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H-Measures Applied to Symmetric Systems

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Abstract

H-measures were recently introduced by Tartar [Thmo] as a tool that might provide much better understanding of propagating oscillations.

Partial differential equations of mathematical physics can (almost always) be written in the form of a symmetric system:

$$\sum_{k=1}^{n} \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \ ,$$

where $\mathbf{A}^k$ and $\mathbf{B}$ are matrix functions, while u is a vector unknown function, and f a known vector function.

In this work we prove a general propagation theorem for H-measures associated to symmetric systems (theorem 3). This result, combined with the localisation property ([Thmo]) is then used to obtain more precise results on the behaviour of H-measures associated to the wave equation and Maxwell's system.

Particular attention is paid to the equations that change type: Tricomi's equation and variants. The H-measure is not supported in the elliptic region; it moves along the characteristics in the hyperbolic region, and bounces of the parabolic boundary, which separates the hyperbolic region from the elliptic region.

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1. Introduction

H-measures

In the study of continuum physics, the equations governing the behaviour of continuous media can be divided into two classes: *balance relations* and *constitutive assumptions.*

While the Young measures were good for the study of *oscillation* effects, they proved inappropriate for the study of *concentration* effects. In a way, as a measure depending on the variable $\mathbf{x}$ only, Young measures were not well suited to describe any effect that depends on a particular direction in space.

The H-measure is a Radon measure on the spherical bundle over the domain Ω in consideration (in general, the base space of the fibre bundle is a manifold Ω, while the fibre is, of course, the unit sphere S^{n-1}). For a single parametrisation (suppose $\Omega \subseteq \mathbf{R}^n$ is an open domain) it is a measure on the product $\Omega \times S^{n-1}$. In order to apply Fourier transform, functions defined on the whole of $\mathbf{R}^n$ should be considered and this can be achieved by extending them by zero outside the domain. After such adjustment, the following theorem can be stated (for details see [Thmo]):

Theorem 1. **(existence of H-measures)** *If* (u^ε) *is a sequence in* $\mathrm{L}^2(\mathbf{R}^n;\mathbf{R}^p)$, *such that* $\mathsf{u}^\varepsilon \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there exists a subsequence* $(\mathsf{u}^{\varepsilon'})$ *and a complex matrix Radon measure* $\boldsymbol{\mu}$ *on* $\mathbf{R}^n \times \mathrm{S}^{n-1}$ *such that for all* $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^n)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{n-1})$*:*

$$
\begin{aligned}
\lim_{\varepsilon' \searrow 0} \int_{\mathbf{R}^n} \mathcal{F}\left(\varphi_1 \mathsf{u}^{\varepsilon'}\right) \otimes \mathcal{F}\left(\varphi_2 \mathsf{u}^{\varepsilon'}\right) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} &= \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \psi \rangle \\
&= \int_{\mathbf{R}^n \times \mathrm{S}^{n-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi}) \, .
\end{aligned}
\tag{1}
$$

∎

The Fourier transform used above is defined in the following way:

$$
\hat{\mathsf{u}}(\boldsymbol{\xi}) := \mathcal{F}\mathsf{u}(\boldsymbol{\xi}) := \int_{\mathbf{R}^n} e^{-2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} \mathsf{u}(\mathbf{x})\, dx \, ,
$$

while its inverse is:

$$
\check{\mathsf{v}}(\mathbf{x}) := \bar{\mathcal{F}}\mathsf{v}(\mathbf{x}) := \int_{\mathbf{R}^n} e^{2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} \mathsf{v}(\boldsymbol{\xi})\, d\boldsymbol{\xi} \, .
$$

The derivative of a Fourier transform and the Fourier transform of a derivative satisfy:

$$
\partial^j(\mathcal{F}\mathsf{u})(\boldsymbol{\xi}) = -2\pi i \mathcal{F}(x^j \mathsf{u}(\mathbf{x}))(\boldsymbol{\xi}) \qquad \mathcal{F}(\partial_j \mathsf{u})(\boldsymbol{\xi}) = 2\pi i \xi_j \mathcal{F}\mathsf{u}(\boldsymbol{\xi}) \, .
$$

Remark. The notation $\mathsf{a} \otimes \mathsf{b}$ denotes the complex tensor product of two vectors. It is defined as a linear operator, acting on a vector v by: $(\mathsf{a} \otimes \mathsf{b})\mathsf{v} := (\mathsf{v} \cdot \mathsf{b})\mathsf{a}$, where $\mathsf{v} \cdot \mathsf{b} = \sum_{i=1}^n v_i \bar{b^i}$ is the complex scalar product.

Variables in Ω are denoted by $\mathbf{x} = (x^1, \ldots, x^n)$ (or $\mathbf{x} = (x^0, x^1, \ldots, x^n)$ when this is more convenient), and $\partial_k = \frac{\partial}{\partial x^k}$. Similarly for the *dual* variable $\boldsymbol{\xi} = (\xi^1, \ldots, \xi^n)$, where derivatives are denoted by $\partial^l = \frac{\partial}{\partial \xi_l}$.

Summation with respect to repeated indices (one upper, another lower) is always assumed over the whole range of indices, except when explicitly stated otherwise.

∎

In a certain sense the H-measure *measures* how far is the given weakly convergent sequence from a strongly convergent one: for strongly convergent sequences the H-measure is zero.

One difference between H-measures and Young measures is readily seen: H-measures depend on the dual variable $\boldsymbol{\xi}$, so if the equation under consideration describes a physical phenomenon that propagates, there is *a priori* hope that the H-measure can *see* the direction of propagation.

A class of symbols and associated operators

The H-measure theory shares some ideas with the linear theory of partial differential equations; namely the theory of *pseudodifferential operators* (see [Hlpd] or [Tipf]). That theory was motivated by the study of differential operators of the form: $Lu = P(\mathbf{x}, D)u$ (of course, the goal is to solve the equation $Lu = f$, and to study the regularity properties of its solutions), where P is given by:

$$P(\mathbf{x},\boldsymbol{\xi}) = \sum_{|\alpha|\leq m} c_\alpha(\mathbf{x})\boldsymbol{\xi}^\alpha ,$$

while D_j is defined to be $D_j = \frac{1}{2\pi i}\partial_j$. This study was extended to the case where P is not a polynomial in $\boldsymbol{\xi}$.

If we apply such an operator L on the function u expressed via Fourier inversion formula:

$$u(\mathbf{x}) = \int_{\mathbf{R}^n} e^{2\pi i\boldsymbol{\xi}\cdot\mathbf{x}}\hat{u}(\boldsymbol{\xi})\,d\boldsymbol{\xi} ,$$

we obtain the expression that makes sense even if P is not a polynomial in $\boldsymbol{\xi}$.:

$$P(\mathbf{x}, D)u(x) = \int_{\mathbf{R}^n} e^{2\pi i\boldsymbol{\xi}\cdot\mathbf{x}} P(\mathbf{x},\boldsymbol{\xi})\hat{u}(\boldsymbol{\xi})\,d\boldsymbol{\xi} .$$

Considering the form of the function P, we can rewrite the above expression in the following form:

$$P(\mathbf{x}, D)u(x) = \sum_{|\alpha|\leq m} c_\alpha(\mathbf{x})\bar{\mathcal{F}}\Big(\boldsymbol{\xi}^\alpha\hat{u}(\boldsymbol{\xi})\Big) .$$

So, the operators just considered can be written as a sum of the terms of the form: $Lu(\mathbf{x}) = c(\mathbf{x})\bar{\mathcal{F}}\Big(d(\boldsymbol{\xi})\hat{u}(\boldsymbol{\xi})\Big)$.

More generally, a *classical pseudodifferential operator* is a linear operator $A : \mathcal{E}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$ such that there is a function a, *an amplitude*, in the space $\mathrm{C}^\infty(\Omega\times\mathbf{R}^n)$, with additional boundedness properties on the derivatives, such that:

$$Au(x) = \int_{\mathbf{R}^n} e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}} a(\mathbf{x},\boldsymbol{\xi})\hat{u}(\boldsymbol{\xi})d\boldsymbol{\xi}$$

(for details, see [Tipf]).

The above approach is suited for the equations written in the form :

$$a^i\partial_i u + bu = f .$$

But, the equations of continuum mechanics are usually written in the conservative form:

$$\partial_i(a^i u) + bu = f \ ,$$

so a slightly different approach seems more natural.

If a is a function in $\boldsymbol{\xi}$, while b is a function in $\mathbf{x}$, we consider the following linear operators on functions defined in $\mathbf{x}$:

$$\begin{aligned} A\mathsf{u}(\mathbf{x}) &:= \bar{\mathcal{F}}\Big(a\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big)\mathcal{F}\mathsf{u}(\boldsymbol{\xi})\Big) \\ B\mathsf{u}(\mathbf{x}) &:= b(\mathbf{x})\mathsf{u}(\mathbf{x}) \ . \end{aligned}$$

Lemma 1. **(first commutation lemma, [Thmo])** *If $a \in \mathrm{C}(S^{n-1})$ and $b \in \mathrm{C}_0(\mathbf{R}^n)$ then the above defined operators belong to $\mathcal{L}(\mathrm{L}^2(\mathbf{R}^n))$ (i.e. they are bounded linear operators on $\mathrm{L}^2(\mathbf{R}^n)$), and their norms coincide with supremum norms of a and b (respectively). Moreover, the commutator $C := [A, B] = AB - BA$ is a compact operator on $\mathrm{L}^2(\mathbf{R}^n)$ (denoted $C \in \mathcal{K}(\mathrm{L}^2(\mathbf{R}^n))$).*

∎

We are now ready to define the symbols and corresponding operators. An *admissible symbol* is a function $P \in \mathrm{C}(\mathbf{R}^n \times S^{n-1})$ that can be written in the form: $P(\mathbf{x}, \boldsymbol{\xi}) = \sum_k b_k(\mathbf{x})a_k(\boldsymbol{\xi})$; with $a_k \in \mathrm{C}(S^{n-1}), b_k \in \mathrm{C}_0(\mathbf{R}^n)$ and such that the following boundedness condition is satisfied: $\sum_k \| a_k \|_\infty \| b_k \|_\infty < \infty$.

We say that an operator $L \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^n))$ has an admissible symbol P if that operator can be written as a sum: $L = \sum_k A_k B_k (\mathrm{mod}\mathcal{K}(\mathrm{L}^2(\mathbf{R}^n)))$; where the operators A_k and B_k are defined as above. Let us denote the space of such operators L by $\mathcal{H}$.

Among all the operators corresponding to a given symbol P we can choose the *standard* one: $L_0 := \sum_k A_k B_k$. It satisfies (for $u \in \mathrm{L}^2(\mathbf{R}^n) \bigcap \mathrm{L}^1(\mathbf{R}^n)$):

$$\begin{aligned} \mathcal{F}(L_0 u)(\boldsymbol{\xi}) &= \sum_k a_k\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \int_{\mathbf{R}^n} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} b_k(\mathbf{x})u(\mathbf{x})\, d\mathbf{x} \\ &= \int_{\mathbf{R}^n} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} P\Big(\mathbf{x}, \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) u(\mathbf{x})\, d\mathbf{x} \ . \end{aligned}$$

Thus, L_0 is well defined—it does not depend on the choice of the representation for P.

The above definitions lead to correspondence between multiplication in $\mathcal{H}/\mathcal{K}$ and the multiplication of symbols.

Remark. If we consider the operator $L := \sum_k B_k A_k$, where A_k and B_k are as in the decomposition of the standard operator L_0, we have for $u \in \mathrm{L}^2(\mathbf{R}^n) \cap \mathcal{F}(\mathrm{L}^1(\mathbf{R}^n))$:

$$\begin{aligned} Lu(\mathbf{x}) &= \sum_k b_k(\mathbf{x}) \int_{\mathbf{R}^n} e^{2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} a_k\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) \hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \int_{\mathbf{R}^n} e^{2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} P\left(\mathbf{x}, \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) \hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi} \end{aligned}$$

and this is exactly the operator with the symbol P in the framework of the linear theory (note an additional assumption in that theory that the symbols have to be smooth).

Let us note that L and L_0 differ only by a compact operator on $\mathrm{L}^2(\mathbf{R}^n)$: $L - L_0 = \sum_k (B_k A_k - A_k B_k) = \sum_k [B_k, A_k]$, because by the first commutation lemma each of the commutators is compact, with the norm less then $2 \| a_k \|_\infty \| b_k \|_\infty$ (we use the fact that a uniform limit of compact operators is compact).

∎

2. Symmetric systems

Localisation property for symmetric systems

We consider the following system of partial differential equations for a vector function $\mathsf{u} : \Omega \longrightarrow \mathbf{R}^p$ $(\Omega \subseteq \mathbf{R}^n)$:

$$\mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \quad . \tag{2}$$

For each $k \in \{1, \dots, n\}$, $\mathbf{A}^k$ is a continuously differentiable *hermitian* $(p \times p)$ matrix function (additional properties that we need will be specified below), while f is a function from Ω into $\mathbf{R}^p$. $\mathbf{B}$ is a continuous matrix function.

In order to apply the theory of H-measures, we consider two sequences of functions (u^ε) and (f^ε) such that for each ε (2) is satisfied. Let us first assume the following convergences:

$$\mathsf{u}^\varepsilon \xrightarrow{\mathrm{L}^2} 0 \quad \text{(weakly)},$$
$$\mathsf{f}^\varepsilon \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \quad \text{(strongly)}.$$

For simplicity, we assume that all the functions $\mathsf{u}^\varepsilon, \mathsf{f}^\varepsilon$ have their supports* in a compact subset of Ω. Now we can easily extend these functions by 0 to the functions defined on the whole $\mathbf{R}^n$. (In order not to unnecessarily complicate the notation, we shall still denote these extensions by $\mathsf{u}^\varepsilon, \mathsf{f}^\varepsilon$.) Having their supports in a compact subset of Ω, the extensions of the functions $\mathsf{u}^\varepsilon, \mathsf{f}^\varepsilon$ converge as above.

