NAMT
92-010

Singularities and Uniqueness of Cylindrically Symmetric Surfaces Moving by Mean Curvature

H. M. Soner
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

and

P. E. Souganidis
Department of Mathematics
University of Wisconsin
Madison, WI 53706

Research Report No. 92-NA-010

April 1992

Singularities and Uniqueness of Cylindrically Symmetric Surfaces Moving by Mean Curvature

H.M. Soner [1]
Department of Mathematics
Carnegie Mellon University
Pittsbourgh, PA 15213
and

P.E. Souganidis [2]
Department of Mathematics
University of Wisconsin
Madison, WI 53706

April 1, 1992

Supported in part by NSF grant DMS-9002249 and by the Center for Nonlinear Analysis at CMU.

Supported in part by NSF grants DMS-9024617 and DMS-8657464 (PYI), ARO contract DAAL03-90-G-0012 and the Sloan Foundation. Part of this work was done while visiting the Institute for Advanced Study at Princeton and the Division of Applied Mathematics at Brown University.

not necessary, conditions on Γ_0, which guarantee empty interior for Γ_t, were given in Barles, Soner and Souganidis [BSS]. On the other hand, DeGiorgi [DG] conjectured that there is a particular torus, whose evolution by mean curvature will develop interior.

We continue by formulating the main results. As long as Γ_t is smooth, (since Γ_0 is smooth, Γ_t will be smooth for small time (cf. Evans, Spruck [ES])), it can be parametrized by

$$r = h(z,t), \tag{0.3}$$

where the, possibly multivalued, smooth function h solves the equation

$$\begin{cases} h_t = \dfrac{h_{zz}}{1+h_z^2} - \dfrac{N-2}{h} \\ h(z,0) = h_0(z), \end{cases} \tag{0.4}$$

in some time dependent domains for each branch of h_0, and, therefore, h. It is, of course, immediate that as long as $h \neq 0$, (0.4) has classical solutions, and, therefore, Γ_t is smooth. The only potential singularities of Γ_t may therefore occur when $h = 0$ for the first time. To formulate our first result, let us specify that h vanishes for the first time at $(0,T)$ and that (0.4) holds in

$$\Omega = (-2A, 2A) \times (0,T), \tag{0.5}$$

for some $A > 0$. Throughout this discussion we will be assuming that

$$\begin{cases} h(\cdot,t) \text{ has only one minimum at } (0,t) \text{ in } (-2A,2A) \times \{t\} \\ \text{for } t \in (0,T). \end{cases} \tag{0.6}$$

This assumption is made only for simplicity. Our arguments can be easily modified to allow for more local minima which vary in time.

Theorem 1: *Assume that (0.6) holds. Then*

$$\lim_{t\uparrow T}(T-t)^{-\frac{1}{2}}h(y(T-t)^{\frac{1}{2}},t) = \sqrt{2(N-2)}, \tag{0.7}$$

with the limit uniform for $|y|$ bounded.

The following corollary is an immediate consequence.

Corollary 2: *Up to a parabolic scaling, at the singularity the surface Γ_t converges to a cylinder.*

A similar result was obtained by Huisken [H] when $N = 3$ under the rather restrictive condition that Γ_0 has positive mean curvature.

Another consequence of Theorem 1 is that Γ_t does not develop interior at $t = T$ and that it continues as a smooth surface for $t > T$ till the time it becomes extinct. We formulate this result for the special case of a torus. The precise statement for other types of surfaces is given in Section 3.

Consider the evolution $t \mapsto \Gamma_t$ by mean curvature of the torus Γ_0, which is given by

$$\Gamma_0\{x \in I\!R^N : (r-1)^2 + z^2 = R^2\} \qquad (0 < R < 1),$$

where $r^2 = \sum_{i=1}^{N-1} x_i^2$.

Proposition 3: *Γ_t never develops interior. Moreover, there exists $R_0 \in (0,1)$ such that for all $R \in (0, R_0), \Gamma_t$ shrinks to a circle and then becomes extinct. For $R \in (R_0, 1), \Gamma_t$ "focuses" at 0 at some time T_R. It then "opens up" and, finally, shrinks to a point and becomes extinct. Finally, for $R = R_0, \Gamma_t$ focuses at exactly the same time it becomes a circle.*

DeGiorgi's conjecture was that there exists $R_0 \in (0,1)$ and $T_{R_0} > 0$ such that $\Gamma_{T_{R_0}}$ has nonempty interior. Numerical evidence suggesting the above described behavior were obtained by Paolini and Verdi [PV].

The paper is organized as follows: In Section 1 we prove a curvature bound. Section 2 is devoted to analyzing the behavior of Γ_t at the focusing point. In Section 3 we show that Γ_t does not develop interior and study its behavior after the focusing occurs. Finally, in the Appendix we state a result about the number of zeroe's of solutions of linear parabolic equations, which we will be using repeatedly throughout the paper.

The results of this paper were announced in [SS]. By the time this paper was completed, the authors learned that Altschuler, Angenent and Giga [AAG1,2] also studied the motion by mean curvature of bodies of rotation. Their set up is and their results are formulated for "barbell" type shapes. On the other hand, they do not need assumption (0.6).

Acknowledgement: The second author wishes to thank Luis Caffarelli for several very interesting discussions about this problem.

1 A curvature bound

In this section we obtion an upper bound for the ratio of the radial and angular components of the curvature of Γ_t near the singularity. This bound plays a fundamental role in the proof of Theorem 1 and its consequences.

More precisely, let

$$\psi = \frac{hh_{zz}}{1+h_z^2} \tag{1.1}$$

and recall that h solves in (0.4) in $(-2A,2A)\times(0,T)$.

Proposition 1.1: *There exist $\varepsilon > 0$ and $c_0 > 0$ such that*

$$|\psi| \le c_0^{-1} \text{ in } [-\frac{A}{2},\frac{A}{2}]\times[0,T) \tag{1.2}$$

and

$$\psi \le (N-2) - c_0 \text{ in } [-\varepsilon,\varepsilon]\times[T-\varepsilon,T). \tag{1.3}$$

The meaning of (1.3) is that near the singularity at $(0,T)$, the radial component of the curvature is strictly dominated by the angular one. As a matter of fact, the ratio between the radial and angular components of the curvature is strictly below of the same ratio of the curvature of the *catenoid*, the stationary solution of (0.4). This, in turn, yields (as we will see in Section 2.4) that $0 \notin \text{ int } \Gamma_T$.

The proof of Proposition 1.1 is based on the existence of an one-parameter family of barriers (q_λ), with $\lambda \in [\lambda_0, \lambda_1]$ for some $0 < \lambda_0 < \lambda_1$, which are defined on

$$\tilde{\Omega}_1 = [-B/2, B/2] \times [0,T),$$

with $0 < B \leq A$.

In the sequel we will need the following notation. For $\lambda \in [\lambda_0, \lambda_1]$, $t \in [0,T)$ and $|\theta| \leq B$ we define:

$$\Omega_1 = (-B/2, B/2) \times (0,T), \tilde{\Omega} = (-2B, 2B) \times [0,T),$$

$$I(t,\lambda,\theta) = \{|z| \leq B/2 \ : \ q_\lambda(z,t) \leq h(z-\theta,t)\},$$

$$n(t,\lambda,\theta) = \#\{|z| \leq B/2 \ : \ q_\lambda(z,t) = h(z-\theta,t)\}.$$

Next we state two lemmas which assert the existence of the barriers with the necessary properties.

Lemma 1.3: *There exist $B > 0, 0 < \lambda_0 < \lambda_1$, an one-parameter family*

$$q_\lambda : \tilde{\Omega}_1 \to (0,\infty) \ , \ \ (\lambda \in [\lambda_0, \lambda_1]),$$

and constants $c, \delta > 0$ such that for every $|\theta| \leq B$ and $t \in [0,T)$:

$$n(t,\lambda_0,\theta) = 2, \tag{1.4}$$

$$I(t,\lambda_1,\theta) = \phi, \tag{1.5}$$

$$n(t,\lambda,\theta) \leq 2, \ \ \forall \lambda \in [\lambda_0, \lambda_1], \tag{1.6}$$

(1.7) (A,*,<) -4 *(q\(z,t),* $\frac{d}{\partial z}$*qx(z,t))* ***is continuous on*** **[A$_0$,Ai] x** $\tilde{\Omega}_1$,

(1.8) ***zqx*{z,t)*** $\geq$ **0,** ***and*** $q_x(z,t) = q_x\{-z,t)$,

(1.9) 9 A (± | , <) ≥ $\sup_{\backslash z\backslash \leq 2B}$ **fc(z,t),**

• ^

and

(1.10) *tl*≥\<*(N-2)*- c, t^/ieneuer ?$_A$ ≤ 5,

where

$$\psi_\lambda = \frac{q_\lambda q_{\lambda,zz}}{1 + (q_{\lambda,z})^2}.$$

Lemma 1.4: *For every (z$_o$,t$_o$)* G fli, *there exists* (A*,0*) G [A$_o$, A^xf—JB/2—z$_0$,5/2 — z$_0$] *such that*

We continue with the proof of Proposition 1.1 and then return to Lemmas **1.3** and **1.4.**

Proof of Proposition 1.1: Let (A*,*0*)* be as in Lemma 1.4. We claim that

$$qx\%zz\{z_0 + 6^m, t_0) \geq h_{Z2}(z_0, to).$$

Indeed, if not, then by (1.9) and (1.11), the difference 9A*(-,<O) - *h('-6\t$_0$)* would have more than two zeroes, which contradicts (1.6).

Therefore

V^O,*O)≤V>A-(*O + 0VO).

