Anisotropic Singular Perturbations - The Vectorial Case

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Abstract. We obtain the r(L*(Q)) - limit of the sequence

$$J_\varepsilon(u) = \frac{1}{\varepsilon} E_\varepsilon(u)$$

where $E_£$ is the family of anisotropic perturbations

$$Ee(u) := j\ W(u(x))\ dx + e^2\ J\ h^*(x.Vu(x))\ dx$$

of the nonconvex functional of vector-valued functions

$$E_0(u) = \int_\Omega W(u(x))\ dx.$$

The proof relies on the blow-up argument introduced by Fonseca and Müller [FMl].

Table of Contents

1. Introduction.

In this paper we obtain the $\Gamma(L^1(\Omega))$ - limit of a family of anisotropic singular perturbations of a nonconvex functional in the vector-valued case. The study of this problem was motivated by the analysis of variational problems for phase transitions.

We consider the nonconvex energy

$$E(u) = \int_\Omega W(u(x))\, dx \tag{1.1}$$

where Ω is an open, bounded, strongly Lipschitz domain of $\mathbb{R}^N$, $u : \Omega \to \mathbb{R}^P$ and W supports two phases. The problem

(P) minimize E(.) subject to the constraint

$$\frac{1}{\text{meas}(\Omega)}\int_\Omega u(x)\, dx = m, \text{ where } m = \theta a + (1-\theta)b \text{ for some } \theta \in (0,1) \tag{1.2}$$

has infinitely many solutions which are piecewise constant functions of bounded variation, $u = \chi_A a + (1-\chi_A)b$ with $\text{meas}(A) = \theta\text{meas}(\Omega)$. In order to determine a selection criterion for resolving this non-uniqueness one studies the properties of the limits of sequences of minimizers for the quasiconvex perturbed problems

$$E_\varepsilon(u) = \int_\Omega [W(u(x)) + \varepsilon^2 h^2(x,\nabla u(x))]\, dx \tag{1.3}$$

where the relevant notion of convergence in this context is Γ-convergence as introduced by De Giorgi [DG] (see [At], [DM], [DD] for more recent expositions). Hence we are lead to the problem of identifying the $\Gamma(L^1(\Omega))$ - limit of the rescaled energies

$$J_\varepsilon(u) := \frac{1}{\varepsilon} E_\varepsilon(u).$$

We show that if W satisfies a certain growth condition and attains the minimum value of zero at exactly two points *a* and *b* and if h grows at most linearly in the last argument and satisfies some technical continuity conditions (see Section 2) then the $\Gamma(L^1(\Omega))$ - limit of $J_\varepsilon(.)$ is given by

$$J_0(u) = \begin{cases} \int_{\Omega\cap\partial^*\{u=a\}} K(x,a,b,\nu(x))\, dH_{N-1}(x) & \text{if } u \in \{a,b\} \text{ a.e., } u \in BV \\ +\infty & \text{otherwise} \end{cases}$$

where $\nu(x)$ is the normal to the interface $\Omega\cap\partial^*\{u=a\}$,

$$K(x, a, b, \nu(x)) := \inf\left\{ \int_{Q_\nu} [LW(\xi(y)) + \tfrac{1}{L}(h^{**})^2(x,\nabla\xi(y))]\, dy : \xi \in \mathcal{A}(a, b, \nu(x)), L > 0 \right\},$$

$\mathcal{A}(a, b, \nu) := \{\xi \in H^1(Q_\nu;\mathbb{R}^P) : \xi(y) = a \text{ if } y.\nu = -1/2, \xi(y) = b \text{ if } y.\nu = 1/2$, and ξ is periodic with period one in the directions of $\nu_1,\dots,\nu_{N-1}\}$,

$\{\nu_1,\ldots,\nu_{N-1},\nu\}$ forms an orthonormal basis of $\mathbb{R}^N$, Q_ν is the open unit cube centered at the origin with two of its faces normal to ν and the *recession function* h^∞ is given by (see [FM2])

$$h^\infty(x,A) := \limsup_{t\to+\infty} \frac{h(x,tA)}{t}.$$

We will also show that a sequence of minimizers of (1.3) will single out the solution of (P) for which

$$\int_{\Omega\cap\partial^*\{u=a\}} K(x,a,b,\nu(x))\, dH_{N-1}(x)$$

is a minimum, recovering the Wulff shape as the preferred equilibrium configuration (see [Fo], [FM3], [T1], [T2], [W]).

As remarked by Gurtin [G2], the assumption that W has two potential wells of equal depth involves no loss of generality; indeed, because of the constraint (1.2) we can always add an affine function of u to the integrand in (1.1) without changing the solution set of (P).

In the isotropic scalar case, i.e. if $u : \Omega \to \mathbb{R}$ and $h = \|.\|$, the $\Gamma(L^1(\Omega))$ - limit of $J_\varepsilon(.)$, $J_0(.)$, was studied by Gurtin [G1], [G2] and Modica [Mo] who showed that

$$J_0(u) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\, dx : u_n \in W^{1,1}(\Omega;\mathbb{R}),\ u_n \to u \text{ in } L^1 \right\}$$

where $f(x,u,A) = 2\sqrt{W(u)}\, h(A)$. This result was generalized by Owen and Sternberg [OS] to anisotropic functions h with linear growth for which h^2 is convex. The isotropic vector valued case, i.e. if $u : \Omega \to \mathbb{R}^p$ $(p>1)$ and $h = \|.\|$, was studied by Kohn and Sternberg [KS], by Sternberg [S] and by Fonseca and Tartar [FT] who obtained the representation

$$J_0(u) := \begin{cases} K\, \mathrm{Per}_\Omega(\{u=a\}) & \text{if } u(x) \in \{a,b\} \text{ a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

where

$$K = 2 \inf \left\{ \int_{-1}^{1} \sqrt{W(g(s))}\, |g'(s)|\, ds : g \text{ is piecewise } C^1,\ g(-1) = a,\ g(1) = b \right\}.$$

The paper is organized as follows; in Section 2 we mention some results on functions of bounded variation and sets of finite perimeter and state the theorem characterizing the Γ - limit of the functionals J_ε (see Theorem 2.9). In Section 3 we obtain a lower bound for the Γ - limit and in Section 4 we conclude the proof of Theorem 2.9 by constructing sequences $u_n \in H^1(\Omega;\mathbb{R}^p)$ and $\varepsilon_n \to 0^+$ such that

$$\lim_{n\to+\infty} J_{\varepsilon_n}(u_n) = J_0(u).$$

The results of Sections 3 and 4 rely on a lemma (cf. Lemma 3.2) which allows us to modify a sequence near the boundary without increasing its total energy. In Section 5 we show that the Γ-limit of a sequence of minimizers of (1.3) selects the solution of (P) which minimizes the integral over the interface of the surface energy density.

2. Preliminaries. Statement of the Theorem.

In what follows $\Omega \subset \mathbb{R}^N$ is an open, bounded, strongly Lipschitz domain, p, N ≥ 1, $\{e_1,...,e_N\}$ is the standard orthonormal basis of $\mathbb{R}^N$ and $M^{p\times N}$ is the vector space of all pxN real matrices. If $A \in M^{p\times N}$ let $\|A\| := (\mathrm{tr}(A^T A))^{1/2}$.

Given $\nu \in S^{N-1} := \{x \in \mathbb{R}^N : \|x\| = 1\}$ we denote by Q_ν the open unit cube centered at the origin with two of its faces normal to ν, i.e. if $\{\nu_1,...,\nu_{N-1},\nu\}$ is an orthonormal basis of $\mathbb{R}^N$ then

$$Q_\nu := \{x \in \mathbb{R}^N : |x.\nu_i| < \frac{1}{2}, |x.\nu| < \frac{1}{2}, i = 1,...,N-1\}.$$

Definition 2.1.([DG]) $J_0(.)$ is the *$\Gamma(L^1(\Omega))$ - limit* of the sequence $J_\varepsilon(.)$ if and only if
i) given any $u \in L^1(\Omega;\mathbb{R}^p)$ and any sequence u_ε such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^p)$

$$J_0(u) \leq \liminf_{\varepsilon\to0^+} J_\varepsilon(u_\varepsilon);$$

ii) given any $u \in L^1(\Omega;\mathbb{R}^p)$ there exists a sequence $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^p)$ such that

$$J_0(u) = \lim_{\varepsilon\to0^+} J_\varepsilon(u_\varepsilon).$$

We recall briefly some facts on functions of bounded variation and sets of finite perimeter which will be of later use in this paper. For more details we refer the reader to Evans and Gariepy [EG], Federer [F], Giusti [G] and Ziemer [Z].

Definition 2.2. A function $u \in L^1(\Omega;\mathbb{R}^p)$ is said to be of *bounded variation*, $u \in BV(\Omega;\mathbb{R}^p)$, if for all $i \in \{1,...,p\}$, $j \in \{1,...,N\}$ there exists a Radon measure μ_{ij} such that

$$\int_\Omega u_i(x)\frac{\partial\phi}{\partial x_j}(x)\, dx = -\int_\Omega \phi(x) d\mu_{ij}$$

for every $\phi \in C_0^1(\Omega)$. The distributional derivative Du is the matrix-valued measure with components μ_{ij}.

Definition 2.3. A set $A \subset \Omega$ is said to be of *finite perimeter* in Ω if $\chi_A \in BV(\Omega)$, where χ_A denotes the characteristic function of A. The *perimeter of A in Ω* is defined by

$$\mathrm{Per}_\Omega(A) := \sup\{ \int_A \mathrm{div}\phi(x)\, dx : \phi \in C_0^1(\Omega;\mathbb{R}^N), \|\phi\|_\infty \leq 1\}. \quad (2.1)$$

The *approximate upper* and *lower limit* of each component u*, for all i € {l,...,p), are given by

u*(x) := inf {t € IR : lim$_{E«>}$o*"^{L} i?$_{N}$[{ui > t) n B(x,e)] * 0}
e^{N}

and

uT(x) := sup {t € IR : lim$_{e}$_>o* «^N [{UI < t} n B(x,e)] « 0}
e^{N}

where B(x,e) is the open ball centered at x and with radius e. The set X(u) is called the *singular set ofu* or *jump set* and is defined as

p
i=1
Ku)=u{x€Q:uT(x)<uf(x)}.

It is well known that £(u) is N-1 rectifiable, i.e.