The equation (2) (for $\mathsf{u}^\varepsilon, \mathsf{f}^\varepsilon$) can be rewritten in divergence form:

$$\partial_k(\mathbf{A}^k \mathsf{u}^\varepsilon) = \mathsf{f}^\varepsilon + (\partial_k \mathbf{A}^k)\mathsf{u}^\varepsilon - \mathbf{B}\mathsf{u}^\varepsilon =: \mathbf{g}^\varepsilon \ ,$$

where $\mathbf{g}^\varepsilon \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0$.

Theorem 2. **(localisation property)** *If the sequence* $\mathsf{u}^\varepsilon \longrightarrow 0$ *in the space* $\mathrm{L}^2(\mathbf{R}^n)^p$ *defines a H-measure* $\boldsymbol{\mu}$, *and if* u^ε *satisfies:*

$$\partial_k(\mathbf{A}^k \mathsf{u}^\varepsilon) \to 0 \ \text{ in the space} \, (\mathrm{H}^{-1}_{\mathrm{loc}}(\Omega))^p \ ,$$

where $\mathbf{A}^k$ *are continuous matrix functions on* $\Omega \subseteq \mathbf{R}^n$; *then for* $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \xi_k \mathbf{A}^k(\mathbf{x})$ *the following identity is satisfied on* $\Omega \times S^{n-1}$*:*

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu} = \mathbf{0} \, .$$

(This result implies that the support of the H-measure $\boldsymbol{\mu}$ *is contained in the set* $\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{n-1} : \det \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = 0\}$ *of points where* $\mathbf{P}$ *is a singular matrix.)*

Dem. For any $\varphi \in \mathrm{C}^1_c(\Omega)$ we have:

$$\partial_k(\varphi \mathbf{A}^k \mathsf{u}^\varepsilon) = (\partial_k \varphi)\mathbf{A}^k \mathsf{u}^\varepsilon + \varphi \partial_k(\mathbf{A}^k \mathsf{u}^\varepsilon) \, .$$

* If this is not the case, one can multiply the equation (2) by a cutoff function $\varphi \in \mathcal{D}(\Omega)$.

As $\mathbf{A}^k \mathsf{u}^\varepsilon \longrightarrow 0$ weakly in the space $(\mathrm{L}^2(\Omega))^p$, therefore strongly in $(\mathrm{H}^{-1}_{\mathrm{loc}}(\Omega))^p$ as well, we have that $\partial_k(\varphi \mathbf{A}^k \mathsf{u}^\varepsilon) \longrightarrow 0$ strongly in the space $(\mathrm{H}^{-1}(\Omega))^p$:

$$\left\| \frac{\mathcal{F}(\partial_k(\varphi \mathbf{A}^k \mathsf{u}^\varepsilon))}{\sqrt{1+|\boldsymbol{\xi}|^2}} \right\|_{\mathrm{L}^2} = \left\| \frac{1}{\sqrt{1+|\boldsymbol{\xi}|^2}} \xi_k \mathcal{F}(\varphi \mathbf{A}^k \mathsf{u}^\varepsilon) \right\|_{\mathrm{L}^2} \longrightarrow 0 \,.$$

This gives that $\frac{\xi_k}{|\xi|}\mathcal{F}(\varphi \mathbf{A}^k \mathsf{u}^\varepsilon) \longrightarrow 0$ strongly in the space $(\mathrm{L}^2(\Omega))^p$. (Note that $\mathcal{F}(\varphi \mathbf{A}^k \mathsf{u}^\varepsilon) \longrightarrow 0$ pointwise.)

We can now multiply (forming the tensor product) this strongly convergent sequence by a weakly convergent one, $\mathcal{F}(\varphi \mathsf{u}^\varepsilon)\psi(\frac{\xi}{|\xi|})$, and pass to the limit, obtaining the claim.

Q.E.D.

Propagation property

By $X^m(\mathbf{R}^n)$ we shall denote the space of functions w with derivatives up to order m belonging to the image by the Fourier transform of the space $\mathrm{L}^1(\mathbf{R}^n)$, equipped with the norm:

$$\| w \|_{X^m} := \int_{\mathbf{R}^n} (1 + |2\pi \boldsymbol{\xi}|^m) |\mathcal{F}w(\boldsymbol{\xi})| \, d\boldsymbol{\xi} \,.$$

Lemma 2. (second commutation lemma, [Thmo]) *Let A and B be standard operators as defined before, with symbols a and b, satisfying one of the following conditions:*

a) $a \in \mathrm{C}^1(S^{n-1})$ and $b \in X^1(\mathbf{R}^n)$.

b) $a \in X^1_{loc}(\mathbf{R}^n \setminus \{0\})$ and $b \in \mathrm{C}^1_0(\mathbf{R}^n)$.

Then the commutator $C := [A, B] \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^n), \mathrm{H}^1(\mathbf{R}^n))$ and, by extending a to a homogeneous (of degree zero) function on $\mathbf{R}^n$, ∇C has symbol $(\nabla_{\boldsymbol{\xi}} a \cdot \nabla_{\mathbf{x}} b)\boldsymbol{\xi}$. ∎

Theorem 3. (propagation property for symmetric systems) *Let the matrices $\mathbf{A}^k$ be of class $\mathrm{C}^1_0(\Omega)$. If for every ε the pair $(\mathsf{u}^\varepsilon, \mathsf{f}^\varepsilon)$ satisfies the system (2), and both sequences (u^ε) and (f^ε) converge to zero weakly in $\mathrm{L}^2(\Omega)$, then for every ψ, function of class C^1_0 on Ω and of class X^1 on S^{n-1}, the H-measure associated to the sequence $(\mathsf{u}^\varepsilon, \mathsf{f}^\varepsilon)$:*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix},$$

satisfies the equation:

$$(3) \qquad \langle \boldsymbol{\mu}_{11}, \{\mathbf{P}, \psi\} + \psi \partial_k \mathbf{A}^k - 2\psi \mathbf{S} \rangle + \langle 2\mathsf{Re}\,\mathsf{tr}\boldsymbol{\mu}_{12}, \psi \rangle = 0 \,,$$

where $\mathbf{S}$ is the hermitian part of the matrix $\mathbf{B}$, i.e. $\mathbf{S} := \frac{1}{2}(\mathbf{B} + \mathbf{B}^*)$.

The H-measure $\boldsymbol{\mu}$ is associated to the pair of sequences $(\mathsf{u}^\varepsilon, \mathsf{f}^\varepsilon)$, with block $\boldsymbol{\mu}_{11}$ corresponding to u^ε and $\boldsymbol{\mu}_{22}$ to f^ε, while nondiagonal blocks correspond to products of u^ε and f^ε.

Dem. Let us act by a scalar pseudodifferential operator $\mathcal{A}$ with a sufficiently smooth symbol a (a is a homogeneous function of the dual variable $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n \setminus \{\mathrm{o}\}$) on the equation (2), with u and f replaced by u^ε and f^ε. The left hand side gives us (derivatives in $\mathbf{x}$ commute with the operator $\mathcal{A}$):

$$\begin{aligned}\mathcal{A}(\mathbf{A}^k\partial_k\mathsf{u}^\varepsilon) &= \mathcal{A}\,\partial_k(\mathbf{A}^k\mathsf{u}^\varepsilon) - \mathcal{A}\,(\partial_k\mathbf{A}^k)\mathsf{u}^\varepsilon \\ &= \partial_k[\mathcal{A}(\mathbf{A}^k\mathsf{u}^\varepsilon)] - \mathcal{A}(\partial_k\mathbf{A}^k)\mathsf{u}^\varepsilon \\ &= \partial_k[\mathcal{A}(\mathbf{A}^k\mathsf{u}^\varepsilon) - \mathbf{A}^k\mathcal{A}\mathsf{u}^\varepsilon] + \partial_k(\mathbf{A}^k\mathcal{A}\mathsf{u}^\varepsilon) - \mathcal{A}(\partial_k\mathbf{A}^k)\mathsf{u}^\varepsilon \\ &= \mathcal{M}'\mathsf{u}^\varepsilon + (\partial_k\mathbf{A}^k)\mathcal{A}\mathsf{u}^\varepsilon + \mathbf{A}^k\partial_k(\mathcal{A}\mathsf{u}^\varepsilon) - \mathcal{A}(\partial_k\mathbf{A}^k)\mathsf{u}^\varepsilon \\ &= \mathcal{M}\mathsf{u}^\varepsilon + \mathbf{A}^k\partial_k(\mathcal{A}\mathsf{u}^\varepsilon) + \mathcal{K}\mathsf{u}^\varepsilon \ ,\end{aligned}$$

where $\mathcal{M}'$ denotes the operator $\partial_k(\mathcal{A}\,\mathbf{A}^k - \mathbf{A}^k\mathcal{A})$, which, by the *second commutation lemma*, can be expressed as the operator $\mathcal{M}$ with the symbol $\xi_k\partial^l a\,\partial_l\mathbf{A}^k$ up to a compact operator, that is included in the compact operator $\mathcal{K}$, as well as the operator $(\partial_k\mathbf{A}^k)\mathcal{A} - \mathcal{A}(\partial_k\mathbf{A}^k)$ (the compactness of the last operator is a consequence of the *first commutation lemma*).

Thus we have got the equality (here we use the first commutation lemma, this time for the term with $\mathbf{B}$, and include the compact part in the operator $\mathcal{K}$):

$$\mathbf{A}^k\partial_k(\mathcal{A}\mathsf{u}^\varepsilon) = -\mathbf{B}\mathcal{A}\mathsf{u}^\varepsilon + \mathcal{A}\mathsf{f}^\varepsilon - \mathcal{M}\mathsf{u}^\varepsilon - \mathcal{K}\mathsf{u}^\varepsilon \ .$$

Multiplying it by u^ε from the right (using complex scalar product), and adding the result to the equation (2) multiplied by $\mathcal{A}\mathsf{u}^\varepsilon$ from the left, we obtain:

$$\tag{4}\begin{aligned}\mathbf{A}^k\partial_k(\mathcal{A}\mathsf{u}^\varepsilon)\cdot\mathsf{u}^\varepsilon + \mathcal{A}\mathsf{u}^\varepsilon\cdot\mathbf{A}^k\partial_k\mathsf{u}^\varepsilon = &-\mathcal{A}\mathsf{u}^\varepsilon\cdot\mathbf{B}\mathsf{u}^\varepsilon - \mathbf{B}\mathcal{A}\mathsf{u}^\varepsilon\cdot\mathsf{u}^\varepsilon \\ &+ \mathcal{A}\mathsf{f}^\varepsilon\cdot\mathsf{u}^\varepsilon + \mathcal{A}\mathsf{u}^\varepsilon\cdot\mathsf{f}^\varepsilon - \mathcal{M}\mathsf{u}^\varepsilon\cdot\mathsf{u}^\varepsilon - \mathcal{K}\mathsf{u}^\varepsilon\cdot\mathsf{u}^\varepsilon \ .\end{aligned}$$

Using the fact that $(\mathbf{A}^k)^* = \mathbf{A}^k$, the left hand side can be written in the form:

$$\partial_k(\mathcal{A}\mathsf{u}^\varepsilon)\cdot\mathbf{A}^k\mathsf{u}^\varepsilon + \mathcal{A}\mathsf{u}^\varepsilon\cdot\mathbf{A}^k\partial_k\mathsf{u}^\varepsilon \ .$$

We would like to write these two terms as a derivative of a product; clearly, that product cannot be a scalar, but should be a matrix (tensor), so that contraction with $\mathbf{A}^k$ (for each k) gives a scalar. A natural candidate is the tensor product of two vectors, whose action on an arbitrary vector v is given by $(\mathsf{a}\otimes\mathsf{b})\mathsf{v} := (\mathsf{v}\cdot\mathsf{b})\mathsf{a}$. If the scalar product of the two matrices is defined to be: $\mathbf{A}\cdot\mathbf{A} := \mathrm{tr}(\mathbf{A}^*\mathbf{A})$ (where tr is the unique linear extension of the map $\mathrm{tr} : \mathsf{a}\otimes\mathsf{b} \longmapsto \mathsf{a}\cdot\mathsf{b}$), then the following identity† is valid: $\mathsf{a}\cdot\mathbf{A}\mathsf{b} = (\mathsf{a}\otimes\mathsf{b})\cdot\mathbf{A}$. Now we can rewrite the left hand side of (4) as:

$$[\partial_k(\mathcal{A}\mathsf{u}^\varepsilon)\otimes\mathsf{u}^\varepsilon + \mathcal{A}\mathsf{u}^\varepsilon\otimes\partial_k\mathsf{u}^\varepsilon]\cdot\mathbf{A}^k = \partial_k(\mathcal{A}\mathsf{u}^\varepsilon\otimes\mathsf{u}^\varepsilon)\cdot\mathbf{A}^k \ .$$

We should do the same with the right hand side; it is the sum of traces of the tensor products (here we use a simple identity $\mathsf{a}\cdot\mathsf{b} = (\mathsf{a}\otimes\mathsf{b})\cdot\mathbf{I}$, with $\mathbf{I}$ being the identity matrix). The right hand side of (4) thus becomes:

$$-(\mathcal{A}\mathsf{u}^\varepsilon\otimes\mathsf{u}^\varepsilon)\cdot(\mathbf{B}+\mathbf{B}^*) + (\mathcal{A}\mathsf{f}^\varepsilon\otimes\mathsf{u}^\varepsilon + \mathcal{A}\mathsf{u}^\varepsilon\otimes\mathsf{f}^\varepsilon)\cdot\mathbf{I} - (\mathcal{M}\mathsf{u}^\varepsilon\otimes\mathsf{u}^\varepsilon)\cdot\mathbf{I} - (\mathcal{K}\mathsf{u}^\varepsilon\otimes\mathsf{u}^\varepsilon)\cdot\mathbf{I} \ .$$

† This can easily be checked using components. On the other hand, if an intrinsic proof is preferred, the general case can be reduced to the case where $\mathbf{A}$ is of the form $\mathsf{c}\otimes\mathsf{d}$ (by linearity).

