} v^{(j)}(x + y_n^j)\|_q \to 0 \\ |y_n^i - y_n^j| \to \infty \quad if \;\; i \neq j \\ \sum_{j=1}^{m} J_\infty(v^{(j)}) = d \end{aligned}\right\} \tag{2.6}$$

PROOF: We denote by $g(v) \equiv |v|^{q-2}v$; then note that

$$J'_\infty(v) = g(v) - L_\infty v \in L^p(\mathbf{R}^N)$$

By standard arguments, the sequence $\{v_n\}$ is uniformly bounded in $L^q(\mathbf{R}^N)$. Using Step 5 in the proof of Lemma 3.1 in [AL], there exist constants $\alpha > 0$ and $R > 0$ and a sequence $\{y_n^1\} \in \mathbf{Z}^N$ such that

$$\int_{B_R(y_n^1)} |v_n|^q \, dx \geq \alpha \tag{2.7}$$

Set $v_n^{(1)}(x) \equiv v_n(x - y_n^1)$. By the $\mathbf{Z}^N$-invariance of J_∞, we have $J_\infty(v_n^{(1)}) = J_\infty(v_n) \to d$ and $J'_\infty(v_n^{(1)}) = J'_\infty(v_n) \to 0$ as $n \to \infty$. Extract a subsequence (which we will still denote as $v_n^{(1)}$) with

$$v_n^{(1)} \rightharpoonup v^{(1)} \text{ in } L^q(\mathbf{R}^N)$$

Next we examine the convergence of $v_n^{(1)}$ and $g(v_n^{(1)})$. First, as $g(v_n^{(1)}) = J'_\infty(v_n^{(1)}) + L_\infty v_n^{(1)}$, clearly there exists $\gamma \in L^p(\mathbf{R}^N)$ with $g(v_n^{(1)}) \rightharpoonup \gamma$ weakly in L^p. Let $\varphi \in C_0^\infty(\mathbf{R}^N)$ with $\operatorname{supp}\varphi \equiv S$. Then

$$\begin{aligned} \left|\int_S \varphi[g(v_n^{(1)}) - g(v_m^{(1)})]\,dx\right| &\leq \left|J'_\infty(v_n^{(1)})\varphi - J'_\infty(v_m^{(1)})\varphi\right| + \left|\int_{\mathbf{R}^N} \chi_S L_\infty(v_n^{(1)} - v_m^{(1)}) \cdot \varphi\,dx\right| \\ &= o(1)\|\varphi\|_{L^p(S)} \end{aligned}$$

as $L_\infty : L^q \to L^p$ is locally compact. Therefore, exhausting $\mathbf{R}^N$ by compact sets $S_m \to \mathbf{R}^N$ (as $m \to \infty$) and taking a diagonal subsequence, there exists $\gamma \in L^p(\mathbf{R}^N)$ with $g(v_n^{(1)}) \to \gamma$ in the L^p_{loc} sense, and consequently $g(v_n^{(1)}(x)) \to \gamma(x)$ and $v_n^{(1)}(x) \to g^{-1}(\gamma(x))$ almost everywhere. As we already have $v_n^{(1)} \rightharpoonup v^{(1)}$, it follows that $\gamma = g(v^{(1)})$. Putting this together yields

$$g(v_n^{(1)}) \rightharpoonup g(v^{(1)}) \text{ weakly in } L^p \text{ and strongly in } L^p_{loc} \tag{2.8}$$

In addition, we have (by the weak L^q and almost everywhere convergence of $v_n^{(1)}$) that for any compact $S \in \mathbf{R}^N$

$$\begin{aligned}\int_S |v_n^{(1)} - v^{(1)}|^q &= \int_S |v_n^{(1)}|^q - |v^{(1)}|^q + o(1) \\ &= \int_S |g(v_n^{(1)})|^p - |g(v^{(1)})|^p + o(1) = o(1)\end{aligned}$$

by (2.8); i.e., $v_n^{(1)} \to v^{(1)}$ in L^q_{loc}.

Now we show that the limit function $v^{(1)} \in \mathcal{C}\backslash\{0\}$. Let $\varphi \in L^q(\mathbf{R}^N)$. Then,

$$\begin{aligned}0 &= \lim_{n\to\infty} J'_\infty(v_n^{(1)})\varphi \\ &= \lim_{n\to\infty} \int \varphi \cdot (g(v_n^{(1)}) - L_\infty v_n^{(1)}) \\ &= \int \varphi \cdot (g(v^{(1)}) - L_\infty v^{(1)})\end{aligned}$$

by applying (2.8) and the boundedness of L_∞, and so $v^{(1)} \in \mathcal{C}$. Suppose that $v^{(1)} \equiv 0$. Let $\varphi \in C_0^\infty(\mathbf{R}^N)$ such that $\varphi|_{B_R(0)} = 1$ (where R is as in (2.7)) and $0 \le \varphi(x) \le 1$. By local compactness, $\varphi L v_n^{(1)} \to 0$ strongly in $L^p(\mathbf{R}^N)$ and hence:

$$\begin{aligned}0 &= \lim_{n\to\infty} J'_\infty(v_n^{(1)})(\varphi v_n^{(1)}) \\ &= \lim_{n\to\infty} \left[\int |v_n^{(1)}|^q \varphi - \int v_n^{(1)} \cdot \varphi L_\infty v_n^{(1)}\right] \\ &\ge \liminf_{n\to\infty} \int_{B_R(0)} |v_n^{(1)}|^q \ge \alpha > 0\end{aligned}$$

by (2.7), and so $v^{(1)} \not\equiv 0$.

Now let's compare the critical value with the value d. As

$$\begin{aligned}d = \lim_{n\to\infty} J_\infty(v_n) &= \lim_{n\to\infty} (J_\infty(v_n) - \frac{1}{2} J'_\infty(v_n)v_n) \\ &= \lim_{n\to\infty} \frac{2-q}{2q} \int |v_n|^q\end{aligned}$$

and by Lemma 2.3,

$$\int |v^{(1)}|^q = \frac{2q}{2-q} J_\infty(v^{(1)})$$

we see that $J_\infty(v^{(1)}) \le d$ with equality if and only if $v_n^{(1)} = v_n(\cdot + y_n^1) \to v^{(1)}$ strongly in L^q. Thus, in the case that $J_\infty(v^{(1)}) = d$, the Lemma is proven, with $m = 1$.

If this is not the case, i.e., if $0 < J_\infty(v^{(1)}) < d$, then define a new sequence

$$v_n^{(2)}(x) = v_n(x - y_n^1) - v^{(1)}(x) = v_n^{(1)}(x) - v^{(1)}(x)$$

We have $v_n^{(2)} \rightharpoonup 0$ in L^q and almost everywhere. We now claim that $v_n^{(2)}$ is again a Palais-Smale sequence for J_∞, but with lower energy than v_n. By the boundedness of L_∞ and the weak convergence of $v_n^{(1)}$, we have

$$\int v_n^{(2)} \cdot L_\infty v_n^{(2)} = \int v_n^{(1)} \cdot L_\infty v_n^{(1)} - \int v^{(1)} \cdot L_\infty v^{(1)} + o(1)$$

In addition, we claim:

$$\int |v_n^{(2)}|^q = \int |v_n^{(1)}|^q - \int |v^{(1)}|^q + o(1)$$
$$\text{and} \quad g(v_n^{(2)}) = g(v_n^{(1)}) - g(v^{(1)}) + o(1)$$

(where the second statement is interpreted in the strong L^p sense.) The first of these statements is a direct application of the Brezis-Lieb Lemma [BL]. For the second one, we use the following basic estimate: there exists $C > 0$ so that for all $a, b \in \mathbf{R}$,

$$|g(a+b) - g(a) - g(b)| \leq C|b|^{q-1}$$

Let $\epsilon > 0$ be given. Fix $R > 0$ so that $\int_{|x|>R} |v^{(1)}|^q < C^{-p}\epsilon^p$. Applying the estimate with $a = v_n^{(1)}$ and $b = v^{(1)}$, we have

$$\left(\int_{|x|>R} |g(v_n^{(1)} - v^{(1)}) - g(v_n^{(1)}) + g(v^{(1)})|^p\right)^{1/p} \leq C\left(\int_{|x|>R} |v^{(1)}|^q\right)^{1/p} < \epsilon$$

On the bounded set $\{|x| \leq R\}$, we use the strong local convergence as in (2.8):

$$\begin{aligned}
&\left(\int_{|x|\leq R} |g(v_n^{(2)}) - g(v_n^{(1)}) + g(v^{(1)})|^p\right)^{1/p} \leq \\
&\leq \left(\int_{|x|\leq R} |g(v_n^{(1)} - v^{(1)})|^p\right)^{1/p} + \left(\int_{|x|\leq R} |g(v_n^{(1)}) - g(v^{(1)})|^p\right)^{1/p} \\
&= \left(\int_{|x|\leq R} |v_n^{(1)} - v^{(1)}|^q\right)^{1/p} + \left(\int_{|x|\leq R} |g(v_n^{(1)}) - g(v^{(1)})|^p\right)^{1/p} = o(1)
\end{aligned}$$

and so we arrive at the second part of the claim as ϵ is arbitrary.

As a result of the claim, we have

$$\begin{aligned}
J_\infty(v_n^{(2)}) &= J_\infty(v_n) - J_\infty(v^{(1)}) + o(1) \\
&= d - J_\infty(v^{(1)}) + o(1)
\end{aligned}$$

and

$$J'_\infty(v_n^{(2)}) = J'_\infty(v_n) - J'_\infty(v^{(1)}) + o(1) = o(1)$$

We now repeat the above argument for the sequence $v_n^{(2)}$, obtaining $\{z_n^2\} \subset \mathbf{Z}^N$ and $v^{(2)} \in \mathcal{C}\backslash\{0\}$ with $v_n^{(2)}(x - z_n^2) \rightharpoonup v^{(2)}(x)$ in L^q and almost everywhere. Note that z_n^2 must be unbounded, as $v_n^{(2)}(x) \rightharpoonup 0$. Set $y_n^2 \equiv y_n^1 + z_n^2$. As before, $J_\infty(v^{(2)}) \leq d - J_\infty(v^{(1)})$, with equality if and only if the convergence $v_n^{(2)}(x - z_n^2) \to v^{(2)}(x)$ is strong. If so, we are done and $m = 2$. If not, we continue the process a third time.

By Lemma 2.2, $J_\infty(v) \geq \beta > 0$ for any $v \in \mathcal{C}\backslash\{0\}$, and so this process must end after a finite number m of iterations, giving $v_n^{(m)}(x - z_n^m) \to v^{(m)}(x)$ strongly in L^q, $z_n^m \in \mathbf{Z}^N$ unbounded and

$$y_n^m = y_n^{m-1} + z_n^m = y_n^1 + \sum_{j=2}^{m} z_n^j$$

with $v_n^m(x) = v_n^{m-1}(x - z_n^{m-1}) - v^{(m-1)}(x)$, i.e.,

$$v_n(x) = \sum_{j=1}^{m} v^{(j)}(x + y_n^j) + o(1)$$

and $J_\infty(v^{(m)}) = d - \sum_{j=1}^{m-1} J_\infty(v^{(j)})$. This proves the Lemma. ∎

REMARK: A similar concentration compactness lemma holds for J: with the same hypotheses as Lemma 2.4, there exists a critical point $v^{(0)}$ of J (possibly trivial), a positive integer $m < \infty$, m critical points $\{v^{(j)}\}_{j=1,\ldots,m} \subset \mathcal{C} \cap (J_\infty)^d$, and m sequences of integer coordinate vectors $\{y_n^j\}_{j=1,\ldots,m} \subset \mathbf{Z}^N$ such that along some subsequence $n \to \infty$:

$$\begin{cases} \|v_n - v^{(0)} - \sum_{j=1}^{m} v^{(j)}(x + y_n^j)\|_q \to 0 \\ |y_n^i| \to \infty \ \ \& \ \ |y_n^i - y_n^j| \to \infty \ if \ i \neq j \\ J(v^{(0)}) + \sum_{j=1}^{m} J_\infty(v^{(j)}) = d \end{cases}$$

Now, as in [AL] we may construct a mountain pass value for J_∞. Let

$$\begin{aligned} \Gamma &= \{\gamma \in C([0,1]; L^q) : \quad \gamma(0) = 0, \quad J(\gamma(1)) < 0\} \\ \Gamma_\infty &= \{\gamma \in C([0,1]; L^q) : \quad \gamma(0) = 0, \quad J_\infty(\gamma(1)) < 0\} \end{aligned} \tag{2.9}$$

and define

$$c = \inf_{\gamma \in \Gamma_\infty} \max_{0 \leq t \leq 1} J_\infty(\gamma(t)) \tag{2.10}$$

From Lemmas 3.11, 3.12 of [AL], we have $c > 0$, and via Theorem 3.1 of [AL] (or Lemma 2.4 above), J_∞ has a critical value less than or equal to c. In fact, for the particular nonlinearity $g(v) = \pm|v|^{q-2}v$, it is easy to show that c is the smallest nontrivial critical value of J_∞. To see this, if $v \in \mathcal{C}\backslash\{0\}$, multiplying the Euler-Lagrange equation $L_\infty v = g(v)$ by v, $\int v \cdot L_\infty v = \int |v|^q > 0$ and so

$$J_\infty(v) = \left(\frac{1}{q} - \frac{1}{2}\right) \int |v|^q$$

As $J_\infty(Rv) < 0$ for $R > 0$ sufficiently large, $\gamma(t) = tRv \in \Gamma$ and so

$$\begin{aligned} c &\leq \max_{0 \leq t \leq R} J_\infty(tv) \\ &= \max_{0 \leq t \leq R} \frac{t^q}{q} \int |v|^q - \frac{t^2}{2} \int v \cdot L_\infty v \\ &= \left(\frac{1}{q} - \frac{1}{2}\right) \frac{\left(\int v \cdot L_\infty v\right)^{q/(q-2)}}{\left(\int |v|^q\right)^{2/(q-2)}} \\ &= \left(\frac{1}{q} - \frac{1}{2}\right) \int |v|^q = J_\infty(v) \end{aligned}$$

We seek the following type of multiplicity result: if J_∞ has finitely many critical points (mod $\mathbf{Z}^N$) near the level c, then then we will construct infinitely many critical points for J near *each* level kc, for $k = 2, 3, \ldots$ In particular this implies that the functional J_∞ will either have infinitely many critical points with energy near the level c, or else it will have infinitely many critical points at each energy kc, $k = 2, 3, \ldots$, and hence in either case it must possess infinitely many of critical points. Henceforth, we shall always assume that we are in the second case, and we introduce the hypothesis on J_∞:

$$\text{there exists } 0 < a < \frac{c}{2} \text{ such that } \mathcal{C}^{c+a} \bmod \mathbf{Z}^N \text{ is finite and } \mathcal{C}^{c+a} \doteq \mathcal{C}^c. \qquad (*)$$

Notice that the assumption $\mathcal{C}^{c+a} = \mathcal{C}^c$ in $(*)$ does not pose any real restriction. The reason is that once $\mathcal{C}_c^{c+a}$ mod $\mathbf{Z}^N$ is finite, the critical values of J_∞ in the interval $[c, c+a]$ will be finite, so we can always make a small enough to insure that c is the only critical value of J in the interval $[c, c+a]$.