X(u)= n=1 u K $_{n}$ u E

where HM.JCE) = 0 and K_n is a compact subset of a C^1 hypersurface.

Theorem 2.4. If u *e* BV(£2;[R^{P}) then for H$_{N}$.! a.e. x € X(u) there exists a unit vector v(x) € S^{N-1}, *normal to £(u) atx,* and there exist vectors u~(x), u+(x) € [R^{p} such that

lim$_{£}$-»o+ $\frac{1}{eN}$ J |u(y) - u+(x)l^{N}/(^{N}-D dy = 0,
{y€B(x$_{f}$E):(y-x).v(x)>0}

lim$_{£}$^$_{0}^{+}$ $\frac{1}{£N}$ J |u(y) - u-(x)|N/(N-i) $_{dy}$ = 0.
{y€B(x,£):(y-x).v(x)<0}

We note that it may happen that u*(x) * (u$^{\pm}$(x))i.

If u_n € BVCQitRP) converges to u in LKQjtRP) then

IDul (Q) ≤ Um inf$_{n}$-^ IDuJ *(Q)* (2.2)

where IDul denotes the total variation measure of Du. If u € BV(Q;[R^{P}) then Du may be represented as

Du = Vu dx + (u+ - u-) ® v dH$_{N}$.! [I(u) + C(u) (2.3)

where Vu is the density of the absolutely continuous part of Du with respect to the N dimensional Lebesgue measure $£_N$ and H$_{N}$.i is the N-1 dimensional Hausdorff measure. The three measures in (2.3) are mutually singular, if H$_{N}$.!(B) < +<» then IC(u)l = 0 and there exists a Bonel set E such that *J£$_{s}$* (E) -0 and IC(u)l(B) = IC(u)l(BnE) for all Borel sets BC D=1^{N}. The following version of the Besicovitch Differentiation Theorem was proven by Ambrosio and Dal Maso, [ADM] Proposition 2.2.

Theorem 2.5. If λ and μ are Radon measures in Ω, $\mu \geq 0$, then there exists a Borel set E $\subset \Omega$ such that $\mu(E) = 0$ and for every $x \in (\text{supp}\, \mu) \setminus E$

$$\frac{d\lambda}{d\mu}(x) := \lim_{\varepsilon \to 0^+} \frac{\lambda(x+\varepsilon C)}{\mu(x+\varepsilon C)}$$

exists and is finite whenever C is a bounded, convex, open set containing the origin.

Theorem 2.6. Let A be a subset of Ω such that $Per_\Omega(A) < +\infty$. There exists a sequence of polyhedral sets $\{A_k\}$ (i.e. A_k are bounded, strongly Lipschitz domains with $\partial A_k = H_1 \cup H_2 \cup \ldots \cup H_p$ where each H_i is a closed subset of a hyperplane $\{x \in \mathbb{R}^N : x.v_i = \alpha_i\}$) satisfying the following properties:

i) $\mathcal{L}_N[((A_k \cap \Omega)\setminus A) \cup (A \setminus (A_k \cap \Omega))] \to 0$ as $k \to +\infty$;

ii) $Per_\Omega(A_k) \to Per_\Omega(A)$ as $k \to +\infty$;

iii) $H_{N-1}(\partial A_k \cap \partial \Omega) = 0$;

iv) $\mathcal{L}_N(A_k) = \mathcal{L}_N(A)$.

It can be shown that (see [FM2], Lemma 2.6) if $Per_\Omega(A) < +\infty$ then for H_{N-1} a.e. $x \in \Omega \cap \partial^* A$

$$\lim_{\delta \to 0^+} \frac{1}{\delta^{N-1}} H_{N-1}((\Omega \cap \partial^* A) \cap (x + \delta Q_{v(x)})) = 1. \tag{2.4}$$

Let $W : \mathbb{R}^p \to [0, +\infty)$ and $h : \Omega \times M^{p \times N} \to [0, +\infty)$ be continuous functions satisfying the following hypotheses:

(H1) $W(u) = 0$ if and only if $u \in \{a,b\}$;

(H2) there exist constants $c_1, c > 0$ such that

$$c_1 \|u\|^q - c \leq W(u) \leq c\,(1 + \|u\|^q)$$

for all $u \in \mathbb{R}^p$ and for some $q \geq 2$;

(H3) there exist constants $C_1, C > 0$ such that

$$C_1 \|A\| - C \leq h(x, A) \leq C\,(1 + \|A\|)$$

for all $x \in \Omega$ and for all $A \in M^{p \times N}$.

Let $h^\infty : \Omega \times M^{p \times N} \to [0, +\infty)$ be the *recession function*, i.e.

$$h^\infty(x, A) := \limsup_{t \to +\infty} \frac{h(x, tA)}{t}.$$

In addition to (H1)-(H3) we will also need the following hypotheses:

(H4) there exist $0 < m < 2$, $C', L > 0$ such that

$$\left| (h^\infty)^2(x, A) - \frac{h^2(x, tA)}{t^2} \right| \leq C' \frac{\|A\|^{2-m}}{t^m}$$

for all $(x, A) \in \Omega \times M^{p \times N}$ and for all $t > 0$ such that $t\,\|A\| > L$;

(H5) for all $x_0 \in \Omega$ and for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$| h^2(x_0, A) - h^2(x, A) | \le \varepsilon C (1 + \|A\|^2)$$

whenever $|x - x_0| < \delta$.

It is an easy consequence of the definition of recession function that

Lemma 2.7. Under the hypotheses (H3) and (H5) the following hold:
i) $C_1 \|A\| \le h^\infty(x, A) \le C \|A\|$, for every $(x, A) \in \Omega \times M^{p\times N}$;
ii) For all $x_0 \in \Omega$ and for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$| (h^\infty)^2(x_0, A) - (h^\infty)^2(x, A) | \le \varepsilon C \|A\|^2$$

whenever $|x - x_0| < \delta$.

Let $(a, b, \nu) \in \mathbb{R}^p \times \mathbb{R}^p \times S^{N-1}$, let $\{\nu_1, ..., \nu_{N-1}, \nu\}$ form an orthonormal basis of $\mathbb{R}^N$ and define the class of admissible functions

$$\mathcal{A}(a, b, \nu) := \{\xi \in H^1(Q_\nu; \mathbb{R}^p) : \xi(y) = a \text{ if } y.\nu = -1/2,\ \xi(y) = b \text{ if } y.\nu = 1/2, \text{ and } \xi \text{ is periodic with period one in the directions of } \nu_1, ..., \nu_{N-1}\},$$

where the boundary values of ξ are understood in the sense of traces. A function ξ is said to be *periodic with period one in the direction of* ν_i if

$$\xi(y) = \xi(y + k\nu_i)$$

for all $k \in \mathbb{Z}$, $y \in Q_\nu$.

Our surface energy density $K : \Omega \times \mathbb{R}^p \times \mathbb{R}^p \times S^{N-1} \to [0, +\infty)$ is defined by

$$K(x, a, b, \nu) := \inf \Big\{ \int_{Q_\nu} [LW(\xi(y)) + \frac{1}{L}(h^\infty)^2(x, \nabla\xi(y))]\, dy : \xi \in \mathcal{A}(a, b, \nu), L > 0 \Big\}.$$

We examine some continuity properties of K. In what follows C denotes a generic constant.

Proposition 2.8. Under the hypotheses (H2), (H3) and (H5) we have:
i) $0 \le K(x, a, b, \nu) \le C (1 + \|a\|^q + \|b\|^q + \|b-a\|^2)$ for all $(x, a, b, \nu) \in \Omega \times \mathbb{R}^p \times \mathbb{R}^p \times S^{N-1}$;
ii) For all $x_0 \in \Omega$ and for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|K(x, a, b, \nu) - K(x_0, a, b, \nu)| < \varepsilon C (1 + \|a\|^q + \|b\|^q + \|b-a\|^2).$$

Proof. We follow here the proof of Fonseca and Rybka [FR].
i) Fix $(x, a, b, \nu) \in \Omega \times \mathbb{R}^p \times \mathbb{R}^p \times S^{N-1}$ and let

$$\xi(y) = (b - a)(y.\nu) + \frac{a+b}{2}.$$

Clearly $\xi \in \mathcal{A}(a, b, \nu)$ so, by (H2) and Lemma 2.7 i),

$$0 \le K(x, a, b, \nu) \le \int_{Q_\nu} [LW(\xi(y)) + \frac{1}{L}(h^\infty)^2(x, \nabla\xi(y))]\, dy \le$$

$$\le \int_{Q_\nu} [LC(1+\|\xi(y)\|^q) + \frac{C}{L}\|\nabla\xi(y)\|^2]\, dy \le \text{const.}\ (1 + \|a\|^q + \|b\|^q + \|b-a\|^2),$$

since $\|\xi(y)\| \le \frac{1}{2}\|b - a\| + \frac{1}{2}\|b + a\| \le \|a\| + \|b\|$.

ii) Fix $x_0 \in \Omega$ and $\varepsilon > 0$. By Lemma 2.7 ii) choose $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|(h^\infty)^2(x_0, A) - (h^\infty)^2(x, A)| \le \varepsilon C \|A\|^2. \tag{2.5}$$

For all $n \in \mathbb{N}$ choose $\xi_n \in \mathscr{A}(a, b, \nu)$, $L_n > 0$ such that

$$\int_{Q_\nu} [L_n W(\xi_n(y)) + \frac{1}{L_n}(h^\infty)^2(x_0, \nabla\xi_n(y))]\, dy \le K(x_0, a, b, \nu) + \frac{1}{n}.$$

By Lemma 2.7 i) it follows that

$$\int_{Q_\nu} \frac{C}{L_n} \|\nabla\xi_n(y)\|^2\, dy \le \int_{Q_\nu} \frac{1}{L_n}(h^\infty)^2(x_0, \nabla\xi_n(y))\, dy \le K(x_0, a, b, \nu) + \frac{1}{n}$$

and so

$$\int_{Q_\nu} \frac{1}{L_n} \|\nabla\xi_n(y)\|^2\, dy \le \frac{K(x_0, a, b, \nu) + 1}{C} \le \text{const.}\ (1 + \|a\|^q + \|b\|^q + \|b-a\|^2).$$