Suppose now that *h(z$_0$, t$_0$)* ≤ *\$*<> where *6* is the constant appearing in (1.10). **Since** ***h(z$_o$,t$_o$)*** **=** ***q$_x$*(z$_0$ + 0*,t$_o$),*** **(1.10) yields**

$$\psi_{\lambda^*}(z_0 + \theta^*, t_0) \leq (N-2) - c.$$

Using (0.4) we conclude that

$$h_t(z_0,t_0)<0 \textit{ and } \psi(z_0,t_0)\le (N-2)-c.$$

A simple iteration of the above argument then yields that, for all $t\in[t_0,T)$,

$$h(z_0,t)\le\delta, h_t(z_0,t)<0 \text{ and } \psi(z_0,t)\le(N-2)-c.$$

Since $h(0,t)\to 0$ as $t\to T$ and $h(\cdot,t)$ is continuous for every $t\in[0,T)$, there exist $\gamma>0$ and $t_0<T$ such that

$$h(\cdot,t_0)\le\delta \qquad \text{in } [-\gamma,\gamma]$$

and

$$\psi\le(N-2)-c \text{ in } [-\gamma,\gamma]\times[t_0,T).$$

Hence (1.3) holds with $\varepsilon=\min(\gamma,T-t_0)$.

To prove (1.2), we first observe that a simple but tedious calculation yields that ψ satisfies the equation

$$\psi_t=\frac{\psi_{zz}}{1+w^2}-\frac{2(1+\psi)w}{h(1+w^2)}\psi_z+\frac{2w^2}{h^2(1+w^2)}(\psi-(N-2))(\psi+1) \tag{1.12}$$

where to simplify the notation we write

$$w=\psi_z. \tag{1.13}$$

Since (0.4) holds in $(-2A,2A)\times(0,T)$ and $\inf\limits_{t\in[0,T)}h(\pm A,t)>0$, interior elliptic regularity yields that $\sup\limits_{t\in[0,T)}|\psi|(\pm A,t)<\infty$. Finally, applying the maximum principle to (1.12), we see that ψ cannot have an interior minimum smaller than -1. Combining all the above yields

$$\psi\ge\min\left[\inf_{|z|\le A}\psi(z,0),\inf_{[0,T)}\psi(\pm A,t),-1\right].$$

Arguing in a similar way and also using (1.3), we obtain an upper bound for ψ.

$\square$

We continue with several preliminaries leading to the proof of Lemma 1.3. The construction of the q_λ's is based on catenoids, which are stationary solutions of (0.4), i.e. functions $z \mapsto \lambda q(z/\lambda)$, where

$$\frac{qq_{zz}}{1+q_z^2} = N-2 \quad \text{and} \quad q(0) = 1.$$

An elementary calculation gives

$$q_z(z) = (q(z)^{2(N-2)} - 1)^{\frac{1}{2}} \text{ and } G(q(z)) = z, \tag{1.14}$$

with the function

$$G(r) = \int_0^r (\rho^{2(N-2)} - 1)^{-\frac{1}{2}} d\rho$$

defined for $r < r_N$ with $r_3 = +\infty$ and $r_N < +\infty$ for $N \geq 4$. Let

$$L = \max\{2T, 2\sqrt{(N-2)T}, 2\|h\|_{L^\infty(\bar{\Omega})} + 2(N-2) + 1\} \tag{1.15}$$

and choose $0 < \lambda^*$ and $B = \min\{r_N, A\}$ such that

$$\begin{cases} (a) & \lambda^* q(\frac{\mu B}{2\lambda^*}) \geq L \text{ for every } \mu \in [\frac{1}{2}, 1] \text{ and } \lambda \in (0, \lambda^*], \\ (b) & \lambda^* \leq [\frac{8}{N-2} \sup\{|h_{zz}(z,0)| : |z| \leq A\}]^{-1}, \\ (c) & (\lambda^*)^2 T \leq \dfrac{3(N-2)}{4} \dfrac{1}{q(\frac{B}{2\lambda^*})} \dfrac{1}{1+q_z^2(\frac{B}{2\lambda^*})}. \end{cases} \tag{1.16}$$

The existence of such a λ^* and B follows from the facts that the function $\lambda \rightarrow \lambda q(\frac{\mu B}{2\lambda})$ is increasing and $q(r_N) = q_z(r_N) = +\infty$.

Next we consider an one-parameter family of solutions $p(\cdot,\cdot;\mu)\,(\mu \in [\frac{1}{2}, 1])$ of

$$\begin{cases} (a) & p_t = \dfrac{p_{zz}}{1+(p_z)^2} - \dfrac{N-2}{p} \text{ on } (-B/2, B/2) \times (0, T_\mu), \\ (b) & p(z, 0; \mu) = \lambda^* q(\dfrac{\mu z}{\lambda^*}) \text{ for } |z| \leq B/2, \\ (c) & p(\pm B, t; \mu) = \lambda^* q(\dfrac{\mu B}{2\lambda^*}) - \alpha(\mu) t \text{ for } t \in [0, T(\mu)), \end{cases}$$

where

$$\alpha(\mu) = \inf_{|z|\le B/2}\Big\{-\Big[\frac{p_{zz}}{1+(p_z)^2} - \frac{N-2}{p}\Big](z,0;\mu)\Big\} \ge 0$$

and $T_\mu \le \infty$ is the "focusing" time of $p(\cdot,\cdot,\cdot\mu)$, i.e., the first time such that $p(0,t;\mu)=0$. A simple calculation yields that there exists $c_0>0$ such that

$$\alpha(\mu) \ge c_0(1-\mu).$$

It is also immediate that the $p(\cdot,\cdot;\mu)$'s and, therefore, the focusing times T_μ's, depend smoothly on μ and that $T_1=+\infty; p(\cdot,\cdot,1)$ being a stationary solution of (0.4). Moreover, the maximum principle yields

$$p_t(z,t;\mu) \le -\alpha(\mu) \le -c_0(1-\mu).$$

Hence $T_\mu<\infty$ for every $\mu<1$. In view of (1.16)(c), there exists $\mu^*\in(\frac{1}{2},1)$ such that

$$T_{\mu^*} = 2T.$$

Set

$$L^* = \lambda^* q(\frac{\mu^* B}{2\lambda^*}) \ge L \tag{1.17}$$

and define, for $\lambda\in(0,\lambda^*]$, the functions

$$q_\lambda : [-\frac{B}{2},\frac{B}{2}]\times[0,\overline{T}_\lambda] \to [0,\infty)$$

so that

$$\begin{cases} (a)\ \ q_\lambda \text{ solves (0.4) on } [-\frac{B}{2},\frac{B}{2}]\times[0,\overline{T}_\lambda), \\ (b)\ \ q_\lambda(z,0)=\min\{L^*,\lambda q(\frac{\mu^* z}{\lambda})\} \text{ on } [-\frac{B}{2},\frac{B}{2}], \\ (c)\ \ q_\lambda(\pm\frac{B}{2},t) = L^*-\alpha t \text{ on } [0,\overline{T}_\lambda), \end{cases} \tag{1.18}$$

where $\overline{T}_\lambda$ is the focusing time and

$$\alpha = \min\left\{\frac{1}{L^*}, \inf_{\substack{\lambda \leq \lambda^* \\ |z| \leq z^*(\lambda)}} \left\{ -\left[\frac{q_{\lambda,zz}}{1+(q_{\lambda,z})^2} - \frac{N-2}{q_\lambda}\right](z,0)\right\}\right\},$$

$z^*(\lambda)$ being the solution of $\lambda q(\frac{\mu^* z}{\lambda}) = L^*$ in $[0, B]$. An explicit computation based on (1.14) and the fact that $q_\lambda(z,0) \leq L^*$ for $|z| \leq z^*(\lambda)$ gives $\alpha > 0$. On the other hand, the maximum principle yields

$$q_{\lambda,t} \leq -\alpha \text{ in } [-B/2, B/2] \times [0, \overline{T}_\lambda), \tag{1.19}$$

and, in particular,

$$0 \leq q_\lambda(0,t) \leq q_\lambda(0,0) - \alpha t \leq \lambda - \alpha t;$$

hence $\overline{T}_\lambda \leq \lambda/\alpha$, i.e. $\overline{T}_\lambda \to 0$ as $\lambda \to 0$.

Recall that μ^* is chosen so that $T_{\mu^*} = 2T$. Moreover, $\alpha \leq \alpha(\mu^*)$ and by the maximum principle, $q_{\lambda^*}(z,t) \geq p(z,t,\mu^*)$ and $\overline{T}_{\lambda^*} \geq \overline{T}_{\mu^*} = 2T$. Define

$$\lambda_0 = \inf\left\{\lambda \leq \lambda^* \text{ such that } \overline{T}_\lambda \geq T \text{ for all } \lambda \in [\lambda, \lambda^*]\right\}.$$

It is clear that $\overline{T}_{\lambda_0} = T$; hence all the q_λ's are defined on $\tilde{\Omega}_1$ for $\lambda \in [\lambda_0, \lambda^*]$.

Finally set $\lambda_1 = \lambda^* + 1$ and define q_λ, for $\lambda \in [\lambda^*, \lambda_1]$, to be the solution of (1.18)(a),(c) with initial datum

$$q_\lambda(z,0) = (\lambda - \lambda^*)L^* + (\lambda_1 - \lambda)q_{\lambda^*}(z,0), \text{ in } [-\frac{B}{2}, \frac{B}{2}]. \tag{1.20}$$

Since

$$q_\lambda(z,0) \geq q_{\lambda^*}(t,0) \text{ for } \lambda \in [\lambda^*, \lambda_1],$$

the maximum principle again yields that

$$q_\lambda \geq q_{\lambda^*} \text{ in } \tilde{\Omega}_1.$$

In particular the focusing time $\overline{T}_\lambda$ for q_λ is larger than T, and, hence, q_λ is defined on $\tilde{\Omega}_1$. Moreover,

$$\delta = \inf\{q_\lambda(z,t) : \lambda \in [\lambda^*, \lambda_1], (z,t) \in \tilde{\Omega}_1\} > 0. \tag{1.21}$$

Proof of Lemma 1.3: It is immediate from the construction described above, that (1.7) and (1.8) hold. Also (1.9) follows from (1.15), (1.17), (1.20), (1.21) and the observation that, for all A € [Ao, Ai],

$$qx(\pm A/2,t) = r - a\ t \geq L * - \pounds \geq ! - ! > \|h\|_{LOO(6)}.$$

To prove (1.10), observe that, for A 6 [A_o, A*],

$$V{>}A = N - 2 + q\backslash q\backslash_{,t}.$$

Hence, by (1.19), there exists a $CQ > 0$ such that

$$\max(\psi_\lambda(z,0), \psi_\lambda(\pm|, 0) \leq \wedge - 2 - co,$$

for every A G [A_0,A*],z G [-f,f] and * € (0,T). Applying the maximum principle to the equation satisfied by the T/>A'S, which similar to (1.12), we conclude that, for all A £ [Ao, A*],

$$\psi_\lambda(z,t) \leq N - 2 - co \quad \text{in } \bar{\Omega}_1.$$

The definition of *6* in (1.21) completes the proof of (1.10).