Multiplying the equation (4) transformed in this way by a scalar test function $w \in C^1_c(\Omega)$ and integrating, we* get:

$$\langle \partial_k(\mathcal{A}u^\varepsilon \otimes u^\varepsilon), \mathbf{A}^k w\rangle = -\langle \mathcal{A}u^\varepsilon \otimes u^\varepsilon, w(\mathbf{B}+\mathbf{B}^*)\rangle + \langle \mathcal{A}f^\varepsilon \otimes u^\varepsilon + \mathcal{A}u^\varepsilon \otimes f^\varepsilon, w\mathbf{I}\rangle$$
$$- \langle \mathcal{M}u^\varepsilon \otimes u^\varepsilon, w\mathbf{I}\rangle - \langle \mathcal{K}u^\varepsilon \otimes u^\varepsilon, w\mathbf{I}\rangle \ .$$

After integrating the left hand side by parts, we can pass to the limit $\varepsilon \searrow 0$, because of the assumption that $f^\varepsilon \xrightarrow{L^2} 0$. We shall denote the H-measure (it is an (2×2) block matrix measure, with $(p\times p)$ blocks) associated to the sequence $(u^\varepsilon, f^\varepsilon)$ by:

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix} .$$

Clearly, $\boldsymbol{\mu}_{11}$ is the already defined H-measure associated to the sequence (u^ε). In the limit (due to the compactness of K, the last term on the right hand side converges to 0) we obtain:

(5) $$\langle \boldsymbol{\mu}_{11}, a\,\partial_k(\mathbf{A}^k w) - aw(\mathbf{B}+\mathbf{B}^*) - \xi_k \partial^l a\, \partial_l \mathbf{A}^k w\rangle + \langle \boldsymbol{\mu}_{12} + \boldsymbol{\mu}_{21}, a\mathbf{I}w\rangle = 0 \ ,$$

where the symbols of the operators $\mathcal{A}$ and M appear.

Let us take $\psi(\mathbf{x}, \boldsymbol{\xi}) := a(\boldsymbol{\xi})w(\mathbf{x})$. Then the Poisson bracket of $\mathbf{P}$ and ψ is:

$$\begin{aligned} \{\mathbf{P}, \psi\} &= \partial^l \mathbf{P}\, \partial_l \psi - \partial^l \psi\, \partial_l \mathbf{P} \\ &= \delta^l{}_k \mathbf{A}^k a\, \partial_l w - \partial^l a\, w\, \xi_k \partial_l \mathbf{A}^k \\ &= a\mathbf{A}^k \partial_k w - \xi_k \partial^l a\, \partial_l \mathbf{A}^k w \ . \end{aligned}$$

We can now write (5) in the form:

(6) $$\langle \boldsymbol{\mu}_{11}, \{\mathbf{P}, \psi\} + \psi \partial_k \mathbf{A}^k - \psi(\mathbf{B}+\mathbf{B}^*)\rangle + \langle 2\mathrm{Re}\,\mathrm{tr}\boldsymbol{\mu}_{12}, \psi\rangle = 0 \ .$$

If $s > 1 + \frac{n}{2}$, then $H^s(\mathbf{R}^n) \subseteq X^1(\mathbf{R}^n)$. Thus the equation (6) has a meaning for any $\psi \in C_0^{[\frac{n+3}{2}]}(\mathbf{R}^n \times S^{n-1})$. By density, the formula can be understood even for ψ of class C^1_0 on $\mathbf{R}^n$ and X^1 on S^{n-1} (and homogeneously extended outside S^{n-1}).

Q.E.D.

Remark. In a similar way, starting from the equation:

$$\partial_k(\mathbf{A}^k u) = f \quad ,$$

instead of (2), one could obtain a formula similar to (6):

$$\langle \boldsymbol{\mu}_{11}, \{\mathbf{P}, \psi\}\rangle + \langle 2\mathrm{Re}\,\mathrm{tr}\boldsymbol{\mu}_{12}, \psi\rangle = 0 \ .$$

∎

Remark. As a notational convenience we decompose H-measure $\boldsymbol{\mu}$ into blocks. In the examples, while discussing the localisation property, $\boldsymbol{\mu}_{11}$ might be denoted by $\boldsymbol{\mu}$ for simplicity. After obtaining additional relations among the components of the H-measure, we shall return to the notation described above. ∎

* We shall denote duality product with the same symbol $\langle .\,, .\rangle$, regardless of the type (scalar or matrix) of the functions. In any case the result is a scalar. If the functions appearing are matrix functions, we assume that their scalar product has been taken before integration.

3. Hyperbolic equations

The wave equation

Let us consider the wave equation in n-dimensional space:

$$(\rho u')' - \operatorname{div}(\mathbf{A}\nabla u) = g \ .$$

We assume that $\rho : \mathbf{R}^n \times \mathbf{R}_0^+ \longrightarrow \mathbf{R}^+$ and $\mathbf{A} : \mathbf{R}^n \times \mathbf{R}_0^+ \longrightarrow \mathrm{Psym}$ (the values of $\mathbf{A}$ are symmetric positive definite matrices). We would like to rewrite the wave equation as a symmetric hyperbolic system. Denoting the time $t = x^0$ and $\partial_0 := \frac{\partial}{\partial t}$, the wave equation can be written in the following form:

$$\partial_0(\rho\partial_0 u) - \sum_{i,j=1}^{n} \partial_i(a^{ij}\partial_j u) = g \ . \tag{7}$$

In order to reduce the second order equation to a first order system we must introduce new variables: $v_j := \partial_j u$, for $j = 0, \ldots, n$.

The previous transformation gives us only one equation. In order to make the system with $n+2$ unknowns formally deterministic, we have to provide $n+1$ more equations. Clearly, adding the definition equations for v^i would lead to a formally deterministic system, which, unfortunately, is not symmetric. Besides these, we have, by the Schwarz's theorem, the following $(n+1)(n+2)/2$ symmetry relations $\partial_i v_j = \partial_j v_i$, for $i, j = 0, \ldots, n$ as well. One choice* of $(n+2)$ equations, that will lead to a symmetric hyperbolic system, requires taking the derivatives of the product in (7) (summation over $i, j = 1, \ldots, n$):

$$\rho\partial_0 v_0 - a^{ij}\partial_i v_j + \partial_0\rho v_0 - (\partial_i a^{ij})v_j = g \ .$$

This will be the second equation of the system. For the first, we shall just take the definition of v_0. The remaining n equations will be the symmetry relations, with one index being 0, but multiplied by the matrix $\mathbf{A}^\top = \mathbf{A}$. So, the system we shall consider is (summation over $i, j = 1, \ldots, n$):

$$\begin{aligned} \partial_0 u - v_0 &= 0 \\ \rho\partial_0 v_0 - a^{ij}\partial_i v_j + b^0 v_0 + b^j v_j &= g \\ a^{ij}\partial_0 v_i - a^{ij}\partial_i v_0 &= 0 \ , \end{aligned} \tag{8}$$

where $b^0 := \partial_0\rho$, $b^j := -\partial_i a^{ij} = [-\operatorname{div}\mathbf{A}^\top]^j$, for $j = 1, \ldots, n$. This system can be written in the required form.

Before writing it down, let us note that the first equation in (8) is the only one where u appears explicitly. Thus, we can solve the system for v_i first, and later use the solution in order to obtain u. This reduces the system to $n+1$ unknowns $\mathsf{v} = (v_0, \ldots, v_n)$ and $n+1$

* The only one I know of.

equations:

$$
(9)\qquad
\begin{bmatrix} \rho & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A} & \\ 0 & & & \end{bmatrix} \partial_0 \mathsf{v} + \sum_{i=1}^{n} \begin{bmatrix} 0 & -a^{i1} & \cdots & -a^{in} \\ -a^{i1} & & & \\ \vdots & & \mathbf{0} & \\ -a^{in} & & & \end{bmatrix} \partial_i \mathsf{v}
+ \begin{bmatrix} b^0 & b^1 & \cdots & b^n \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{bmatrix} \mathsf{v} = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \end{bmatrix} .
$$

It is clear that $\mathbf{A}^i$ are all symmetric, $\mathbf{A}^0$ is even positive definite (because $\rho > 0$ and $\mathbf{A}$ is positive definite). Thus, we have written the wave equation in the form of a symmetric hyperbolic system.

Remark. Such a system is *symmetric hyperbolic* (see [Fshl]) if there is a vector $\tilde{\boldsymbol{\xi}}$ such that $\tilde{\xi}_k \mathbf{A}^k$ is a positive definite matrix. Clearly, $\tilde{\boldsymbol{\xi}} := (1, 0, \ldots, 0)$ gives $\begin{bmatrix} \rho & 0 \\ 0 & \mathbf{A} \end{bmatrix}$, which is positive definite.

In particular, the system to which we reduced the wave equation is *hyperbolic in the sense of Petrovski:* for every vector $\boldsymbol{\xi}$ the matrices: $\mathbf{A}(\boldsymbol{\xi}, \lambda) := \xi_k \mathbf{A}^k - \lambda \tilde{\xi}_k \mathbf{A}^k$ have simple elementary divisors, and $\det \mathbf{A}(\boldsymbol{\xi}, \lambda) = 0$ has real eigenvalues λ. ∎

If we assume that the initial data were given for the wave equation by $u(0, .) = u_0$ and $u'(0, .) = u_1$, we can take:

$$
\begin{aligned}
v_0(0, .) &= u_1 \\
v_i(0, .) &= \partial_i u_0 \quad , \text{for } i = 1, \ldots, n
\end{aligned}
$$

as the initial data for the system (9). The relation $u(0, .) = u_0$ determines the initial condition for the time derivative of u.

Due to the fact that u_0 is defined on $\mathbf{R}^n$, we can compute its derivatives in the spatial directions. We should still check whether the identities defining v_n (and therefore the symmetry relations) are valid.

For any $i = 1, \ldots, n$ we have:

$$
\partial_0 v_i = \partial_i v_0 = \partial_i \partial_0 u = \partial_0 \partial_i u .
$$

(The first equality follows from the regularity of $\mathbf{A}^\top$, because $\mathbf{A}^\top (\partial_0 \mathsf{v} - \nabla_{\mathbf{x}'} v_0) = 0$ implies $\partial_0 v_i = \partial_i v_0$.) Now, we have the fact that $\partial_0 (v_i - \partial_i u) = 0$, and $v_i - \partial_i u = 0$ at $t = 0$, and we conclude that the last identity holds for any $t > 0$.

Let us now apply the general result for H-measures to the system (9) (note that for notational convenience we denote $\mathbf{x} = (x^0, \mathbf{x}') = (x^0, x^1, \ldots, x^n)$ and $\boldsymbol{\xi} = (\xi_0, \boldsymbol{\xi}') = (\xi_0, \xi_1, \ldots, \xi_n)$).

The symbol of the differential operator is:

$$
\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = \xi_k \mathbf{A}^k(\mathbf{x}) = \begin{bmatrix} \xi_0 \rho & -(\mathbf{A}\boldsymbol{\xi}')^\top \\ -\mathbf{A}\boldsymbol{\xi}' & \xi_0 \mathbf{A} \end{bmatrix} .
$$

We assume that $\mathsf{v}^\varepsilon \longrightarrow 0$ weakly in the space $\mathrm{L}^2(\mathbf{R}_0^+ \times \mathbf{R}^n)$, satisfy the system (9) and define the H-measure:

$$
\boldsymbol{\mu} = \begin{bmatrix} \mu_{00} & \mu_{01} \\ \mu_{10} & \mu_{11} \end{bmatrix}
$$

(where μ_{00} is a 1×1 block, while $\boldsymbol{\mu}_{11}$ is a $n \times n$ block).

The localisation property gives us:

$$\mathbf{0} = \mathbf{P}\boldsymbol{\mu} = \begin{bmatrix} \xi_0 \rho \mu_{00} - \mathbf{A}\boldsymbol{\xi}' \cdot \bar{\boldsymbol{\mu}}_{10} & \rho \xi_0 \boldsymbol{\mu}_{01} - (\boldsymbol{\mu}_{11}^\top \mathbf{A}\boldsymbol{\xi}')^\top \\ -\mu_{00}\mathbf{A}\boldsymbol{\xi}' + \xi_0 \mathbf{A}\boldsymbol{\mu}_{10} & -\mathbf{A}\boldsymbol{\xi}' \otimes \boldsymbol{\mu}_{01}^\top + \xi_0 \mathbf{A}\boldsymbol{\mu}_{11} \end{bmatrix} .$$

This gives us the following relations between the components of the H-measure $\boldsymbol{\mu}$:

$$\begin{aligned} \rho\xi_0\mu_{00} &= \mathbf{A}\boldsymbol{\xi}' \cdot \bar{\boldsymbol{\mu}}_{10} \\ \rho\xi_0\boldsymbol{\mu}_{01} &= (\boldsymbol{\mu}_{11}^\top \mathbf{A}\boldsymbol{\xi}')^\top \\ \mu_{00}\mathbf{A}\boldsymbol{\xi}' &= \xi_0\mathbf{A}\boldsymbol{\mu}_{10} \\ \mathbf{A}\boldsymbol{\xi}' \otimes \boldsymbol{\mu}_{01}^\top &= \xi_0\mathbf{A}\boldsymbol{\mu}_{11} \end{aligned}$$

The first identity is between scalars, the last between matrices, while the remaining two are between vectors.

The second equality gives us (after taking the hermitian conjugate of the matrices, and using the hermitian property of H-measures) $\rho\xi_0\boldsymbol{\mu}_{10} = \boldsymbol{\mu}_{11}\mathbf{A}\boldsymbol{\xi}'$. If we multiply the last equality by $\xi_0\rho\mathbf{A}^{-1}$ and use the relation obtained from the second inequality, we obtain:

$$\boldsymbol{\xi}' \otimes \boldsymbol{\mu}_{11}\mathbf{A}\boldsymbol{\xi}' = \xi_0^2\rho\boldsymbol{\mu}_{11} .$$

Taking into account the hermitian character of the H-measure $\boldsymbol{\mu}_{11}$, as well as the (real) symmetry of $\mathbf{A}$, we finally obtain:

$$(\mathbf{A}\boldsymbol{\xi}' \otimes \boldsymbol{\xi}' - \rho\xi_0^2\mathbf{I})\boldsymbol{\mu}_{11} = \mathbf{0} ,$$

so $\boldsymbol{\mu}_{11}$ is supported on the set where $\det(\mathbf{A}\boldsymbol{\xi}' \otimes \boldsymbol{\xi}' - \rho\xi_0^2\mathbf{I}) = 0$. Similarly, from the first and third equality we obtain: $(\mathbf{A}\boldsymbol{\xi}' \cdot \boldsymbol{\xi}' - \rho\xi_0^2)\mu_{00} = 0$.