Hence, if $|x - x_0| < \delta$, by (2.5) we have

$$K(x, a, b, \nu) - K(x_0, a, b, \nu) \le \int_{Q_\nu} \frac{1}{L_n}(h^\infty)^2(x, \nabla\xi_n(y))\, dy - \int_{Q_\nu} \frac{1}{L_n}(h^\infty)^2(x_0, \nabla\xi_n(y))\, dy + \frac{1}{n} \le$$

$$\le \int_{Q_\nu} \frac{1}{L_n} |(h^\infty)^2(x, \nabla\xi_n(y)) - (h^\infty)^2(x_0, \nabla\xi_n(y))|\, dy + \frac{1}{n} \le$$

$$\le \int_{Q_\nu} \frac{1}{L_n} \varepsilon C \|\nabla\xi_n(y)\|^2\, dy + \frac{1}{n} \le \varepsilon C\, (1 + \|a\|^q + \|b\|^q + \|b-a\|^2) + \frac{1}{n}.$$

Let $n \to \infty$ to obtain

$$K(x, a, b, \nu) - K(x_0, a, b, \nu) \le \varepsilon C\, (1 + \|a\|^q + \|b\|^q + \|b-a\|^2).$$

In a similar way we obtain

$$K(x_0, a, b, \nu) - K(x, a, b, \nu) \le \varepsilon C\, (1 + \|a\|^q + \|b\|^q + \|b-a\|^2). \qquad \blacksquare$$

The main result of this paper is the following

Theorem 2.9. Let (H1)-(H5) hold and let

$$J_\varepsilon(u) = \int_\Omega [\frac{1}{\varepsilon}W(u(x)) + \varepsilon h^2(x,\nabla u(x))]\, dx.$$

Then the $\Gamma(L^1(\Omega))$-limit of the sequence $J_\varepsilon(.)$ is given by

$$J_0(u) = \begin{cases} \int_{\Omega\cap\partial^*\{u=a\}} K(x,a,b,\nu(x))\, dH_{N-1}(x) & \text{if } u \in \{a,b\} \text{ a.e., } u \in BV \\ +\infty & \text{otherwise} \end{cases}$$

We divide the proof of Theorem 2.9 into two parts. The first part, corresponding to item i) of Definition 2.1, will be shown in Section 3 and the second part is proven in Section 4.

3. A lower bound for the Γ - limit.

In this section we prove that the $\Gamma(L^1(\Omega))$ - limit of $J_\varepsilon(.)$ is bounded below by $J_0(.)$.

Proposition 3.1. Let (H1)-(H5) hold and let $u \in L^1(\Omega;\mathbb{R}^P)$ be given. If $\varepsilon_n \to 0^+$ and if $u_n \in H^1(\Omega;\mathbb{R}^P)$ is such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^P)$ then

$$\liminf_{n\to\infty} \int_\Omega [\frac{1}{\varepsilon_n}W(u_n(x)) + \varepsilon_n h^2(x,\nabla u_n(x))]\, dx \geq J_0(u).$$

The proof relies on the following lemma which allows us to modify a sequence near the boundary without increasing its total energy.

Lemma 3.2. Assume that (H1), (H2) and (H3) hold and let

$$u_0(y) = \begin{cases} b & \text{if } y.\nu(x) > 0 \\ a & \text{if } y.\nu(x) < 0. \end{cases}$$

Let ρ be a symmetric mollifier and set $v_n = \rho_{1/\varepsilon_n} * u_0$ where $\rho_{1/\varepsilon_n}(x) = (\frac{1}{\varepsilon_n})^N \rho(\frac{x}{\varepsilon_n})$ and $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \to 0^+$. If $\{u_n\}$ is a sequence in $H^1(Q_\nu;\mathbb{R}^P)$ converging in $L^1(Q_\nu;\mathbb{R}^P)$ to u_0 then there exists a subsequence $\{\varepsilon_{n_k}\}$ and a sequence $\{w_k\}$ in $H^1(Q_\nu;\mathbb{R}^P)$ such that $w_k \to u_0$ in $L^1(Q_\nu;\mathbb{R}^P)$, $w_k = v_{n_k}$ on ∂Q_ν and

$$\limsup_{k\to+\infty} \int_{Q_\nu} \left[\frac{1}{\varepsilon_k} W(w_k(y)) + \varepsilon_k h^2(y,\nabla w_k(y))\right] dy \leq$$

$$\leq \liminf_{n\to+\infty} \int_{Q_\nu} \left[\frac{1}{\varepsilon_n} W(u_n(y)) + \varepsilon_n h^2(y,\nabla u_n(y))\right] dy.$$

Proof. *Step 1.* Assume, without loss of generality, that

$$\liminf_{n\to+\infty} \int_{Q_\nu} \left[\frac{1}{\varepsilon_n} W(u_n(y)) + \varepsilon_n h^2(y,\nabla u_n(y))\right] dy =$$

$$= \lim_{n\to+\infty} \int_{Q_\nu} \left[\frac{1}{\varepsilon_n} W(u_n(y)) + \varepsilon_n h^2(y,\nabla u_n(y))\right] dy < +\infty. \tag{3.1}$$

We begin by showing that $u_n \to u_0$ in $L^q(Q_\nu;\mathbb{R}^p)$. Indeed, after extracting a subsequence, we have $u_n(y) \to u_0(y)$ a.e. and by (3.1),

$$\int_{Q_\nu} W(u_n(y))\, dy = \varepsilon_n \int_{Q_\nu} \frac{1}{\varepsilon_n} W(u_n(y))\, dy \to 0 \text{ as } n\to+\infty.$$

By (H2),

$$\|u_n(y)-u_0(y)\|^q \leq C(W(u_n(y)) + 1)$$

and so by Fatou's Lemma,

$$\int_{Q_\nu} C\, dy = \int_{Q_\nu} \liminf_{n\to+\infty} \left[C(W(u_n(y)) + 1) - \|u_n(y) - u_0(y)\|^q\right] dy \leq$$

$$\leq \liminf_{n\to+\infty} \left[\int_{Q_\nu} C\, W(u_n(y))\, dy + \int_{Q_\nu} C\, dy - \int_{Q_\nu} \|u_n(y) - u_0(y)\|^q\, dy\right]$$

Therefore,

$$\limsup_{n\to+\infty} \int_{Q_\nu} \|u_n(y) - u_0(y)\|^q\, dy = 0.$$

Also, as $q \geq 2$, we conclude that $u_n \to u_0$ in $L^2(Q_\nu;\mathbb{R}^p)$ as $n \to +\infty$.

Step 2. For simplicity assume that $\nu = e_N$ and denote $Q_\nu = Q$. Notice that

$$u_0(y) = \begin{cases} b & \text{if } y_N > \varepsilon_n \\ a & \text{if } y_N < -\varepsilon_n \end{cases}$$

and

$$v_n \in \mathcal{A}(a,b,e_N),\ \|\nabla v_n\|_\infty = O(1/\varepsilon_n) \text{ and } \operatorname{supp} \nabla v_n \subset \{|y_N| < \varepsilon_n\}. \tag{3.2}$$

Also $v_n \to u_0$ in $L^2(Q;\mathbb{R}^p)$. Let $T_n = Q \setminus \sigma_n Q$ where $\sigma_n = (1 - \|u_n - v_n\|_2^{1/3})^{1/N}$ so that

$$\text{meas } T_n = \|u_n - v_n\|_2^{1/3} \to 0 \text{ as } n \to +\infty.$$

Let $M = c\,(1 + 2\|u_0\|_q^q)$ where c is the constant appearing in the growth condition of W and define $k_n \in \mathbb{Z}^+$ as $k_n = [\frac{2M}{\varepsilon_n \sqrt{\|u_n - v_n\|_2}}] + 1$, where [y] denotes the integer part of y. We divide T_n into k_n slices of measure $\frac{\text{meas } T_n}{k_n}$, $T_n = \bigcup_{j=1}^{k_n} S_j^n$ and S_j^n are of the form $\lambda_{j+1}^n Q \setminus \lambda_j^n Q$ where $0 < \lambda_j^n < 1$, $\lambda_1^n = \alpha_n$ and $\lambda_{k_n+1}^n = 1$. Consider a family of smooth cut–off functions $\varphi_j^n \in C_0^\infty(\alpha_n Q \cup \bigcup_{k=1}^{j} S_k^n)$ such that $0 \le \varphi_j^n \le 1$, $\varphi_j^n = 1$ on $\alpha_n Q \cup \bigcup_{k=1}^{j-1} S_k^n$ and $\|\nabla\varphi_j^n\|_\infty = O\left(\frac{k_n}{\|u_n - v_n\|_2^{1/3}}\right)$ for $j = 1,\dots,k_n$. Using these functions φ_j^n we will consider convex combinations of u_n with v_n across the slices S_j^n. We claim that there exists $m \in \mathbb{N}$ such that, for all $n > m$ there exists $j \in \{1,\dots,k_n\}$ such that

$$\int_{S_j^n} [\frac{1}{\varepsilon_n} W(\varphi_j^n(y)u_n(y) + (1-\varphi_j^n(y))v_n(y)) + \varepsilon_n h^2(y, \varphi_j^n(y)(\nabla u_n(y) - \nabla v_n(y)) + \nabla v_n(y) +$$

$$+ (u_n(y) - v_n(y)) \otimes \nabla\varphi_j^n(y))]\, dy \le \sqrt{\|u_n - v_n\|_2}. \tag{3.3}$$