Next observe that ^(2,0) = *L**. We will prove (1.5) by constructing an appropriate subsolution to (0.4). To this end define

$$v(z,t) = L^* - \frac{2(N-2)}{L^*}t, \quad \left((z,t) \in \tilde{\Omega}_1\right).$$

Then, by (1.15), (1.17),

$$(L')^2 \geq 4(N - 2)T.$$

Therefore, for all (z,t) € Cl_u

$$v\{z,t) > L\_ - \frac{\qquad}{L\, m} T - L - \frac{\quad}{2} y \frac{\qquad}{(1\cdot)2} J \geq y.$$

The above inequality implies that v is a subsolution of (0.4). Also the choice of α yields

$$v(\pm B,t) \leq q_{\lambda_1}(\pm B,t) \text{ for } t \in [0,T).$$

Using again the maximum principle and (1.15), (1.17) we obtain

$$q_{\lambda^*} \geq v \geq \|h\|_{L^\infty(\tilde{\Omega})} + 1;$$

(1.5) follows from the above inequality.

To verify (1.4) and (1.6), we fix $|\theta| \leq B$ and define

$$W_\lambda(z,t) = q_\lambda(z,t) - h(z-\theta,t), \quad \Big((z,t) \in \tilde{\Omega}_1\Big).$$

We now claim that $z \mapsto W_\lambda(z,t)$ has at most two zeroes for each t and exactly two, when $\lambda = \lambda_0$.

Indeed, if $\lambda \geq \lambda^* + \frac{1}{2}$, then (1.20), (1.15) and (1.17) yield

$$W_\lambda(z,0) \geq \frac{L^*}{2} - \|h\|_{L^\infty(\tilde{\Omega})} > 0.$$

If $\lambda \in [\lambda^*, \lambda^* + \frac{1}{2}]$, then

$$W_{\lambda,zz}(z,0) = (\lambda_1 - \lambda)\frac{(\mu^*)^2}{\lambda^*} q_{zz}(\frac{\mu^* z}{\lambda^*}) - h_{zz}(z,0).$$

Since $\mu^* \geq \frac{1}{2}$ and $q_{zz} = (N-2)(1+q_z^2)q^{-1} = (N-2)q^{2N-5} \geq (N-2)$,

$$W_{\lambda,zz}(z,0) \geq \frac{(N-2)}{8\lambda^*} - h_{zz}(z-\theta,0),$$

and, in view of (1.16)(b), $W_{\lambda,zz}(z,0) > 0$. But $W_\lambda(\pm B,0) > 0$. Therefore $W_\lambda(z,0)$ has at most two zeroes.

When $\lambda \in [\lambda_0, \lambda^*]$, (1.18)(b) yields either $q_\lambda(z,0) = L^*$ or

$$W_{\lambda,zz}(z,0) = \frac{(\mu^*)^2}{\lambda} q_{zz}(\frac{\mu^* z}{\lambda}) - h_{zz}(z-\theta,0) \geq \frac{N-2}{4\lambda^*} - h_{zz}(z-\theta,0) > 0,$$

if $q_\lambda(z,0) < L^*$. On the other hand, $W_\lambda(z,0) > 0$, whenever $q_\lambda(z,0) = L^*$ (by the choice of L^*). Hence $W_\lambda(z,0)$ has at most two zeroes. Finally,

$W_{\lambda_0}(z,0)$ has exactly two zeroes, if there is a z such that $W_{\lambda_0}(z,0) < 0$. Suppose $W_{\lambda_0}(z,0) \geq 0$ for all $|z| \leq B/2$. Since $W_{\lambda_0}(\pm\frac{B}{2},t) > 0$ for all $t \in [0,T)$, the strong maximum principle implies that $W_{\lambda_0}(z,t) > 0$ for all $|z| \leq B/2, t \in (0,T]$. Recall, however, that λ_0 is chosen so that $\overline{T}_{\lambda_0} = T$. Hence $\lim\limits_{t\to T} W_{\lambda_0}(0,t) = 0$. Consequently $W_{\lambda_0}(z,0)$ must be strictly negative for some $|z| < B/2$.

Finally, since $W_\lambda(\pm B,t) > 0$, the maximum principle yields that the number of zeroes of W_λ is a nondecreasing function of time and therefore less or equal to two, i.e. (1.8). Arguing as above we conclude that for every $t < T, W_{\lambda_0}(z,t)$ must be strictly negative for some $|z| \leq B/2$.

□

We conclude this section with the proof of Lemma 1.4, which is similar to an argument used by Angenent in [A2, pg 192].

Proof of Lemma 1.4. For fixed $(z_0,t_0) \in \tilde{\Omega}_1$ set

$$A(z_0) = [-\frac{B}{2} - z_0, \frac{B}{2} + z_0]$$

and define $\omega : [\lambda_0, \lambda_1] \times A(z_0) \to \mathbb{R}^2$ by

$$\omega(\lambda,\theta) = (q_\lambda(z_0+\theta), t_0) - h(z_0,t_0), q_{\lambda,z}(z_0+\theta,t_0) - h_z(z_0,t_0)).$$

We will prove the lemma by showing that the winding number of ω is one. To this end, let $\omega_1(\lambda,\theta), \omega_2(\lambda,\theta)$ denote the first and second components of ω respectively.

First, observe that by the definitions of $I(t,\lambda,\theta)$ and $\omega_1(\lambda,\theta)$,

$$\omega_1(\lambda,\theta) \leq 0 \text{ iff } z_0 + \theta \in I(t_0,\lambda,\theta).$$

Hence (1.5) yields

$$\omega_1(\lambda_1,\theta) > 0 \qquad \text{for all } \theta \in A(z_0). \tag{1.22}$$

Also

$$w_1(\lambda, \pm\frac{B}{2} - z_0) = q_\lambda(\pm\frac{B}{2}, t_0) - h(z_0,t_0),$$

and, by (1.9),

$$w_1(\lambda, \pm\frac{B}{2}, -z_0) > 0, \qquad (\forall\lambda \in [\lambda_0, \lambda_1]). \tag{1.23}$$

On the other hand, since $n(t_0, \lambda_0, \theta) = 2$ for all $|\theta| \leq B$, the equation

$$f(z, \theta) = q_{\lambda_0}(z, t_0) - h(z - \theta, t_0) = 0$$

has exactly two solutions in $[-\frac{B}{2}, \frac{B}{2}]$ for all $|\theta| \leq B$, which we denote by $\mu(\theta)$ and $z(\theta)$, with $\mu(\theta) \leq z(\theta)$. It is immediate that μ and z depend continuously on $\theta \in [-B, B]$. Moreover, since

$$z(B) - B \leq -\frac{B}{2} \text{ and } z(-B) + B \geq \frac{B}{2},$$

there exists $\theta_1 \in [-B, B]$ such that

$$z(\theta_1) - \theta_1 = z_0;$$

in other words

$$\omega_1(\lambda_0, \theta_1) = 0 \text{ and } \theta_1 \in A(z_0). \tag{1.24}$$

Similarly there exists $\theta_2 \in A(z_0)$ such that $\mu(\theta_2) - \theta_2 = z_0$.

Now suppose that there exists $\hat{\theta} \in A(z_0)$ such that $\omega_1(\lambda_0, \hat{\theta}) = 0$. Then either $z(\hat{\theta}) - \hat{\theta} = z_0$ or $\mu(\hat{\theta}) - \hat{\theta} = z_0$. Since (1.9) yields $f(\pm B/2, \hat{\theta}) > 0$ and $f(z(\hat{\theta}), \hat{\theta}) = f(\mu(\hat{\theta}), \hat{\theta}) = 0$, we have

$$f_z(z(\hat{\theta}), \hat{\theta}) \geq 0 \geq f_z(\mu(\hat{\theta}), \hat{\theta}).$$

We have proved the following: if $\omega_1(\lambda_0, \hat{\theta}) = 0$ for some $\hat{\theta} \in A(z_0)$, then

$$\begin{cases} (a) \quad \omega_2(\lambda_0, \hat{\theta}) \geq 0 \quad \text{if} \quad \hat{\theta} + z_0 = z(\hat{\theta}), \\ (b) \quad \omega_2(\lambda_0, \hat{\theta}) \leq 0 \quad \text{if} \quad \hat{\theta} + z_0 = \mu(\hat{\theta}). \end{cases} \tag{1.25}$$

Suppose now that u>i(A_o,$\bar{0}$) = 0 for some $\bar{0}$ G *A(ZQ).* Then

$$?Ao(*o + \hat{0}, *o) = 9A_O(*O + \bar{0}, t_0) = h\{zo, t_0).$$

Using (1.8), we conclude that

$$z_0 + \hat{\theta} = \pm(z_0 + \bar{\theta}).$$

In summary we find that there are at most two 0's such that u>i(Ao,0) = 0. But we also know that there are 0i,0_2 € *A(zo)* such that

$$z_0 = \mu(\theta_1) - \theta_1 = z(\theta_2) - \theta_2.$$

Hence $0_X \leq 0_2$ and consequently *w* has the following properties,

(<*) k>i(Ao,0)>O, for *0* £ $(0_U 0_2)$,

a o£\ ) (v\ . . t\ fi \ _ n / « ^ \ /j _/ n

(c) cJi(Ao,tf_2) = 0, u>$_2$(A_o,0_2) $\geq$ 0.