When hypothesis $(*)$ holds, we may deduce further properties of our functional $J(v)$ for v nearby those points which are of the form $\sum_{j=1}^m v^{(j)}(\cdot + y^j)$ where $v^{(j)} \in \mathcal{C}_c^{c+a}$, $y^j \in \mathbf{Z}^N$, $|y^i - y^j|$ is large for $i \neq j$. The following lemma is due to Séré [S]:

LEMMA 2.5. *Assume $(*)$ holds.*

(a) *There exists r_0 with $0 < r_0 < \nu/3$ (where ν is as in Lemma 2.2) such that for all $u \in \mathcal{C}^{c+a}, v \in \mathcal{C}$ with $u \neq v$,*

$$\|u - v\|_q \geq 3r_0.$$

(b) *For all* $\rho < r_0$,

$$\inf\{\|J'_\infty(w)\|_p : \quad w \in \bigcup_{v\in\mathcal{C}^{c+a}} B(v,r_0)\backslash B(v,\rho)\} > 0$$

(c) *If* $v^{(1)},\ldots,v^{(k)} \in \mathcal{C}^{c+a}$, *and* $y_n^j \in \mathbf{Z}^N$ *with* $\lim_{n\to\infty}|y_n^i| \to \infty$ *(for all* i*) and* $\lim_{n\to\infty}|y_n^i - y_n^j| \to \infty$ $(i \neq j)$*, then*

$$\liminf_{n\to\infty} \inf\{\|J'(w)\|_p : \quad w \in B(\sum_{j=1}^k v^{(j)}(\cdot + y_n^j), r_0)\backslash B(\sum_{j=1}^k v^{(j)}(\cdot + y_n^j), \rho)\} > 0$$

PROOF: Although the proof follows the same lines as in [S], this result is central to all of the ideas in the paper, so we provide full details here.
(a) As $\mathcal{C}^{c+a}/\mathbf{Z}^N$ is finite, we write $\mathcal{C}^{c+a}/\mathbf{Z}^N = \{\bar{u}^{(1)},\ldots,\bar{u}^{(m)}\}$.
For $i \neq j$, we have

$$\lim_{y\in\mathbf{Z}^N,|y|\to\infty} \|\bar{u}^{(i)} - \bar{u}^{(j)}(\cdot + y)\|_q = \|\bar{u}^{(i)}\|_q + \|\bar{u}^{(j)}\|_q \geq 2\nu > 0$$

by Lemma 2.2. Therefore there exists a large constant $R > 0$, such that,

$$\inf_{y\in\mathbf{Z}^N,|y|\geq R} \|\bar{u}^{(i)} - \bar{u}^{(j)}(\cdot + y)\|_q \geq \nu.$$

As $\{\bar{u}^{(j)}(\cdot + y) : \quad j = 1,\ldots,m, \ y \in \mathbf{Z}^N, \ |y| \leq R\}$ is discrete, we can choose an $0 < r_1 < \nu/3$ such that for any $u, v \in \mathcal{C}^{c+a}$ with $u \neq v$, we have $\|u - v\| \geq 3r_1$. To allow v to belong to $\mathcal{C}$, we only have to notice that, if we choose $0 < r_0 < r_1$ small enough, we will have $B(u, 3r_0) \cap \mathcal{C} = B(u, 3r_0) \cap \mathcal{C}^{c+a}$ for all $u \in \mathcal{C}^{c+a}(= \mathcal{C}^c)$. Hence we have proved (a).
(b) Note that

$$\bigcup_{v\in\mathcal{C}^{c+a}} B(v,r_0)\backslash B(v,\rho) = \bigcup_{\xi\in\mathbf{Z}^N} \xi * \left(\bigcup_{i=1}^m B(\bar{u}^{(i)},r_0)\backslash B(\bar{u}^{(i)},\rho)\right) \tag{2.11}$$

Now, suppose for some fixed i there is a sequence $v_n \in B(\bar{u}^{(i)}, r_0)\backslash B(\bar{u}^{(i)}, \rho)$ with $J'_\infty(v_n) \to 0$. By Lemma 2.4, there exist integer translations y_n^j, $(1 \leq j \leq k$ for some integer $k)$ with

$$\lim_{n\to\infty} |y_n^l - y_n^j| = +\infty, \text{ for } l \neq j \tag{2.12}$$

(as $n \to \infty$) and critical points $v^{(j)}$ with $\|v_n - \sum_{j=1}^k v^{(j)}(\cdot - y_n^j)\|_q \to 0$. It follows from the triangle inequality and $v_n \in B(\bar{u}^{(i)}, r_0)\backslash B(\bar{u}^{(i)}, \rho)$ that (for n sufficiently large)

$$\rho/2 \leq \|\sum_{j=1}^k v^{(j)}(\cdot - y_n^j) - \bar{u}^{(i)}\| \leq 2r_0 \tag{2.13}$$

First, we must have $k = 1$. Indeed, if $k > 1$, then from (2.12) we have (say) $\lim_{n\to\infty} |y_n^j| = \infty$ for $j = 2, \dots, k$ and we see:

$$\begin{aligned}
&\liminf_{n\to\infty} \|\sum_{j=1}^{k} v^{(j)}(\cdot - y_n^j) - \bar{u}^{(i)}\| \\
&\qquad = \liminf_{n\to\infty} \left(\|v^{(1)}(x - y_n^1) - \bar{u}^{(i)}\| + \sum_{j=2}^{k} \|v^{(j)}(x - y_n^j)\| \right) \\
&\qquad \geq (k-1)\nu > 2r_0
\end{aligned}$$

which contradicts (2.13), and so necessarily $k = 1$. But when $k = 1$, we have from (2.13) that

$$\rho/2 \leq \|v^{(1)}(\cdot - y_n^1) - \bar{u}^{(i)}\| \leq 2r_0$$

which contradicts part (a).

Thus, $\|J'_\infty\|_p$ must be bounded below on the finite union $\bigcup_{i=1}^m B(\bar{u}^{(i)}, r_0)\backslash B(\bar{u}^{(i)}, \rho)$. Using (2.11) and translation invariance, we obtain (b).

(c) The proof follows the same lines as (b). Define $w_n \equiv \sum_{j=1}^k v^{(j)}(\cdot + y_n^j)$, with $v^{(j)}$, y_n^j as in the statement of (c). Suppose for a contradiction that there exists a sequence of functions $v_n \in B(w_n, r_0)\backslash B(w_n, \rho)$ with $J'(v_n) \to 0$. Applying concentration compactness (see the Remark following Lemma 2.4), there exists an integer m, $u^{(1)}, \dots, u^{(m)} \in \mathcal{C}$, $z_n^1, \dots, z_n^m \in \mathbf{Z}^N$ with $|z_n^i| \to \infty$ and $|z_n^i - z_n^j| \to \infty$ $(i \neq j)$, and a critical point $u^{(0)}$ of J such that

$$\|v_n - u^{(0)} - \sum_{i=1}^{m} u^{(i)}(\cdot - z_n^i)\| \to 0$$

As $v_n \in B(w_n, r_0)\backslash B(w_n, \rho)$, we have

$$\|\sum_{j=1}^{k} v^{(j)}(\cdot - y_n^j) - u^{(0)} - \sum_{i=1}^{m} u^{(i)}(\cdot - z_n^i)\| \in [\frac{\rho}{2}, 2r_0]$$

We now sketch a proof of why this is not possible.

First, if $u^{(0)} \not\equiv 0$, then

$$\begin{aligned}
&\|\sum_{j=1}^{k} v^{(j)}(\cdot - y_n^j) - u^{(0)} - \sum_{i=1}^{m} u^{(i)}(\cdot - z_n^i)\| \\
&\qquad = \|u^{(0)}\| + \|\sum_{j=1}^{k} v^{(j)}(\cdot - y_n^j) - \sum_{i=1}^{m} u^{(i)}(\cdot - z_n^i)\| + o(1) \geq \nu > 2r_0
\end{aligned}$$

by Lemma 2.2 (contradiction).

The next possibility we treat is that one of the sequences y_n^j remains far from the z_n^i for all i; ie, we suppose (without loss of generality) that

$$|y_n^1 - z_n^i| \to \infty \qquad \text{for all } i = 1, \dots, m$$

(In particular, this must occur if $k > m$). If so, then the term $v^{(1)}(\cdot - y_n^1)$ splits from the sum as $N \to \infty$,

$$\|\sum_{j=1}^{k} v^{(j)}(\cdot - y_n^j) - \sum_{i=1}^{m} u^{(i)}(\cdot - z_n^i)\| = \|\sum_{j=2}^{k} v^{(j)}(\cdot - y_n^j) - \sum_{i=1}^{m} u^{(i)}(\cdot - z_n^i)\| + \|v^{(1)}\| + o(1)$$
$$\geq \nu > 2r_0$$

as in the previous case.

Of course, another possibility is that one of the sequences z_n^i remains far from the y_n^j for all j. (This situation must occur if $m > k$.) If so, then we reach the same contradiction as above.

The final possibility is that $m = k$ and (renumbering our sequences if necessary,) there is a constant C so that

$$|y_n^j - z_n^j| \leq C \quad \text{for all } j = 1, \dots, k$$

If so, then using $|y_n^j - y_n^i| \to \infty$, the sum splits by pairs,

$$\|\sum_{j=1}^{k} v^{(j)}(\cdot - y_n^j) - \sum_{i=1}^{m} u^{(i)}(\cdot - z_n^i)\| = \sum_{j=1}^{k} \|v^{(j)}(\cdot - y_n^j) - u^{(j)}(\cdot - z_n^j)\| + o(1)$$
$$\geq 3kr_0$$

using part (a), and again arriving at a contradiction. This proves (c). ∎

REMARK: Following [CES], [S], we say that a functional F satisfies the $(\overline{PS})$ condition if whenever there is a sequence v_n with $F(v_n)$ convergent, $F'(v_n) \to 0$, and $\|v_n - v_{n-1}\| \to 0$, then v_n has a convergent subsequence. Lemma 2.4 implies that (under the hypothesis $(*)$), J satisfies the $(\overline{PS})$ condition. Furthermore, a functional which satisfies $(\overline{PS})$ gives rise to a deformation theorem (see [CES]).

REMARK 2.6: As in section 5 of [CR1] we may carry out the analysis for our problem by replacing condition $(*)$ with the following weaker hypothesis:

$$\text{there exists } 0 < a < \frac{c}{2} \text{ and } v_0 \in \mathcal{C}^c \text{ such that } v_0 \text{ is isolated in } \mathcal{C}^{c+a}. \qquad (2.14)$$

Using (2.14) instead of $(*)$ requires replacing $\mathcal{C}^{c+a}$ by $\{v_0\}$ in Lemma 2.5 and later in Lemmas 3.3, 4.2, and 4.3. We should point out, though, that while the methods described here under the hypothesis $(*)$ extend to more general nonlinearities (ie, non-homogenous), using (2.14) relies on more restrictive conditions on the nonlinear term in the functional– see [CR1] or our discussion in section 5.

3. THE MIN-MAX SCHEME

The goal in this section is to construct a min-max argument leading to critical points for J at the level kc, for $k \geq 2$ an integer.