Assuming (3.3) holds, for each $n > m$ we obtain a slice $S^{(n)} \in \{S_j^n : j = 1,\dots,k_n\}$, $S^{(n)} = S_{j(n)}^n$ such that, setting

$$w_n(y) = \varphi^{(n)}(y)u_n(y) + (1 - \varphi^{(n)}(y))v_n(y)) = \begin{cases} v_n & \text{if } y \in B^{(n)} \\ u_n & \text{if } y \in A^{(n)} \end{cases}$$

where $B^{(n)} = \bigcup_{j=j(n)+1}^{k_n} S_j^n$ and $A^{(n)} = Q \setminus (B^{(n)} \cup S^{(n)})$, then

$$\lim_{n\to+\infty} \int_Q [\frac{1}{\varepsilon_n} W(u_n(y)) + \varepsilon_n h^2(y, \nabla u_n(y))]\, dy \ge$$

$$\limsup_{n\to+\infty} \int_{A^{(n)}} [\frac{1}{\varepsilon_n} W(u_n(y)) + \varepsilon_n h^2(y, \nabla u_n(y))]\, dy =$$

$$= \limsup_{n\to+\infty} [\int_Q [\frac{1}{\varepsilon_n} W(w_n(y)) + \varepsilon_n h^2(y, \nabla w_n(y))]\, dy - \int_{B^{(n)}} [\frac{1}{\varepsilon_n} W(v_n(y)) + \varepsilon_n h^2(y, \nabla v_n(y))]\, dy$$

$$- \int_{S^{(n)}} [\frac{1}{\varepsilon_n} W(\varphi^{(n)}(y)u_n(y) + (1-\varphi^{(n)}(y))v_n(y)) + \varepsilon_n h^2(y, \varphi^{(n)}(y)(\nabla u_n(y) - \nabla v_n(y)) + \nabla v_n(y) +$$

$$+(u_n(y)-v_n(y))\otimes\nabla\varphi^{(n)}(y))]\,dy\Big].$$

By (3.3) and Step 1 the last term is bounded by $\sqrt{\|u_n - v_n\|_2}$ which goes to zero as $n\to+\infty$. We show that the second term also goes to zero. Indeed, by (H1), (H2), (H3), (3.2) and since $\{v_n\}$ is bounded in L^∞ we have

$$\int_{B^{(n)}} [\frac{1}{\varepsilon_n}W(v_n(y)) + \varepsilon_n h^2(y,\nabla v_n(y))]\,dy \le \int_{T_n} [\frac{1}{\varepsilon_n}W(v_n(y)) + \varepsilon_n h^2(y,\nabla v_n(y))]\,dy \le$$

$$\le \frac{1}{\varepsilon_n}\int_{T_n\cap\{|y_N|<\varepsilon_n\}} C\,dy + \int_{T_n\cap\{|y_N|<\varepsilon_n\}} \varepsilon_n C\,\|\nabla v_n(y)\|^2\,dy + \int_{T_n} C\,\varepsilon_n\,dy \le$$

$$\le C[2(\text{meas}T_n)^{(N-1)/N} + \varepsilon_n \text{meas}T_n] \to 0 \text{ as } n\to+\infty.$$

Hence,

$$\lim_{n\to+\infty}\int_Q [\frac{1}{\varepsilon_n}W(u_n(y)) + \varepsilon_n h^2(y,\nabla u_n(y))]\,dy \ge$$

$$\ge \limsup_{n\to+\infty}\int_Q [\frac{1}{\varepsilon_n}W(w_n(y)) + \varepsilon_n h^2(y,\nabla w_n(y))]\,dy.$$

On the other hand,

$$\lim_{n\to+\infty}\int_Q \|w_n(y) - u_0(y)\|\,dy = \lim_{n\to+\infty}\Big[\Big[\int_{A^{(n)}}\|u_n(y) - u_0(y)\|\,dy + \int_{B^{(n)}}\|v_n(y) - u_0(y)\|\,dy +$$

$$+ \int_{S^{(n)}}\|\varphi^{(n)}(y)u_n(y) + (1-\varphi^{(n)}(y))v_n(y) - u_0(y)\|\,dy\Big] \le$$

$$\le \lim_{n\to+\infty}[\int_Q \|u_n(y) - u_0(y)\|\,dy + \int_Q \|v_n(y) - u_0(y)\|\,dy] = 0$$

since $u_n \to u_0$ and $v_n \to u_0$ in $L^1(Q;\mathbb{R}^p)$.

Step 3. It remains to show (3.3). We begin by proving that

$$\limsup_{n\to+\infty}\sum_{j=1}^{k_n}\int_{S_j^n} W(\varphi_j^n(y)u_n(y)+(1-\varphi_j^n(y))v_n(y))\,dy \le M. \tag{3.4}$$

Indeed, by (H2) and since $u_n \to u_0$ and $v_n \to u_0$ in $L^q(Q;\mathbb{R}^p)$

$$\limsup_{n\to+\infty}\sum_{j=1}^{k_n}\int_{S_j^n} W(\varphi_j^n(y)u_n(y)+(1-\varphi_j^n(y))v_n(y))\,dy \le$$

$$\le \limsup_{n\to+\infty}\int_{T_n} C(1+\|\varphi_j^n(y)u_n(y)+(1-\varphi_j^n(y))v_n(y)\|^q)\,dy \le$$

$$\leq \limsup_{n\to+\infty} \int_{T_n} C(1+\|u_n(y)\|^q+\|v_n(y)\|^q)\,dy \leq C(1+2\|u_0\|_q^q) = M.$$

To show (3.3) we argue by contradiction. If (3.3) were false then for all $m \in \mathbb{N}$ there would exist $n > m$ such that for all $j \in \{1,\ldots,k_n\}$

$$\int_{S_j^n} [\frac{1}{\varepsilon_n} W(\varphi_j^n(y)u_n(y)+(1-\varphi_j^n(y))v_n(y)) + \varepsilon_n h^2(y,\varphi_j^n(y)(\nabla u_n(y)-\nabla v_n(y))+\nabla v_n(y)+$$

$$+(u_n(y)-v_n(y))\otimes\nabla\varphi_j^n(y))]\,dy > \sqrt{\|u_n - v_n\|_2}.$$

Then by (H3), for this subsequence

$$\int_{S_j^n} W(\varphi_j^n(y)u_n(y)+(1-\varphi_j^n(y))v_n(y))\,dy > \varepsilon_n \sqrt{\|u_n - v_n\|_2} -$$

$$-\varepsilon_n^2 \int_{S_j^n} h^2(y,\varphi_j^n(y)(\nabla u_n(y)-\nabla v_n(y))+\nabla v_n(y)+(u_n(y)-v_n(y))\otimes\nabla\varphi_j^n(y))\,dy \geq$$

$$\geq \varepsilon_n \sqrt{\|u_n - v_n\|_2} - C\varepsilon_n^2 \text{meas} S_j^n - C\,\varepsilon_n^2 \int_{S_j^n} \|\nabla u_n(y)\|^2\,dy - C\,\varepsilon_n^2 \int_{S_j^n} \|\nabla v_n(y)\|^2\,dy -$$

$$- C\frac{\varepsilon_n^2 k_n^2}{\|u_n - v_n\|_2^{2/3}} \int_{S_j^n} \|u_n(y)-v_n(y)\|^2\,dy\ .$$

Summing the above inequality from $j = 1$ to $j = k_n$ we obtain

$$\sum_{j=1}^{k_n} \int_{S_j^n} W(\varphi_j^n(y)u_n(y)+(1-\varphi_j^n(y))v_n(y))\,dy \geq$$

$$\geq k_n\,\varepsilon_n \sqrt{\|u_n - v_n\|_2} - C\varepsilon_n^2 \text{meas} T_n - C\,\varepsilon_n^2 \int_{T_n} \|\nabla u_n(y)\|^2\,dy - C\,\varepsilon_n^2 \int_{T_n} \|\nabla v_n(y)\|^2\,dy -$$

$$- C[\varepsilon_n k_n \|u_n-v_n\|_2^{2/3}]^2.$$

By choice of k_n,

$$k_n\,\varepsilon_n \sqrt{\|u_n - v_n\|_2} \geq 2M$$

and

$$C\,[\varepsilon_n k_n\,\|u_n - v_n\|_2^{2/3}]^2 \leq C\,[2M\,\|u_n - v_n\|_2^{1/6} + \varepsilon_n\,\|u_n - v_n\|_2^{2/3}]^2 \to 0 \text{ as } n \to +\infty$$

and by properties of v_n,

$$C\, zf\backslash\, \|\nabla v_n(y)\|^2\, dy \; £ \; C\, ej \qquad f\, \frac{1}{}\, dy = Ce\text{^}measTn)\text{^}-^1)\text{^} \to 0 \text{ as } n \to +\infty.$$

$T_n \cap \{|y_N| < \varepsilon_n\}$

Therefore, since En J $H\nabla u_n(y)H^2$ dy is bounded, it follows that

$$\text{Um supn}\text{^}o \sum_{j=1}^{k_n} \int_{S_j^n} fW(9]^l(y)u_n(y)+(l.(p;(y))v_{ll}(y))\, dy \;*\; 2M$$

contradicting (3.4). •

Proof of Proposition 3.1. *Step 1.* We begin by proving the proposition in the case where u = XA00a+(l-3CA00)b ^{w}^h Per^(A) = +©o. As Jo(u) = +©© it suffices to show that for any sequence £n -» 0+ and for any u_n *e* $H^!(\text{^};tR^p)$ such that u_n -» u in LKQ;$[R^P)$ we have

$$J\, [\frac{1}{\varepsilon_n} W(u_n(x)) + e_n h2(x,\nabla u_n(x))]\, dx \; -4 \; +oo.$$

We argue by contradiction. Suppose that there exists a subsequence (which we continue to denote by En and u_n for convenience) such that

$$J_n\, [\text{^}W(u_n(x)) + E_n h2(x,\nabla u_n(x))]\, dx \; £ \; \text{const},$$

Then, by the growth condition on h, we have

$$\int [\frac{1}{} W(u_n(x)) + EnC\|\nabla u_n(x)\|2]\, dx \;\text{^}\; \text{const.}$$

which, by the Cauchy-Schwartz inequality, implies

$$J\sqrt{W(u_n(x))}\; \|\nabla u_n(x)\|\, dx \;\text{^}\; \text{const.} \tag{3.5}$$

Let

$$f(r) := \text{infiu^isr}\; \sqrt{W(u)},$$

where

$$\frac{a+b}{2}$$

If we set

$$r_0 := \left| \frac{a-b}{2} \right|,$$

then by (H2) there exists $r_1 > r_0$ such that

$$\int_{r_0}^{r_1} f(r)\, dr > \frac{K}{2}$$

where

$$K = 2 \inf \left\{ \int_{-1}^{1} \sqrt{W(g(s))}\, |g'(s)|\, ds : g \text{ is piecewise } C^1,\ g(-1) = a,\ g(1) = b \right\}.$$

Let

$$M := \max_{|u-c| \le r_1} \sqrt{W(u)}$$

and define

$$\varphi(v) := \inf \left\{ \int_{-1}^{1} T(\gamma(s))\, \|\gamma'(s)\|\, ds : \gamma \text{ is a piecewise } C^1,\ \gamma(-1) = a,\ \gamma(1) = v \right\}$$

where

$$T(u) := \min \{ \sqrt{W(u)}, M \}.$$

This function was studied by Fonseca and Tartar [FT] where they showed that

i) $\varphi: \mathbb{R}^p \to [0, +\infty)$ is a Lipschitz function

ii) if $u \in H^1(\Omega;\mathbb{R}^p)$ then $\varphi \circ u \in H^1(\Omega;\mathbb{R}^p)$ and $\|\nabla(\varphi \circ u)(x)\| \le \sqrt{W(u(x))}\, \|\nabla u(x)\|$ a.e. $x \in \Omega$.