Using (1.22), (1.23) and (1.26), we conclude that the winding number of w is one.

D

2 Behavior of T_t at the focusing point.

In this section we prove Theorem 1 and discuss some of its immediate consequences. As mentioned in the Introduction, a result similar to (0.7) was proved, for iV = 3, by Huisken [H], under several assumptions, the most important one being a bound on the blow-up rate of the curvature, which was verified in [H] only for barbell-type surfaces, which have strictly positive mean curvature. A similar result was also obtained by Dziuk and Kawohl **[DK].**

We will organize this section in several subsections where we will explain the basic steps of the proof of Theorem 1.

2.1 Scaling and preliminary estimates.

One of the main estimates in our analysis is

$$\liminf_{t\to T}(T-t)^{-\frac{1}{2}}h(0,t) > 0. \tag{2.1}$$

which is essentially equivalent to the assumption in [H,(2) page 286].

To obtain (2.1) we rewrite (0.4) as

$$(h^2)_t = 2[\psi - (N-2)] \tag{2.2}$$

and observe that (0.6) yields

$$\psi(0,t) \geq 0 \text{ for all } t \in [0,T).$$

Combining this with $h(0,T) = 0$ we obtain an easy upper bound

$$h(0,t) \leq \sqrt{2(N-2)(T-t)}, \qquad (t \in [0,T)). \tag{2.3}$$

The lower bound follows from the nontrivial curvature estimate (1.3), which yields

$$h(0,t) \geq \sqrt{2c_0(T-t)} \qquad (t \in [T-\varepsilon,T)). \tag{2.4}$$

Actually (1.3) yields a more general result than (2.4). Indeed, for $0 < |z| \leq \varepsilon$ and $t \in [T-\varepsilon,T)$ we have

$$\frac{\partial}{\partial z}\log(1+(h_z(z,t))^2) = 2\psi(z,t)\frac{h_z(z,t)}{h(z,t)} \leq \gamma\frac{z}{|z|}\frac{\partial}{\partial z}\log(h(z,t))^2, \tag{2.5}$$

where

$$\gamma = (N-2) - c_0. \tag{2.6}$$

and, therefore,

$$|h_z(z,t)| \leq ((\frac{h(z,t)}{h(0,t)})^{2\gamma} - 1)^{\frac{1}{2}} \text{ in } [-\varepsilon,\varepsilon]\times[T-\varepsilon,T). \tag{2.7}$$

Next we follow some of the ideas in [H] and the techniques developed by Giga and Kohn [GK] to study the blow-up of the solution of semilinear heat equation.

To this end, we define

$$v(y,s) = (T-t)^{-\frac{1}{2}} h(y\sqrt{T-t}, t) \text{ with } s = -\log(T-t).$$

Observe that $v(y,s)$ is defined for $(ye^{-\frac{s}{2}}, T-e^{-s})$ in $(-2A, 2A) \times (0,T) \subset \Omega$, i.e. v is defined on

$$K(A) = \{(y,s) \ : \ s > -\log T, |y| \leq 2Ae^{s/2}\}.$$

We now write

$$v(y,s) = e^{\frac{s}{2}} h(ye^{-\frac{s}{2}}, T-e^{-s})$$

and calculate

$$v_s(y,s) = \frac{1}{2}v(y,s) - yh_z(ye^{-\frac{s}{2}}, T-e^{-s}) - e^{-\frac{s}{2}}h_t(ye^{-\frac{s}{2}}, T-e^{-s}),$$

$$v_y(y,s) = h_z(ye^{-s/2}, T-e^{-s}),$$

$$v_{yy}(y,s) = e^{-s/2}h_{zz}(ye^{-s/2}, T-e^{-s}).$$

Using (2.4) and (2.7) we get

$$v(0,s) \geq \sqrt{2c_0}, \tag{2.8}$$

and

$$|v_y(y,s)| \leq ((\frac{v(y,s)}{\sqrt{2c_0}})^{2\gamma} - 1)^{\frac{1}{2}}. \tag{2.9}$$

To obtain more pointwise estimates for v, v_y, v_{yy} and v_s we use the equation which is satisfied by v, namely

$$v_s = \frac{v_{yy}}{1+(v_y)^2} - \frac{N-2}{v} + \frac{1}{2}(yv_y - v) \text{ in } K(A), \tag{2.10}$$

and recall that (1.10) holds in $[-\varepsilon,\varepsilon]\times[T-\varepsilon,T)$. Finally, denote $K(\varepsilon)=\{(y,s)\in K(A)$ such that $|y|\le\varepsilon e^{s/2}$ and $s>-\log\varepsilon\}$.

Proposition 2.1: *There exists* $c_i>0$ $(i=1,2,3)$ *such that for all* $(y,s)\in K(\varepsilon)$,

$$\tag{2.11}\begin{cases}(a)\quad \sqrt{2c_0}\le v(y,s)\le c_1(|y|+1),\\ (b)\quad |v_{yy}(y,s)|\le c_2(|y|^{2\gamma}+1),\\ (c)\quad |v_s(y,s)|\le c_3(|y|^{2\gamma}+1).\end{cases}$$

Proof: For $c\ge 0$ set

$$v^*(y,s)=c(|y|+1)$$

and observe that

$$v^*_s-\frac{v^*_{yy}}{1+(v^*_y)^2}-\frac{1}{2}(yv^*_y-v^*)\ge 0 \text{ in } K(A)\backslash\{0\}\times(-\log T,\infty).$$

On the other hand, (2.3) and the definition of v yield

$$v(0,s)\le\sqrt{2(N-2)},$$

hence,

$$v(0,s)\le v^*(0,s) \text{ in } (\log T,+\infty)$$

provided $c\ge\sqrt{2(N-2)}$. Also

$$v(+2Ae^{s/2},s)\le e^{s/2}\|h\|_\infty\le v^*(+2Ae^{s/2},s)$$

if $c\ge\|h\|_\infty/2A$. Applying the maximum principle we get $v\le v^*$ in $K(A)$, and, therefore, (2.11)(a).

For the second derivative estimate we have

$$|v_{yy}(y,s)|=\frac{1}{v(y,s)}((1+h_z^2)|\psi|)(ye^{-s/2},T-e^{-s})\le\frac{1}{c_0v(0,s)}\left(\frac{v(y,s)}{v(0,s)}\right)^{2\gamma},$$

with the inequality following from (1.2) and (2.7).

Hence, by (2.11)(a),

$$|v_{yy}(y,s)| \le \frac{1}{c_0(\sqrt{2c_0})^{2\gamma}}(c_1(|y|+1))^{2\gamma} \le c_2(|y|^{2\gamma}+1),$$

for some C2 > 0.

Finally,

$$= \frac{1}{2} v \quad \left(- yv_y(y,s) - \frac{\psi(ye^{s/2}, T - e^{-s}) - (N-2)}{, \quad) \quad \wedge} \right.$$

We now obtain (2.11)(c) using *v(y,s)* ≥ r(0,s), (1.2), (2.7) and (2.11)(a), **(b).**

●

2,2 A monotonicity formula.

As we will explain in the next subsection, Proposition 2.1 yields the local uniform compactness of *v* as *s* —> oo. To show that the whole family converges, it is sufficient (cf. [GK]) to come up with an "energy-type" functional, which will play the role of a Lyaponov function as *e* —* oo. To this end, following [H] we define

$$E(v(\cdot,s)) = \int^{s)} \rho(y,s)v(y,s)(1+(v_y(y,s))^2)^{\frac{1}{2}}dy,$$

where

$$p(y,s) = \exp(-\wedge(v^2(y,s) + y^2)) \text{ and } R(s) = ee''^2.$$

Proposition 2.2:

$$(2.12) \qquad \frac{}{ds} iw^*;.))) = - \int_{-R(s)} \frac{\rho(y,s)v(y,s)}{[1+(v_y(y,s))^2]^{\frac{1}{2}}} v_s^2(y,s)ds + e(s),$$

and

$$\lim_{s\to\infty} e(s) = 0. \tag{2.13}$$

We will prove (2.12) by an elementary computation using (2.10). The error $e(s)$ is due to the boundary $|y| = \varepsilon e^{s/2}$ we are imposing in the formula for E. Finally, we refer the reader to [H,Section 3] for a more elegant proof (using differential geometric arguments).

Proof: We directly calculate

$$\frac{d}{ds}E(v(\cdot,s)) = A(s) + B(s) + C(s) + e(s),$$

where

$$A(s) = \int_{-R(s)}^{R(s)} \rho_s v(1+v_y^2)^{\frac{1}{2}}dy,$$

$$B(s) = \int_{-R(s)}^{R(s)} \rho v_s(1+v_y^2)^{\frac{1}{2}}dy,$$

$$C(s) = \int_{-R(s)}^{R(s)} \rho v[(1+v_y^2)^{\frac{1}{2}}]v_{sy}dy,$$

and

$$e_1(s) = Ae^{s/2}\{(\rho v(1+v_y^2)^{\frac{1}{2}})(2Ae^{s/2},s) - (\rho v(1+v_y^2)^{\frac{1}{2}})(-2Ae^{s/2},s)\}.$$

Since $\rho(\pm\varepsilon e^{s/2},s) \leq \exp(-\frac{\varepsilon^2}{4}s)$, (2.11) yields

$$\lim_{s\to\infty} e_1(s) = 0. \tag{2.14}$$

Next we will show that

$$A(s) + B(s) + C(s) = -\int_{-R(s)}^{R(s)} \frac{\rho v}{(1+v_y^2)^{\frac{1}{2}}}v_s^2 + e_2(s) \tag{2.15}$$

with $e_2(s) \to 0$ as $s \to \infty$. This will conclude that proof with $e = e_1 + e_2$.