Let $k \geq 2$ be a fixed integer. We choose k sequences of points in $\mathbf{Z}^N$, $\{x_{n,j}\}$ ($j = 1, \ldots, k$ and $n = 1, 2, \ldots$) and an increasing sequence of real numbers $R_n > 0$ ($n = 1, 2, \ldots$) satisfying:

$$\lim_{n \to \infty} |x_{n,i}| = \infty \tag{3.1}$$

$$\lim_{n \to \infty} |x_{n,i} - x_{n,j}| = \infty, \quad \text{if } i \neq j \tag{3.2}$$

$$R_n \leq \frac{1}{2} \min_{i \neq j} (x_{n,i}, |x_{n,i} - x_{n,j}|) \quad \text{for all } n \geq 1 \tag{3.3}$$

$$\lim_{n \to \infty} R_n = +\infty \tag{3.4}$$

We then decompose a function $v \in L^q(\mathbf{R}^N)$ as follows:

$$\left. \begin{aligned} [v]_{n,i} &\equiv \chi_{B_{R_n}(x_{n,i})} v, \qquad 1 \leq i \leq k \\ [v]_{n,k+1} &\equiv v - \sum_{i=1}^{k} [v]_{n,i} \end{aligned} \right\} \tag{3.5}$$

Note that (3.2)–(3.4) together with (2.3) yield the following useful estimate:
Given $\epsilon > 0$ there exists $N = N(\epsilon)$ such that for all $n \geq N$ and for all $v \in L^q(\mathbf{R}^N)$

$$\|(L - L_\infty)[v]_{n,i}\|_{L^p} \leq \epsilon \|[v]_{n,i}\|_{L^q} \quad i = 1, \ldots, k \tag{3.6}$$

Next we introduce a sequence of auxilliary functionals, $\tilde{J}_n$. As in [S], we wish to separate regions of space and remove the "interaction term" connecting these regions. In [S] this is done by splitting the real line into two half-lines. However, in order to treat the asymptotically periodic case (and also to construct solutions at energy kc for $k \geq 3$ in the periodic case in dimension $N = 1$,) we need to consider sequences of "multibump" functions whose components move away from the origin and away from each other as $n \to \infty$ (this being the basis for the work of [CR1], [CR2].) Hence, in building our auxilliary functionals we use restrictions to expanding balls which travel far apart as $n \to \infty$, and remove the interaction term between these balls. By building the support properties into the auxilliary functional (as in [S]), we will be able to use a deformation theorem without relying on an additional variational problem as in [CR1], [CR2]; by following expanding balls we force our minmax sequences to approach the periodic problem near infinity, and can arrive at "multibump" functions with arbitrarily many bumps. So we define our auxilliary functionals as:

$$\begin{aligned} \tilde{J}_n(v) &= \sum_{i=1}^{k} J([v]_{n,i}) + \frac{1}{q} \int |[v]_{k,n+1}|^q \\ &= \frac{1}{q} \int |v|^q - \sum_{i=1}^{k} \frac{1}{2} \int [v]_{n,i} L [v]_{n,i} \end{aligned} \tag{3.7}$$

We also define a class of paths,

$$\Gamma_n^j = \{\gamma \in \Gamma : \quad \operatorname{supp} \gamma(t) \subset B_{R_n}(x_{n,j}) \text{ and } \gamma(t) \in J^{3c/2} \text{ for all } t \in [0,1]\} \tag{3.8}$$

where Γ is defined in (2.9).

The basic idea is to glue together k paths $\gamma_j \in \Gamma_n^j$ in order to obtain suitable sets for a min-max value at kc. The following lemma introduces a scheme for cutting off and translating which will be used again in the proof of Lemma 3.3.

LEMMA 3.1.

(a) *Let $\gamma \in \Gamma_\infty$. Then there exists $N = N(\gamma)$ such that for all $n \geq N$, $[x_{n,j} * \gamma]_{n,j} \in \Gamma_n^j$ and*

$$\max_{0\leq t\leq 1} J_\infty(\gamma(t)) = \lim_{n\to\infty} \max_{0\leq t\leq 1} J([x_{n,j} * \gamma(t)]_{n,j})$$

(b) *For each $j = 1, \dots, k$,*

$$c \equiv \inf_{\gamma\in\Gamma_\infty} \max_{0\leq t\leq 1} J_\infty(\gamma(t)) = \lim_{n\to\infty} \inf_{\bar{\gamma}\in\Gamma_n^j} \max_{0\leq t\leq 1} J(\bar{\gamma}(t))$$

PROOF: First, take $\gamma \in \Gamma_\infty$. Then we have $\gamma(0) = 0$ and $J_\infty(\gamma(1)) < 0$. Using (3.1) and (3.4), we obtain $N_1 = N_1(\gamma)$ so that for all $n \geq N_1$ we have $J_\infty([x_{n,j} * \gamma(1)]_{n,j}) < 0$. Then, (3.6) implies that there exists $N = N(\gamma) \geq N_1(\gamma)$ so that for all $n \geq N$ we have (setting $\gamma_{n,j}(t) \equiv [x_{n,j} * \gamma(t)]_{n,j}$)

$$|J(\gamma_{n,j}(1)) - J_\infty(\gamma_{n,j}(1))| = \left|\int \gamma_{n,j}(1) \cdot (L - L_\infty)\gamma_{n,j}(1)\right| \leq \frac{1}{2}|J_\infty(\gamma_{n,j}(1))| \tag{3.9}$$

Then, for $n \geq N$, $\gamma_{n,j}(t)$ satisfies $\gamma_{n,j}(0) = 0$ and $J(\gamma_{n,j}(1)) < 0$, so $\gamma_{n,j} \in \Gamma_n^j$.

Next we show that (a) holds if we replace J by J_∞ on the right hand side, ie, for each $\gamma \in \Gamma_\infty$,

$$\max_{0\leq t\leq 1} J_\infty(\gamma(t)) = \lim_{n\to\infty} \max_{0\leq t\leq 1} J_\infty([x_{n,j} * \gamma(t)]_{n,j}) \tag{3.10}$$

holds for all $n \geq N_2$ for some $N_2 = N_2(\gamma)$.

To prove (3.10), we first fix j and $\gamma \in \Gamma_\infty$, and let $\epsilon > 0$ be given. It is easy to see that J_∞ satisfies the following estimate: for any $M > 0$, there exists some positive constant $A = A(M) > 0$ such that $||J'_\infty(u)|| \leq A(M)$ for any $||u|| \leq M$. Now we choose $M > 0$, such that, $\max_{0\leq t\leq 1}\{||\gamma(t)||\} \leq M/2$. We then take $\delta_1 > 0$ so small that $A(M)\delta_1 < \epsilon$. As $J_\infty(\gamma(t))$ is uniformly continuous for $t \in [0,1]$ there is a $\delta_2 > 0$ with $|J_\infty(\gamma(t_2)) - J_\infty(\gamma(t_1))| < \delta_1/2$ whenever $|t_2 - t_1| < \delta_2$. Cover $[0,1]$ with (finitely many) open intervals I_i of width δ_2. Call t_i the midpoint of each I_i, and denote $u_i \equiv \gamma(t_i)$. For each i, there exists $R(i)$ such that

$$\left(\int_{|x|>R(i)} |u_i|^q\, dx\right)^{1/q} < \frac{\delta_1}{2}$$

Set $R = \max R(i)$. As each $t \in I_i$ for some choice of i, we have

$$\left(\int_{|x|>R} |\gamma(t)|^q\, dx\right)^{1/q} \leq \left(\int_{|x|>R(i)} |u_i|^q\, dx\right)^{1/q} + \|\gamma(t) - u_i\|_q < \delta_1 \tag{3.11}$$

Now choose $N \geq 1$ so that $R_n \geq R$ for all $n \geq N$. Then

$$\begin{aligned}\|x_{n,j} * \gamma(t) - [x_{n,j} * \gamma(t)]_{n,j}\|_q &\leq \|(1 - \chi_{B_R(x_{n,j})})(x_{n,j} * \gamma(t))\|_q \\ &= \|(1 - \chi_{B_R(0)})\gamma(t)\|_q < \delta_1\end{aligned}$$

via (3.11). We therefore obtain, from the mean value theorem and the choice of M,

$$\begin{aligned}|J_\infty([x_{n,j} * \gamma(t)]_{n,j}) - J_\infty(\gamma(t))| &= |J_\infty([x_{n,j} * \gamma(t)]_{n,j}) - J_\infty(x_{n,j} * \gamma(t))| \\ &\leq A(M)\|[x_{n,j} * \gamma(t)]_{n,j} - x_{n,j} * \gamma(t)\|_q \\ &\leq A(M)\delta_1 \\ &< \epsilon\end{aligned}$$

for each $t \in [0,1]$; taking the max and noting that ϵ was arbitrary, we see that (3.10) holds.

Now to conclude the proof of part (a), we only need to show that

$$\lim_{n\to\infty} \max_{t\in[0,1]} |J(\gamma_{n,j}(t)) - J_\infty(\gamma_{n,j}(t))| = 0$$

But, as $\|\gamma_{n,j}(t)\| \leq C(\gamma)$ uniformly in t, and using (3.6) (and arguing as in (3.9)) we obtain

$$|J(\gamma_{n,j}(t)) - J_\infty(\gamma_{n,j}(t))| = \left|\int \gamma_{n,j}(t) \cdot (L - L_\infty)\gamma_{n,j}(t)\right| \leq C^2\epsilon$$

uniformly in $t \in [0,1]$, and so (a) holds.

From (a), we see that (for n sufficiently large)

$$\inf_{\gamma\in\Gamma_\infty} \max_{0\leq t\leq 1} J_\infty(\gamma(t)) \leq \inf_{\bar\gamma\in\Gamma_n^j} \max_{0\leq t\leq 1} J(\bar\gamma(t)) + \epsilon$$

To obtain the opposite inequality, observe that for any $\bar\gamma \in \Gamma_n^j$, we have $\bar\gamma \in \Gamma_\infty$ for all n sufficiently large. This proves (b). ∎

Next we introduce min-max constructions which give rise to a mountain-pass situation at level kc. First define

$$\begin{aligned}\hat\Gamma_n = \{\hat\gamma : &[0,1]^k \to \bigoplus_{j=1}^k L^q(B_{R_n}(x_{n,j})), \\ &(t_1, \ldots, t_k) \to \sum_{j=1}^k \gamma^j(t_j),\ \gamma^j \in \Gamma_n^j\}\end{aligned}$$

The advantage of these paths $\hat\gamma$ are that

$$\tilde J_n(\hat\gamma(t)) = \sum_{j=1}^k J(\gamma^j(t)), \qquad 0 \leq t \leq 1$$

and so Lemma 3.1 yields

$$kc = \lim_{n\to\infty} \inf_{\hat\gamma\in\hat\Gamma_n} \max_{0\le t\le 1} \tilde J_n(\hat\gamma(t)) \tag{3.12}$$

The problem with the class $\hat\Gamma_n$, however, is that it is not invariant under the pseudo-gradient flow for either J or $\tilde J_n$.

Following Séré, we define yet another class $\tilde\Gamma_n$. First, denote by 0_j $(1\le j\le k)$ the set of vectors $t\in[0,1]^k$ with jth component $t_j=0$ and similarly 1_j $(1\le j\le k)$ the set of vectors $t\in[0,1]^k$ with jth component $t_j=1$. Then define $\tilde\Gamma_n$ to be the collection of all $\tilde\gamma: [0,1]^k\to L^q$ satisfying the following conditions on $\partial[0,1]^k$:

$$\left.\begin{aligned} [\tilde\gamma(0_j)]_{n,j} &= 0 && (j=1,\dots,k)\\ J\circ[\tilde\gamma(1_j)]_{n,j} &< 0 && (j=1,\dots,k)\\ J\circ[\tilde\gamma(0_j)]_{n,i} &\le (k-\tfrac{1}{2})c && (i,j=1,\dots,k)\\ J\circ[\tilde\gamma(1_j)]_{n,i} &\le (k-\tfrac{1}{2})c && (i,j=1,\dots,k)\end{aligned}\right\} \tag{3.13}$$

Note that $\hat\Gamma_n\subset\tilde\Gamma_n$. Note also that although $\tilde\Gamma_n$ is defined (as is Séré's analogous $\tilde\Gamma$) by conditions on the boundary of the cube $\partial[0,1]^k$, it is not true that $\tilde\gamma\in\tilde\Gamma_n$ satisfies $\tilde J_n(\tilde\gamma(\partial[0,1]^k))\le(k-\frac12)c$, due to the uncontrolled term involving $[v]_{n,k+1}$ which appears in (3.7). In constructing our deformation arguments, we will need to exploit the fact that this term is non-negative, and choose paths for which it tends to zero as $n\to\infty$.

LEMMA 3.2. *(Lemma 6 of [S]; Prop. 3.4 of [CR1])*

$$kc = \lim_{n\to\infty} \inf_{\tilde\gamma\in\tilde\Gamma_n} \max_{0\le t\le 1} \tilde J_n(\tilde\gamma(t))$$

PROOF: As $\hat\Gamma_n\subset\tilde\Gamma_n$, we have

$$\inf_{\hat\gamma\in\hat\Gamma_n} \max_{0\le t\le 1} \tilde J_n(\hat\gamma(t)) \ge \inf_{\tilde\gamma\in\tilde\Gamma_n} \max_{0\le t\le 1} \tilde J_n(\tilde\gamma(t))$$

for each $n\ge 1$, and so (3.12) and Lemma 3.1 yield

$$kc \ge \limsup_{n\to\infty} \inf_{\tilde\gamma\in\tilde\Gamma_n} \max_{0\le t\le 1} \tilde J_n(\tilde\gamma(t)) \tag{3.14}$$

To prove the lemma, we will show that for any $\tilde\gamma\in\tilde\Gamma_n$, there exists $\bar t\in[0,1]^k$ such that $\tilde J_n(\tilde\gamma(\bar t))\ge kc$.

Fix $\tilde\gamma\in\tilde\Gamma_n$ and let $\epsilon>0$ be given. We remark that (using (3.6)), there exists $N_1=N_1(\tilde\gamma)$ such that for all $n\ge N_1$

$$\max_{t\in[0,1]} |J([\tilde\gamma(t)]_{n,j}) - J_\infty([\tilde\gamma(t)]_{n,j})| < \epsilon \tag{3.15}$$

and so there exists $N = N(\tilde{\gamma})$ such that (3.13) holds with J_∞ replacing J throughout. So take $n \geq N$ in what follows.

Consider $\tilde{J}_n \circ [\tilde{\gamma}]_{n,1}$. As $[\tilde{\gamma}(0_1)]_{n,1} = 0$ and $J_\infty([\tilde{\gamma}(1_1)]_{n,1}) < 0$, then any curve $\sigma(s)$ joining the edges $\{t_1 = 0\}$ and $\{t_1 = 1\}$ in $[0,1]^k$ is such that $[\tilde{\gamma}(\sigma(s))]_{n,1}$ belongs to Γ_∞. By the mountain-pass structure of J_∞, this curve must intersect the level set $(J_\infty)^{-1}(c)$. Thus, $(J_\infty)^{-1}(c)$ separates the edges $\{t_1 = 0\}$ and $\{t_1 = 1\}$ in $[0,1]^k$. Let δ be such that $J_\infty([\tilde{\gamma}]_{n,1}) \geq c - \epsilon$ for $\tilde{\gamma}$ in an δ-neighborhood of $(J_\infty)^{-1}(c)$. As $[0,1]^k$ is compact, this neighborhood has finitely many components. Choose a component, D_1 which separates the edges $\{t_1 = 0\}$ and $\{t_1 = 1\}$; D_1 is connected and joins the edges $\{t_i = 0\}$ and $\{t_i = 1\}$ for each $i = 2, \ldots, k$.