Hence, since $\varphi \circ u_n \to \varphi \circ u = (1-\chi_A)\varphi(b)$ in L^1 strong, from the lower semicontinuity formula (2.2), ii) and (3.5) we have

$$\varphi(b)\mathrm{Per}_\Omega(A) = |D(1-\chi_A)\varphi(b)|\,(\Omega) \le \liminf_{n\to+\infty} \int_\Omega \|\nabla(\varphi \circ u_n)(x)\|\, dx \le \text{const.}$$

contradicting the fact that $\mathrm{Per}_\Omega(A) = +\infty$.

Step 2. We now turn to the case where $u = \chi_A(x)a + (1-\chi_A(x))b$ with $\mathrm{Per}_\Omega(A) < +\infty$. Assume, without loss of generality, that

$$\liminf_{n\to+\infty} \int_\Omega \left[\frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x,\nabla u_n(x))\right] dx = \lim_{n\to+\infty} \int_\Omega \left[\frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x,\nabla u_n(x))\right] dx < +\infty.$$

We must show that

$$\lim_{n\to+\infty} \int_\Omega \left[\frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x,\nabla u_n(x))\right] dx \ge \int_{\Omega\cap\partial^* A} K(x,a,b,\nu(x))\, dH_{N-1}(x). \quad (3.6)$$

Using the blow up method introduced by Fonseca and Müller [FM1] we reduce the problem to verifying the pointwise inequality (3.8) below. As the integrands $\frac{1}{\varepsilon_n}W(u_n(x))+\varepsilon_n h^2(x,\nabla u_n(x))$ form a sequence of nonnegative functions bounded in L^1 there exists a subsequence (still denoted by ε_n and u_n) and a nonnegative Radon measure μ such that

$$\frac{1}{\varepsilon_n}W(u_n(.)) + \varepsilon_n h^2(.,\nabla u_n(.)) \rightarrow \mu \text{ weakly * in the sense of measures}$$

i.e. for all $\varphi \in C_0(\Omega)$

$$\int_\Omega \varphi(x) \left[\frac{1}{\varepsilon_n}W(u_n(x)) + \varepsilon_n h^2(x,\nabla u_n(x))\right] dx \rightarrow \int_\Omega \varphi d\mu \text{ as } n \rightarrow +\infty. \qquad (3.7)$$

Using the Radon-Nikodym theorem we may write μ as a sum of two mutually singular nonnegative measures $\mu = \mu_a H_{N-1}\lfloor(\Omega \cap \partial^* A) + \mu_s$. We claim that

$$\mu_a(x_0) \geq K(x_0,a,b,\nu(x_0)) \text{ for } H_{N-1} \text{ a.e. } x_0 \in \Omega \cap \partial^* A. \qquad (3.8)$$

Assuming that (3.8) holds we consider an increasing sequence of smooth cut-off functions, $\varphi_k \in C_0(\Omega)$, with $0 \leq \varphi_k \leq 1$ and $\sup_k \varphi_k(x) = 1$ in Ω and we obtain

$$\lim_{n\to+\infty} \int_\Omega \left[\frac{1}{\varepsilon_n}W(u_n(x)) + \varepsilon_n h^2(x,\nabla u_n(x))\right] dx \geq$$

$$\geq \lim_{n\to+\infty} \int_\Omega \varphi_k(x) \left[\frac{1}{\varepsilon_n}W(u_n(x)) + \varepsilon_n h^2(x,\nabla u_n(x))\right] dx = \int_\Omega \varphi_k(x) d\mu(x)) \geq$$

$$\geq \int_\Omega \varphi_k(x)\mu_a(x)\, dH_{N-1}\lfloor(\Omega \cap \partial^* A)(x) \geq \int_{\Omega\cap\partial^* A} \varphi_k(x) K(x,a,b,\nu(x))\, dH_{N-1}(x).$$

Letting $k \rightarrow +\infty$ and using the Monotone Convergence Theorem we conclude (3.6).

Step 3. It remains to show inequality (3.8). By Theorems 2.4 and 2.5 for H_{N-1} a.e. $x \in \Omega \cap \partial^* A$ we have

$$\text{i) } \lim_{\delta\to 0^+} \frac{1}{\delta^N} \int_{\{y\in B(x,\delta):(y-x).\nu(x)>0\}} |u(y) - b|\, dy = 0$$

$$\lim_{\delta\to 0^+} \frac{1}{\delta^N} \int_{\{y\in B(x,\delta):(y-x).\nu(x)<0\}} |u(y) - a|\, dy = 0$$

and

$$\text{ii) } \mu_a(x) = \lim_{\delta\to 0^+} \frac{\mu(x+\delta Q_{\nu(x)})}{H_{N-1}\lfloor(\Omega\cap\partial^* A)(x+\delta Q_{\nu(x)})}.$$

Choose a point $x \in \Omega \cap \partial^* A$ such that i) and ii) hold. Let $0 < \eta < 1$ and let $\varphi \in C_0^\infty(Q_{\nu(x)})$ be such that $0 \le \varphi \le 1$ and $\varphi = 1$ on $\eta Q_{\nu(x)}$. Using ii) and (2.4) we have

$$\mu_a(x) \ge \limsup_{\delta\to 0^+} \frac{\mu(x+\delta Q_{\nu(x)})}{\delta^{N-1}} \ge \limsup_{\delta\to 0^+} \frac{1}{\delta^{N-1}} \int_{x+\delta Q_{\nu(x)}} \varphi(\frac{y-x}{\delta})\, d\mu(y) =$$

$$= \limsup_{\delta\to 0^+} \frac{1}{\delta^{N-1}} \lim_{n\to+\infty} \int_{x+\delta Q_{\nu(x)}} \varphi(\frac{y-x}{\delta})\, [\frac{1}{\varepsilon_n} W(u_n(y)) + \varepsilon_n h^2(y,\nabla u_n(y))]\, dy =$$

$$= \limsup_{\delta\to 0^+} \delta \lim_{n\to+\infty} \int_{Q_{\nu(x)}} \varphi(y)\, [\frac{1}{\varepsilon_n} W(u_n(x+\delta y)) + \varepsilon_n h^2(x+\delta y,\nabla u_n(x+\delta y))]\, dy \ge$$

$$\ge \limsup_{\delta\to 0^+} \delta \limsup_{n\to+\infty} \int_{\eta Q_{\nu(x)}} [\frac{1}{\varepsilon_n} W(u_n(x+\delta y)) + \varepsilon_n h^2(x+\delta y,\nabla u_n(x+\delta y))]\, dy. \quad (3.9)$$

Let

$$w_{n,\delta}(y) = u_n(x+\delta y),\ u_0(y) = \begin{cases} b & \text{if } y.\nu(x) > 0 \\ a & \text{if } y.\nu(x) < 0. \end{cases}$$

Note that, since $u_n \to u$ in L^1 and by i), we have

$$\lim_{\delta\to 0^+} \lim_{n\to+\infty} \| w_{n,\delta} - u_0 \|_{L^1(Q_{\nu(x)})} =$$

$$= \lim_{\delta\to 0^+} \lim_{n\to+\infty} \Big[\int_{Q_{\nu(x)}\cap\{y:y.\nu(x)>0\}} \|u_n(x+\delta y) - b\|\, dy + \int_{Q_{\nu(x)}\cap\{y:y.\nu(x)<0\}} \|u_n(x+\delta y) - a\|\, dy \Big] =$$

$$= \lim_{\delta\to 0^+} \Big[\int_{Q_{\nu(x)}\cap\{y:y.\nu(x)>0\}} \|u(x+\delta y) - b\|\, dy + \int_{Q_{\nu(x)}\cap\{y:y.\nu(x)<0\}} \|u(x+\delta y) - a\|\, dy \Big] = 0.$$

Since $\nabla w_{n,\delta}(y) = \delta \nabla u_n(x+\delta y)$, from (3.9) we get

$$\mu_a(x) \ge \limsup_{\delta\to 0^+} \limsup_{n\to+\infty} \Big[\int_{\eta Q_{\nu(x)}} [\frac{\delta}{\varepsilon_n} W(w_{n,\delta}(y)) + \frac{\varepsilon_n}{\delta} (h^\infty)^2(x+\delta y,\nabla w_{n,\delta}(y))]\, dy +$$

$$+ \int_{\eta Q_{\nu(x)}} [\delta\varepsilon_n h^2(x+\delta y,\frac{1}{\delta}\nabla w_{n,\delta}(y)) - \frac{\varepsilon_n}{\delta} (h^\infty)^2(x+\delta y,\nabla w_{n,\delta}(y))]\, dy \Big]. \quad (3.10)$$