To obtain (2.15) we calculate

$$\begin{aligned} C(s) &= \int \rho v \frac{w}{(1+w^2)^{\frac{1}{2}}} v_{sy} = -\int \Big(\frac{\rho v w}{(1+w^2)^{\frac{1}{2}}}\Big)_y v_s + e_2(s) \\ &= -\int \Big[(\rho v)_y \frac{w}{(1+w^2)^{\frac{1}{2}}} + \frac{\rho v}{(1+w^2)^{\frac{1}{2}}} \frac{v_{yy}}{1+w^2}\Big] v_s + e_2(s) \\ &= -\int \Big\{(\rho v)_y w + \rho v\Big[v_s + \frac{1}{v} + \frac{1}{2}(yw - v)\Big]\Big\} \frac{v_s}{(1+w^2)^{\frac{1}{2}}} + e_2(s), \end{aligned}$$

with

$$e_2(s) = \Big(\frac{\rho v w v_s}{(1+w^2)^{\frac{1}{2}}}\Big)(2Ae^{s/2}, s) - \Big(\frac{\rho v w v_s}{(1+w^2)^{\frac{1}{2}}}\Big)(-2Ae^{s/2}, s).$$

where all the integrals are over the interval $[-R(s), R(s)], w = v_y$ and we used that

$$\rho_y = -\frac{1}{2}(vw + y)\rho, \rho_s = -\frac{1}{2} vw\rho.$$

Now a straightforward computation gives (2.15). The fact that $e_2 \to 0$ as $\varepsilon \to \infty$, follows again from (2.11) and the form of ρ.

□

2.3 Blow-up

In view of (2.7) and (2.11), there exists $s_j \to \infty$ with $s_{j+1} - s_j \geq 2$ such that

$$v_j(y, s) \equiv v(y, s + s_j) \to v_\infty(y, s) \text{ as } j \to \infty,$$

with the limit uniform on compact subsets of $\mathbb{R}^2$ and

$$v_{j,y} \rightharpoonup v_{\infty,y} \text{ in } L^\infty - w^*.$$

By passing to a further subsequence, which we again denote by s_j, we also have that

$$v_{j,y} \to v_{\infty,y} \text{ for almost every } (y, s).$$

If for some $s, v_{j,y}(y,s) \to v_\infty(y,s)$ for almost every y, then (2.11)(b) yields that this convergence is uniform for bounded y's. Hence for every integer k, there is a $n_k \in [k, k+1)$ such that

$$v_{j,y}(y,n_k) \to v_{\infty,y}(y,n_k)$$

uniformly for bounded y for each n_k.

Applying the dominated convergence theorem and using the exponential decay of ρ we get, that, for each n_k,

$$\lim_{j\to\infty} E(v_j(\cdot,n_k)) = E(v_\infty(\cdot,n_k)), \tag{2.16}$$

On the other hand, $v_j(\cdot,n_k) = v(\cdot, s_j + n_k)$. Therefore,

$$E(v_{j+1}(\cdot,n_k)) - E(v_j(\cdot,n_k)) = -\int_{n_k}^{n_k+s_{j+1}-s_j} (\ell(j,s) + e(s+s_j))ds,$$

where

$$\ell(j,s) = \int_{-R(s+s_j)}^{R(s+s_j)} [\frac{\rho v_j}{(1+(v_{j,y})^2)^{\frac{1}{2}}} v_{j,s}^2]dy.$$

Finally, in view of (2.7) and (2.11)(a),

$$|v_{j,y}|(y,s) \le g(y,s) = c(|y|^{2\gamma}+1)$$

for some $c > 0$. Since $v_j > 0$, this yields

$$\ell(j,s) \ge \int_{-R(s+s_j)}^{R(s+s_j)} \frac{\rho v_j}{1+g^2}(v_{j,s})^2 dy.$$

Using the exponential decay of ρ, the uniform convergence of v_j (2.11) and the fact that by construction $s_{j+1} - s_j \ge 2$, we get, for each n_k,

$$\liminf_{j\to\infty} \int_{n_k}^{n_k+2} \ell(j,s) \ge \int_{n_k}^{n_k+2} \int_{-\infty}^{\infty} \frac{\rho v_\infty}{1+g^2}(v_{\infty,s})^2 dy$$

and

$$\lim_{j\to\infty} \int_{n_k}^{n_k+s_{j+1}-s_j} e(s+s_j)ds = 0.$$

Hence

$$\int_{n_k}^{n_k+2}\int_{-\infty}^{\infty}\frac{\rho v_\infty}{1+g^2}(v_{\infty,s})^2dy=0.$$

But, in view of (2.11)(a),

$$v_\infty \geq \sqrt{2c_0};$$

therefore, for almost every y,

$$v_{\infty,s}(y,s)=0 \text{ for almost every } s,$$

which, by (2.11)(c), yields

$$v_\infty(y,s)\equiv v_\infty(y) \text{ for all } y\in I\!R.$$

Passing to the limit in (2.10) we get that v_∞ solves

$$\frac{v_{\infty,yy}}{1+v_{\infty,y}^2}-\frac{N-2}{v_\infty}+\frac{1}{2}(yv_{\infty,y}-v_\infty)=0 \text{ in } I\!R. \tag{2.17}$$

As a matter of fact, the estimates of Proposition 2.1 yield that v_∞ is a classical solution of (2.17) and, moreover,

$$\sqrt{2c_0}\leq v_\infty(y)\leq c_1(|y|+1) \text{ and } |v_{\infty,yy}(y)|\leq c_2(|y|^{2\gamma}+1). \tag{2.18}$$

A direct calculation also shows that if

$$\Psi=v_\infty v_{\infty,yy}(1+v_{\infty,y}^2)^{-1},$$

then

$$\frac{(\Psi^2)_{yy}}{1+(v_{\infty,y})^2}-\frac{2(1+\Psi)}{1+(v_{\infty,y})^2v_\infty}(\Psi^2)_y+\frac{4(v_{\infty,y})^2}{(v_\infty)^2(1+(v_{\infty,y})^2)}\Psi(\Psi-(N-2))(\Psi+1)$$

$$-\frac{1}{2}y(\Psi^2)_y=\frac{2(\psi_y)^2}{1+(v_{\infty,y})^2}.$$

In view of (1.3), if Ψ^2 has an interior maximum at y_0, then

$$-1\leq\Psi(y_0)\leq 0.$$

We now claim that \$ <_0 in *JR*. Indeed suppose that there exists y* G JR such that \P(j/*) > 0. Without any loss of generality we may assume that *ty$_y$(y*)* > 0, since, else we consider the point —*y*~* But then

«(y) ≥ *(y*) > 0 in (y*,oo),

since, in view of the discussion above, \t cannot have a positive interior maximum. This, however, contradicts (2.18). Hence V? ≤ 0 in JR. Hence Voo is concave and, in view of (2.18), constant. Using (2.17) we see that the only constant solution is $\sqrt{2(N-2)}$. Since any limit of r(y, *s* + *Sj)* is equal to $\sqrt{2(N-2)}$, we have concluded the proof of Theorem 1.

Corollary 2.3: *For* ***any*** *e* > **0** *there exists 6* > 0 *satisfying*

(2.19) $\lim_{t\to T}$ fc(M) ≤ *e*\z\ for 2 € [-£,<\$].

Note that in view of (1.3) the above limit exists for sufficiently small \z\.

Proof: Suppose that for a given *e* there is a sequence z_n —* 0 such that

(2.20) $\lim_{\to T} h(z_n,t) > e|z_n|$.

If

$$s_n = 2\log\left(\frac{2(N-2)}{\varepsilon|z_n|}\right),$$

then

$$v\left(\frac{2(N-2)z_n}{\varepsilon|z_n|}, s_n\right) = e^{\frac{s_n}{2}} h\left(\frac{2(N-2)z_n}{\varepsilon|z_n|}e^{-\frac{s_n}{2}}, T - e^{s_n}\right)$$

$$= \frac{2(N-2)}{\varepsilon|z_n|} h(z_n, T - e^{-s_n}).$$

Since z_n —* 0, s_n —» oo and, by (2.20),

$$\liminf_{n\to\infty} v\left(\frac{2(N-2)z_n}{e\backslash{*}{*}\backslash}, {*}n\right) \geq 2(N-2) > \sqrt{2(N-2)},$$

which contradicts Theorem 1.

$\square$

Corollary 2.4: *As* $t \to T, h(\cdot,t)$ *converges, uniformly in* $[-2A,2A]$, *to* $h(\cdot,T)$, *which is smooth for* $z \neq 0$ *and differentiable at* $z = 0$ *and* $h(\cdot,T) > 0$ *for all* $y \neq 0$.

Proof. In view of (1.3),

$$h_t(z,t) \leq 0 \text{ for } |z| \leq \varepsilon \text{ and } T - t \leq \varepsilon.$$

Hence $h(z,t)$ converges to a limit which we call $h(z,T)$ for $|z| \leq \varepsilon$. On the other hand, (0.6) implies that $h(\cdot,T)$ is nondecreasing on $[0,\varepsilon)$ and nonincreasing on $(-\varepsilon,0]$.

Suppose now that $h(2z_0,T) = 0$ for some $2z_0 \in (0,\varepsilon)$. Then $h(z,T) \equiv 0$ for $z \in [0,2z_0]$ and by Dini's Theorem $h(z,t) \to 0$ uniformly on $[0,2z_0]$.