Now repeat the process, using $[\tilde{\gamma}]_{n,2}$ to obtain a connected set $D_2 \subset D_1$ separating $D_1 \cap \{t_2 = 0\}$ and $D_1 \cap \{t_2 = 1\}$, but joining the edges $\{t_i = 0\}$ and $\{t_i = 1\}$ for each $i = 3, \ldots, k$. On D_2, $J_\infty([\tilde{\gamma}]_{n,2}) \geq c - \epsilon$. Continuing, we obtain $D_1 \supset D_2 \supset \cdots \supset D_{k-1}$ with $J_\infty([\tilde{\gamma}]_{n,i}) \geq c - \epsilon$ $(1 \leq i \leq k-1)$ in D_{k-1}, and D_{k-1} connects the edges $\{t_k = 0\}$ and $\{t_k = 1\}$. Thus we can find $\bar{t}_\epsilon \in D_{k-1}$ with $J_\infty([\tilde{\gamma}(\bar{t}_\epsilon)]_{n,k}) \geq c - \epsilon$. For this $\bar{t}_\epsilon$,

$$\begin{aligned}
\tilde{J}_n(\tilde{\gamma}(\bar{t}_\epsilon)) &= \sum_{j=1}^{k} J([\tilde{\gamma}(\bar{t}_\epsilon)]_{n,j}) + \frac{1}{q}\int |[\tilde{\gamma}(\bar{t}_\epsilon)]_{n,k+1}|^q \\
&\geq \sum_{j=1}^{k} J([\tilde{\gamma}(\bar{t}_\epsilon)]_{n,j}) \\
&\geq \sum_{j=1}^{k} (J_\infty([\tilde{\gamma}(\bar{t}_\epsilon)]_{n,j}) - \epsilon) \quad \text{(using (3.15))} \\
&\geq k(c - 2\epsilon)
\end{aligned}$$

Letting $\epsilon \to 0$, we get the desired inequality. ∎

Next we choose some special paths in $\tilde{\Gamma}_n$ which stay close to translates of critical points of J. We follow Lemma 9 of [S] and also Prop. 2.22 of [CR1].

LEMMA 3.3. *Assume that $(*)$ holds. Let r_0 be as in Lemma 2.5 and $0 < r < r_0$. There exist $v^{(1)}, \ldots, v^{(m)} \in \mathcal{C}^c$ $(m = m(r) < \infty)$ and $\epsilon_1 = \epsilon_1(r) > 0$ such that for any $0 < \epsilon < \epsilon_1$ there exists a sequence $\{\tilde{\gamma}_n^\epsilon\}$ and $n_\epsilon > 0$ such that for all $n \geq n_\epsilon$:*

(i) $\tilde{\gamma}_n^\epsilon \in \tilde{\Gamma}_n$

(ii) $\tilde{\gamma}_n^\epsilon(t) \cap (\tilde{J}_n)_{kc-\epsilon} \subset \bigcup_{\alpha \in [1,m]^k} B(\sum_{i=1}^k v^{(\alpha_i)}(x - x_{n,i}), r)$ *for all* $t \in [0,1]^k$

(iii) $\tilde{\gamma}_n^\epsilon(t) \subset (\tilde{J}_n)^{kc+\epsilon}$ *for all* $t \in [0,1]^k$

PROOF: Again the modifications of Séré's proof are minor, but we provide some details for the reader's convenience.

Define

$$\mu = \inf\{\|J'_\infty(w)\| : w \in \bigcup_{v \in \mathcal{C}_c^{c+a}} B(v, r/2) \backslash B(v, r/4)\}$$

By Lemma 2.5 (b), $\mu > 0$. By (*) there exists $\delta > 0$ with $\delta < \min(\frac{\mu r}{16}, a)$ such that $J_\infty(\mathcal{C}) \cap [c, c+\delta] = \{c\}$.

Fix a $\gamma \in \Gamma$ such that

$$J_\infty(\gamma(t)) \le c + \delta/2 < c + a \tag{3.16}$$

As $\{\gamma(t) : 0 \le t \le 1\}$ is compact, there is a finite number $v^{(1)}, \dots, v^{(m)} \in \mathcal{C}^c$ of critical points such that $\{\gamma(t) : 0 \le t \le 1\}$ intersects $B(v^{(i)}, r/2)$, $i = 1, \dots, m$.

Now, let $\epsilon_1 = \delta/4$, and suppose that $\epsilon < \epsilon_1$. Fix ρ, $0 < \rho < r/4$ such that

$$J_\infty\left(\bigcup_{u \in \mathcal{C}^c} B(u, \rho)\right) \le c + \frac{\epsilon}{2k} \tag{3.17}$$

Consider a pseudo-gradient vector field V_ρ for J_∞, with

$$J'_\infty(v) \cdot V_\rho(v) \le -1 \quad \text{if} \quad v \in (J_\infty)^{c+\delta/2}_{c-\delta/2} \setminus \left(\bigcup_{u \in \mathcal{C}^c} B(u, \rho)\right) \tag{3.18}$$

$$\|V_\rho(v)\| \le 2\|J'_\infty(v)\|^{-1} \quad \text{for all } v \in (J_\infty)^{c+\delta}_{c-\delta} \setminus \left(\bigcup_{u \in \mathcal{C}^c} B(u, \rho)\right) \tag{3.19}$$

$$J'_\infty(v) \cdot V_\rho(v) \le 0 \quad \text{for all } v \in L^q \tag{3.20}$$

$$V_\rho(v) = 0 \quad \text{if} \quad v \in (J_\infty)^{c-\delta} \cup (J_\infty)_{c+\delta} \cup \left(\bigcup_{u \in \mathcal{C}^c} B(u, \rho/2)\right) \tag{3.21}$$

We will use the flow

$$\begin{cases} \frac{\partial}{\partial s}\eta(s, v) = V_\rho(\eta(s, v)) \\ \eta(0, v) = v \end{cases}$$

to deform our curve $\gamma(t)$ to lower values of J_∞.

Arguing as in [CR1] or [CES], the flow $\eta(s, v)$ is defined on all of $(s, v) \in [0, \infty) \times L^q$. We define

$$\gamma^\epsilon(t) \equiv \eta(\delta, \gamma(t)) \quad \text{for all } t \in [0, 1]$$

Step 1: We will show that γ^ϵ satisfies:

$$\gamma^\epsilon \in \Gamma \tag{3.22}$$

$$\gamma^\epsilon(t) \cap (J_\infty)_{c-\frac{\epsilon}{2k}} \subset \bigcup_{i=1}^m B(v^{(i)}, r/2), \quad \text{for all } 0 \le t \le 1 \tag{3.23}$$

$$\gamma^\epsilon(t) \subset (J_\infty)^{c+\frac{\epsilon}{2k}} \quad \text{for all } t \in [0, 1] \tag{3.24}$$

By (3.20), $J_\infty(\eta(s, v))$ is non-increasing in s, and so $\gamma^\epsilon \in \Gamma$. Now fix $t \in [0, 1]$, and consider $v = \gamma(t)$. By (3.16), (3.18), there are two possibilities: either (i) $\eta(s, \gamma(t))$ must intersect $(J_\infty)^{c-\delta/2}$ or (ii) $\eta(s,\gamma(t))$ remains in $(J_\infty)_{c-\delta/2}$ but intersects $\bigcup_{u\in\mathcal{C}^c} B(u,\rho)$ in time $s \le \delta$. If $\gamma(t)$ belongs to case (i), we have $\gamma^\epsilon(t) \in (J_\infty)^{c-\delta/2}$, and (3.23), (3.24) are satisfied at such t. When $\gamma(t)$ belongs to case (ii), there exists $u_0 \in \mathcal{C}^c$ and $s_0 \in [0,\delta]$ such that $\eta(s_0,\gamma(t)) \in B(u_0,\rho)$. By (3.17), for such t we have

$$J_\infty(\gamma^\epsilon(t)) = J_\infty(\eta(\delta,\gamma(t)) \le J_\infty(\eta(s_0,\gamma(t)) \le c + \frac{\epsilon}{2k}$$

and so (3.24) is satisfied for such t. We claim now that in this case,

$$\gamma^\epsilon(t) = \eta(\delta,\gamma(t)) \in B(v^{(i)}, r/2)$$

for some $i = 1,\ldots,m$ (and therefore (3.23) holds for such t). First, we show that $\gamma^\epsilon(t) = \eta(\delta,\gamma(t)) \in B(u_0, r/2)$ for the same u_0 as above. If not, then the trajectory $\eta(s,\gamma(t))$ must cross $\partial B(u_0,r/4)$ and $\partial B(u_0,r/2)$ at times $s = s_1, s_2$ (respectively), with $s_0 < s_1 < s_2 \le \delta$. If so, then (using (3.19)),

$$\begin{aligned}\frac{r}{4} &\le \|\eta(s_2,\gamma(t)) - \eta(s_1,\gamma(t))\| \\ &= \|\int_{s_1}^{s_2} \eta'(s,\gamma(t))\,ds\| = \|\int_{s_1}^{s_2} V_\rho(\eta(s,\gamma(t)))\,ds\| \\ &\le 2\delta \sup_{s\in[s_1,s_2]} \|J'_\infty(\eta(s,\gamma(t))\|^{-1} \\ &\le 2\mu^{-1}\delta \le 2\mu^{-1}\frac{\mu r}{16} = \frac{r}{8}\end{aligned}$$

and so the trajectory must stay within $B(u_0,r/2)$. Now, u_0 must necessarily belong to the collection $\{v^{(i)}\}$. If not, then by the choice of the $v^{(i)}$, $\eta(0,\gamma(t)) = \gamma(t) \notin B(u_0,r/2)$, so again there must be times $0 < s_1 < s_2 \le \delta$ for which the trajectory $\eta(s,\gamma(t))$ intersects $\partial B(u_0,r/2)$ and $\partial B(u_0,r/4)$ (respectively). The same calculation as above leads again to a contradiction, and so the claim holds, and step 1 is finished.

Note that in this construction the choice of the critical points $v^{(i)}$ depends on the value of r, but *not* on the value of $\epsilon < \epsilon_1$. This will be important later, when we will fix a value of r but "squeeze" ϵ, and it will be essential that our choice of $v^{(i)}$ does not vary.

Step 2: Cutting and pasting.

To construct $\gamma_n^\epsilon \in \tilde{\Gamma}_n$, we translate and cut off as in the proof of Lemma 3.1: we can find n_ϵ sufficiently large such that for $n \ge n_\epsilon$, $[x_{n,j} * \gamma^\epsilon]_{n,j} \in \Gamma^j$ and satisfies

$$J([x_{n,j} * \gamma^\epsilon]_{n,j}) \le c + \frac{\epsilon}{k}$$

$$[x_{n,j} * \gamma^\epsilon]_{n,j} \cap J_{c-\frac{\epsilon}{2k}} \subset \bigcup_{i=1}^{m} B(v^{(i)}(x_{n,j}), r)$$

for all $0 \le t \le 1$. Then, summing over j,

$$\tilde{\gamma}_n^\epsilon(t) \equiv \sum_{j=1}^{k} [x_{n,j} * \gamma^\epsilon(t_j)]_{n,j}$$

(where, as usual, we write $t = (t_1, \ldots, t_k)$). is the desired path. ∎

REMARK 3.4: Note that for each $t \in [0,1]$ $\tilde{\gamma}_n^\epsilon(t)$ is supported exactly in the union of the k balls, $\bigcup_{i=1}^k B_{R_n}(x_{n,i})$, and so $\tilde{\gamma}_n^\epsilon \in \hat{\Gamma}_n$ and hence $[\tilde{\gamma}_n^\epsilon(t)]_{n,k+1} = 0$ for all $t \in [0,1]^k$. In particular, we have for these particular paths that

$$\tilde{\gamma}_n^\epsilon\left(\partial[0,1]^k\right) \subset (\tilde{J}_n)^{(k-\frac{1}{2})c} \tag{3.25}$$

This fact will be crucial in applying our deformation lemma to $\tilde{\gamma}_n^\epsilon$, as the flow we will define will preserve $(\tilde{J}_n)^{(k-\frac{1}{2})c}$, and hence the deformed path will remain in $\hat{\Gamma}_n$ (see the proof of Theorem 4.1).

REMARK 3.5: The proof of Lemma 3.3 above does not at all use the homogeneous structure of our nonlinearity, and so generalizes to certain other nonlinearities (see for example [S] or section 3 of [AL].) We remark that in the homogenous case (or in the case where the nonlinearity satisfies a condition such as (f.6) of section 5), Lemma 3.3 simplifies somewhat. In particular, a similar result may be proven with the weaker assumption (2.14) instead of $(*)$. In the statement, we would remove the reference to the critical points $v^{(1)}, \ldots, v^{(m)} \in \mathcal{C}^c$, and replace (ii) by
(ii') $\tilde{\gamma}_n^\epsilon(t) \cap (\tilde{J}_n)_{kc-\epsilon} \subset B(\sum_{i=1}^k v_0(\cdot - x_{n,i}), r)$ for all $t \in [0,1]^k$.
where v_0 is the isolated critical point from (2.14). In the proof, choose $\gamma \in \Gamma$ in (3.16) to be $\gamma(t) = tRv_0$, where R is chosen large enough so that $J(Rv_0) < 0$. Then $J(\gamma(t)) \leq c$ for all $t \in [0,1]$, and there exists an $\epsilon_1 = \epsilon_1(r)$ with

$$\gamma(t) \cap (J_\infty)_{c-\frac{\epsilon}{2k}} \subset B(v_0, r/2), \quad \text{for all } 0 \leq t \leq 1$$

(cf (3.23)) holding for all $\epsilon < \epsilon_1$. Thus, the deformation step can be avoided in this case.