Now,

$$\int_{\eta Q_{\nu(x)}} |\delta\varepsilon_n h^2(x+\delta y,\frac{1}{\delta}\nabla w_{n,\delta}(y)) - \frac{\varepsilon_n}{\delta} (h^\infty)^2(x+\delta y,\nabla w_{n,\delta}(y))|\, dy =$$

$$= \frac{\varepsilon_n}{\delta} \int_{\eta Q_{v(x)}} |\delta^2 h^2(x+\delta y, \frac{1}{\delta}\nabla w_{n,\delta}(y)) - (h^\infty)^2(x+\delta y, \nabla w_{n,\delta}(y))|\, dy =$$

$$= \frac{\varepsilon_n}{\delta} \int_{\eta Q_{v(x)} \cap \{\|\nabla w_{n,\delta}\| \le \delta L\}} |\delta^2 h^2(x+\delta y, \frac{1}{\delta}\nabla w_{n,\delta}(y)) - (h^\infty)^2(x+\delta y, \nabla w_{n,\delta}(y))|\, dy +$$

$$+ \frac{\varepsilon_n}{\delta} \int_{\eta Q_{v(x)} \cap \{\|\nabla w_{n,\delta}\| > \delta L\}} |\delta^2 h^2(x+\delta y, \frac{1}{\delta}\nabla w_{n,\delta}(y)) - (h^\infty)^2(x+\delta y, \nabla w_{n,\delta}(y))|\, dy =: I_1 + I_2$$

where, by (H3) and Lemma 2.7 i),

$$I_1 \le \text{const.} \frac{\varepsilon_n}{\delta} \text{meas}(\eta Q_{v(x)} \cap \{\|\nabla w_{n,\delta}\| \le \delta L\})\, \delta^2 \le \text{const.}\, \varepsilon_n \delta \to 0 \text{ as } \delta \to 0 \text{ and } n \to +\infty$$

and, by (H4) (with $t=1/\delta$), Hölder's inequality and (H3),

$$I_2 \le \frac{\varepsilon_n}{\delta} \int_{\eta Q_{v(x)} \cap \{\|\nabla w_{n,\delta}\| > \delta L\}} C' \|\nabla w_{n,\delta}(y)\|^{2-m} \delta^m \, dy \le$$

$$\le C' \delta^{m-1} \int_{\eta Q_{v(x)} \cap \{\|\nabla w_{n,\delta}\| > \delta L\}} \varepsilon_n \|\nabla w_{n,\delta}(y)\|^{2-m} \, dy \le$$

$$\le C \delta^{m-1} \varepsilon_n^{m/2} \Big[\int_{\eta Q_{v(x)} \cap \{\|\nabla w_{n,\delta}\| > \delta L\}} \varepsilon_n \|\nabla w_{n,\delta}(y)\|^2 \, dy \Big]^{1-m/2} =$$

$$= C \delta^{m-1} \varepsilon_n^{m/2} \Big[\int_{\eta Q_{v(x)} \cap \{\|\nabla w_{n,\delta}\| > \delta L\}} \varepsilon_n \delta^2 \|\nabla u_n(x+\delta y)\|^2 \, dy \Big]^{1-m/2} \le$$

$$\le C \delta^{m-1} \varepsilon_n^{m/2} \Big[\varepsilon_n \delta^2 + \int_{\eta Q_{v(x)} \cap \{\|\nabla w_{n,\delta}\| > \delta L\}} \delta^2 \varepsilon_n h^2(x+\delta y, \nabla u_n(x+\delta y)) \, dy \Big]^{1-m/2} =$$

$$= C (\delta \varepsilon_n)^{m/2} \Big[\varepsilon_n \delta + \int_{\eta Q_{v(x)} \cap \{\|\nabla w_{n,\delta}\| > \delta L\}} \delta \varepsilon_n h^2(x+\delta y, \nabla u_n(x+\delta y)) \, dy \Big]^{1-m/2} \to 0 \text{ as } \delta \to 0,\ n \to +\infty$$

since by (3.9) $\Big\{ \int_{\eta Q_{v(x)}} \delta \varepsilon_n h^2(x+\delta y, \nabla u_n(x+\delta y))\, dy \Big\}$ remains bounded. So (3.10) reduces to

$$\mu_a(x) \ge \limsup_{\delta \to 0^+} \limsup_{n \to +\infty} \int_{\eta Q_{v(x)}} [\frac{\delta}{\varepsilon_n} W(w_{n,\delta}(y)) + \frac{\varepsilon_n}{\delta} (h^\infty)^2(x+\delta y, \nabla w_{n,\delta}(y))]\, dy =$$

$$= \limsup_{\delta \to 0^+} \limsup_{n \to +\infty} \Big[\int_{\eta Q_{v(x)}} [\frac{\delta}{\varepsilon_n} W(w_{n,\delta}(y)) + \frac{\varepsilon_n}{\delta} (h^\infty)^2(x, \nabla w_{n,\delta}(y))]\, dy +$$

$$+ \quad \int [\wedge(h\text{-})2(x+8y,Vw_{n},_{5}(y))\text{-}\wedge(h\text{-})2(x,Vw_{n},8(y))]\ dy]. \qquad (3.11)$$

Fix e > 0. By Lemma 2.7 ii), (H3) and (3.9) we have for 8 small enough,

$$\underline{f}\ *\text{-}f\ l(h\text{-})2(x+8y,Vw_{n},_{8}(y))\ \text{-}\ (h\text{-})2(x,Vw_{n},_{5}(y))l\ dy\ \pounds\ \wedge\quad J\ eC\ IIVw_{n},_{8}(y)ll^{2}\ dy\ =$$

$\eta Q_{v(x)}$

$$=: eC\quad J\ \pounds n8\ IIVu_{n}(x+8y)l|2\ dy\ \wedge\ eC\ [e«8 + \quad |\ e_{n}8\ h2(x+8y,Vu_{n}(x+8y))\ dy]\ =$$

•nQvd) ¹iQv(x)

$$= O(e) \text{ as } 8\text{-}{*}0,\ n\text{-}{*}{+}«>.$$

Hence,

$$»i.(x)\ \wedge\ \lim\ sups\wedge o^*\ Km\ sup_{n}\_\wedge.\ \int_{TjQv(x)} [\frac{\delta}{\wedge n} W(w_{Bf}«(y)) + \%h\text{-})2(x,Vw_{n},_{5}(y))]\ dy + O(e). \qquad (3.12)$$

Let

$$\lim\ sups\text{-}{*}\text{-}\ \lim\ sup_{n}\_\wedge \int_{\eta Q_{v(x)}} [\frac{\delta}{\varepsilon_n} W(w».{}_{8}(y)) + \wedge\ (h\text{-})^{2}(x,Vwn,_{5}(y))]\ dy + O(e) =$$

$$= \lim_{k\to+\infty} \lim\ sup_{n\to+\infty} \int_{\eta Q_{v(x)}} [\frac{\delta_k}{En} W(w_{n,\delta_k}(y)) + \frac{\varepsilon_n}{\delta_k}(h^{-})^{2}(x,\nabla w_{n,\delta_k}(y))]\ dy + O(\varepsilon) \qquad (3.13)$$

where 8k -> 0+ as k -^ +*». Choose n(k) large enough so that, setting $a_k = \frac{\varepsilon}{\ }\wedge$, we have $0 < a_k <$ 1/k, II wnoc),^ - uo $H_{L,(Qv(it))}$ ^ 1/k and

$$\lim\ sup_{n\to+\infty} \int_{\eta Q_{v(x)}} W(w_{n,\delta_k}(y)) + \frac{\varepsilon_n}{\delta_k}(h^{-})^{2}(x,\nabla w_{n,\delta_k}(y))]\ dy =$$

$$\int_{liQv(x)} I.\ W(w_{n(k)\ 6k}(y)) + a_{k}\ (h\text{-})^{2}(x,Vw_{n(k)}\ \wedge(y))]\ dy + O(l/k). \qquad (3.14)$$

Thus, defining $v_k(y) = Wn\wedge\wedge y)$, it follows from (3.12), (3.13), and (3.14) that

$$\mu_a(x) \geq \lim_{k\to+\infty} \int_{\eta Q_{\nu(x)}} [\frac{1}{\alpha_k} W(v_k(y)) + \alpha_k (h^\infty)^2(x,\nabla v_k(y))]\, dy + O(\varepsilon)$$

where $v_k \to u_0$ in $L^1(Q_{\nu(x)})$ and $\alpha_k \to 0^+$ as $k \to +\infty$. Changing variables we obtain

$$\mu_a(x) \geq \lim_{k\to+\infty} \eta^N \int_{Q_{\nu(x)}} [\frac{1}{\alpha_k} W(v_k(\eta z)) + \alpha_k (h^\infty)^2(x,\nabla v_k(\eta z))]\, dz + O(\varepsilon) =$$

$$= \eta^{N-1} \lim_{k\to+\infty} \int_{Q_{\nu(x)}} [\frac{1}{\bar\alpha_k} W(\bar u_k(z)) + \bar\alpha_k (h^\infty)^2(x,\nabla \bar u_k(z))]\, dz + O(\varepsilon) \qquad (3.15)$$

where $\bar\alpha_k = \frac{\alpha_k}{\eta}$, $\bar\alpha_k \to 0^+$ and $\bar u_k(z) = v_k(\eta z)$. Applying Lemma 3.2 to h^∞ and to the sequences $\bar u_k$ and $\bar\alpha_k$ we conclude that there exists a subsequence $\{\bar\alpha_i\} \subset \{\bar\alpha_k\}$ and a sequence $\{\xi_i\} \in H^1(Q_{\nu(x)};\mathbb{R}^P)$ such that $\xi_i \to u_0$ in $L^1(Q_{\nu(x)};\mathbb{R}^P)$, $\xi_i \in \mathcal{A}(a,b,\nu(x))$ and

$$\liminf_{i\to+\infty} \int_{Q_{\nu(x)}} [\frac{1}{\bar\alpha_i} W(\xi_i(z)) + \bar\alpha_i (h^\infty)^2(x,\nabla \xi_i(z))]\, dz \leq$$

$$\leq \lim_{k\to+\infty} \int_{Q_{\nu(x)}} [\frac{1}{\bar\alpha_k} W(\bar u_k(z)) + \bar\alpha_k (h^\infty)^2(x,\nabla \bar u_k(z))]\, dz. \qquad (3.16)$$

Thus, by (3.15) and (3.16) we have

$$\mu_a(x) \geq \eta^{N-1} \liminf_{i\to+\infty} \int_{Q_{\nu(x)}} [\frac{1}{\bar\alpha_i} W(\xi_i(z)) + \bar\alpha_i (h^\infty)^2(x,\nabla \xi_i(z))]\, dz + O(\varepsilon) \geq$$

$$\geq \eta^{N-1} K(x,a,b,\nu(x)) + O(\varepsilon).$$

(3.8) now follows if we let $\eta \to 1^-$ and $\varepsilon \to 0^+$. ■

4. An upper bound for the Γ - limit.

We now prove the second part of Theorem 2.9.