Set

$$\Phi(z,t) = h(z+z_0,t) - h(z,t) \ , \ \ (z,t) \in O,$$

where

$$O = (0,z_0) \times (T-\varepsilon,T).$$

It is immediate that Φ satisfies

$$\Phi_t = a\Phi_{zz} + b\Phi_z + c\Phi \text{ in } O, \tag{2.21}$$

where

$$\begin{aligned} a(z,t) &= (1 + (h_z(z,t))^2)^{-1}, \\ c(z,t) &= (h(z+z_0,t)h(z,t))^{-1}, \\ b(z,t) &= h_{zz}(z+z_0,t)[h_z(z+z_0,t)h_z(z,t)]a(z+z_0,t)a(z,t). \end{aligned}$$

But $0 \leq a \leq 1$ and $|b(z,t)| \leq |h_{zz}(z+z_0,t)| \leq h^{-1}(z+z_0,t)|\psi|(z+z_0,t)(h(z+z_0,t)h^{-1}(0,t))^{2\gamma}$. Using (1.3), we conclude that, for $(z,t) \in O$,

$$(2.22) \qquad \begin{cases} (a) \quad |b(z,t)| \leq c(T-t)^{-\frac{1}{2}} \\ (b) \quad |c(z,t)| \leq c(T-t)^{-1}. \end{cases}$$

Moreover, again using (1.3), we see that, for some appropriate constant K,

$$(2c_0(T-t))^{\frac{1}{2}} \leq -h_t \leq (K(T-t))^{-\frac{1}{2}} \text{ in } [-\varepsilon,\varepsilon] \times [T-\varepsilon,t).$$

Integrating we obtain

$$(2.23) \qquad 0 \leq \Phi(z,t) \leq K\sqrt{T-t}, \text{ in } 0$$

For $\mu > 0$ consider the auxiliary function

$$\hat{\Phi}(z,t) = \Phi(z,t) + \frac{\mu}{T-t} + \frac{\mu}{2}(z-z_0)^2.$$

A direct calculation yield

$$\hat{\Phi}_t - a\hat{\Phi}_{yy} - b\hat{\Phi}_y = c\Phi + \mu\{\frac{1}{(T-t)^2} - a - b(z-z_0)\}.$$

Using (2.22) and (2.23), we conclude that, there is $t_o < T$ such that for all $\mu \geq 0$,

$$\hat{\Phi}_t - a\hat{\Phi}_{yy} - b\hat{\Phi}_y \geq 0, (z,t) \in (0,z_0) \times (t_0,T).$$

Moreover, for all sufficiently small μ there is $\hat{t} < T$ such that

$$\inf_{t\in[\hat{t},T)} \hat{\Phi}(z_0,t) < \inf\left\{\hat{\Phi}(z,t) : (z,t) \in \{0,2z_0\} \times [\hat{t},T) \cup [0,2z_0] \times \{\hat{t}\}\right\}.$$

Hence $\hat{\Phi}$ has an interior minimum, which contradicts with the fact that $\hat{\Phi}$ is a supersolution to a linear equation.

In summary we have shown that

$$h(z,T) > 0 \text{ in } (0,\varepsilon).$$

Similarly we show that

$$h(z,T) > 0, \text{ in } [-\varepsilon, 0).$$

Using (0.6), we

$$\inf\{h(z,t) \,:\, |z| \geq \varepsilon, t \in [0,T)\} > 0.$$

Finally, equation (0.4) for $|z| > \varepsilon$, we can easily show that $h(z,t)$ has a limit as $t \longrightarrow T$.

□

2.4 No interior at the focusing point

We conclude this long section with a brief discussion, without any proofs, of why the focusing point 0 cannot be in the interior of $\Gamma_{T+\rho}$ for $\rho > 0$ and very small. This will be a consequence of (1.3).

To this end, we consider the solution u of (0.2) with initial datum g, such that $\{g = 0\} = \{r = h_0(z)\}$. It follows (cf. [ES]) that, for $t \leq T$,

$$\Gamma_t = \{u(\cdot,t) = 0\} = \{r = h(z,t)\},$$

where here, as usual, we denote by h the solutions of (0.4) which correspond to the different branches of h_0. Assume now that $0 \in I\!R^N$ belongs to the interior of $\Gamma_{T+\rho}$ for $\rho > 0$. This implies that there exist $R(\rho) > 0$ such that $B(0, R(\rho)) \subset \Gamma_{T+\rho}$. On the other hand, (1.3) yields that we can bound the part of h which focuses by catenoids as close to $(0,T)$ as we want. Recall that catenoids are stationary solutions to (0.2). Finally, we recall that the distance between two surfaces which move by mean curvature is a nondecreasing function in time (cf. [ES]).

To conclude this heuristic discussion we argue as follows. If $0 \in \Gamma_{T+\rho}$, choose $\varepsilon > 0$ so small that $h(0, T-\varepsilon) = \frac{1}{2}R(\rho)$ and bound h by the catenoid passing through $(\frac{1}{2}R(\rho), 0)$. In view of the above discussion, the set

$\{u(\cdot, T+\rho) = 0\}$ cannot touch the catenoid which contradicts the choice of ε. This argument can be made rigorous at the expense of technical arguments. We choose, therefore, to omit the details.

3 No interior - Motion after the focusing

Our goal here is to show that, under certain assumtions, if Γ_t "focuses at $(0,T)$, then, for $t > T, \Gamma_t$: (i) does not develop interior and (ii) "opens" up.

We begin with the definition of non-interior.

Definition 3.1: Γ_t has no interior iff $\partial\{u(\cdot,t) \geq 0\} = \partial\{u(\cdot,t) > 0\}$, where u is the solution of (0.2).

As mentioned in the Introduction, in general, Γ_t will develop interior, (see for example: Soner [S]). On the other hand, [BSS] gives a fairly general geometric condition on Γ_0, which yields no interior. We next state this result of [BSS] as it applies to the case of cylindrically symmetric surfaces moving by mean curvature.

Theorem 3.2 ([BSS]): *Assume that Γ_0 is C^2 surface and that there exists a constant C such that*

$$x \cdot Dd + C\Delta d \neq 0 \text{ on } \Gamma_0, \tag{3.1}$$

where d is the signed distance to Γ_0. Then Γ_t has empty interior in $(0, +\infty)$.

Condition (3.1) has a geometric meaning, since the left hand side is the generator of dilations and translations in (x,t) evaluated at $t = 0$ on Γ_0.

If Γ_0 is smooth, then Γ_t is smooth for $t \in [0,t_1)(t_1 > 0)$ and, therefore, has empty interior in $(0,t_1)$. As remarked in [BSS], if the solution u of (0.2) which defines Γ_t satisfies, for some $t_0 \in (0,t_1)$,

$$\frac{x \cdot Du + Cu_t}{|Du|} \neq 0 \text{ on } \Gamma_{t_0} \tag{3.2}$$

then Γ_t has empty interior in $(0, \infty)$. Indeed let $\bar{d}(\cdot,t)$ be the signed distance to Γ_t. Then at t_0

$$\bar{d}_t = u_t/|Du|, \quad D\bar{d} = Du/|Du|.$$

Also

$$\bar{d}_t = \Delta\bar{d}/(N-1),$$

since Γ_t is a classical solution of the mean curvature flow and $\Delta\bar{d}$ is equal to $(N-1)$ times the mean curvature.

Since in this paper we assume that Γ_0 is smooth, the main goal in this section will be to show that, under some additional assumptions on Γ_0, (3.2) holds near the focusing time T, although it may not hold at $t = 0$.

Throughout the discussion below we will need to go back and forth to parametrizing Γ_t, for $t \in (0, T)$, in terms of both z and r. More precisely, we will need the existence of positive numbers $z(t), r_i(t)$ $(i = 1,2,3)$ with $t \in [0,T)$ and smooth functions $h,H : [-z(t),z(t)] \to [0,\infty)$ and $g : [r_1(t),r_2(t)] \to [0,\infty)$ such that:

$$r_1(t) = h(0,t), r_2(t) = H(0,t) \quad \text{and} \quad z(t) = g(r_3(t),t), \tag{3.3}$$

$$\begin{aligned} \Gamma_t &= \{r = h(z,t) : |z| \le z(t)\} \cup \{r = H(z,t) : |z| \le z(t)\} \\ &= \{|z| = g(r,t) : r \in [r_1(t),r_2(t)]\}, \end{aligned} \tag{3.4}$$

$$h_t = \frac{h_{zz}}{1+h_z^2} - \frac{N-2}{h} \quad \text{and} \quad H_t = \frac{H_{zz}}{1+H_z^2} - \frac{N-2}{H} \quad \text{in } [-z(t),z(t)], \tag{3.5}$$

$$g_t = \frac{g_{rr}}{1+g_r^2} + \frac{(N-2)g_r}{r} \quad \text{in } [r_1(t),r_3(t)], \tag{3.6}$$

$$\begin{cases} (i) & g(h(z,t),t) = g(H(z,t),t) = z \quad \text{in } [-z(t),z(t)], \\ (ii) & h(g(r,t),t) = r \quad \text{in } [r_1(t),r_2(t)], \\ (iii) & H(g(r,t),t) = r \quad \text{in } [r_2(t),r_3(t)], \end{cases} \tag{3.7}$$

and

$$(3.8) \qquad \begin{cases} (r - r_2(t))g_r(r,t) < 0 \text{ in } [r_1(t), r_2(t)] \\ \text{and} \\ g_r(r_2(t),t) = 0. \end{cases}$$

It is immediate that if (3.3)-(3.8) hold, then

$$(3.9) \qquad \begin{cases} g_r = h_z^{-1} = H_z^{-1}, g_t = -h_t h_z^{-1} = -H_t H_z^{-1} \\ \text{and} \\ g_{rr} = -h_{zz} h_z^{-3} = -H_{zz} H_z^{-3}. \end{cases}$$

The existence of such $z(t), r_i(t)(i = 1,2,3)$ and h, H and g with the above properties follows from the next proposition.