4. THE DEFORMATION ARGUMENT

In this section, we study the deformation of our auxilliary functionals $\tilde{J}_n$ in order to prove our multiplicity assertion below.

THEOREM 4.1. *Suppose $(*)$ holds. Then for all $k \geq 2$, $v^{(j)} \in \mathcal{C}^c$ $(1 \leq j \leq k)$, there exists N_k, such that, J has critical points lying in neighborhoods*

$$B\left(\sum_{j=1}^k v^{(j)}(x - x_{n,j}), r_0\right)$$

for all but finitely many n.

Applying Theorem 4.1 to the periodic functional J_∞, we recover the results of [CR1], [S] on the existence of infinitely many solutions for (2.5), with or without the condition $(*)$:

COROLLARY. *(2.5) possesses infinitely many ($\mathbf{Z}^N$-distinct) solutions in $H^2(\mathbf{R}^N)$.*

To prove Theorem 4.1, we argue by contradiction: suppose that for some $k \geq 2$, $v^{(j)} \in \mathcal{C}^c$, $(1 \leq j \leq k)$,

$$\left.\begin{array}{c}\text{there exists a sequence of } n \text{ going to infinity such that} \\ J \text{ has no critical points lying in } B(\sum_{j=1}^{k} v^{(j)}(x - x_{n,j}), r_0)\end{array}\right\} \qquad (**_k)$$

Note that in [S] attention is focussed on the level $2c$, and the hypothesis $(**_k)$ is replaced with the stronger assumption that $\mathcal{C}$ mod $\mathbf{Z}^N$ is finite. A different minmax procedure is introduced in [CR1] and [CR2] which allows one to work at levels kc $(k \geq 2)$. Working with our auxillary functional also allows us to reach higher levels kc, $k \geq 2$. The conclusion above coincides with that of [CR1] and [CR2]; however, their method is substantially different. In particular, we were not able to push through their localization argument ([CR1] section 4, steps 3,4) with our indefinite dual functional. Here we introduce our auxillary functional which can take care of the asymptotically periodic case.

LEMMA 4.2. *Assume both $(*)$ and $(**_k)$ hold. Then there exists $r_1 \in (0, r_0)$ such that for any $0 < \rho < \frac{r_1}{2}$ and any collection $v^{(1)}, \ldots, v^{(k)} \in \mathcal{C}^c$ there is a constant $\mu_\rho > 0$ and an integer $A_\rho > 0$ so that for all $n \geq A_\rho$,*

$$v \in B(\sum_{j=1}^{k} v^{(j)}(\cdot - x_{n,j}), r_1) \backslash B(\sum_{j=1}^{k} v^{(j)}(\cdot - x_{n,j}), \rho)$$

there exists $V_v \in L^q$ with

$$J'(v) \cdot V_v > \mu_\rho, \qquad \tilde{J}'_n \cdot V_v > \mu_\rho$$
$$\|V_v\|_q \leq 1$$

LEMMA 4.3. *Assume that $(*)$ and $(**_k)$ hold. Let $r_1 < r_0$ be the number in Lemma 4.2. Suppose that $v^{(1)}, \ldots, v^{(k)} \in \mathcal{C}$ with $J(v^{(j)}) = c$, $j = 1, \ldots, k$. Then there exist*

$$\delta = \delta(v^{(1)}, \ldots, v^{(k)}), \qquad N = N(v^{(1)}, \ldots, v^{(k)})$$

such that for all $n \geq N$ and for all $\epsilon \leq \delta$ there exists a homeomorphism φ_n (depending on $v^{(1)}, \ldots, v^{(k)}$) with:

$$\varphi_n\left((\tilde{J}_n)^{kc+\epsilon} \cap B(\sum_{j=1}^{k} v^{(j)}(x - x_{n,j}), \frac{r_1}{2})\right) \subset (\tilde{J}_n)^{kc-\epsilon} \qquad (4.1)$$

$$\text{if } v \notin (\tilde{J}_n)_{kc-2\epsilon} \cap \left(B(\sum_{j=1}^{k} v^{(j)}(x - x_{n,j}), r_1)\right) \text{ then } \varphi_n(v) = v \qquad (4.2)$$

The proof of the above two lemmas are essentially due to Sére. However we need to make some modifications to fit our auxillary functional $\tilde{J}_n$, so we skech the proofs in the Appendix.

In the case where the hypothesis (*) is weakened (as in Remark 2.6) to (2.14), the proofs are essentially the same, with C^{c+a} replaced by $\{v_0\}$ (where v_0 is the isolated solution referred to in (2.14)).

We are now ready to finish the contradiction argument which proves Theorem 4.1.

PROOF OF THEOREM 4.1: Applying Lemma 2.5, we obtain an $r_0 > 0$. Then take $r_1 \leq r_0$ as in Lemmas 4.2, 4.3. Apply Lemma 3.3 to get $\epsilon_1 = \epsilon_1(r_0) > 0$ and $v^{(1)}, \ldots, v^{(m)} \in C_c^c$ ($m = m(r) < \infty$) such that for any $\epsilon < \epsilon_1$ there exist $n_\epsilon > 0$ and $\tilde{\gamma}_n^\epsilon \in \hat{\Gamma}_n \subset \tilde{\Gamma}_n$ with

$$\tilde{\gamma}_n^\epsilon(t) \cap (\tilde{J}_n)_{kc-\epsilon} \subset \bigcup_{\alpha \in [1,m]^k} B(\sum_{i=1}^{k} v^{(\alpha_i)}(x - x_{n,j}), r) \text{ for all } t \in [0,1]^k,\ n \geq n_\epsilon \tag{4.4}$$

and $\tilde{\gamma}_n^\epsilon(t) \subset (\tilde{J}_n)^{kc+\epsilon}$ for all $t \in [0,1]^k$ and $n \geq n_\epsilon$.

For $n \geq N_k'$ chosen sufficiently large, apply the Deformation Lemma 4.3 to obtain, for each collection of k critical points $\vec{v} = (v^{(\alpha_1)}, \ldots, v^{(\alpha_k)})$, $\alpha \in [1,m]^k$ and $\epsilon < \min\{\delta, \frac{c}{2}\}$ a diffeomorphism $\varphi_n^{\vec{v}}$ satisfying (4.1), (4.2). Let Φ_n be the composition of these diffeomorphisms $\varphi_n^{\vec{v}}$ where the composition ranges over all the $\vec{v} = (v^{(\alpha_1)}, \ldots, v^{(\alpha_k)})$ and $\alpha \in [1,\ell]^k$. Using (4.2) and Lemma 2.5(c), we see that the diffeomorphisms $\varphi_n^{\vec{v}}$ commute, and so Φ is well-defined. Recalling the discussion in Remark 3.4 we see that for this choice of $\tilde{\gamma}_n^\epsilon$, (3.25) holds, and so by (4.2) we see that

$$\varphi_n^{\vec{v}}\left(\tilde{\gamma}_n^\epsilon(\partial[0,1]^k)\right) = \tilde{\gamma}_n^\epsilon(\partial[0,1]^k)$$

for each choice of $n, \vec{v}$. Recalling also the definition of $\tilde{\Gamma}_n$ via conditions on $\partial[0,1]^k$ (in (3.13)), we see that $\varphi_n^{\vec{v}}(\tilde{\gamma}_n^\epsilon) \subset \tilde{\Gamma}_n$ for each choice of $n, \vec{v}$, and so

$$\Phi_n(\tilde{\gamma}_n^\epsilon) \subset \tilde{\Gamma}_n$$

Now, by (4.4) and (4.1) we have $\Phi(\tilde{\gamma}_n^\epsilon(t)) \in (\tilde{J}_n)^{kc-\epsilon}$ for all $t \in [0,1]^k$. This contradicts Lemma 3.2. ▮

5. A VARIATIONAL PROBLEM IN $H^1(\mathbf{R}^N)$

In this section, we give a sketch of how to apply the method of the previous sections to treat problems posed in $H^1(\mathbf{R}^N)$, and thus give a different proof of a slightly more general result than that of Coti-Zelati and Rabinowitz ([CR2]).

We consider the following equation for $u \in H^1(\mathbf{R}^N)$:

$$-\Delta u + u = f(x,u) \tag{5.1}$$

Here we assume that $f(x,u) = \nabla_u F(x,u)$ satisfies the properties:

(f.1) $\quad f \in C^2(\mathbf{R}^N \times \mathbf{R}; \mathbf{R})$

(f.2) $F(x,0) = f(x,0) = f_u(x,0) = 0$

(f.3) There exist constants $a_1, a_2 > 0$ and $s > 1$ such that

$$|f_u(x,u)| \leq a_1 + a_2|u|^{s+1}$$

where we restrict $1 < s < \frac{N+2}{N-2}$ if $N \geq 3$.

(f.4) There exist constants $\theta > 2$ and $R > 0$ so that for every $|x| \geq R$ and $u \in H^1\backslash\{0\}$,

$$0 < \theta F(x,u) \leq f(x,u)u$$

Note that (f.4) implies that there exists a constant $a_3 > 0$ so that for all $|u|$ large,

$$F(x,u) \geq a_3|u|^\theta \tag{5.2}$$

Also note that (f.2), (f.3) together imply that there exists a constant C such that

$$|f(x,u)u| \leq \frac{1}{2}|u|^2 + C|u|^{s+1} \tag{5.3}$$

In [CR2], it is assumed that $f(x,u)$ is periodic in x; we make the more general assumption that f and F are asymptotically periodic:

(f.5) There exists a periodic function $F_\infty(x,u) \in C^3(\mathbf{R}^N \times \mathbf{R}; \mathbf{R})$, $F_\infty(x+z,u) = F_\infty(x,u)$ for all $z \in \mathbf{Z}^N$, such that F_∞ and $f_\infty = \nabla_u F_\infty$ satisfy the following relations: given any $\epsilon > 0$ there exists $R > 0$ so that for all $|x| \geq R$ and $u \in \mathbf{R}$

$$|F(x,u) - F_\infty(x,u)| \leq \epsilon(|u|^2 + |u|^{s+1})$$
$$|f(x,u) - f_\infty(x,u)| \leq \epsilon(|u| + |u|^s)$$

Next we introduce the functionals

$$J(u) = \int_{\mathbf{R}^N} [\frac{1}{2}(|\nabla u|^2 + |u|^2) - F(x,u)]\,dx \tag{5.4}$$

$$J_\infty(u) = \int_{\mathbf{R}^N} [\frac{1}{2}(|\nabla u|^2 + |u|^2) - F_\infty(x,u)]\,dx \tag{5.5}$$

In [CR2] it is shown that $J, J_\infty \in C^1$ and that critical points of J, J_∞ correspond to solutions of (5.4) and (5.5) respectively. We denote by $\mathcal{C}$ the critical point set of J_∞, and we use J^t, J_s, etc. as in the previous sections.

Following [CR2], we introduce a mountain-pass value for J_∞

$$c = \inf_{\gamma \in \Gamma_\infty} \max_{t \in [0,1]} J_\infty(\gamma(t)) \tag{5.6}$$

where

$$\Gamma_\infty = \{\gamma \in C([0,1]; H^1) : \quad \gamma(0) = 0, \quad J_\infty(\gamma(1)) < 0\} \tag{5.7}$$

Again we introduce the hypothesis $(*)$,

$$\text{there exists } 0 < a < \frac{c}{2} \text{ such that } \mathcal{C}^{c+a} \bmod \mathbf{Z}^N \text{ is finite and } \mathcal{C}^{c+a} = \mathcal{C}^c. \tag{$*$}$$

and prove the following multiplicity result:

THEOREM 5.1. *Suppose* $(*)$ *holds, and* f *satisfies (f.1)–(f.5). Then for all* $k \geq 2$, $v^{(j)} \in \mathcal{C}^c$ $(1 \leq j \leq k)$ J *has critical points lying in neighborhoods*

$$B(\sum_{j=1}^{k} v^{(j)}(x - x_{n,j}), r_0)$$

for all but finitely many n.

Again, in the periodic case, we remark (as in [CR2]) that, whether $(*)$ holds or not, Theorem 5.1 shows that J_∞ has infinitely many critical points.

COROLLARY. *If* f *satisfies (f.1)–(f.4) and is* $\mathbf{Z}^N$ *periodic, then (5.1) possesses infintely many* ($\mathbf{Z}^N$*-distinct) solutions in* $H^1(\mathbf{R}^N)$.

REMARK: As it is remarked in section 5 of [CR1], if we add the hypothesis

(f.6) For all $w \neq 0$, $w \in \mathbf{R}$, $t^{-1} f(x, tw)w$ is an increasing function of $t > 0$;

then $(*)$ may be weakened to (2.14). (See [CR1] for a complete exposition.) If one does wish to substitute (2.14) for $(*)$, the immediate consequence for the theorems we state is to replace $\mathcal{C}^{c+a}$ with $\{v_0\}$ where v_0 is the isolated solution from (2.14).

To prove Theorem 5.1, we follow the same steps as in the previous sections:

LEMMA 5.2.

(i) 0 *is an isolated critical point of both* J *and* J_∞, *i.e., there exists* $\nu > 0$ *so that for any* v *which is a nontrivial critical point of either* J *or* J_∞, $\|v\|_{H^1} \geq \nu > 0$

(ii) *There exists* $\beta > 0$ *so that* $J(u) \geq \beta$ *for all* $u \in \mathcal{C}\backslash\{0\}$.