From the third equality, due to the invertibility of $\mathbf{A}$, we get: $\mu_{00}\boldsymbol{\xi}' = \xi_0\boldsymbol{\mu}_{10}$. If we introduce a (scalar nonnegative) measure ν such that $\mu_{00} = \xi_0^2\nu$ (if $\xi_0 = 0$, then $\mu_{00} = 0$, because $\boldsymbol{\xi} \neq 0$), we can express $\boldsymbol{\mu}_{10} = \xi_0\boldsymbol{\xi}'\nu$ (again, if $\xi_0 = 0$, then $\mu_{00} = 0$, and because of the nonnegative hermitian property of H-measures, $\boldsymbol{\mu}_{10} = 0$).

From the last equality we get: $\xi_0\boldsymbol{\mu}_{11} = \boldsymbol{\xi}' \otimes \boldsymbol{\mu}_{01}^\top = \xi_0\boldsymbol{\xi}' \otimes \boldsymbol{\xi}'\nu$ (we use the algebraic identity $\mathbf{A}\boldsymbol{\xi}' \otimes \boldsymbol{\mu}_{10} = \mathbf{A}(\boldsymbol{\xi}' \otimes \boldsymbol{\mu}_{10})$). This gives us the simple expression $\boldsymbol{\mu} = \boldsymbol{\xi} \otimes \boldsymbol{\xi}\nu$ (at least when $\xi_0 \neq 0$).

From the first equality we finally get: $\xi_0(\xi_0^2\rho - \mathbf{A}\boldsymbol{\xi}' \cdot \boldsymbol{\xi}')\nu = 0$, which in the case when $\xi_0 \neq 0$ gives us that the support of ν is contained in the *light cone* in the dual space.

Remark. At this point we have lost some information contained in the wave equation, because we discarded a number of symmetry relations. Using them all, there would be no preference given to ξ_0, and we would be able to conclude that $\boldsymbol{\mu} = \boldsymbol{\xi} \otimes \boldsymbol{\xi}\nu$. ∎

In order to write down the propagation property, we first need to compute the Poisson bracket:

$$\{\mathbf{P}, \psi\} = \partial^l\mathbf{P}\partial_l\psi - \partial^l\psi\partial_l\mathbf{P} = \begin{bmatrix} \rho\partial_0\psi - \xi_0\partial_l\rho\partial^l\psi & -(\mathbf{A}\nabla_{\mathbf{x}'}\psi)^\top + ((\partial_l\mathbf{A}\partial^l\psi)\boldsymbol{\xi}')^\top \\ -\mathbf{A}\nabla_{\mathbf{x}'}\psi + (\partial_l\mathbf{A}\partial^l\psi)\boldsymbol{\xi}' & \partial_0\psi\mathbf{A} - \xi_0\partial_l\mathbf{A}\partial^l\psi \end{bmatrix} .$$

Now, adding the term:

$$\psi\partial_k \mathbf{A}^k - 2\psi \mathbf{S} = \psi \begin{bmatrix} \partial_0 \rho & -(\operatorname{div}_{\mathbf{x}'}\mathbf{A})^\top \\ -\operatorname{div}_{\mathbf{x}'}\mathbf{A} & \partial_0 \mathbf{A} \end{bmatrix} - \psi \begin{bmatrix} 2\partial_0 \rho & -(\operatorname{div}_{\mathbf{x}'}\mathbf{A})^\top \\ -\operatorname{div}_{\mathbf{x}'}\mathbf{A} & \mathbf{0} \end{bmatrix}$$
$$= \begin{bmatrix} -\partial_0 \rho\psi & \mathbf{0}^\top \\ \mathbf{0} & \partial_0 \mathbf{A}\psi \end{bmatrix},$$

we obtain:

$$\{\mathbf{P},\psi\} + \psi\partial_k \mathbf{A}^k - 2\psi\mathbf{S} = \begin{bmatrix} \partial_0(\rho\psi) - 2\partial_0\rho\psi - \xi_0\partial_l\rho\partial^l\psi & ((\partial_l \mathbf{A}\partial^l\psi)\boldsymbol{\xi}')^\top \\ (\partial_l \mathbf{A}\partial^l\psi)\boldsymbol{\xi}' & \partial_0(\psi\mathbf{A}) - \xi_0\partial_l\mathbf{A}\partial^l\psi \end{bmatrix}.$$

Taking into account the form of $\boldsymbol{\mu}$, we have:

$$\begin{aligned}\langle \boldsymbol{\mu}, \{\mathbf{P},\psi\} + \psi\partial_k\mathbf{A}^k - 2\psi\mathbf{S}\rangle &= \left\langle \nu, \xi_0^2(\partial_0(\rho\psi) - 2\psi\partial_0\rho - \xi_0\partial_l\rho\partial^l\psi)\right\rangle \\ &\quad + \left\langle \nu, \partial_0(\psi\mathbf{A})\boldsymbol{\xi}'\cdot\boldsymbol{\xi}' + \xi_0(\partial_l\mathbf{A}\partial^l\psi)\boldsymbol{\xi}'\cdot\boldsymbol{\xi}'\right\rangle \\ &= \left\langle \nu, \partial_0\Big((\rho\xi_0^2 + \mathbf{A}\boldsymbol{\xi}'\cdot\boldsymbol{\xi}')\psi\Big) - 2\xi_0^2\psi\partial_0\rho\right\rangle \\ &\quad - \left\langle \nu, \xi_0\partial^l\psi\partial_l(\xi_0^2\rho - \mathbf{A}\boldsymbol{\xi}'\cdot\boldsymbol{\xi}')\right\rangle .\end{aligned}$$

Let us now take into account the right hand side as well. Assume that $g^\varepsilon \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^n)$. So, the sequence $(\mathsf{v}^\varepsilon, g^\varepsilon, 0)$ converges weakly to zero, and defines a H-measure, that is a (2×2) block matrix measure, with $(n+1)\times(n+1)$ blocks. The upper left block is just discussed $\xi\otimes\xi\nu$, while the right upper block has all but the first column zero, so its trace is equal to its upper left element:

$$\langle \gamma, \varphi_1\bar\varphi_2\psi\rangle := \lim_{\varepsilon\searrow 0}\int_{\mathbf{R}^2} \mathcal{F}(\varphi_1 v_0^\varepsilon)(\boldsymbol{\xi})(\mathcal{F}(\varphi_2 g^\varepsilon)(\boldsymbol{\xi}))^*\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)d\boldsymbol{\xi}\,.$$

After placing the derivatives on ν, the propagation property takes the form:

$$(\rho\xi_0^2 + \mathbf{A}\boldsymbol{\xi}'\cdot\boldsymbol{\xi}')\partial_0\nu + \partial_l(\mathbf{A}\boldsymbol{\xi}'\cdot\boldsymbol{\xi}' - \rho\xi_0^2)\partial^l(\xi_0\nu) - 2\operatorname{div}_{\mathbf{x}'}\mathbf{A}\cdot\mathbf{x}'\xi_0\nu = 2\operatorname{Re}\gamma\,.$$

Remark. The result we obtained is a generalisation of the result in Tartar [Thmo, 3.3]. Under a stronger assumption that ρ and $\mathbf{A}$ do not depend on $t = x^0$, the term $\psi\partial_k\mathbf{A}^k - 2\psi\mathbf{S}$ is zero, and the two results coincide (up to a factor ξ_0).

∎

Example. Let us consider the one dimensional wave equation (in one space dimension):

$$\rho u_{tt} - (a u_x)_x = g\,.$$

This equation leads to the symmetric system:

$$\begin{bmatrix} \rho & 0 \\ 0 & a\end{bmatrix}\mathsf{v}_t - \begin{bmatrix} 0 & a \\ a & 0\end{bmatrix}\mathsf{v}_x + \begin{bmatrix} \rho_t & -a_x \\ 0 & 0\end{bmatrix}\mathsf{v} = \begin{bmatrix} g \\ 0\end{bmatrix}.$$

The corresponding symbol is: $\mathbf{P}(\mathbf{x},\boldsymbol{\xi}) = \xi_k \mathbf{A}^k(\mathbf{x}) = \begin{bmatrix} \rho\tau & -a\xi \\ -a\xi & \tau a \end{bmatrix}$, where we use more convenient notation: $\mathbf{x} = (t,x)$ and $\boldsymbol{\xi} = (\tau,\xi)$.

For a sequence $\mathsf{v}^\varepsilon \longrightarrow 0$ weakly in $\mathrm{L}^2(\mathbf{R}^2)$, defining the H-measure $\boldsymbol{\mu}$, localisation property gives us:

$$0 = \mathbf{P}\boldsymbol{\mu} = \begin{bmatrix} \rho\tau\mu_{00} - a\xi\mu_{01} & \rho\tau\mu_{01} - a\xi\mu_{11} \\ -a\xi\mu_{00} + \tau a\mu_{10} & -a\xi\mu_{10} + \tau a\mu_{11} \end{bmatrix},$$

so (note that $a > 0$) $\tau^2\mu_{11} = \tau\xi\mu 10 = \xi^2\mu_{00}$. On the other hand $\xi^2\mu_{11} = \frac{\rho}{a}\tau\xi\mu_{01} = \frac{\rho^2}{a^2}\tau^2\mu_{00}$. If either τ or ξ is zero, both μ_{00} and μ_{11} are zero, so $\boldsymbol{\mu}$ is supported on the set where $\tau\xi \neq 0$. On the support of $\boldsymbol{\mu}$ we have: $\rho\tau^2 = a\xi^2$. If we define $\tau^2\nu = \mu_{00}$, we obtain the following expression for $\boldsymbol{\mu} = \boldsymbol{\xi} \otimes \boldsymbol{\xi}\nu$, where ν satisfies: $(\rho\tau^2 - a\xi^2)\nu = 0$.

Let us now take into account the right hand side as well. Assume that $g^\varepsilon \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^2)$. So, the sequence $(\mathsf{v}^\varepsilon, g^\varepsilon, 0)$ converges weakly to zero, and defines a H-measure, that is a (2×2) block matrix measure, with (2×2) blocks. The upper left block is just discussed $\xi\otimes\xi\nu$, while the right upper block has the second column zero, so its trace is equal to its upper left element:

$$\langle \gamma, \varphi_1\bar{\varphi}_2\psi\rangle := \lim_{\varepsilon\searrow 0}\int_{\mathbf{R}^2} \mathcal{F}(\varphi_1 v_0^\varepsilon)(\boldsymbol{\xi})(\mathcal{F}(\varphi_2 g^\varepsilon)(\boldsymbol{\xi}))^*\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi}\,.$$

The propagation property reads:

$$\langle \boldsymbol{\xi}\otimes\boldsymbol{\xi}\nu, \{\mathbf{P},\psi\} + \psi\partial_k\mathbf{A}^k - 2\psi\mathbf{S}\rangle + \langle 2\mathrm{Re}\,\gamma,\psi\rangle = 0\,.$$

The term with Poisson bracket is:

$$\{\mathbf{P},\psi\} = \begin{bmatrix} \rho\psi_t - \tau(\rho_t\psi_\tau + \rho_x\psi_\xi) & -a\psi_x + a_x\psi_\xi\xi \\ -a\psi_x + a_x\psi_\xi\xi & \psi_t a - \tau(a_t\psi_\tau + a_x\psi_\xi) \end{bmatrix},$$

while the other term is:

$$\psi\partial_k\mathbf{A}^k - 2\psi\mathbf{S} = \psi\begin{bmatrix} \rho_t & -a_x \\ -a_x & a_t \end{bmatrix} - \psi\begin{bmatrix} 2\rho_t & -a_x \\ -a_x & 0 \end{bmatrix} = \begin{bmatrix} -\rho_t\psi & 0 \\ 0 & a_t\psi \end{bmatrix}.$$

After we perform the matrix operations, the propagation property reads:

$$\left\langle \nu, \left((\rho\tau^2 + a\xi^2)\psi\right)_t + \tau\nabla_{\tau,\xi}\psi\cdot\nabla_{t,x}(\xi^2 a - \tau^2\rho) - 2\tau\xi a_x\psi\right\rangle = 2\left\langle \mathrm{Re}\,\gamma,\psi\right\rangle.$$

Thus, the measure ν satisfies the following equation in the weak sense:

$$(\tau^2\rho + \xi^2 a)\partial_t\nu + \tau\nabla_{\tau,\xi}\nu\cdot\nabla_{t,x}(\xi^2 a - \tau^2\rho) - 2\tau\xi a_x\nu = 2\mathrm{Re}\,\gamma\,.$$

∎

Maxwell's system

We shall now present a more complicated example—the system of Maxwell's equations in a material with electric permeability ϵ, conductivity $\boldsymbol{\sigma}$ and magnetic susceptibility $\boldsymbol{\mu}$. The system reads:

$$\begin{aligned} \mathsf{D}' &= \mathrm{rot}\,\mathsf{H} - \mathsf{J} + \mathsf{F} \\ \mathsf{B}' &= -\mathrm{rot}\,\mathsf{E} + \mathsf{G}\,, \end{aligned} \tag{10}$$

together with $\operatorname{div} \mathsf{D} = \rho$ and $\operatorname{div} \mathsf{B} = 0$, and with the constitutive laws:

$$\begin{aligned} \mathsf{D}(.,t) &= \boldsymbol{\epsilon}\mathsf{E}(.,t) \\ \mathsf{J}(.,t) &= \boldsymbol{\sigma}\mathsf{E}(.,t) \\ \mathsf{B}(.,t) &= \boldsymbol{\mu}\mathsf{H}(.,t) . \end{aligned}$$

Choosing E and H as variables and introducing $\mathsf{u} := \begin{bmatrix} \mathsf{E} \\ \mathsf{H} \end{bmatrix}$, the system (10) can be written in the form of a symmetric system:

$$\sum_{i=0}^{3} \mathbf{A}^i \partial_i \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \ ,$$

where:

$$\mathbf{A}^0 = \begin{bmatrix} \epsilon & 0 \\ 0 & \mu \end{bmatrix} \ , \ \mathbf{A}^1 := \begin{bmatrix} 0 & \mathbf{Q}_1^\top \\ \mathbf{Q}_1 & 0 \end{bmatrix}, \mathbf{A}^2 := \begin{bmatrix} 0 & \mathbf{Q}_2^\top \\ \mathbf{Q}_2 & 0 \end{bmatrix}, \mathbf{A}^3 := \begin{bmatrix} 0 & \mathbf{Q}_3^\top \\ \mathbf{Q}_3 & 0 \end{bmatrix} .$$

The constant antisymmetric matrices $\mathbf{Q}_k$ are given by:

$$\mathbf{Q}_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{Q}_2 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \mathbf{Q}_3 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

The matrix $\mathbf{B}$ is of the form: $\mathbf{B} = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}$, while the right hand side is $\mathsf{f} = \begin{bmatrix} \mathsf{F} \\ \mathsf{G} \end{bmatrix}$. In the above we have used the fact that the rotator (curl) of a vector field E can be written as:

$$\operatorname{rot} \mathsf{E} = \begin{bmatrix} \partial_2 E^3 - \partial_3 E^2 \\ \partial_3 E^1 - \partial_1 E^3 \\ \partial_1 E^2 - \partial_2 E^1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \partial_1 \mathsf{E} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \partial_2 \mathsf{E} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \partial_3 \mathsf{E} .$$

If we assume the uniform boundedness and symmetry of the permeability and susceptibility tensors, the above system is even symmetric hyperbolic.