Proposition 3.2: *Assume that there exist positive numbers* $z_0, r_{0,i}(i = 1,2,3)$ *and smooth functions* $h_0, H_0 : [-z_0, z_0] \to [0,\infty)$ *and* $g_0 : [r_{0,1}, r_{0,3}] \to [0,+\infty)$ *such that (3.3), (3.4), (3.7) and (3.8) hold at* Γ_0. *Then there exist smooth* $z, r_i : [0,T) \to (0,+\infty)(i = 1,2,3)$ *and* $h(\cdot,t), H(\cdot,t) : [-z(t), z(t)] \to (0,\infty), g : [r_1(t), r_3(t)] \to [0,+\infty)(t \in [0,T))$ *satisfying (3.3)-(3.8) for all* $t \in (0,T)$.

Proof: Consider the solution u of (0.2) with initial data u_0 satisfying

$$u_0(x) = g_0(r) - |z| \qquad (r \in [r_{0,1}, r_{0,3}]),$$

where, for $x = (x_1, \ldots, x_n) \in I\!R^N, z = x_N$ and $r^2 = \sum_{i=1}^{N-1} x_i^2$. Since u_0 is smooth, (0.2) has a, local in time, smooth solution. The resulting smooth Γ_t can be parametrized as in (3.4) where h, H and g solve (3.5) and (3.6) with initial data h_0, H_0 and g_0 respectively. Moreover, u satisfies

$$(3.10) \qquad u(x,t) = g(r,t) - |z| \ (r \in [r_1(t), r_3(t)]).$$

On the other hand, (3.5) admits a smooth solution as long as the solution stays positive. This yields that u is smooth as long as Γ_t does not focus i.e. for $t \in (0,T)$. Finally (3.8) follows from analyzing the properties of the number of zeroe's of g_r as in the Appendix.

□

Next we use (3.10) to write the expresion in (3.2) as

$$\frac{x \cdot Du + Cu_t}{|Du|} = \frac{rg_r - g + Cg_t}{\sqrt{1+g_r^2}} \quad \text{on } \Gamma_t. \tag{3.11}$$

The first important result in this section is:

Proposition 3.3: *Assume that* $\#\{r \in [r_{01}, r_{03}] : rg_{0r} - g_0 - 2Tg_{0t} = 0\} \leq 2$. *Then there exists* $t_0 \in [0,T)$ *and* $B > 0$ *such that*

$$rg_r - g - 2Bg_t < 0 \qquad \text{in } (r_1(t_0), r_3(t_0)) \times \{t_0\}. \tag{3.12}$$

In particular Γ_t *has no interior for all* $t \geq 0$.

Before we begin with some preliminaries which will lead to the proof of Proposition 3.3, let us first comment on why (3.12) seems reasonable to hold. Indeed that analysis in Section 2 yields that $h_t(0,t) \to -\infty$ as $t \to T$, which, in turn, suggests, by (3.9), that $g_t(r_0(t),t) \to +\infty$ as $t \to T$. This would yield (3.12), provided one is able to control, away from the singularity, the term $rg_r - g$. Keeping this in mind, we define

$$K : \cup_{t\in[0,T)}((r_1(t), r_3(t)) \times \{t\}) \to \mathbb{R}$$

by

$$K(r,t) = rg_r(r,t) - g(r,t) - 2(T-t)g_t(r,t). \tag{3.13}$$

Using (3.6) we obtain

$$K_t = \mathcal{L}(K) \text{ in } \cup_{t\in[0,T)}((r_1(t), r_3(t)) \times \{t\}), \tag{3.14}$$

where

$$\mathcal{L}\psi(r) = \frac{1}{1+g_r^2}\psi_{rr} - \frac{2g_{rr}g_r}{(1+g_r^2)^2}\psi_r + \frac{N-2}{r}\psi_r.$$

We will also need to define the functions
$I, J : \cup_{t\in[0,T)}((-z(t), z(t)) \times \{t\}) \to \mathbb{R}$ by

$$(3.15)\qquad \begin{cases} I(z,t) = h(z,t) - zh_z(z,t) + 2(T-t)h_t(z,t), \\ J(z,t) = H(z,t) - zH_z(z,t) + 2(T-t)H_t(z,t). \end{cases}$$

It follows from (3.7) and (3.9) that

$$I(g(r,t),t) = g_r^{-1}(r,t)K(r,t) \qquad (r \in [r_1(t), r_2(t)))$$

and

$$J(g(r,t),t) = g_r^{-1}(r,t)K(r,t) \qquad (r \in (r_2(t), r_3(t)]).$$

Hence, (3.8) yields

$$(3.16)\qquad \begin{cases} \operatorname{sign}(K(r,t)) = \operatorname{sign}(I(g(r,t),t)) & (r \in (r_1(t), r_2(t))) \\ \operatorname{sign}(K(r,t)) = -\operatorname{sign}(J(g(r,t),t)) & (r \in (r_2(t), r_3(t))). \end{cases}$$

Finally, another direct computation gives

$$(3.17)\qquad I_t = \tilde{\mathcal{L}}(h, I) \text{ and } J_t = \tilde{\mathcal{L}}(H, J) \text{ in } \cup_{t\in[0,T)}((-z(t), z(t)) \times \{t\}),$$

where

$$\tilde{\mathcal{L}}(f,\phi) = \frac{1}{1+f_z^2}\phi_{zz} - \frac{2f_z f_{zz}}{(1+f_z^2)^2}\phi_z + \frac{(N-2)}{f^2}\phi \text{ in } \cup_{t\in[0,T)}(-z(t), z(t)) \times \{t\}.$$

To state the next result we define $n : [0,T) \to \mathbb{Z}^+$ by

$$\begin{aligned} n(t) \ &= \#\{r \in (r_1(t)+\delta, r_3(t)-\delta) \ : \ K(r,t) = 0\} \\ &+\#\{|z| \le g(r_1(t)+\delta) \ : \ I(z,t) = 0\} \\ &+\#\{|z| \le g(r_3(t)-\delta) \ : \ J(z,t) = 0\}, \end{aligned}$$

where $0 < \delta < \min\{r_3(t) - r_2(t), r_2(t) - r_1(t)\}$ is arbitrary. In view of (3.16) the above definition is independent of δ.

Lemma 3.4: *Assume* n(0)) < oo. *Then n(t)* ≤ **n(0)** *and t* »+ *n(t) is* ***nonincreasing in [0,T).***

Lemma 3.4 follows by applying the lemma in the Appendix about the number of zeroe's of solutions to linear parabolic equations in one dimension. Of course, special care has to be taken for the fact that, in principle, the boundary U$_{t€}$[o,r)({ri(<),r$_2$(i)} ^{x} {*}) °f ^^{e} domain where *K* satisfies (3.14) may generate new zeroe's. This difficulty, however, may be overcome, in view of (3.16), by applying the aforementioned lemma to *K,I* and *J* and the equations they satisfy. The proof is long but rather standard, we, therefore, omit it.

Next we will extend the statement of Lemma 3.4 up to T. This is not immediate, since the coefficients of *C* and $\tilde{C}$ are no longer bounded at *t* = *T*. But Corollary 2.4 asserts that *h* and, therefore, by (3.7) and Proposition 3.2, *H* and *g* are defined in a continuous way up to *t* = *T*. It follows that *K, I* and *J* can be extended up to *t* = *T* away from their respective singularities. Finally, Proposition 2.1 and Corollary 2.4 also yield that

/(*, T) = h(z, T) - zh$_z$(z, T)   (z € (-z(T), *{T})))

and

$$I(0,T) = 0.$$

Moreover / is continuous on its domain possibly except at (0, *T).*

The next result asserts that *z* t-» *I(z,T)* is negative in a neighborhood of **2 = 0.**

Lemma 3.5: ***Ifn(0)* < oo, *then there exists e* > 0 *such that***

I(z,T)<0in [-e,e]\{0}.

Proof: If 7(-,T) ≥ 0 in *(z$_u$z$_2$)* G *(0,z(T)),* then

$$\frac{\partial}{\partial z}(\frac{h(z,T)}{z}) = -\frac{1}{z^2}I(z,T) \leq 0.$$

Hence

$$\frac{h(z_2,T)}{z_2} \le \frac{h(z_1,T)}{z_1}. \tag{3.18}$$

Let

$$z^* = \begin{cases} \inf\{z \in (0,z(T)) : I(z,T) = 0\} \\ z(T)\,, \text{if } I(\cdot,T) \neq 0 \text{ in } (0,z(T)). \end{cases}$$

Since $n(0) < \infty$, Lemma 3.4 yields $z^* > 0$. Hence $I(\cdot,T)$ is either negative or positive in $(0,z^*)$. If the latter holds, then

$$\frac{h(z^*,T)}{z^*} \le \frac{h(z,T)}{z} \text{ for all } z \in (0,z^*)$$

and, since $h_z(0,T) = 0$,

$$h(z^*,T) = 0$$

which contradicts the positivity of $h(\cdot,T)$ in $(0,z(T))$.

□

Lemma 3.6: *If $n(0) < \infty$, then*

$$n(T) = \#\{z \ge 0 : I(z,T) = 0 \text{ or } J(z,T) = 0\} \le n(0) \tag{3.19}$$

and

$$n^* = \#\{r > 0 : K(r,T) = 0\} \le n(0) - 1. \tag{3.20}$$

Proof: In view of (3.16) and $I(0,T) = 0$, (3.19) and (3.20) are equivalent. To prove (3.20), we first claim that, for sufficiently small $\gamma > 0$,

$$n_\gamma(T) \le \varliminf_{t \uparrow T} n_\gamma(t), \tag{3.21}$$

where, for $t \in [0,T]$,

$$n_\gamma(t) = \#\{r \in (h(\gamma,t), r_3(t)) : K(r,t) = 0\}.$$

To this end, choose $\gamma > 0$ sufficiently small so that

$$I(\gamma, T) < 0.$$

Then, by (3.16), $K(h(\gamma,t),t) < 0$ for t near T, hence (3.21) follows from (3.14) and an application of Lemma A. As in the proof of Lemma 3.4, again we need to apply Lemma A to K, I and J.