(ii) *If* $v \in \mathcal{C}\backslash\{0\}$ *and* $J_\infty(v) = b$, *then*

$$\|v\|_{H^1} \leq \left(\frac{2\theta b}{\theta - 2}\right)^{1/2}$$

PROOF: Suppose v is any nontrivial critical point of J_∞ (the case for J is identical). Then, applying (5.3), (and denoting the $H^1(\mathbf{R}^N)$ norm simply by $\|\cdot\|$)

$$\begin{aligned} \|v\|^2 = \int f(x,v)v &\leq \frac{1}{2}\int |v|^2 + C\int |v|^{s+1} \\ &\leq \frac{1}{2}\|v\|^2 + C\|v\|^{s+1} \end{aligned}$$

which yields (i), as $s > 1$.

To prove (ii), we have

$$\begin{aligned} 0 = J'_\infty(v)v = \|v\|^2 - \int f(x,v)v & \\ \leq \|v\|^2 - \theta \int F(x,v) & \quad \text{via (f.4)} \\ = \frac{2-\theta}{2}\|v\|^2 + \theta J_\infty(v) & \end{aligned}$$

Hence we see that

$$J_\infty(v) \geq \frac{\theta-2}{2\theta}\|v\|^2 \geq \left(\frac{\theta-2}{2\theta}\right)\nu^2$$

by part (i). Both (ii) and (iii) follow from the above inequality. ∎

As for our dual functional, J_∞ also satisfies the concentration compactness property:

LEMMA 5.3. *Suppose $\{v_n\}$ is a sequence in $H^1(\mathbf{R}^N)$ such that (as $n \to \infty$)*

$$J_\infty(v_n) \to d > 0$$
$$J'_\infty(v_n) \to 0 \qquad (in\ H^{-1}(\mathbf{R}^N))$$

Then, there is a positive integer $m < \infty$, m critical points $\{v^{(j)}\}_{j=1,\ldots,m} \subset \mathcal{C} \cap (J_\infty)^d$, and m sequences of integer coordinate vectors $\{y_n^j\}_{j=1,\ldots,m} \subset \mathbf{Z}^N$ such that along some subsequence $n \to \infty$:

$$\begin{cases} \|v_n - \sum_{j=1}^m v^{(j)}(\cdot + y_n^j)\|_{H^1} \to 0 \\ |y_n^i - y_n^j| \to \infty \\ \sum_{j=1}^m J_\infty(v^{(j)}) = d \end{cases}$$

The proof of this lemma is contained in [CR2].
Now, under the hypothesis $(*)$ we have Lemma 2.5 holding for our functional J_∞:

LEMMA 5.4. *Assume $(*)$ holds.*

(a) *There exists r_0 with $0 < r_0 < \nu/3$ (where ν is as in Lemma 5.2) such that for all $u \in \mathcal{C}^{c+a}, v \in \mathcal{C}$ with $u \neq v$,*

$$\|u - v\|_{H^1} \geq 3r_0.$$

(b) *For all $\rho < r_0$,*

$$\inf\{\|J'_\infty(w)\|_{H^{-1}} : \quad w \in \bigcup_{v \in \mathcal{C}^{c+a}} B(v, r_0)\backslash B(v,\rho)\} > 0$$

(c) *If $v^{(1)}, \ldots, v^{(k)} \in \mathcal{C}^{c+a}$, and $y_n^j \in \mathbf{Z}^N$ with $\lim_{n\to\infty} |y_n^i| \to \infty$ (for all i) and $\lim_{n\to\infty} |y_n^i - y_n^j| \to \infty$ $(i \neq j)$, then*

$$\liminf_{n\to\infty} \inf\{\|J'(w)\|_{H^{-1}} : \quad w \in B(\sum_{j=1}^k v^{(j)}(\cdot + y_n^j), r_0)\backslash B(\sum_{j=1}^k v^{(j)}(\cdot + y_n^j), \rho)\} > 0$$

(The proof is identical to that of Lemma 2.5)

Now we construct the min-max argument (as in section 3) leading to critical points for J at the level kc, for $k \geq 2$ an integer.

Let $k \geq 2$ be a fixed integer. We choose k sequences of points in $\mathbf{Z}^N$, $\{x_{n,j}\}$ ($j = 1, \dots, k$ and $n = 1, 2, \dots$) and an increasing sequence of real numbers $R_n > 0$ ($n = 1, 2, \dots$) as in (3.1)–(3.4). We must now define smooth cut-offs to replace the characteristic functions in (3.5). Let $\beta_{n,i} \in C_0^\infty(\mathbf{R}^N)$, $i = 1, \dots, k$ be such that:

$$0 \leq \beta_{n,i}(x) \leq 1 \tag{5.8}$$

$$\beta_{n,i}(x) = 1 \quad \text{if} \quad x \in B(x_{n,i}, R_n) \tag{5.9}$$

$$\operatorname{supp} \beta_{n,i} \subset B(x_{n,i}, 2R_n) \tag{5.10}$$

$$\|\nabla \beta_{n,i}\|_\infty \leq \frac{C}{R_n} \tag{5.11}$$

where $C > 0$ is a fixed constant, independent of n, i. We also define

$$\beta_{n,k+1}(x) \equiv \sqrt{1 - \sum_{i=1}^{k} \beta_{n,i}^2} \tag{5.12}$$

Now, we set

$$[v]_{n,i} \equiv \beta_{n,i} \cdot v, \qquad i = 1, \dots, k+1 \tag{5.13}$$

Note that each $[v]_{n,i} \in H^1(\mathbf{R}^N)$, for $i = 1, \dots, k+1$, and in fact $[v]_{n,i} \in H_0^1(B(x_{n,i}, 2R_n))$ for $i = 1, \dots, k$.

Note that (3.2)–(3.4) together with (f.5) yield the following useful estimates: *Given $\epsilon > 0$ there exists $N = N(\epsilon)$ such that for all $n \geq N$ and for all $v \in H^1(\mathbf{R}^N)$*

$$\begin{aligned} &\left|\int F(x, [v]_{n,i}) - F_\infty(x, [v]_{n,i})\, dx\right| \leq \epsilon \max\{\|[v]_{n,i}\|_{H^1}^{s+1}, 1\} \\ &\|f(x, [v]_{n,i}) - f_\infty(x, [v]_{n,i})\|_{H^{-1}} \leq \epsilon \max\{\|[v]_{n,i}\|_{H^1}^{s+1}, 1\} \end{aligned} \tag{5.12}$$

Next we introduce a sequence of auxilliary functionals

$$\tilde{J}_n(v) = \sum_{i=1}^{k} J([v]_{n,i}) + \frac{1}{2}\|[v]_{k,n+1}\|_{H^1}^2 \tag{5.13}$$

and sets Γ_n^j, $\hat{\Gamma}_n$, and $\tilde{\Gamma}_n$ as in section 3. It is easy to see that (5.12) allows us to repeat the argument for Lemmas 3.1, 3.2, and 3.3 to obtain the following two Lemmas:

LEMMA 5.5.

$$kc = \lim_{n \to \infty} \inf_{\tilde{\gamma} \in \tilde{\Gamma}_n} \max_{0 \leq t \leq 1} \tilde{J}_n(\tilde{\gamma}(t))$$

LEMMA 5.6. *Assume that $(*)$ holds. Let r_0 be as in Lemma 5.3 and $0 < r < r_0$. There exist $v^{(1)}, \dots, v^{(m)} \in \mathcal{C}^c$ ($m = m(r) < \infty$) and $\epsilon_1 = \epsilon_1(r) > 0$ such that for any $0 < \epsilon < \epsilon_1$ there exists a sequence $\{\tilde{\gamma}_n^\epsilon\}$ and $n_\epsilon > 0$ such that for all $n \geq n_\epsilon$:*

(i) $\tilde{\gamma}_n^\epsilon \in \tilde{\Gamma}_n$

(ii) $\tilde{\gamma}_n^\epsilon(t) \cap (\tilde{J}_n)_{kc-\epsilon} \subset \bigcup_{\alpha \in [1,m]^k} B(\sum_{i=1}^k v^{(\alpha_i)}(x - x_{n,i}), r)$ *for all* $t \in [0,1]^k$

(iii) $\tilde{\gamma}_n^\epsilon(t) \subset (\tilde{J}_n)^{kc+\epsilon}$ *for all* $t \in [0,1]^k$

REMARK 5.7:

(a) The proofs of Lemmas 5.4 and 5.5 are essentially the same as those of Lemmas 3.2 and 3.3, and so they are omitted.
(b) The observations noted in Remark 3.4 remain valid here.

Now, the only step which remains is to prove Lemma 4.2 for our functional; once this is done, then Lemma 4.3 and Theorem 5.1 will follow exactly as before, without any revision. In particular, we again introduce the following hypothesis in order to derive a contradiction: suppose that for some $k \geq 2$, $v^{(j)} \in \mathcal{C}^c$ $(1 \leq j \leq k)$,

$$\left.\begin{array}{c}\text{there exists a sequence of } n \text{ going to infinity,} \\ J \text{ has no critical points lying in } B(\sum_{j=1}^{k} v^{(j)}(x - x_{n,j}), r_0)\end{array}\right\} \qquad (**_k)$$

LEMMA 5.8. *Under hypotheses $(*)$ and $(**_k)$, Lemma 4.2 holds for J, $\tilde{J}_n$.*

A proof of Lemma 5.8 is given in the appendix.

Now the proof of Theorem 5.1 follows exactly the argument given in section 4.

APPENDIX

In this section, we prove Lemmas 4.3 and 4.2. The proofs follow [S]. The main idea is to compare the dynamics of the pseudo-gradient flows associated with J' and $\tilde{J}_n'$.

DEFINITION A.1. *Let $\mu(x)$ be an upper semicontinuous function on an open set $\mathcal{O}$ of a Banach space. We say that a locally Lipschitz vector field V is a μ-gradient of a C^1 functional F on $\mathcal{O}$ if for every $x \in \mathcal{O}$, $F'(x) \cdot V(x) \geq \mu(x)$ and $\|V(x)\| \leq 1$*

We will also need a technical lemma from [S]: the proof is standard, and involves introducing an appropriate partition of unity.

LEMMA A.2. *Let $\mathcal{O}$ be an open subset of a Banach space, F, G two C^1 functionals on $\mathcal{O}$, and μ, ν two upper semicontinuous functions on $\mathcal{O}$. Assume that for any $x \in \mathcal{O}$ there is a $V_x \in E$ with*

$$F'(x) \cdot V_x > \mu(x), \qquad G'(x) \cdot V_x > \nu(x), \qquad \|V_x\| \leq 1$$

Then there exists V, a μ-gradient of F and a ν-gradient of G.

Now we present the proofs of Lemma 4.3 and 4.2.

PROOF OF LEMMA 4.2:

It is clear, with the choice of the auxillary functional $\tilde{J}_n$, that there exists some constant $K > 0$, such that, for any $u \in L^q$,

$$J'(u)u \geq ||u||^q - K||u||^2$$

$$\tilde{J}_n'(u)u \geq ||u||^q - K||u||^2$$

We choose $r_1 \in (0, r_0)$ with the property $s^q \geq 2Ks^2$ for any $0 \leq s \leq 2r_1$. With this choice of r_1, it is clear from the above bounds that

$$J'(u)u \geq K||u||^2, \ \tilde{J}'_n(u)u \geq K||u||^2, \quad \forall \ ||u|| \leq 2r_1 \tag{A.0}$$

Now let $\rho > 0$ and $v^{(1)}, \dots v^{(k)} \in \mathcal{C}^c$ be given. We claim that there is no sequence $(v_l)_{l \geq 0}$ satisfying:

$$v_l \in B(\sum_{j=1}^{k} v^{(j)}(\cdot - x_{n_l,j}), r_1) \backslash B(\sum_{j=1}^{k} v^{(j)}(\cdot - x_{n_l,j}), \rho), \quad n_l \to \infty \tag{A.1}$$

$$\forall V \in L^q, \ \ J'(v_l) \cdot V \leq \frac{1}{l}||V|| \ \text{ or } \tilde{J}'_{n_l}(v_l) \cdot V \leq \frac{1}{l}||V||. \tag{A.2}$$

If such sequence exists, we set $u_l = v_l - \sum_{j=1}^{k} v^{(j)}(\cdot - x_{n_l,j})$. There are only two cases to consider:

Case One: For any $R > 0$,

$$\lim_{l \to \infty} ||u_l \cdot \chi_{\{\cup_{i=1}^{k} B(x_{n_l,i}, R)\}}|| = 0. \tag{A.3}$$

In the following, we let $\bar{J}$ denote either J or $\tilde{J}_n$. Clearly

$$\bar{J}'(v_l) = \bar{J}'(v_l \cdot \chi_{\{\cup_{i=1}^{k} B(x_{n_l,i}, R)\}}) + \bar{J}'(v_l \cdot \chi_{\{\mathbf{R}^N \backslash \cup_{i=1}^{k} B(x_{n_l,i}, R)\}}).$$

For any $\epsilon > 0$, using (A.3), (3.6) and the continuity of J' at $v^{(i)}$, we have $||\bar{J}'(v_l \cdot \chi_{\{\cup_{i=1}^{k} B(x_{n_l,i}, R)\}})|| \leq \epsilon$ for $R > R_\epsilon$, $l > l(\epsilon, R)$.