In order to apply the H-measure theory we should consider a sequence $\mathsf{u}^\varepsilon \rightharpoonup 0$ weakly in the space $\mathrm{L}^2(\Omega)$ (i. e. $\mathsf{E}^\varepsilon, \mathsf{H}^\varepsilon \rightharpoonup 0$). The right hand side term f is allowed to oscillate as well; so take $\mathsf{f}^\varepsilon \rightharpoonup 0$ weakly in the space $\mathrm{L}^2(\Omega)$.

The H-measure corresponding to (a subsequence of) the sequence (u^ε) will be denoted by:

$$\boldsymbol{\mu}_{11} = \begin{bmatrix} \boldsymbol{\nu}_e & \boldsymbol{\nu}_{em} \\ \boldsymbol{\nu}_{me} & \boldsymbol{\nu}_m \end{bmatrix} .$$

The Radon measure $\boldsymbol{\mu}_{11}$ is a 2×2 block matrix measure, with each block of size 3×3.

In order to express the localisation property we should compute the symbol $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{i=0}^{3} \xi_i \mathbf{A}^i(\mathbf{x})$ of the differential operator in (10):

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = \begin{bmatrix} \xi_0 \epsilon & -\boldsymbol{\Xi} \\ \boldsymbol{\Xi} & \xi_0 \mu \end{bmatrix} ,$$

where $\boldsymbol{\Xi}$ denotes (the matrix of) linear operator defined by its action on a vector v: $\boldsymbol{\Xi}\mathsf{v} = \boldsymbol{\xi}' \times \mathsf{v}$. In components $[\boldsymbol{\Xi}]_{ik} = \epsilon^{ijk}\xi_j$, or:

$$\boldsymbol{\Xi} = \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix} .$$

Clearly, $\boldsymbol{\Xi}$ is antisymmetric ($\boldsymbol{\Xi}^\top = -\boldsymbol{\Xi}$), so $\mathbf{P}$ is a symmetric matrix.

The localisation property states that $\mathbf{P}\boldsymbol{\mu}_{11} = \mathbf{0}$. In our case this takes the form:

$$\begin{bmatrix} \xi_0\epsilon & -\boldsymbol{\Xi} \\ \boldsymbol{\Xi} & \xi_0\mu \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_e & \boldsymbol{\nu}_{em} \\ \boldsymbol{\nu}_{me} & \boldsymbol{\nu}_m \end{bmatrix} = \begin{bmatrix} \xi_0\epsilon\boldsymbol{\nu}_e + \boldsymbol{\Xi}^\top\boldsymbol{\nu}_{me} & \xi_0\epsilon\boldsymbol{\nu}_{em} + \boldsymbol{\Xi}^\top\boldsymbol{\nu}_m \\ \boldsymbol{\Xi}\boldsymbol{\nu}_e + \xi_0\mu\boldsymbol{\nu}_{me} & \boldsymbol{\Xi}\boldsymbol{\nu}_{em} + \xi_0\mu\boldsymbol{\nu}_m \end{bmatrix} = \mathbf{0} .$$

Thus, the following matrix equalities must be satisfied:

$$\begin{aligned} \xi_0\epsilon\boldsymbol{\nu}_e + \boldsymbol{\Xi}^\top\boldsymbol{\nu}_{me} &= \mathbf{0} \\ \xi_0\epsilon\boldsymbol{\nu}_{em} + \boldsymbol{\Xi}^\top\boldsymbol{\nu}_m &= \mathbf{0} \\ \boldsymbol{\Xi}\boldsymbol{\nu}_e + \xi_0\mu\boldsymbol{\nu}_{me} &= \mathbf{0} \\ \boldsymbol{\Xi}\boldsymbol{\nu}_{em} + \xi_0\mu\boldsymbol{\nu}_m &= \mathbf{0} . \end{aligned}$$

In order to simplify the above equations, let us first study the case where $\xi_0 = 0$, so $\boldsymbol{\xi}' \neq 0$. In this special case we have the following equations:

$$\boldsymbol{\Xi}\boldsymbol{\nu}_{me} = \boldsymbol{\Xi}\boldsymbol{\nu}_m = \boldsymbol{\Xi}\boldsymbol{\nu}_e = \boldsymbol{\Xi}\boldsymbol{\nu}_{em} = \mathbf{0} .$$

In doing that, we shall use the following simple fact from linear algebra:

Lemma 3. *If $\boldsymbol{\Xi}\mathbf{A} = \mathbf{0}$, then the matrix $\mathbf{A}$ is of the form: $\mathbf{A} = \boldsymbol{\xi}' \otimes \mathsf{a}$, for some vector $\mathsf{a} \in \mathbf{R}^3$.*

Dem. First, let us denote the columns of the matrix $\mathbf{A}$ as vectors: $\mathbf{A} = [\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3]$. By the linearity and definition of $\boldsymbol{\Xi}$, we have the following:

$$\boldsymbol{\Xi}\mathbf{A} = [\boldsymbol{\Xi}\mathsf{a}_1\ \boldsymbol{\Xi}\mathsf{a}_2\ \boldsymbol{\Xi}\mathsf{a}_3] = [\boldsymbol{\xi}' \times \mathsf{a}_1\ \ \boldsymbol{\xi}' \times \mathsf{a}_2\ \ \boldsymbol{\xi}' \times \mathsf{a}_3] = \mathbf{0} .$$

Thus, all the columns of the matrix $\mathbf{A}$ are parallel to the vector $\boldsymbol{\xi}'$, so we can write: $\mathsf{a}_i = \alpha_i\boldsymbol{\xi}'$, and by arranging these numbers α_i as components of the vector a we obtain the claim.

Q.E.D.

Using the lemma, as well as the hermitian property of the H-measures, we can conclude that the blocks of the matrix H-measure satisfy: $\boldsymbol{\nu}_e = m_e\boldsymbol{\xi}' \otimes \boldsymbol{\xi}'$, $\boldsymbol{\nu}_m = m_m\boldsymbol{\xi}' \otimes \boldsymbol{\xi}'$, $\boldsymbol{\nu}_{em} = m_{em}\boldsymbol{\xi}' \otimes \boldsymbol{\xi}' = \boldsymbol{\nu}^*_{me}$; where m_e and m_m are real, while $m_{em} = \bar{m}_{me}$ is a complex scalar function:

$$\boldsymbol{\mu}_{11} = \begin{bmatrix} m_e\boldsymbol{\xi}' \otimes \boldsymbol{\xi}' & m_{em}\boldsymbol{\xi}' \otimes \boldsymbol{\xi}' \\ m_{me}\boldsymbol{\xi}' \otimes \boldsymbol{\xi}' & m_m\boldsymbol{\xi}' \otimes \boldsymbol{\xi}' \end{bmatrix} .$$

In the case $\xi_0 \neq 0$, we can try to simplify the calculations by dividing $\boldsymbol{\xi}$ by ξ_0. By this transformation ξ_0 is replaced by 1, while $\boldsymbol{\xi}'$ takes arbitrary values in $\mathbf{R}^3$.

The matrix equations can be rewritten as:

$$\begin{aligned}\boldsymbol{\epsilon}\boldsymbol{\nu}_e &= \boldsymbol{\Xi}\boldsymbol{\nu}_{me}\\ \boldsymbol{\epsilon}\boldsymbol{\nu}_{em} &= \boldsymbol{\Xi}\boldsymbol{\nu}_m\\ \boldsymbol{\Xi}\boldsymbol{\nu}_e &= -\boldsymbol{\mu}\boldsymbol{\nu}_{me}\\ \boldsymbol{\Xi}\boldsymbol{\nu}_{em} &= -\boldsymbol{\mu}\boldsymbol{\nu}_m\,.\end{aligned}$$

Using the fact that $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are invertible, we can express $\boldsymbol{\nu}_e$ using $\boldsymbol{\nu}_{me}$, $\boldsymbol{\nu}_m$ using $\boldsymbol{\nu}_{em}$; and vice versa. Substituting, we can obtain the following relations that have to be satisfied:

$$\begin{aligned}(\boldsymbol{\epsilon}+\boldsymbol{\Xi}\boldsymbol{\mu}^{-1}\boldsymbol{\Xi})\boldsymbol{\nu}_e &= \mathbf{0}\\ (\boldsymbol{\epsilon}+\boldsymbol{\Xi}\boldsymbol{\mu}^{-1}\boldsymbol{\Xi})\boldsymbol{\nu}_m &= \mathbf{0}\\ (\boldsymbol{\mu}+\boldsymbol{\Xi}\boldsymbol{\epsilon}^{-1}\boldsymbol{\Xi})\boldsymbol{\nu}_{me} &= \mathbf{0}\\ (\boldsymbol{\mu}+\boldsymbol{\Xi}\boldsymbol{\epsilon}^{-1}\boldsymbol{\Xi})\boldsymbol{\nu}_{em} &= \mathbf{0}\,.\end{aligned}$$

In order for this system to have a nontrivial solution, it is necessary that matrices multiplying the unknowns are singular. Thus, the support of the H-measure is contained in the set of solutions of the equations:

$$\begin{aligned}\det(\boldsymbol{\epsilon}+\boldsymbol{\Xi}\boldsymbol{\mu}^{-1}\boldsymbol{\Xi}) &= 0\\ \det(\boldsymbol{\mu}+\boldsymbol{\Xi}\boldsymbol{\epsilon}^{-1}\boldsymbol{\Xi}) &= 0\,.\end{aligned}$$

4. Equations of mixed type

Tricomi's equation

Let us consider the Tricomi's equation:

$$y\partial_x^2 u + \partial_y^2 u = 0\,. \tag{11}$$

(Some authors, like F. John, call the equation $y\partial_x^2 u - \partial_y^2 u = 0$ by the name of Tricomi. There is no significant difference in taking the reflection with respect to x-axis, so I study the equation as written in Tricomi's *Equazioni a derivate parziali*, called *the equation $\mathcal{T}$* there.)

The Tricomi's equation is of mixed type. The standard procedure for classification of the second order linear partial differential equations in two variables, with highest derivatives part in the form $a\partial_x^2 u + 2b\partial_x\partial_y u + c\partial_y^2 u$, for the Tricomi's equation gives us $ac - b^2 = y$, so the equation is *elliptic* for $y > 0$, parabolic on the line $y = 0$ and hyperbolic in the lower half plane $y < 0$.

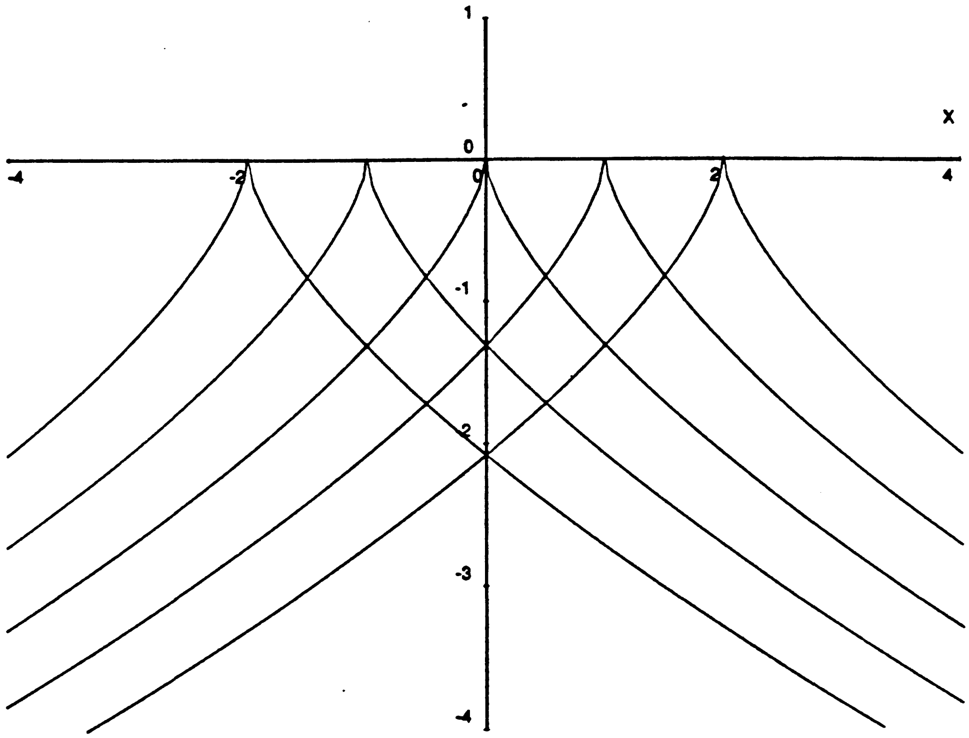
The characteristics of the Tricomi's equation (in the closed lower half plane only) are the solutions of the ordinary differential equation: $ydy^2 + dx^2 = 0$; or equivalently ($-y \geq 0$):

$$\frac{dx}{dy} = \pm\sqrt{-y}\,.$$

The solutions can be written explicitly in the form:

$$x(y) = \pm \int_0^y \sqrt{-\eta}\, d\eta + x_0 = \mp \frac{2}{3} y \sqrt{-y} + x_0 \; ,$$

where x_0 is the point of intersection of the characteristic curve and the x-axis ($x_0 = x(0)$). Alternatively, we can express $y(x) = -\sqrt[3]{\frac{9(x-x_0)^2}{4}}$; and draw the graph:



Remark. It is a trivial exercise to repeat the proceeding arguments for the coefficient $k(y)$ instead of y, as long as k is a sufficiently smooth function, having the *same sign* as y (Garabedian). A similar equation was studied by Lavrent'ev and Bicadze (1950): $\partial_x^2 u + \vartheta(y)\partial_y^2 u = 0$, where $\vartheta(y) = \operatorname{sign} y$ (a discontinuous function). ∎

Let us try to rewrite the Tricomi's equation as an equivalent first order system. Certainly, we have to introduce two unknown functions:

$$\begin{aligned} v &:= \partial_x u \\ w &:= \partial_y u \; . \end{aligned}$$

With this notation, the equation can be rewritten in the form: $y\partial_x v - \partial_y w = 0$. These three equations form a formally deterministic system. Unfortunately, it is not symmetric (so it is not in the framework for the application of H-measures).