Proposition 4.1. Under the hypotheses (H1)-(H5) given any $u \in L^1(\Omega;\mathbb{R}^P)$ there exist sequences $\varepsilon_n \to 0^+$ and $u_n \in H^1(\Omega;\mathbb{R}^P)$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^P)$ and

$$\lim_{n\to+\infty} \int_{\Omega} [\frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x,\nabla u_n(x))] \, dx = J_0(u).$$

It is clear that it suffices to consider the case where $u = \chi_A(x)a+(1-\chi_A(x))b$ with $Per_\Omega(A) < +\infty$, since

$$\lim_{n\to+\infty} \int_{\Omega} [\frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x,\nabla u_n(x))] \, dx < +\infty$$

implies that

$$\lim_{n\to+\infty} \int_{\Omega} W(u_n(x)) \, dx = 0$$

and so, as $u_n \to u$ in $L^1(\Omega;\mathbb{R}^P)$ and due to the continuity of W, we conclude that

$$u(x) \in \{a,b\} \text{ a.e. } x \in \Omega.$$

Also, as in Step 1 of the proof of Proposition 3.1, we obtain $Per_\Omega(A) < +\infty$. We begin by considering the simpler case where $u = \chi_A(x)a+(1-\chi_A(x))b$ has planar interface and h and K do not depend explicitly on x.

Lemma 4.2. Let (H1)-(H5) hold, let $\Omega = Q_\nu$ and

$$u(y) = \begin{cases} b & \text{if } y.\nu > 0 \\ a & \text{if } y.\nu < 0. \end{cases}$$

Then there exist sequences $\varepsilon_n \to 0^+$ and $u_n \in \mathcal{A}(a,b,\nu)$ such that $u_n \to u$ in $L^1(Q_\nu;\mathbb{R}^P)$ and

$$\lim_{n\to+\infty} \int_{Q_\nu} [\frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(\nabla u_n(x))] \, dx = K(a,b,\nu) = J_0(u).$$

Proof. Assume, without loss of generality, that $\nu = e_N$ so that

$$u(y) = \begin{cases} b & \text{if } y.e_N > 0 \\ a & \text{if } y.e_N < 0. \end{cases}$$

Denote Q_ν by Q and let Q' be the projection of Q on $\mathbb{R}^{N-1}$: $Q' = \{y \in Q : y_N = 0\}$. Let $L_n > 0$ and $\xi_n \in \mathcal{A}(a,b,e_N)$ be such that

$$\lim_{n\to+\infty} \int_{Q} [L_n W(\xi_n(y)) + \frac{1}{L_n} (h^\infty)^2(\nabla \xi_n(y))] \, dy = K(a,b,e_N). \tag{4.1}$$

For n fixed, define

$$v_\varepsilon^n(y) = v_\varepsilon^n(y',y_N) = \begin{cases} b & \text{if } y_N > \varepsilon/2 \\ \xi_n(y',\frac{y_N}{\varepsilon}) & \text{if } -\varepsilon/2 \le y_N \le \varepsilon/2 \\ a & \text{if } y_N < -\varepsilon/2. \end{cases}$$

Clearly $v_\varepsilon^n \in \mathcal{A}(a,b,e_N)$ for all $n \in \mathbb{N}$, $\varepsilon > 0$. Also,

$$\|v_\varepsilon^n - u^+\|_{L^1(Q^+)} = \|v_\varepsilon^n - b\|_{L^1(Q^+)} = \int_0^{\varepsilon/2}\int_{Q'} \|\xi_n(y',\frac{y_N}{\varepsilon}) - b\| \, dy =$$

$$= \varepsilon \int_0^{1/2}\int_{Q'} \|\xi_n(y',y_N) - b\| \, dy = \varepsilon \int_{Q^+} \|\xi_n(y) - b\| \, dy \to 0 \text{ as } \varepsilon \to 0^+$$

for n fixed. Likewise $\|v_\varepsilon^n - u^-\|_{L^1(Q^-)} = \|v_\varepsilon^n - a\|_{L^1(Q^-)} \to 0$ as $\varepsilon \to 0^+$ and so $v_\varepsilon^n \to u$ in $L^1(Q;\mathbb{R}^p)$ as $\varepsilon \to 0^+$. On the other hand

$$\int_Q [\frac{L_n}{\varepsilon} W(v_\varepsilon^n(y)) + \frac{\varepsilon}{L_n} h^2(\nabla v_\varepsilon^n(y))] \, dy =$$

$$= \int_Q [\frac{L_n}{\varepsilon} W(v_\varepsilon^n(y)) + \frac{\varepsilon}{L_n} (h^\infty)^2(\nabla v_\varepsilon^n(y))] \, dy + \int_Q [\frac{\varepsilon}{L_n} h^2(\nabla v_\varepsilon^n(y)) - \frac{\varepsilon}{L_n} (h^\infty)^2(\nabla v_\varepsilon^n(y))] \, dy =$$

$$=: I_1 + I_2$$

where

$$I_1 = \int_{-\varepsilon/2}^{\varepsilon/2}\int_{Q'} [\frac{L_n}{\varepsilon} W(v_\varepsilon^n(y)) + \frac{\varepsilon}{L_n} (h^\infty)^2(\nabla v_\varepsilon^n(y))] \, dy =$$

$$= \int_{-\varepsilon/2}^{\varepsilon/2}\int_{Q'} [\frac{L_n}{\varepsilon} W(\xi_n(y',\frac{y_N}{\varepsilon})) + \frac{1}{L_n\varepsilon} (h^\infty)^2(\nabla \xi_n(y',\frac{y_N}{\varepsilon}))] \, dy =$$

$$= \int_Q [L.w<Uy)> + £ (h^{\circ\circ})^2(^v\wedge(y))| \, dy \qquad (4.2)$$

and

$$|I_2| \le \int_Q \frac{\varepsilon}{L_n} |h^2(\nabla v^n_\varepsilon(y)) - (h^\infty)^2(\nabla v^n_\varepsilon(y))| \, dy =$$

$$i \text{ jnj}_Q^{r} \quad e \quad e \quad e^2 \, (h^\infty)^2(\nabla\xi_n(y',\frac{y_N}{\varepsilon}))| \, dy =$$

$$= \frac{1}{L_n\varepsilon} \int_{Q\cap\{\|\nabla\xi_n(y',\frac{y_N}{\varepsilon})\|>L\varepsilon\}} |\varepsilon^2 h^2(I\nabla\xi_n(y',\frac{y_N}{\varepsilon})) - (h^\infty)^2(\nabla\xi_n(y',\frac{y_N}{\varepsilon}))| \, dy +$$

$$+ \frac{J}{Lne} \int_{QndIV\wedge Cy \, \varepsilon} |\varepsilon^2 h^2(\frac{1}{e}\nabla uy',\frac{y_N}{e})) - (h^\infty)^2(\nabla\xi_n(y',\frac{y_N}{\varepsilon}))| \, dy =: I^1_2 + I^2_2.$$

By (H3) and Lemma 2.7 i)

$$I^2_2 \le \frac{1}{Ue} \text{meas}Q \; Ce^2 = \text{const.}?\frac{\varepsilon}{L_n}$$

and by (H4), HSlder's inequality and Lemma 2.7 i),

$$I^1_2 \le \frac{C'}{L_n\varepsilon} \int_{Q\cap\{\|\nabla\xi_n(y',\frac{y_N}{\varepsilon})\|>L\varepsilon\}} \|\nabla\xi_n(y',\frac{y_N}{\varepsilon})\|^{2-m} \varepsilon^m \, dy \le \frac{C}{L_n^{m/2}} \varepsilon^{m-1} [\int_{Q\cap\{\|\nabla\xi_n(y',\frac{y_N}{\varepsilon})\|>L\varepsilon\}} \frac{1}{L_n} \|\nabla\xi_n(y',\frac{y_N}{\varepsilon})\|^2 \, dy]^{1-m/2} \le$$

$$\le -\wedge e''\text{-}i \, [\int_Q \wedge \|V\wedge_n(y)\|^2 \, dy]^{1,1''72} \wedge C(\wedge r^{/2}[\int_Q q; (h\text{-})^2(V\wedge_n(y)) \, dy]^{1''172}$$

where by (4.1) $\{ \int_Q \frac{1}{L_n} (h^\sim)^2(V\wedge_n(y)) \, dy \}$ is a bounded sequence. Choose E(n) such that e(n) $\to$ 0+as n-4+«, ^ - ^ 0+, $\|v\wedge_{(n)} - u\|_{L},$-»0 and

$$\left| \int_Q [L_n W(\xi_n(y)) + \frac{1}{L_n}(h^{\infty})^2(\nabla\xi_n(y))]\, dy - \int_Q [\frac{L_n}{\varepsilon(n)} W(v^n_{\varepsilon(n)}(y)) + \frac{\varepsilon(n)}{L_n} h^2(\nabla v^n_{\varepsilon(n)}(y))]\, dy \right| < \frac{1}{n}.$$

Let $u_n = v^n_{\varepsilon(n)}$ and $\delta_n = \frac{\varepsilon(n)}{L_n}$. Then $\delta_n \to 0^+$, $u_n \to u$ in $L^1(Q;\mathbb{R}^P)$ and

$$\lim_{n\to+\infty} \int_Q [\frac{1}{\delta_n} W(u_n(y)) + \delta_n h^2(\nabla u_n(y))]\, dy = \lim_{n\to+\infty} \int_Q [L_n W(\xi_n(y)) + \frac{1}{L_n}(h^{\infty})^2(\nabla\xi_n(y))]\, dy$$

$$= K(a,b,e_N) = \int_{Q\cap\{y_N=0\}} K(a,b,e_N)\, dH_{N-1}(y). \qquad \blacksquare$$

Lemma 4.3. Let (H1)-(H5) hold and let

$$u(x) = \begin{cases} b & \text{if } (x-a_0).\nu > 0 \\ a & \text{if } (x-a_0).\nu < 0 \end{cases}$$

for some $a_0 \in \mathbb{R}^N$. Define

$\mathcal{B}(a_0,a,b,\nu,\eta) := \{u \in H^1(a_0+\eta Q;\mathbb{R}^P) : u(x) = b$ if $(x-a_0).\nu = \eta/2$, $u(x) = a$ if $(x-a_0).\nu = -\eta/2$ and u is periodic with period η in the directions of $\nu_1,\ldots,\nu_{N-1}\}$.