Choosing l large enough, we have $||v_l \cdot \chi_{\{\mathbf{R}^N \backslash \cup_{i=1}^{k} B(x_{n_l,i}, R)\}}|| \in [\rho/2, 2r_1]$. Hence it follows from (A.0) that

$$\begin{aligned} \bar{J}'(v_l \cdot \chi_{\{\mathbf{R}^N \backslash \cup_{i=1}^{k} B(x_{n_l,i}, R)\}}) \cdot (v_l \cdot \chi_{\{\mathbf{R}^N \backslash \cup_{i=1}^{k} B(x_{n_l,i}, R)\}}) &\geq K||v_l \cdot \chi_{\{\mathbf{R}^N \backslash \cup_{i=1}^{k} B(x_{n_l,i}, R)\}})||^2 \\ &\geq K\frac{\rho}{2}||v_l \cdot \chi_{\{\mathbf{R}^N \backslash \cup_{i=1}^{k} B(x_{n_l,i}, R)\}})|| \end{aligned}$$

Choosing

$$V_l = v_l \cdot \chi_{\{\mathbf{R}^N \backslash \cup_{i=1}^{k} B(x_{n_l,i}, R)\}}$$

and $\epsilon \leq K\frac{\rho}{4}$, we have

$$\begin{aligned} \bar{J}'(v_l) \cdot V_l =& \bar{J}'(v_l \cdot \chi_{\{\mathbf{R}^N \backslash \cup_{i=1}^{k} B(x_{n_l,i}, R)\}}) \cdot V_l + \bar{J}'(v_l \cdot \chi_{\{\cup_{i=1}^{k} B(x_{n_l,i}, R)\}}) \cdot V_l \\ \geq& K\frac{\rho}{2}||V_l|| - \epsilon||V_l|| \\ \geq& K\frac{\rho}{4}||V_l||. \end{aligned}$$

This contradicts (A.2).

Case Two: If Case One does not hold, then there exists $\bar{m} \in (0, r_0)$ and two sequences $R_l^1, R_l^2 > 0$, such that, $R_l^1 \to \infty$, $(R_l^2 - R_l^1) \to \infty$, $R_l^2 < n_l$, and

$$||v_l \cdot \chi_{\{\cup_{i=1}^k B(x_{n_l,i}, R_l^2) \setminus \cup_{i=1}^k B(x_{n_l,i}, R_l^1)\}})|| \to 0, \ ||u_l \chi_{\{\cup_{i=1}^k B(x_{n_l,i}, R_l^1)\}}|| \to \bar{m}.$$

Without loss of generality, we can assume that

$$||u_l \chi_{B(x_{n_l,i}, R_l^1)}|| \to m \in (0, r_0). \tag{A.4}$$

for a fixed i.

Let

$$\mu = \frac{1}{2} \inf \{||J'_\infty(w)||: \ w \in \cup_{u \in \mathcal{C}} B(u, r_0) \setminus B(u, \frac{m}{2})\} \tag{A.5}$$

From Lemma 2.5 we know that $\mu > 0$.

Set

$$\begin{aligned} v_l^1 &= v_l \chi_{\{B(x_{n_l,i}, R_l^1)\}}, \\ v_l^2 &= v_l \chi_{\{B(x_{n_l,i}, R_l^2) \setminus B(x_{n_l,i}, R_l^1)\}}, \\ v_l^3 &= v_l \chi_{\{\mathbf{R}^N \setminus B(x_{n_l,i}, R_l^2)\}}. \end{aligned}$$

Then $\bar{J}'(v_l)$ splits,

$$\bar{J}'(v_l) = \bar{J}'(v_l^1) + \bar{J}'(v_l^2) + \bar{J}'(v_l^3). \tag{A.6}$$

(Recall that $\bar{J}$ still denotes either J or $\tilde{J}_n$.) Clearly we see that

$$\lim_{l \to \infty} ||v_l^2|| = 0, \tag{A.7}$$

$$\lim_{l \to \infty} ||v_l^1 - v^{(1)}(\cdot - x_{n_l,i})|| = \lim_{l \to \infty} ||u_l \chi_{B(x_{n_l,i}, R_l^1)}|| = m \in (0, r_0), \tag{A.8}$$

$$||v_l^3|| \le C. \tag{A.9}$$

where C in (A.9) depends only on k and is independent of l.

Next we claim that (for all l sufficiently large) $||J'(v_l^1)|| \ge \frac{3}{2}\mu$. Indeed, for l sufficiently large, we know that $||v_l^1 - v^{(1)}(\cdot - x_{n_l,i})|| \in [\frac{m}{2}, r_0]$. Hence $||J'_\infty(v_l^1)|| \ge 2\mu$ because of (A.5). But for all l large,

$$||J'(v_l^1) - J'_\infty(v_l^1)|| = ||(L - L_\infty)v_l^1|| \le \frac{\mu}{2}$$

by (3.6), using the support property of v_l^1. Thus the claim holds.

As a consequence of the above claim, there exists $V_l \in L^q$ such that $J'(v_l^1) \cdot V_l \ge \mu$, $||V_l|| = 1$. Set

$$W_l = V_l \chi_{\{B(x_{n_l,i}, \frac{R_l^1 + R_l^2}{2})\}}.$$

Obviously, $||W_l|| \le 1$.

Using estimate (2.2) from Lemma 2.1, we have

$$|J'(v_l^1)(V_l - W_l)| \le C \exp\big(-\kappa \frac{R_l^2 - R_l^1}{2})\big) \to 0 \text{ as } l \to \infty,$$

and therefore if l is sufficiently large

$$J'(v_l^1)W_l \ge \frac{\mu}{2}.$$

Moving to v_l^2, it follows from (A.7) that

$$J'(v_l^2)W_l \to 0 \text{ as } l \to \infty.$$

And, using the exponential decay of the kernel (Lemma 2.1, (2.2)) once again, we have

$$|J'(v_l^3)W_l| \le C \exp\big(-\kappa \frac{R_l^2 - R_l^1}{2})\big) \to 0 \text{ as } l \to \infty. \tag{A.10}$$

Therefore for l large enough, we have

$$J'(v_l) \cdot W_l \ge \frac{\mu}{4} \ge \frac{\mu}{4}||W_l||.$$

To treat $\tilde{J}_n$, notice that

$$\tilde{J}_n'(v_l^1) \cdot W_l = J'(v_l^1) \cdot W_l,$$

$$\lim_{l \to \infty} \tilde{J}_n'(v_l^2) \cdot W_l = 0 \quad \text{(due to (A.7))},$$

and in the same way as (A.10),

$$|\tilde{J}_n'(v_l^3) \cdot W_l| \le C \exp\big(-\kappa \frac{R_l^2 - R_l^1}{2})\big) \to 0 \text{ as } l \to \infty.$$

Therefore we have, for l large enough, that

$$\tilde{J}_n'(v_l) \cdot W_l \ge \frac{\mu}{4} \ge \frac{\mu}{4}||W_l||.$$

This again contradicts (A.2).

Proof of Lemma 4.2: We are given $v^{(1)}, \ldots, v^{(k)} \in \mathcal{C}$ with $J(v^{(j)}) = c$, $j = 1, \ldots, k$. Define $w_n \equiv \sum_{j=1}^k v^{(j)}(\cdot - x_{n_l,j})$. Applying Lemma 4.3 (with $\rho = r_1/2$), there exists a constant $\mu > 0$ and an integer A such that for all $n \ge A$ and for all $v \in B(w_n, r_1) \backslash B(w_n, r_1/2)$ there exists V_v for which

$$J'(v) \cdot V_v > \mu, \qquad \tilde{J}_n' \cdot V_v > \mu, \qquad ||V_v||_p \le 1 \tag{B.1}$$

Set

$$\delta = \min\left(\frac{\mu r_1}{4}, \frac{c}{2}\right) \tag{B.2}$$

and let $\epsilon < \delta$. Set

$$M = \sup\{\|\tilde{J}_n'(v)\| : \|v\| \le r_1 + \sum_{j=1}^{k} \|v^{(j)}\|\} < \infty$$

As $\lim_{n\to\infty} \tilde{J}_n(w_n) = kc$ and $M < \infty$, we can choose $\rho = \rho(\epsilon) > 0$ and $A' \ge A$ such that

$$2c - \frac{\epsilon}{2} \le \tilde{J}_n(B(w_n, \rho)) \le 2c + \frac{\epsilon}{2} \tag{B.3}$$

for all $n \ge A'$. Now set

$$\mu_n = \frac{1}{2}\inf\{\|J'(v)\| : \quad v \in B(w_n, r_1/2)\} \tag{B.4}$$

CLAIM 1: There exists an integer $A'' \ge A'$ such that $\mu_n > 0$ for all $n \ge A''$.

Suppose not. Then there exists a sequence $n_i \to \infty$ with $\mu_{n_i} = 0$ for each $i = 1, 2, \ldots$ Fixing i, there must be a sequence $\{v_\ell^i\} \subset B(w_{n_i}, r_1/2)$ with $J'(v_\ell^i) \to 0$ as $\ell \to \infty$. Extracting a subsequence (still denoted $\{v_\ell^i\}$), we have $v_\ell^i \rightharpoonup v^i$ with $J'(v^i) = 0$. As $\|v_\ell^i - w_{n_i}\| < \frac{r_1}{2}$, we have $\|v^i - w_{n_i}\| < \frac{r_1}{2}$ also, so there exists a (non-trivial) critical point of J in each ball $B(w_{n_i}, r_1/2)$, $i = 1, 2, \ldots$, which contradicts $(**_k)$.

CLAIM 2: There exists an integer $N \ge A''$ such that for all $n \ge N$ and for all $v \in B(w_n, r_1/2)$ there exists V_v with

$$J'(v) \cdot V_v > \mu_n, \qquad \tilde{J}_n'(v) \cdot V_v > -(M+1), \qquad \|V_v\| \le 1 \tag{B.5}$$

and such that

$$\tilde{J}_n'(v) \cdot V_v > 0 \quad \text{for all} \quad v \in B(w_n, r_1/2) \backslash B(w_n, \rho) \tag{B.6}$$

First, for $v \in B(w_n, r_1/2) \backslash B(w_n, \rho)$, apply Lemma 4.2 to obtain $\hat{\mu} > 0$ and V_v with

$$J'(v) \cdot V_v > \hat{\mu}, \qquad \tilde{J}_n'(v) \cdot V_v > \hat{\mu}, \qquad \|V_v\| \le 1 \tag{B.7}$$

As $J'(w_n) \to 0$ we have $\mu_n \to 0$ as $n \to \infty$, and so $\hat{\mu} \ge \mu_n$ for n sufficiently large.

Now for $v \in B(w_n, \rho)$, choose V_v, $\|V_v\| = 1$ so that

$$J'(v) \cdot V_v > \frac{1}{2}\|J'(v)\| \ge \mu_n$$

By the definition of M, we must have $|\tilde{J}_n'(v) \cdot V_v| \le M$, and so $\tilde{J}_n'(v) \cdot V_v \ge -M > -(M+1)$. Thus all of the conditions in (B.5) and (B.6) are met.

Next we define the (upper semi-continuous) functions

$$\left.\begin{aligned} \mu_0(x) &= \mu\chi_{B(w_n,r_1)} - (\mu-\mu_n)\chi_{B(w_n,r_1/2)} \\ \nu_0(x) &= \mu\chi_{B(w_n,r_1)} - \mu\chi_{B(w_n,r_1/2)} - (M+1)\chi_{B(w_n,\rho)} \end{aligned}\right\} \tag{B.8}$$

Note that for $v \in B(w_n, r_1)$ and $n \geq N$,

$$J'(v)\cdot V_v > \mu_0(x), \qquad \tilde{J}'_n(v)\cdot V_v > \nu_0(x), \qquad \|V_x\| \leq 1 \tag{B.9}$$

Applying Lemma A.2, there exists a Lipschitz vector field V_0 which is simultaneously a μ_0-gradient for J and a ν_0-gradient for $\tilde{J}_n$ on $B(w_n, r_1)$ for each $n \geq N$.

We will now choose a suitable cut-off for our vector fields. Define a (Lipschitz) function ψ by:

$$\psi(v) = \begin{cases} 1, & \text{if } v \in B(w_n, \frac{3r_1}{4}) \cap (\tilde{J}_n)_{kc-\epsilon} \\ 0, & \text{if } v \in B^C(w_n, r_1) \cup (\tilde{J}_n)^{kc-2\epsilon} \\ 0 < \psi(v) < 1 & \text{otherwise} \end{cases} \tag{B.10}$$

and set $V = \psi V_0$, $\mu = \psi\mu_0$, $\nu = \psi\nu_0$. V remains a μ-gradient of J and a ν-gradient of $\tilde{J}_n$ on $B(w_n, r_1)$ for all $n \geq N$.

We use V to determine a flow

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi(t,\cdot) = -V(\varphi(t,\cdot)) \\ \varphi(0,v) = v \end{cases}$$

As $\|V\| \leq 1$, the flow is globally defined on $\mathbf{R}_+ \times L^q$.)

We now consider cases corresponding to the statements (4.1) and (4.2):

Case I: $v \notin (\tilde{J}_n)_{kc-2\epsilon} \cap B(w_n, r_1)$.

In this case, $\psi(v) = 0$, and so $\varphi(t, v) = v$ for all $t \geq 0$.

Case II: $v \in (\tilde{J}_n)^{kc+\epsilon} \cup B(w_n, r_1/2)$.

Define

$$S_n = \sup\{|J(u) - J(v)| : \quad u, v \in B(w_n, r_1)\} \tag{B.12}$$

CLAIM 3: There exists $t_1 \in [0, \frac{2S_n}{\mu_n}]$ so that one of the following two possibilities holds:

(a) $\varphi(t_1, v) \in (\tilde{J}_n)^{kc-\epsilon}$, or,

(b) $\|\varphi(t_1, v) - v\| = \frac{3r_1}{4}$

Suppose neither possibility holds. Then $\psi(\varphi(t,v)) \equiv 1$ for all $t \in [0, \frac{2S_n}{\mu_n}]$, and hence

$$\begin{aligned} J(v) - J(\varphi(\frac{2S_n}{\mu_n}, v)) &= -\int_0^{\frac{2S_n}{\mu_n}} \frac{\partial}{\partial t} J(\varphi(t,v))\,dt \\ &\geq \mu_n \cdot \frac{2S_n}{\mu_n} \end{aligned} \tag{B.13}$$

which contradicts the definition of S_n in (B.12).