We will conclude by showing

$$\lim_{t\uparrow T} n_\gamma(t) \le \lim_{t\uparrow T} n(t) - 1. \tag{3.22}$$

Since $n_\gamma(t) \le n(t)$, if

$$\lim_{t\uparrow T} n_\gamma(t) = \lim_{t\uparrow T} n(t),$$

there must exist $\alpha > 0$ such that

$$I < 0 \text{ in } [0,\alpha] \times [T-\alpha, T) \cup (0,\alpha] \times \{T\} \text{ and } I(0,T) = 0.$$

Hence, I has an interior nonpositive maximum, which contradicts the maximum as it applies to the equation (3.17) satisfied by I.

$\square$

Lemma 3.7: *If* $n^* > 0$, *then* $n^* \ge 2$. *In particular, if* $n(0) \le 2$, *then* $K(r,T) < 0$ *for all* $r > 0$.

Proof: Since $J(0,T) = H(0,T) > 0$, there exists $\bar{\varepsilon} > 0$ such that $J(z,T) > 0$ in $[-\bar{\varepsilon}, \bar{\varepsilon}]$. Combining this with Lemma 3.5 and (3.16) we get

$$K(r,T) < 0 \text{ for } r \in (0,\varepsilon] \cup [r_3(T) - \varepsilon, r_3(T)].$$

Hence, if $K(r,T)$ has any zeroe's for $r > 0$, there have to be at least two.

Finally, (3.20) and $n(0) \le 2$ yield $n^* \le 1$, which in view of the previous discussion implies that $n^* = 0$.

$\square$

We are now in a position to prove Proposition 3.3.

Proof: In view of (1.3) and Lemma 3.5, there exists $R_1 \in (0, r_2(T))$ such that, for all $C > 0$,

$$\overline{\lim_{t\uparrow T}} \sup_{|z|\le g(R_1,T)} (I(z,t) + 2Ch_t(z,t)) < 0$$

and, therefore,

$$\overline{\lim_{t\uparrow T}} \sup_{r\in(r(t),R_1)} [K(r,t) - 2Cg_t(r,t)] < 0. \tag{3.23}$$

Next, choose $R_2 \in (r_2(T), r_3(T))$ and set

$$k = \sup_{r\in[R_1,R_2]} K(r,T) < 0.$$

Hence, for any $C < k \sup\limits_{r\in[R_1,R_2]} |g_t(r,T)|^{-1}$,

$$\overline{\lim_{t\uparrow T}} \sup_{r\in[R_1,R_2]} [K(r,t) - 2Cg_t(r,t)] < 0. \tag{3.24}$$

Also

$$J(z,T) = H(z,T) - zH_z(z,T) > 0 \text{ in } [-z(T), z(T)].$$

In fact

$$k_1 = \inf_{|z|\le z(T)} J(z,T) > 0$$

and

$$k_2 = \sup_{|z|\le g(R_2,T)} |H_t(z,T)| < \infty.$$

Hence, if $C < k_1/k_2$,

$$\underline{\lim}_{t\uparrow T} \inf_{|z|\le g(R_2,T)} [J(z,t) + 2CH_t(z,t)] > 0,$$

and, consequently,

$$\overline{\lim_{t\uparrow T}} \sup_{r\in(R_2,r_3(T))} [K(r,t) - 2Cg_t(r,t)] < 0. \tag{3.25}$$

Combining (3.23), (3.24) and (3.25) we obtain (3.12).

$\square$

In view of the discussion at this beginning of this section, Proposition 3.3 yields

$$\frac{-x \cdot Du + \overline{B}u_t}{|Du|} > 0 \text{ on } \Gamma_{t_0}$$

with $\overline{B} = 2[(T - t_0) + C]$. Since u is a smooth function, there exists a $K > 0$ and $\gamma > 0$ such that

$$\text{(3.26)} \qquad -x \cdot Du + \overline{B}u_t + K|u| \geq \gamma \text{ on } I\!R^N \times \{t_0\}.$$

The maximum principle and the properties of equation (0.2) then yield

$$\text{(3.27)} \qquad -x \cdot Du + 2[(T - t) + C]u_t + K|u| \geq \gamma \text{ on } I\!R^N \times [t_0, \infty).$$

The last inequality may be rewritten as

$$\text{(3.28)} \qquad \frac{d}{ds}\{u(x(s), t(s)) \exp[K(sgnu(x(s), t(s)))s]\} \geq \gamma$$

where, for every (x, t) and $s \geq 0$,

$$\text{(3.29)} \qquad \begin{cases} x(s) = xe^{-s} \\ t(s) = T + C - (T - t + C)e^{-2s} \end{cases}$$

Although (3.28) was derived under the assumption that u is smooth, it follows easily that it holds in the viscosity sense for all s as long as u exists. But then, for $x = 0$ and $t = T$, (3.29) yield $x(s) \equiv 0, t(s) = T + C - Ce^{-2s}$. Hence by (3.28), for $\varepsilon > 0$,

$$u(0, T + \varepsilon) > 0$$

i.e. Γ_T "opens up". (Recall that $u(0, t) < 0$ for all $t \in [0, T)$ and $u(0, T) = 0$.)

Finally, in order to show that Γ_t is smooth after the singularity, it suffices to show that the equation

$$G_t = \frac{G_{rr}}{1 + G_r^2} + \frac{(N - 2)G_r}{r}$$

admits a smooth solution for $t > 0$, even if G(-,0) has a singularity like the one of $g(\cdot,T)$. This can be shown by a number of approximations using standard parabolic regularity and the stability properties of surfaces moving by mean curvature (cf. [S]). As a matter of fact, such arguments can be used from the beginning to show that I\ never develops interior. This is the approach of [AAG] for "barbell" type domains. In this paper, however, we chose to follow the approach described above since it gives rise to (3.27), which has a very nice geometric interpretation.

We now combine all the above to state the main result of the section.

Theorem 3.8: *Suppose that the assumptions of Proposition 3.2 hold and the* n(0) $\leq$ 2. *Then the evolution t* H+ T_t *never develops interior. Moreover, it "opens up" instanteneously after the focusing time and continuous moving as a smooth surface.*

We conclude this section by checking that a torus

$$(r-l)^2 + z^2 = R^2 \qquad (0 < /?< 1),$$

whose evolution t »-* T_t by mean curvature focuses at (0, T), satisfies n(0) $\leq$ 2. This will yield Proposition 3 in the Introduction.

A simple calculation yields

$$K(r,0) = \frac{}{rg_0^{?}(r)} W r^2 + r(R^2 - 1 - 2T(N-1)) + T(N-1)],$$

where

$$g_0(r) = [R^2 - (r-1)^2]^{\frac{1}{2}}.$$

The above claim is then obvious.

References

[AAG1] S.J. Altschuler, S.B. Angenent, and Y. Giga, Generalized motion by mean curvature for surfaces of rotation, Advanced Studies in Pure Mathematics, to appear.

[AAG2] S.J. Altschuler, S.B. Angenent, and Y. Giga, Mean curvature flow through singularities for surfaces of rotation, Hokkaido Un. Preprint Series #130, December 1991.

[A1] S.B. Angenent, The zero set of a solution of a parabolic equation, J. Reine Ang. Math., **390** (1988), 79-96.

[A2] S.B. Angenent, Parabolic equations for curves on surfaces - part 2, Annals Math., **133** (1991), 171-215.

[BSS] G. Barles, H.M. Soner, and P.E. Souganidis, Front propagation and phase field theory, SIAM J. Cont. Optim., to appear.

[B] K.A. Brakke, The motion of a surface by its mean curvature, Math. Notes, Princeton, New Jersey, Princeton Univ, Press (1978).

[CGG] Y.-G. Chen, Y. Giga, and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Diff. Geom., **33** (1991), 749-786.

[CIL] M.G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, preprint.

[DG] E. DeGiorgi, Some conjectures on flow by mean curvature, preprint.

[DK] G. Dziuk and B. Kawohl, On rotationally symmetric mean curvature flow, J. Diff. Eqns. **93** (1991), 142-149.

Appendix.

In this Appendix we state a result of Angenent for the convenience of the reader. The statement of this lemma is taken from [A2]. However its proof is in [A1].

Lemma A. *Let* $u : [x_0, x_1] \times (0, t_0) \to R$ *be a continuous classical solution of*

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u$$

with $u(x_0, t) \neq 0, u(x_1, t) \neq 0$ *for all* $t \in (0, t_0)$.

Assume that a, b, c *satisfy*

- *(i)* $\delta \leq a(x,t) \leq \delta^{-1}$ *for some* $\delta > 0$,
- *(ii)* $a, a_t, a_x, a_{xx}, b, b_t, b_x$ *and* c *are bounded measurable functions of* $[x_0, x_1] \times (0, t_0)$.

Then the number of zerocs of $u(\cdot, t)$

$$z(t) = \#\{x \in (x_0, x_1) | u(x,t) = 0\}$$

is finite and non-increasing in t.

[ES] L.C. Evans, and J. Spruck, Motion of level sets by mean curvature I, J. of Diff. Geom. **33** (1991), 635-681, II, Trans. AMS, to appear, III, preprint.

[GK] Y. Giga, and R.V. Kohn, Asymptotically self-similar blow-up of semilinear heat equation, Comm. Pure and Appl. Math., **38** (1985), 297-319.

[H] G. Huisken, Asymptotic behaviour for singularities of the mean curvature flow, J. Diff. Geom., **31** (1991), 285-299.

[I] T. Ilmanen, Generalized flow of sets by mean curvature on a manifold, preprint.

[PV] Paolini and Verdi, Asymptotic and numerical analysis of the mean curvature flow with a space-dependent relaxation parameter, preprint.

[S] H.M. Soner, Motion of a set by the curvature of its boundary, J. Diff. Eqns., to appear.

[SS] H.M. Soner and P.E. Souganidis, Uniqueness and singularities of cylindrically symmetric surfaces moving by mean curvature, LCDS preprint, April 1991.