As in other examples, we can make use of the Schwarz symmetry for second derivatives: $\partial_y v = \partial_x w$. This gives us one more equation, and we are free to chose three out of four. The following choice leads to a symmetric system:

$$\begin{aligned} \partial_x u - v &= 0 \\ -y\partial_x v - \partial_y w &= 0 \\ \partial_x w - \partial_y v &= 0\,. \end{aligned}$$

The unknown u appears only in the first equation; so we can take this equation as its definition (assuming that the initial condition for u is given), and try to solve the system of two remaining equations, with unknows v and w.

We introduce the vector notation in the following way for the unknowns: $u_1 := v, u_2 := w$. For variables, we occasionally write $\mathbf{x}$ for (x, y) and $\boldsymbol{\xi}$ for (ξ, η). Now, any solution of the equation satisfies the symmetric hyperbolic system:

$$\mathbf{A}^1\partial_x \mathsf{u} + \mathbf{A}^2\partial_y \mathsf{u} = 0\,, \tag{12}$$

where the matrices are given by:

$$\mathbf{A}^1 := \begin{bmatrix} -y & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{A}^2 := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Clearly, $\mathbf{A}^1$ and $\mathbf{A}^2$ are symmetric, and for $y < 0$ the matrix $\mathbf{A}^1$ is positive definite (its (simple) eigenvalues are 1 and $-y$). Thus, a symmetric *hyperbolic* system corresponds to the Tricomi's equation in the *lower* half plane.

Let us try to see what we can learn about it by using H-measures.

In order to apply the H-measure theory, let us consider a sequence of solutions, such that $\mathsf{u}^\varepsilon \rightharpoonup 0$ in the space $\mathrm{L}^2(\Omega)$ (weakly). The H-measure corresponding to a subsequence of $(u_1^\varepsilon, u_2^\varepsilon)$ is a 2×2 Radon matrix measure; denote it by:

$$\boldsymbol{\mu} := \begin{bmatrix} \mu^{11} & \mu^{12} \\ \mu^{21} & \mu^{22} \end{bmatrix}.$$

Let us next write down the symbol, 2×2 matrix function $\mathbf{P} := \xi\mathbf{A}^1 + \eta\mathbf{A}^2$ defined on the spherical bundle (product) $\Omega \times S^1$:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = \begin{bmatrix} -y\xi & -\eta \\ -\eta & \xi \end{bmatrix}.$$

Lemma 4. *The H-measure $\boldsymbol{\mu}$ corresponding to a subsequence of a L^2 weakly convergent sequence of solutions of the symmetric system (12), associated to the Tricomi's equation (11), can be written as:*

$$\boldsymbol{\mu} = \begin{bmatrix} \xi^2 & \xi\eta \\ \eta\xi & \eta^2 \end{bmatrix} \nu\,,$$

where ν is a nonnegative (scalar) Radon measure on the spherical bundle $\Omega \times S^1$, supported inside the set $N = \{(x, y, \xi, \eta) \in \Omega \times S^1 : \eta^2 + y\xi^2 = 0\}$.

Dem. The localisation property for H-measures, in the case of a symmetric system, can be expressed in the form: $\mathbf{P}\boldsymbol{\mu} = \mathbf{0}$.

Let us first note that $\boldsymbol{\mu}$ is supported inside the set where $\det\mathbf{P} = 0$, and this reduces to the set N. Writing out all the terms of the product explicitly we get:

$$\begin{bmatrix} -y\xi & -\eta \\ -\eta & \xi \end{bmatrix} \begin{bmatrix} \mu^{11} & \mu^{12} \\ \mu^{21} & \mu^{22} \end{bmatrix} = \begin{bmatrix} -y\xi\mu^{11} - \eta\mu^{21} & -y\xi\mu^{12} - \eta\mu^{22} \\ -\eta\mu^{11} + \xi\mu^{21} & -\eta\mu^{12} + \xi\mu^{22} \end{bmatrix} = \mathbf{0} .$$

This gives us the following equations for the components of the measure $\boldsymbol{\mu}$:

$$\begin{aligned} -y\xi\mu^{11} &= \eta\mu^{21} \\ \eta\mu^{11} &= \xi\mu^{21} \\ -y\xi\mu^{12} &= \eta\mu^{22} \\ \eta\mu^{12} &= \xi\mu^{22} . \end{aligned} \tag{13}$$

Besides these relations given by the localisation property of H-measures, we know that $\boldsymbol{\mu}$ is hermitian, so that diagonal components are real, while $\mu^{21} = \overline{\mu^{12}}$.

From the second and the fourth equation we can get: $\xi^2\mu^{22} = \eta^2\mu^{11}$. Thus, it is natural to express the matrix measure $\boldsymbol{\mu}$ using only one scalar measure ν. As $\xi \neq 0$ on the set N, we define ν by: $\mu^{11} = \xi^2\nu$, and then follows that $\mu^{22} = \eta^2\nu$.

On the other hand, from the first two equations we directly conclude that the measures μ^{21} and μ^{12} are real (because μ^{11} is) and absolutely continuous (on any compact set) with respect to the measure μ^{11}. Clearly, they can be expressed using ν as stated above.

Q.E.D.

The lemma gives us the simple form of the localisation principle (for ν):

$$(y\xi^2 + \eta^2)\nu = 0 .$$

Clearly, for $y > 0$ (upper half plane, elliptic region), $\nu = 0$. At the coordinate line x (parabolic *region*), ν is supported on two opposite points on the circle S^1, namely for $\eta = 0$ (and thus $\xi = \pm 1$). For $y < 0$ (lower half plane, hyperbolic region) ν is supported at the null set of $\eta^2 - (-y)\xi^2$ or, for given y, on the intersection of the circle S^1 with the lines $\eta - \sqrt{-y}\xi = 0$ and $\eta + \sqrt{-y}\xi = 0$. Notice that these two lines have the same slope as the normals to the characteristics at the same point. Clearly, other entries of the original matrix measure are supported for $y < 0$ only, and there they are different from zero wherever ν is not zero.

Furthermore, we know that ξ and η are just the coordinates of a point on the unit circle S^1. This means that there exists an angle ϑ such that: $\xi = \cos\vartheta$ and $\eta = \sin\vartheta$. But, $\mathrm{tg}\vartheta = \frac{\eta}{\xi} = \pm\sqrt{-y}$, so ϑ is naturally restricted to the interval $\langle -\frac{\pi}{2}, \frac{\pi}{2}\rangle$ (at the boundary of that interval ν is zero).

More precisely, we have established that the measure ν (and then the rest of $\boldsymbol{\mu}$ as well) is supported on a three dimensional manifold $\mathbf{R}^2 \times S^1$. We can choose a parametrisation of the circle by the angle ϑ, so that the measure ν *will be zero outside two open submanifolds:* $\mathbf{R}^2 \times U_1$ and $\mathbf{R}^2 \times U_2$; where U_1 corresponds to $\vartheta \in \langle -\frac{\pi}{2}, \frac{\pi}{2}\rangle$ (while U_2 means $\vartheta \in \langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle$). Thus, we have reduced our study of the problem on the manifold to two charts, that are diffeomorphic to (open) layers in three dimensional space.

After simplifying our problem using the localisation property, let us apply the propagation property.

Theorem 4. *The scalar measure ν satisfies the equation:*

$$2y\xi^2\partial_x\nu + 2\eta\xi\partial_y\nu - \xi^3\partial_\eta\nu = 0$$

on its support. The projection of the characteristics of this equation to the x, y plane are the characteristics of the original (Tricomi's) equation.

Dem. The general formula for symmetric systems:

$$\langle \boldsymbol{\mu}, \{\mathbf{P}, \psi\} + \psi\partial_k\mathbf{A}^k\rangle + \langle 2\mathrm{Re}\,\mathrm{tr}\boldsymbol{\mu}_{12}, \psi\rangle = 0\ ,$$

where the test function ψ is homogeneous of degree zero in the variables ξ, η, reduces to:

$$\langle \boldsymbol{\mu}, \{\mathbf{P}, \psi\}\rangle = 0\ .$$

(the right hand side of the equation (11) is zero).

Let us compute the Poisson bracket:

$$\begin{aligned}\{\mathbf{P}, \psi\} &= \partial^l\mathbf{P}\,\partial_l\psi - \partial^l\psi\,\partial_l\mathbf{P}\\ &= \begin{bmatrix} -y & 0\\ 0 & 1\end{bmatrix}\partial_x\psi + \begin{bmatrix} 0 & -1\\ -1 & 0\end{bmatrix}\partial_y\psi - \begin{bmatrix} -\xi & 0\\ 0 & 0\end{bmatrix}\partial_\eta\psi\\ &= \begin{bmatrix} -y\partial_x\psi + \xi\partial_\eta\psi & -\partial_y\psi\\ -\partial_y\psi & \partial_x\psi\end{bmatrix}\ .\end{aligned}$$

Next take the scalar product of the matrices:

$$\begin{aligned}\langle \boldsymbol{\mu}, \{\mathbf{P}, \psi\}\rangle &= \left\langle \begin{bmatrix} \xi^2\nu & \xi\eta\nu\\ \eta\xi\nu & \eta^2\nu\end{bmatrix}, \begin{bmatrix} -y\partial_x\psi + \xi\partial_\eta\psi & -\partial_y\psi\\ -\partial_y\psi & \partial_x\psi\end{bmatrix}\right\rangle\\ &= \langle \nu, -\xi^2(y\partial_x\psi - \xi\partial_\eta\psi) - 2\xi\eta\partial_y\psi + \eta^2\partial_x\psi\rangle\\ &= \langle \nu, (\eta^2 - y\xi^2)\partial_x\psi - 2\xi\eta\partial_y\psi + \xi^3\partial_\eta\psi\rangle = 0\ .\end{aligned}$$

Thus, the measure ν satisfies the following equation:

$$\partial_x((\eta^2 - y\xi^2)\nu) - 2\xi\eta\partial_y\nu + \xi^3\partial_\eta\nu = 0\ ,$$

or, after taking into account the localisation identity $(\eta^2 + y\xi^2)\nu = 0$:

$$2y\xi^2\partial_x\nu + 2\eta\xi\partial_y\nu - \xi^3\partial_\eta\nu = 0\ .$$

The equations of the characteristics are given by:

$$\begin{aligned}\frac{dx}{d\tau} &= 2y\xi^2\\ \frac{dy}{d\tau} &= 2\eta\xi\\ \frac{d\eta}{d\tau} &= -\xi^3 = \mp(1-\eta^2)^{3/2}\ ,\end{aligned}$$

which gives us that $\frac{dx}{dy} = \pm\sqrt{-y}$, that is the equation satisfied by the characteristics of the Tricomi's equation.

Q.E.D.

Remark. The last equation for characteristics is separable, and gives: $\frac{\pm d\eta}{(1-\eta^2)^{3/2}} = d\tau$.

The substitution $\eta = \sin\varphi$, $d\eta = \cos\varphi d\varphi$ would give us $\int \frac{d\varphi}{\cos^2\varphi} = \pm\int d\tau$, or $\mathrm{tg}\varphi = \tau + c$. We can solve the other two equations, but the computations would be tedious. In the next remark a parametrisation is obtained with simpler computations. ∎

Remark. The equation we obtained in the theorem involves ξ and η as variables. Let us write the equation involving the parameter ϑ instead.

First note that we have already obtained during the proof of the theorem the following relation:

$$\langle \mu, \{\mathbf{P}, \psi\}\rangle = \langle \nu, (\eta^2 - y\xi^2)\partial_x\psi - 2\xi\eta\partial_y\psi + \xi^3\partial_\eta\psi\rangle = 0\ . \tag{14}$$

Note that the function ψ is homogeneous of degree zero in ξ and η, so for any $\alpha \in \mathbf{R}^+$ we have that $\psi(x, y, \alpha\xi, \alpha\eta) = \psi(x, y, \xi, \eta)$; so, after taking the derivative of both sides in α, we obtain: $\xi\partial_\xi\psi + \eta\partial_\eta\psi = 0$.