We will now show that in case II, only possibility (a) can hold. So suppose that v is in case II and satisfies (b) in Claim 3. If so then clearly there is a (minimal) $t_0 \in [0, t_1)$ with

$$\varphi([t_0, t_1], v) \cap B(w_n, \rho) = \emptyset$$

Either $t_0 = 0$ (in which case, $v = \varphi(t_0, v) \in (\tilde{J}_n)^{kc+\epsilon}$) or $t_0 > 0$ and $\|\varphi(t_0, v) - w_n\| = \rho$ (in which case we also have $\varphi(t_0, v) \in (\tilde{J}_n)^{kc+\epsilon}$, by applying (B.3)). Fix a (maximal) interval $[\alpha, \beta] \subset [t_0, t_1]$ so that

$$\varphi([\alpha, \beta]) \subset \bar{B}(w_n, \frac{3r_1}{4}) \backslash B(w_n, \frac{r_1}{2}) \tag{B.14}$$

As $\|V\| \leq 1$, we must have

$$\beta - \alpha \geq \frac{r_1}{4} \tag{B.15}$$

and using (B.8) we see

$$\frac{\partial}{\partial t} \tilde{J}_n(\varphi(t, v)) = -\tilde{J}_n'(\varphi(t, v)) \cdot V \leq -\mu < 0$$

and using the above estimate with (B.15),

$$\begin{aligned} \tilde{J}_n(\varphi(t_1, v)) &= \tilde{J}_n(\varphi(t_0, v)) + \int_{t_0}^{t_1} \frac{\partial}{\partial t} \tilde{J}_n(\varphi(t, v))\, dt \\ &\leq (kc + \epsilon) - \mu \cdot \frac{r_1}{4} \\ &< kc - \epsilon \end{aligned}$$

via (B.2). Thus, even assuming case II(b), we arrive at $\varphi(t_1, v) \in (\tilde{J}_n)^{kc-\epsilon}$, i.e., all v in case II satisfy (a) of Claim 3.

CLAIM 4: If $\varphi(t_1, v) \in (\tilde{J}_n)^{kc-\epsilon}$ for some $t_1 \geq 0$, then so is $\varphi(t, v) \in (\tilde{J}_n)^{kc-\epsilon}$ for all $t \geq t_1$.

To see this, note that $\frac{\partial}{\partial t} \tilde{J}_n(\varphi(t, v)) = -\tilde{J}_n'(\varphi(t, v)) \cdot V(\varphi(t, v))$ which is non-positive unless $\varphi(t, v)$ enters $B(w_n, \rho)$ for $t \geq t_1$. But by (B.3), $B(w_n, \rho) \subset (\tilde{J}_n)_{kc-\epsilon/2}^{kc+\epsilon/2}$, and so the trajectory $\varphi(t, v)$ cannot cross into $B(w_n, \rho)$ for $t \geq t_1$.

Combining the results from cases I and II with Claim 4, we take

$$\varphi_n(v) \equiv \varphi\left(\frac{2S_n}{\mu_n}, v\right)$$

and (4.1), (4.2) will hold for the map φ_n, $n \geq N$.

PROOF OF LEMMA 5.8: The proof is very similar to that for Lemma 4.2.

Our first task is to prove an analogous result to (A.0): there exists $r_1 < r_0$ such that (throughout, we denote by $\|\cdot\|$ the norm in $H^1(\mathbf{R}^N)$,)

$$J'(u)u \geq K\|u\|^2, \ \tilde{J}_n'(u)u \geq K\|u\|^2, \quad \forall\ \|u\| \leq 2r_1 \tag{C.0}$$

To see this, we apply (f.4) to obtain:

$$\tilde{J}_n'(u)u = \sum_{i=1}^{k}\left\{\|[u]_{n,i}\|^2 - \int f(x,[u]_{n,i})[u]_{n,i}\right\} + \|[u]_{n,k+1}\|^2$$

$$\geq \sum_{i=1}^{k}\left\{\frac{1}{2}\|[u]_{n,i}\|^2 - C\int |[u]_{n,i}|^{s+1}]\right\} + \|[u]_{n,k+1}\|^2 \tag{C.1}$$

As a consequence, there exists $\delta > 0$ so that for all u with $\|[u]_{n,i}\| < \delta$ $(i = 1,\dots,k)$

$$\tilde{J}_n'(u)u \geq \sum_{i=1}^{k+1}\frac{1}{4}\|[u]_{n,i}\|^2 \tag{C.2}$$

Moreover, (using (5.8)–(5.11))

$$\|[u]_{n,i}\|^2 = \|\nabla(\beta_{n,i}u)\|^2 \leq \left(1 + \frac{C}{R_n}\right)\|u\|^2$$

for all $i = 1,\dots,k+1$. So, there exists N so that (C.2) holds in fact for all $\|u\| < \delta/2$ and $n \geq N$.

Finally, we compare the right hand side of (C.2) with $\|u\|$:

$$\sum_{i=1}^{k+1}\|\beta_{n,i}u\|^2 = \sum_{i=1}^{k+1}\int (\beta_{n,i}^2|\nabla u|^2 + 2\beta_{n,i}u\nabla\beta_{n,i}\cdot\nabla u + |u|^2|\nabla\beta_{n,i}|^2 + \beta_{n,i}^2|u|^2)$$

$$\geq \|u\|^2 + \sum_{i=1}^{k+1} 2\int \beta_{n,i}u\nabla\beta_{n,i}\cdot\nabla u$$

but

$$\left|\int \beta_{n,i}u\nabla\beta_{n,i}\cdot\nabla u\right| \leq \frac{c}{R_n}\|\nabla u\|_2\|u\|_2 < \frac{1}{2}\|u\|^2$$

for n sufficiently large, and so we obtain

$$\tilde{J}_n'(u)u \geq \frac{1}{8}\|u\|^2$$

for all n large and for all $\|u\| < \delta/2 \equiv 2r_1$.

We now introduce a family of smooth cut-offs $\varphi_{\xi,R} \in C_0^\infty(\mathbf{R}^N)$, where $\xi \in \mathbf{R}^N$, $R > 0$ and

$$\begin{gathered}\operatorname{supp}\varphi_{\xi,R} \subset B(\xi, R+1)\\ \varphi_{\xi,R}(x) = 1 \quad \text{if} \quad x \in B(\xi,R)\\ 0 \leq \varphi_{\xi,R}(x) \leq 1, \quad |\nabla\varphi_{\xi,R}(x)| \leq C\end{gathered}$$

for some constant C independent of ξ, R. Set also

$$\psi_{l,R} \equiv \sum_{i=1}^{k} \varphi_{x_{n_l,i},R}(x)$$

Now we continue as in the proof of Lemma 4.2: let $\rho > 0$ and $v^{(1)}, \dots v^{(k)} \in \mathcal{C}^c$ be given and assume that there is a sequence v_l satisfying (A1) and (A2). Again we set $u_l = v_l - \sum_{j=1}^{k} v^{(j)}(\cdot - x_{n_l,j})$.

Again, there are only two cases.

Case One: For any $R > 0$,

$$\lim_{l \to \infty} ||u_l \psi_{l,R}|| = 0 \tag{C.3}$$

As before, $\bar{J}$ denotes either J or $\tilde{J}_n$. Let $\epsilon > 0$ be a small number. By (C.3), (5.12), and the continuity of J' at $v^{(i)}$, there exist constants $R_0 > 0$ and $l_0 > 0$ such that for all $R \geq R_0$ and $l \geq l_0$

$$||\bar{J}'(v_l \psi_{l,R})|| \leq \frac{\epsilon}{2} \tag{C.4}$$

Now, given any $\delta > 0$, we can choose $R \geq R_0$ so that

$$||(\psi_{l,R+1} - \psi_{l,R-1}) \sum_{j=1}^{k} v^{(j)}(\cdot - x_{n_l,j})|| \leq \frac{\delta}{2} \tag{C.5}$$

Then (using (C.3) and (C.5)) there exists $l_1 = l_1(R)$ such that for all $l \geq l_1$

$$\begin{aligned} ||(\psi_{l,R+1} - \psi_{l,R-1})v_l|| &\leq ||(\psi_{l,R+1} - \psi_{l,R-1}) \sum_{j=1}^{k} v^{(j)}(\cdot - x_{n_l,j})|| + ||(\psi_{l,R+1} - \psi_l, R-1)u_l|| \\ &\leq \delta \end{aligned} \tag{C.6}$$

Now, $\bar{J}$ splits into pieces,

$$\bar{J}'(v_l) = \bar{J}'(v_l \psi_{l,R}) + \bar{J}'(v_l(1 - \psi_{l,R})) + E_{l,R}$$

where $E_{l,r}$ is supported in the union of annuli $\bigcup_{i=1}^{k} B(x_{n_l,i}, R+1) \backslash B(x_{n_l,i}, R)$. By (C.6), v_l is small in this region, and so (using the form of the functionals $J, \tilde{J}_n$) we see that $J, \tilde{J}_n$ will be small there: there exists $l_2 > 0$ such that

$$||E_{l,R}|| < \frac{\epsilon}{2} \tag{C.7}$$

for all $l \geq l_2$.

Choosing l large enough, we have $||v_l(1-\psi_{l,R})|| \in [\rho/2, 2r_1]$. Hence it follows from (C.0) that

$$\begin{aligned}\bar{J}'(v_l(1-\psi_{l,R}))\cdot(v_l(1-\psi_{l,R})) \geq K||v_l(1-\psi_{l,R}))||^2 \\ \geq K\frac{\rho}{2}||v_l(1-\psi_{l,R}))||\end{aligned}$$

Choosing

$$V_l = v_l(1-\psi_{l,R})$$

and $\epsilon \leq K\frac{\rho}{4}$, we have

$$\begin{aligned}\bar{J}'(v_l)\cdot V_l =& \bar{J}'(v_l(1-\psi_{l,R}))\cdot V_l + \bar{J}'(v_l\psi_{l,R})\cdot V_l + E_{l,R}\cdot V_l \\ \geq& K\frac{\rho}{2}||V_l|| - \epsilon||V_l|| \quad \text{(from (C.4) and (C.7))} \\ \geq& K\frac{\rho}{4}||V_l||.\end{aligned}$$

This contradicts (A.2).

Case Two: If Case One does not hold, then there exists $\bar{m} \in (0, r_0)$ and two sequences $R_l^1, R_l^2 > 0$, such that, $R_l^1 \to \infty$, $(R_l^2 - R_l^1) \to \infty$, $R_l^2 < n_l$, and

$$||v_l(\psi_{l,R_l^2} - \psi_{l,R_l^1})|| \to 0, \; ||u_l\psi_{l,R_l^1}|| \to \bar{m}$$

Without loss of generality, we can assume that

$$||u_l\psi_{l,R_l^1}|| \to m \in (0, r_0). \tag{C.8}$$

for a fixed i.

Let

$$\mu = \frac{1}{2}\inf\{||J'_\infty(w)||: \; w \in \cup_{u\in C}B(u,r_0)\setminus B(u,\frac{m}{2})\} \tag{C.9}$$

From Lemma 5.3 we know that $\mu > 0$.

Set

$$\left.\begin{aligned}v_l^1 &= v_l\psi_{l,R_l^1+1}, \\ v_l^2 &= v_l(\psi_{l,R_l^2-1} - \psi_{l,R_l^1+1}), \\ v_l^3 &= v_l(1-\psi_{l,R_l^2-1})\end{aligned}\right\} \tag{C.10}$$

As in Case One,

$$\bar{J}'(v_l) = \bar{J}'(v_l^1) + \bar{J}'(v_l^2) + \bar{J}'(v_l^3) + E_l$$

where E_l is supported in annuli,

$$\operatorname{supp} E_l \subset \bigcup_{i=1}^{k}(B(x_{n_l,i}, R_l^2)\backslash B(x_{n_l,i}, R_l^2-1) \cup B(x_{n_l,i}, R_l^1+2)\backslash B(x_{n_l,i}, R_l^1+1))$$

and so the conditions for Case Two imply that $||E_l|| \to 0$ as $l \to \infty$.

Clearly we still have

$$\lim_{l\to\infty} ||v_l^2|| = 0, \tag{C.11}$$

$$\lim_{l\to\infty} ||v_l^1 - v^{(1)}(\cdot - x_{n_l,i})|| = m \in (0, r_0), \tag{C.12}$$

$$||v_l^3|| \le C. \tag{C.13}$$

where C in (C.13) depends only on k and is independent of l.

Next we claim that (for all l sufficiently large) $||J'(v_l^1)|| \ge \frac{3}{2}\mu$. (The proof of this claim is identical to its counterpart in the proof of Lemma 4.2, and relies on the estimate (5.12).) As a consequence of the above claim, there exists $V_l \in H^1$ such that $J'(v_l^1) \cdot V_l \ge \mu, ||V_l|| = 1$. Set

$$W_l = V_l \psi_{l,(\frac{R_l^1+R_l^2}{2})}$$

Obviously, $||W_l|| \le C_0$., for a constant C_0 independent of l. We have (using (C.10))

$$J'(v_l^1)W_l = J'(v_l^1)V_l \ge \mu$$

In addition, it follows from (C.11) that

$$J'(v_l^2)W_l \to 0 \ \text{ as } l \to \infty.$$

and $J'(v_l^3)W_l = 0$ by (C.10) again. Therefore for l large enough, we have

$$J'(v_l) \cdot W_l \ge \frac{\mu}{2} \ge \frac{\mu}{2C_0}||W_l||.$$

For $\tilde{J}_n$, we have

$$\tilde{J}_n'(v_l^1) \cdot W_l = J'(v_l^1) \cdot W_l,$$

$$\lim_{l\to\infty} \tilde{J}_n'(v_l^2) \cdot W_l = 0 \quad \text{(due to (C.11))},$$

$$\tilde{J}_n'(v_l^3) \cdot W_l = 0$$

Therefore we have, for l large enough, that

$$\tilde{J}_n'(v_l) \cdot W_l \ge \frac{\mu}{2} \ge \frac{\mu}{2}||W_l||.$$

This again contradicts (A.2).

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