The function ψ can be written either in variables x, y, ξ and η, or in x, y and ϑ. Formally: $\psi(x, y, \xi, \eta) = \tilde\psi(x, y, \vartheta)$. From the relations: $\xi = \cos\vartheta$ and $\eta = \sin\vartheta$, we obtain:

$$\partial_\vartheta\tilde\psi = -\eta\partial_\xi\psi + \xi\partial_\eta\psi\ .$$

After combining with the above consequence of the homogeneity we get:

$$\begin{aligned}\xi\partial_\vartheta\tilde\psi &= \partial_\eta\psi\\ \eta\partial_\vartheta\tilde\psi &= -\partial_\xi\psi\ .\end{aligned} \tag{15}$$

Now, we can change the variables in (14) and obtain:

$$\langle \nu, (\sin^2\vartheta - y\cos^2\vartheta)\partial_x\tilde\psi - 2\cos\vartheta\sin\vartheta\partial_y\tilde\psi + \cos^4\vartheta\partial_\vartheta\tilde\psi\rangle = 0\ . \tag{16}$$

After placing the derivatives on the measure ν:

$$\partial_x\big((\sin^2\vartheta - y\cos^2\vartheta)\nu\big) - 2\cos\vartheta\sin\vartheta\partial_y\nu + \partial_\vartheta(\cos^4\vartheta\nu) = 0\ ,$$

or, taking into account the localisation identity $(\sin^2\vartheta + y\cos^2\vartheta)\nu = 0$:

$$2\sin^2\vartheta\partial_x\nu - 2\cos\vartheta\sin\vartheta\partial_y\nu + \cos^4\vartheta\partial_\vartheta\nu + 4\cos^3\vartheta\nu = 0\ .$$

The equations that the characteristics satisfy are:

$$\begin{aligned}\frac{dx}{d\tau} &= 2\sin^2\vartheta\\ \frac{dy}{d\tau} &= -2\cos\vartheta\sin\vartheta\\ \frac{d\vartheta}{d\tau} &= \cos^4\vartheta\ .\end{aligned}$$

Note that the last equation implies that ϑ increases with τ increasing.

If we introduce a new variable $\sigma = \mathrm{tg}\vartheta$, we get $d\sigma = \frac{d\vartheta}{\cos^2\vartheta} = \cos^2\vartheta d\tau$. As $\frac{1}{\cos^2\vartheta} = \mathrm{tg}^2\vartheta + 1 = \sigma^2 + 1$, after integration we get: $\frac{\sigma^3}{3} + \sigma = \tau + c$. As τ is only a parameter in parametrisation, we can choose $c = 0$. Thus, σ, ϑ and τ have the same sign.

Expressing the first two equations of characteristics in variable σ, we have:

$$\begin{aligned}\frac{dx}{d\tau} &= \frac{dx}{d\sigma}\frac{d\sigma}{d\tau} = 2\sin^2\vartheta\\ \frac{dy}{d\tau} &= \frac{dy}{d\sigma}\frac{d\sigma}{d\tau} = -2\cos\vartheta\sin\vartheta\ ,\end{aligned}$$

so, from $\frac{d\sigma}{d\tau} = \cos^2\vartheta$ we conclude that:

$$\frac{dx}{d\sigma} = 2\sigma^2$$
$$\frac{dy}{d\sigma} = -2\sigma\ .$$

This gives us $x(\sigma) = \frac{2}{3}\sigma^3 + x_0$ and $y(\sigma) = -\sigma^2 + y_0$. Choosing $y(0) = 0$ for $\vartheta = 0$ (coming from the initiall condition), we have finally got:

$$x(\vartheta) = \frac{2}{3}\mathrm{tg}^3\vartheta + x_0$$
$$y(\vartheta) = -\mathrm{tg}^2\vartheta\ .$$

∎

Remark. Let us construct a measure ν that is a solution of the equation given by the theorem 4. Using the parametrisation of the support, we take the following *ansatz* for ν:

$$w(\vartheta)\delta(\frac{2}{3}\mathrm{tg}^3\vartheta, -\mathrm{tg}^2\vartheta, \vartheta)\ .$$

Taking $\varphi \in \mathrm{C}_c(\mathbf{R}^2 \times \langle\frac{-\pi}{2}, \frac{\pi}{2}\rangle)$, the action of ν on φ can be written as:

$$\langle\nu, \varphi\rangle = \int w(\vartheta)\varphi\left(\frac{2}{3}\mathrm{tg}^3\vartheta, -\mathrm{tg}^2\vartheta, \vartheta\right) d\vartheta\ .$$

Assuming that $\tilde{\psi} \in \mathrm{C}_c(\mathbf{R}^2 \times \langle\frac{-\pi}{2}, \frac{\pi}{2}\rangle)$, we have:

$$\begin{aligned}\frac{d}{d\vartheta}\tilde{\psi}\left(\frac{2}{3}\mathrm{tg}^3\vartheta, -\mathrm{tg}^2\vartheta, \vartheta\right) &= \partial_x\tilde{\psi}\left(\frac{2}{3}\mathrm{tg}^3\vartheta, -\mathrm{tg}^2\vartheta, \vartheta\right)\frac{2\mathrm{tg}^2\vartheta}{\cos^2\vartheta} \\ &\quad - \partial_y\tilde{\psi}\left(\frac{2}{3}\mathrm{tg}^3\vartheta, -\mathrm{tg}^2\vartheta, \vartheta\right)\frac{2\mathrm{tg}\vartheta}{\cos^2\vartheta} + \partial_\vartheta\tilde{\psi}\left(\frac{2}{3}\mathrm{tg}^3\vartheta, -\mathrm{tg}^2\vartheta, \vartheta\right) \\ &= \frac{1}{\cos^4\vartheta}\left[(\sin^2\vartheta - y\cos^2\vartheta)\partial_x\tilde{\psi} - 2\cos\vartheta\sin\vartheta\partial_y\tilde{\psi} + \cos^4\vartheta\partial_\vartheta\tilde{\psi}\right]\ .\end{aligned}$$

Thus, the equation (16) reduces to:

$$\int w(\vartheta)\cos^4\vartheta\frac{d}{d\vartheta}\tilde{\psi}\left(\frac{2}{3}\mathrm{tg}^3\vartheta, -\mathrm{tg}^2\vartheta, \vartheta\right) d\vartheta = 0\ .$$

The last equation is certainly true for: $w(\vartheta)\cos^4\vartheta = C$, where C is an arbitrary constant. So, the measure ν, given by:

$$\langle\nu, \varphi\rangle = \int \frac{C}{\cos^4\vartheta}\varphi\left(\frac{2}{3}\mathrm{tg}^3\vartheta, -\mathrm{tg}^2\vartheta, \vartheta\right) d\vartheta\ ,$$

is a solution of the equation in Theorem 4. It moves along a characteristics, starting from $(-\infty, -\infty, -\frac{\pi}{2})$, bounces at the x axis, and continues towards $(\infty, -\infty, \frac{\pi}{2})$.

∎

A variant of Tricomi's equation

In order to get more insight into the above method, let us consider another equation, which is merely an academic example (with no physical meaning):

$$xy\partial_x^2 u + \partial_y^2 u = 0 \tag{17}$$

in the plane $\mathbf{R}^2$. Proceeding in the same way as before, using the standard method for classification of the second order equations, we see that $ac - b^2 = xy$, so the equation is elliptic for x and y of the same sign (thus, in the I. and the III. quadrant), hyperbolic for x and y of different sign (in the II. and IV. quadrant); while it is parabolic on both coordinate axes.

In the hyperbolic regions the characteristics satisfy the following ordinary differential equation:

$$xy(dy)^2 + (dx)^2 = 0 ,$$

that can be written for $x > 0, y < 0$ as $\frac{dx}{\sqrt{x}} = \pm\sqrt{-y}\,dy$; and for $x < 0, y > 0$ as: $\frac{dx}{\sqrt{-x}} = \pm\sqrt{y}\,dy$.

Restricting our attention to the first case only, let us determine the equation of the characteristics passing through the point (x_0, y_0) in the IV. quadrant:

$$\int_{x_0}^{x} \frac{d\xi}{\sqrt{\xi}} = \pm \int_{y_0}^{y} \sqrt{-\eta}\, d\eta \ .$$

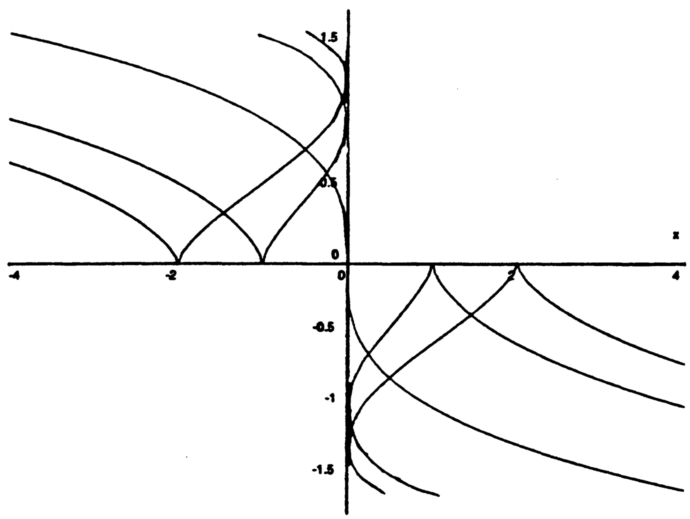
Integration leads to: $2(\sqrt{x} - \sqrt{x_0}) = \mp\frac{2}{3}(y\sqrt{-y} - y_0\sqrt{-y_0})$. Taking the square of both sides we finally obtain:

$$x(y) = [\frac{1}{3}(y\sqrt{-y} - y_0\sqrt{-y_0}) \mp \sqrt{x_0}]^2 \ .$$

In particular, we are interested in the points (x_0, y_0) on the x–axis as initial conditions, so $y_0 = 0$, and the equation of the characteristics reads:

$$x(y) = \Big(\frac{y\sqrt{-y}}{3} \mp \sqrt{x_0}\Big)^2 \ .$$

(It is possible to express y as a function of x locally, but that would not serve any purpose.) In the II. quadrant we have: $x(y) = -(y\sqrt{y}/3 \mp \sqrt{-x_0})^2$. We have the following picture:



Next we should write the equation as a symmetric system. In order to do that we introduce: $v := \partial_x u$ and $w := \partial_y u$. The equation reads: $xy\partial_x v + \partial_y w = 0$. The same choice of the equations as for the Tricomi's problem leads to the symmetric system:

$$\begin{aligned} \partial_x u - v &= 0 \\ -xy\,\partial_x v - \partial_y w &= 0 \\ \partial_x w - \partial_y v &= 0\,. \end{aligned}$$

If we introduce the same notation as before (for vector unknown and variables), and take the first equation as the definition for u, we get the following system:

$$\begin{bmatrix} -xy & 0 \\ 0 & 1 \end{bmatrix} \partial_x \mathsf{u} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \partial_y \mathsf{u} = 0\,. \tag{18}$$

The matrices $\mathbf{A}^1$ and $\mathbf{A}^2$ (named as before) are symmetric, and in the II. and IV. quadrant $\mathbf{A}^1$ is positive definite; so the system is hyperbolic there.

Proceeding as before, we consider a sequence of solutions, and a subsequence defining the H-measure $\boldsymbol{\mu}$. The function $\mathbf{P}$ reads now:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \begin{bmatrix} -xy\xi & -\eta \\ -\eta & \xi \end{bmatrix}\;;$$

while the localisation principle $\mathbf{P}\boldsymbol{\mu} = \mathbf{0}$ gives us:

$$\begin{bmatrix} -xy\xi & -\eta \\ -\eta & \xi \end{bmatrix} \begin{bmatrix} \mu^{11} & \mu^{12} \\ \mu^{21} & \mu^{22} \end{bmatrix} = \begin{bmatrix} -xy\xi\mu^{11} - \eta\mu^{21} & -xy\xi\mu^{12} - \eta\mu^{22} \\ -\eta\mu^{11} + \xi\mu^{21} & -\eta\mu^{12} + \xi\mu^{22} \end{bmatrix} = \mathbf{0}\,.$$

So, the equations for the components of the measure $\boldsymbol{\mu}$ are:

$$\begin{aligned} -xy\xi\mu^{11} &= \eta\mu^{21} \\ \eta\mu^{11} &= \xi\mu^{21} \\ -xy\xi\mu^{12} &= \eta\mu^{22} \\ \eta\mu^{12} &= \xi\mu^{22}\,. \end{aligned}$$

As for the Tricomi's equation, we can prove:

Lemma 5. *The H-measure $\boldsymbol{\mu}$ corresponding to a subsequence of a L^2 weakly convergent sequence of solutions of the symmetric system (18), associated to the equation (14), can be written as:*

$$\boldsymbol{\mu} = \begin{bmatrix} \xi^2 & \xi\eta \\ \eta\xi & \eta^2 \end{bmatrix} \nu\,,$$

where ν is a nonnegative (scalar) Radon measure on the spherical bundle $\Omega \times S^1$, satisfying: $(xy\xi^2 + \eta^2)\nu = 0$, and is supported inside the set $N = \{(x, y, \xi, \eta) \in \Omega \times S^1 : \eta^2 + xy\xi^2 = 0\}$.

∎

Clearly, for $xy > 0$ (I. and III. quadrant), $\nu = 0$. At the coordinate lines ν has the support contained in the set where $\eta = 0$ (thus $\xi = \pm 1$). For $xy < 0$ (II. and IV. quadrant), ν is supported on the null set of the polynomial $\eta^2 - (-xy)\xi^2$ or, for a given point (x, y), on the intersection of the circle S^1 with the lines $\eta - \sqrt{-xy}\xi = 0$ and $\eta + \sqrt{-xy}\xi = 0$. As before, these lines have the same slope as the normals to the characteristics at the same point. Clearly, other components of the H-measure are supported for $xy < 0$ only, and are different than zero wherever ν is.

Again, we can parametrise the circle by the angle ϑ; then $\mathrm{tg}\vartheta = \frac{\eta}{\xi} = \pm\sqrt{-xy}$; thus reducing the problem from the manifold to two charts.

Let us now apply the propagation property (given by the general formula):

$$\langle \boldsymbol{\mu}, \{\mathbf{P}, \psi\} + \psi \partial_k \mathbf{A}^k \rangle + \langle 2\mathrm{Re}\,\mathrm{tr}\boldsymbol{\mu}_{12}, \psi \rangle = 0 \,.$$

First, we have:

$$\partial_x \mathbf{A}^1 = \begin{bmatrix} -y & 0 \\ 0 & 0 \end{bmatrix} .$$

The Poisson bracket will have more terms as well:

$$\begin{aligned} \{\mathbf{P}, \psi\} &= \partial^l \mathbf{P}\, \partial_l \psi - \partial^l \psi\, \partial_l \mathbf{P} \\ &= \begin{bmatrix} -xy & 0 \\ 0 & 1 \end{bmatrix} \partial_x \psi + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \partial_y \psi - \begin{bmatrix} -y\xi & 0 \\ 0 & 0 \end{bmatrix} \partial_\xi \psi - \begin{bmatrix} -x\xi & 0 \\ 0 & 0 \end{bmatrix} \partial_\eta \psi \\ &= \begin{bmatrix} -xy\partial_x\psi + y\xi\partial_\xi\psi + x\xi\partial_\eta\psi & -\partial_y\psi \\ -\partial_y\psi & \partial_x\psi \end{bmatrix} . \end{aligned}$$

We can now compute:

$$\begin{aligned} \langle \boldsymbol{\mu}, \{\mathbf{P}, \psi\} \rangle &= \left\langle \begin{bmatrix} \xi^2\nu & \xi\eta\nu \\ \eta\xi\nu & \eta^2\nu \end{bmatrix}, \begin{bmatrix} -xy\partial_x\psi + y\xi\partial_\xi\psi + x\xi\partial_\eta\psi & -\partial_y\psi \\ -\partial_y\psi & \partial_x\psi \end{bmatrix} \right\rangle \\ &= \langle \nu, (\eta^2 - xy\xi^2)\partial_x\psi - 2\xi\eta\partial_y\psi + \xi^3(y\partial_\xi\psi + x\partial_\eta\psi) \rangle . \end{aligned}$$

After adding the term $-y\xi^2\psi$, coming from $\partial_x \mathbf{A}^1$, we obtain:

$$\langle \nu, (\eta^2 - \xi^2 yx)\partial_x\psi - 2\xi\eta\partial_y\psi + \xi^3(y\partial_\xi\psi + x\partial_\eta\psi) - y\xi^2\psi \rangle = 0 \,. \tag{19}$$

This gives the following equation (in the sense of distribution) satisfied by ν:

$$\partial_x((\eta^2 - xy\xi^2)\nu) - 2\xi\eta\partial_y\nu + \partial_\xi(\xi^3 y\nu) + \partial_\eta(\xi^3 x\nu) + y\xi^2\nu = 0 \,.$$

Thus, we have proven the following:

Theorem 5. *The scalar measure ν satisfies the equation:*

$$2\eta^2\partial_x\nu - 2\xi\eta\partial_y\nu + \xi^3(y\partial_\xi\nu + x\partial_\eta\nu) - 2\xi^2 y\nu = 0 \,.$$

The projection of the characteristics of this equation to the x, y plane are the characteristics of the original equation (16). ∎

Remark. The equations of the characteristics are given by:

$$
\begin{aligned}
\frac{dx}{d\tau} &= 2\eta^2 \\
\frac{dy}{d\tau} &= -2\xi\eta \\
\frac{d\xi}{d\tau} &= \xi^3 y \\
\frac{d\eta}{d\tau} &= \xi^3 x \,.
\end{aligned}
$$

∎

Remark. Replacing ξ and η by the parameter ϑ in the equation (18), using the change of variables formule obtained in the remark following Theorem 4, we get:

$$\Big\langle \nu, (\sin^2\vartheta - xy\cos^2\vartheta)\partial_x\tilde{\psi} - 2\cos\vartheta\sin\vartheta\partial_y\tilde{\psi} + \cos^3\vartheta(-y\sin\vartheta + x\cos\vartheta)\partial_\vartheta\tilde{\psi} - y\cos^2\vartheta\tilde{\psi}\Big\rangle = 0\,,$$

or in the sense of distributions:

$$\partial_x\big((\sin^2\vartheta - xy\cos^2\vartheta)\nu\big) - 2\cos\vartheta\sin\vartheta\partial_y\nu + \partial_\vartheta\big(\cos^3\vartheta(x\cos\vartheta - y\sin\vartheta)\nu\big) + y\cos^2\vartheta\nu = 0\,.$$

After using the localisation property we obtain:

$$2\sin^2\vartheta\partial_x\nu - 2\cos\vartheta\sin\vartheta\partial_y\nu + \cos^3\vartheta(x\cos\vartheta - y\sin\vartheta)\partial_\vartheta\nu + 4\cos^2\vartheta\sin\vartheta(x\cos\vartheta - y\sin\vartheta)\nu = 0\,.$$

The characteristics satisfy:

$$
\begin{aligned}
\frac{dx}{d\tau} &= 2\sin^2\vartheta \\
\frac{dy}{d\tau} &= -2\cos\vartheta\sin\vartheta \\
\frac{d\vartheta}{d\tau} &= \cos^3\vartheta(x\cos\vartheta - y\sin\vartheta)\,.
\end{aligned}
$$

∎

Linear second order equation of mixed type

Next we shall try to state the main results, which we have obtained for two special cases, in a general setting. For a function $a \in \mathrm{C}^1(\Omega; \mathbf{R})$ we study the equation:

$$a\partial_x^2 u + \partial_y^2 u = 0\,. \tag{20}$$

The characteristics of (20) satisfy following ordinary differential equation:

$$\frac{dx}{dy} = \pm\sqrt{-a}\,,$$

and they are real for $a \leq 0$.

Introducing the same two unknown functions (first partial derivatives of u) as before, this equation can be reduced to the following symmetric system:

$$\begin{bmatrix} -a & 0 \\ 0 & 1 \end{bmatrix} \partial_x \mathsf{u} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \partial_y \mathsf{u} = \mathbf{0} \,. \tag{21}$$

Obviously, this system is symmetric, and for $a < 0$ it is even symmetric hyperbolic.

We consider a sequence of solutions $\mathsf{u}^\varepsilon \longrightarrow 0$ in L^2, and the associated H-measure $\boldsymbol{\mu}$. Using the same notation as before, we obtain the expression for the symbol:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \begin{bmatrix} -a(x,y)\xi & -\eta \\ -\eta & \xi \end{bmatrix} .$$

Lemma 6. *The H-measure $\boldsymbol{\mu}$ corresponding to a subsequence of L^2 weakly convergent sequence of solutions of the symmetric system (21), associated to the equation (17), can be written as:*

$$\boldsymbol{\mu} = \begin{bmatrix} \xi^2 & \xi\eta \\ \eta\xi & \eta^2 \end{bmatrix} \nu \,,$$

where ν is a nonnegative (scalar) Radon measure on the spherical bundle $\Omega \times S^1$, supported inside the set $N = \{(x,y,\xi,\eta) \in \Omega \times S^1 : \eta^2 + a\xi^2 = 0\}$.

Dem. The H-measure $\boldsymbol{\mu}$ is supported inside the set where $\det \mathbf{P} = 0$, and this is exactly the set N. Writing out all the terms of the product explicitly we get:

$$\begin{bmatrix} -a\xi & -\eta \\ -\eta & \xi \end{bmatrix} \begin{bmatrix} \mu^{11} & \mu^{12} \\ \mu^{21} & \mu^{22} \end{bmatrix} = \begin{bmatrix} -a\xi\mu^{11} - \eta\mu^{21} & -a\xi\mu^{12} - \eta\mu^{22} \\ -\eta\mu^{11} + \xi\mu^{21} & -\eta\mu^{12} + \xi\mu^{22} \end{bmatrix} = \mathbf{0} \,.$$

Besides these relations given by the localisation property of H-measures, we know that $\boldsymbol{\mu}$ is hermitian, so that diagonal components are real, while $\mu^{21} = \overline{\mu^{12}}$.

From the second row we get: $\xi^2\mu^{22} = \eta^2\mu^{11}$. Thus, it is natural to express the matrix measure $\boldsymbol{\mu}$ using only one scalar measure ν. As $\xi \neq 0$ on the set N, we define ν by: $\mu^{11} = \xi^2\nu$, and then follows that $\mu^{22} = \eta^2\nu$.

On the other hand, from the first column we directly conclude that the measures μ^{21} and μ^{12} are real (because μ^{11} is) and absolutely continuous (on any compact set) with respect to the measure μ^{11}. Clearly, they can be expressed using ν as stated above.

Q.E.D.

The lemma gives us a simple form of the localisation principle (for ν):

$$(a(x,y)\xi^2 + \eta^2)\nu = 0 \,.$$

From the lemma we can conclude that in the elliptic region ($a > 0$) $\nu = 0$. In the *parabolic region* ($a = 0$) the support of ν is contained in the set where $\eta = 0$ (and thus $\xi = \pm 1$).

In the hyperbolic region ($a < 0$) we have, as before, that ν is possibly supported on the null set of the function $\eta^2 + a(x,y)\xi^2$. For the given point (x,y), this turns out to be at the intersection of the circle S^1 with the lines $\eta - \sqrt{-a(x,y)}\xi = 0$ and $\eta + \sqrt{-a(x,y)}\xi = 0$. As before, these lines are in the normal direction to the characteristics ($\frac{dx}{dy} = \pm\sqrt{-a(x,y)}$).

Theorem 6. *Under the above assumptions on a, the H-measure ν corresponding to a subsequence of solutions of the equation (20) satisfies the following equation (in the sense of distributions):*

$$2\eta^2\partial_x\nu - 2\xi\eta\partial_y\nu + \xi^3(\nabla_{\mathbf{x}}a\cdot\nabla_\xi\nu) + 4\xi^2\partial_x a\nu = 0\ .$$

The projection of the characteristics of this equation to the x, y plane are the characteristics of the original equation (17).

Dem. Let us first compute the Poisson's bracket:

$$\begin{aligned}\{\mathbf{P},\psi\} &= \begin{bmatrix}-a & 0\\ 0 & 1\end{bmatrix}\partial_x\psi + \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}\partial_y\psi - \begin{bmatrix}-\xi\partial_x a & 0\\ 0 & 0\end{bmatrix}\partial_\xi\psi - \begin{bmatrix}-\xi\partial_y a & 0\\ 0 & 0\end{bmatrix}\partial_\eta\psi\\ &= \begin{bmatrix}-a(x,y)\partial_x\psi + \xi(\partial_x a\,\partial_\xi\psi + \partial_y a\,\partial_\eta\psi) & -\partial_y\psi\\ -\partial_y\psi & \partial_x\psi\end{bmatrix}.\end{aligned}$$

Next we note that $\partial_k\mathbf{A}^k = \begin{bmatrix}-\partial_x a & 0\\ 0 & 0\end{bmatrix}$. The general formula for symmetric systems (the right hand side of the equation (17) is zero) gives us:

(22)
$$\begin{aligned}\langle\mu, \{\mathbf{P},\psi\} + \psi\partial_k\mathbf{A}^k\rangle &= \left\langle\nu, \begin{bmatrix}\xi^2 & \xi\eta\\ \eta\xi & \eta^2\end{bmatrix}\begin{bmatrix}-a\partial_x\psi - \partial_x a\psi + \xi(\nabla_{\mathbf{x}}a\cdot\nabla_\xi\psi) & -\partial_y\psi\\ -\partial_y\psi & \partial_x\psi\end{bmatrix}\right\rangle\\ &= \langle\nu, (\eta^2 - a\xi^2)\partial_x\psi - 2\xi\eta\partial_y\psi + \xi^3(\nabla_{\mathbf{x}}a\cdot\nabla_\xi\psi) - \partial_x a\xi^2\psi\rangle = 0\ .\end{aligned}$$

Placing the derivatives on the measure ν we get:

$$\partial_x((\eta^2 - a\xi^2)\nu) - 2\xi\eta\partial_y\nu + \partial_\xi(\partial_x a\xi^3\nu) + \partial_\eta(\partial_y a\xi^3\nu) + \partial_x a\xi^2\nu = 0\ ,$$

which, after taking into account the localisation property $(a\xi^2 + \eta^2)\nu = 0$, becomes:

$$2\eta^2\partial_x\nu - 2\xi\eta\partial_y\nu + \xi^3(\nabla_{\mathbf{x}}a\cdot\nabla_\xi\nu) + 4\xi^2\partial_x a\nu = 0\ .$$

The characteristics satisfy the system:

$$\begin{aligned}\frac{dx}{d\tau} &= 2\eta^2\\ \frac{dy}{d\tau} &= -2\xi\eta\\ \frac{d\xi}{d\tau} &= \xi^3\partial_x a\\ \frac{d\eta}{d\tau} &= \xi^3\partial_y a\ ,\end{aligned}$$

and the projections to the (x, y) plane are clearly the characteristics of the equation (17).

Q.E.D.

Remark. Replacing ξ and η by the parameter ϑ in the equation (22), using the change of variables formula obtained in the remark following Theorem 4, we get:

$$\left\langle \nu, \left(\sin^2\vartheta - xy\cos^2\vartheta\right)\partial_x\tilde{\psi} - 2\cos\vartheta\sin\vartheta\partial_y\tilde{\psi} + \cos^3\vartheta\left(-\partial_x a\sin\vartheta + \partial_y a\cos\vartheta\right)\partial_\vartheta\tilde{\psi} - \partial_x a\cos^2\vartheta\tilde{\psi}\right\rangle = 0\ ,$$

or in the sense of distributions:

$$\partial_x\left((\sin^2\vartheta - xy\cos^2\vartheta)\nu\right) - 2\cos\vartheta\sin\vartheta\partial_y\nu + \partial_\vartheta\left(\cos^3\vartheta(-\partial_x a\sin\vartheta + \partial_y a\cos\vartheta)\nu\right) + \partial_x a\cos^2\vartheta\nu = 0\ .$$

After using the localisation property we obtain:

$$2\sin^2\vartheta\partial_x\nu - 2\cos\vartheta\sin\vartheta\partial_y\nu + \cos^3\vartheta(-\partial_x a\sin\vartheta + \partial_y a\cos\vartheta)\partial_\vartheta\nu + 4\cos^2\vartheta\sin\vartheta(\partial_x a\sin\vartheta - \partial_y a\cos\vartheta)\nu = 0\ .$$

The characteristics satisfy:

$$\begin{aligned}\frac{dx}{d\tau} &= 2\sin^2\vartheta\\ \frac{dy}{d\tau} &= -2\cos\vartheta\sin\vartheta\\ \frac{d\vartheta}{d\tau} &= \cos^3\vartheta(-\partial_x a\sin\vartheta + \partial_y a\cos\vartheta)\ .\end{aligned}$$

∎

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