Given a sequence $\varepsilon_n \to 0^+$ there exists a subsequence $\{\varepsilon_{n_k}\}$ and a sequence $\{v_k\}$ in $\mathcal{B}(a_0,a,b,\nu,\eta)$ such that $v_k \to u$ in $L^1(a_0+\eta Q;\mathbb{R}^P)$ and

$$\lim_{k\to+\infty} \int_{a_0+\eta Q_\nu} [\frac{1}{\varepsilon_{n_k}} W(v_k(x)) + \varepsilon_{n_k} h^2(\nabla v_k(x))]\, dx = \eta^{N-1} K(a,b,\nu).$$

Proof. For simplicity, we assume that $\nu = e_N$ and we denote Q_ν by Q.

Case 1. Suppose first that $a_0 = 0$ and $\eta = 1$. By Lemma 4.2, consider $\alpha_k \to 0^+$ and $u_k \in \mathcal{A}(a,b,e_N)$ such that $u_k \to u$ in $L^1(Q;\mathbb{R}^P)$ and

$$\lim_{k\to+\infty} \int_Q [\frac{1}{\alpha_k} W(u_k(x)) + \alpha_k h^2(\nabla u_k(x))]\, dx = K(a,b,e_N). \qquad (4.3)$$

Fix $k \in \mathbb{N}$ and define

$$v_{k,n}(x) := \begin{cases} b & \text{if } \frac{\varepsilon_n}{2\alpha_k} < x_N < \frac{1}{2} \\ u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N) & \text{if } |x_N| < \frac{\varepsilon_n}{2\alpha_k} \\ a & \text{if } -\frac{1}{2} < x_N < -\frac{\varepsilon_n}{2\alpha_k}. \end{cases}$$

Clearly $v_{k,n} \in \mathcal{A}(a,b,e_N)$ and

$$\int_Q [\frac{1}{\varepsilon_n} W(v_{k,n}(x)) + \varepsilon_n h^2(\nabla v_{k,n}(x))]\, dx =$$

$$= \int_Q [\frac{1}{\varepsilon_n} W(v_{k,n}(x)) + \varepsilon_n (h^\infty)^2(\nabla v_{k,n}(x))]\, dx + \int_Q [\varepsilon_n h^2(\nabla v_{k,n}(x)) - \varepsilon_n (h^\infty)^2(\nabla v_{k,n}(x))]\, dx =:$$

$$=: I_1 + I_2 \tag{4.4}$$

where

$$I_1 = \int_{-\frac{\varepsilon_n}{2\alpha_k}}^{\frac{\varepsilon_n}{2\alpha_k}} \int_{Q'} [\frac{1}{\varepsilon_n} W(v_{k,n}(x)) + \varepsilon_n (h^\infty)^2(\nabla v_{k,n}(x))]\, dx =$$

$$= \int_{-1/2}^{1/2} \int_{Q'} [\frac{1}{\alpha_k} W(u_k(x)) + \alpha_k (h^\infty)^2(\nabla u_k(x))]\, dx =$$

$$= \int_Q [\frac{1}{\alpha_k} W(u_k(x)) + \alpha_k h^2(\nabla u_k(x))]\, dx + \int_Q [\alpha_k (h^\infty)^2(\nabla u_k(x)) - \alpha_k h^2(\nabla u_k(x))]\, dx =$$

$$= I_1^1 + I_1^2. \tag{4.5}$$

By (H4), Hölder's inequality and (H3)

$$I_1^2 = \int_{Q \cap \{\|\nabla u_k\| \le L\}} \alpha_k [(h^\infty)^2(\nabla u_k(x)) - h^2(\nabla u_k(x))]\, dx + \int_{Q \cap \{\|\nabla u_k\| > L\}} \alpha_k [(h^\infty)^2(\nabla u_k(x)) - h^2(\nabla u_k(x))]\, dx$$

$$\le C\alpha_k + \int_{Q \cap \{\|\nabla u_k\| > L\}} \alpha_k C \, \|\nabla u_k(x)\|^{2-m}\, dx \le$$

$$\leq O(\alpha_k) + (\alpha_k)^{m/2} C \Big[\int_{Q \cap \{\|\nabla u_k\| > L\}} \alpha_k \|\nabla u_k(x)\|^2 \, dx \Big]^{1-m/2} \leq$$

$$\leq O(\alpha_k) + (\alpha_k)^{m/2} C \Big[\int_Q \alpha_k C' + \alpha_k h^2(\nabla u_k(x)) \, dx \Big]^{1-m/2} = O(\alpha_k) \tag{4.6}$$

since, by (4.3) $\{ \int_Q \alpha_k h^2(\nabla u_k(x)) \, dx \}$ remains bounded. On the other hand

$$|I_2| \leq \int_Q \varepsilon_n \, | \, h^2(\frac{\alpha_k}{\varepsilon_n} \nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)) - \frac{\alpha_k^2}{\varepsilon_n^2} (h^\infty)^2 (\nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)) \, | \, dx =$$

$$= \frac{\alpha_k^2}{\varepsilon_n} \int_Q | \, \frac{\varepsilon_n^2}{\alpha_k^2} h^2(\frac{\alpha_k}{\varepsilon_n} \nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)) - (h^\infty)^2 (\nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)) \, | \, dx =$$

$$= \frac{\alpha_k^2}{\varepsilon_n} \int_{Q \cap \{\|\nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)\| \leq L\varepsilon_n/\alpha_k\}} | \, \frac{\varepsilon_n^2}{\alpha_k^2} h^2(\frac{\alpha_k}{\varepsilon_n} \nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)) - (h^\infty)^2 (\nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)) \, | \, dx +$$

$$+ \frac{\alpha_k^2}{\varepsilon_n} \int_{Q \cap \{\|\nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)\| > L\varepsilon_n/\alpha_k\}} | \, \frac{\varepsilon_n^2}{\alpha_k^2} h^2(\frac{\alpha_k}{\varepsilon_n} \nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)) - (h^\infty)^2 (\nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)) \, | \, dx =: I_2^1 + I_2^2. \tag{4.7}$$

By (H3) and Lemma 2.7 i),

$$I_2^1 \leq \frac{\alpha_k^2}{\varepsilon_n} \int_{Q \cap \{\|\nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)\| \leq L\varepsilon_n/\alpha_k\}} C \, [\frac{\varepsilon_n^2}{\alpha_k^2} (1 + \|\frac{\alpha_k}{\varepsilon_n} \nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)\|^2) + \|\nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)\|^2] \, dx \leq C \frac{\alpha_k^2 \varepsilon_n^2}{\varepsilon_n \, \alpha_k^2} =$$

$$= O(\varepsilon_n) \tag{4.8}$$

and, by (H4),

$$I_2^2 \leq C \frac{\alpha_k^2}{\varepsilon_n} \int_Q \frac{\varepsilon_n^m}{\alpha_k^m} \|\nabla u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N)\|^{2-m} dx \leq C \frac{\varepsilon_n^m}{\alpha_k^{m-1}} \int_Q \|\nabla u_k(x)\|^{2-m} dx =$$

$$=: \varepsilon_n^m \, h(\alpha_k). \tag{4.9}$$

Hence, by (4.4)-(4.9),

$$\limsup_{n\to+\infty}\int_Q [\frac{1}{\varepsilon_n}W(v_{k,n}(x)) + \varepsilon_n h^2(\nabla v_{k,n}(x))]\,dx \le \int_Q [\frac{1}{\alpha_k}W(u_k(x)) + \alpha_k h^2(\nabla u_k(x))]\,dx + $$

$$+ O(\alpha_k) \text{ for all } k$$

and, by definition of $K(a,b,e_N)$

$$K(a,b,e_N) \le \liminf_{n\to+\infty}\int_Q [\frac{1}{\varepsilon_n}W(v_{k,n}(x)) + \varepsilon_n h^2(\nabla v_{k,n}(x))]\,dx.$$

Also,

$$\|v_{k,n} - u\|_{L^1(Q;\mathbb{R}^p)} = \int_{Q\cap\{|x_N|<\varepsilon_n/2\alpha_k\}} |u_k(x',\frac{\alpha_k}{\varepsilon_n}x_N) - u(x)|\,dx = \frac{\varepsilon_n}{\alpha_k}\int_Q |u_k(x) - u(x)|\,dx.$$

Thus, for all k, choose ε_{n_k} such that, setting $v_k := v_{k,n_k}$, we have $\frac{\varepsilon_{n_k}}{\alpha_k} \le 1$ and

$$\limsup_{n\to+\infty}\int_Q [\frac{1}{\varepsilon_n}W(v_{k,n}(x)) + \varepsilon_n h^2(\nabla v_{k,n}(x))]\,dx = \int_Q [\frac{1}{\varepsilon_{n_k}}W(v_k(x)) + \varepsilon_{n_k} h^2(\nabla v_k(x))]\,dx + $$

$$+ O(1/k).$$

Then,

$$\|v_k - u\|_{L^1(Q;\mathbb{R}^p)} \le \|u_k - u\|_{L^1(Q;\mathbb{R}^p)} \to 0 \text{ as } k \to +\infty$$

and

$$\lim_{k\to+\infty}\int_Q [\frac{1}{\varepsilon_{n_k}}W(v_k(x)) + \varepsilon_{n_k} h^2(\nabla v_k(x))]\,dx = K(a,b,e_N).$$

Case 2. We now take $\Omega = a_0+\eta Q$ for some $a_0 \in \mathbb{R}^N$ and $\eta > 0$ and we define

$$h_\eta(A) = h(\frac{A}{\eta}).$$

Setting

$$u_0(x) = \begin{cases} b & \text{if } x.e_N > 0 \\ a & \text{if } x.e_N < 0 \end{cases}$$

by case 1, given $\varepsilon_n \to 0^+$, there exist a subsequence $\{\varepsilon_{n_k}\}$ and a sequence $\{v_k\}$ in $\mathcal{A}(a,b,e_N)$ such that $v_k \to u_0$ in $L^1(Q;\mathbb{R}^p)$ and

$$\lim_{k\to+\infty}\int_Q \left[\frac{1}{\varepsilon_{n_k}} W(v_k(y)) + \varepsilon_{n_k} h_\eta^2(\nabla v_k(y))\right] dy = K_\eta(a,b,e_N),$$

where

$$K_\eta(a,b,e_N) = \inf\left\{ \int_Q \left[LW(\varphi(x)) + \frac{1}{L} h_\eta^2(\nabla\varphi(x))\right] dx : \varphi \in \mathcal{A}(a, b, e_N),\ L > 0 \right\}.$$

Note that, due to the homogeneity of h^∞,

$$K_\eta(a,b,e_N) = \frac{1}{\eta}\inf\left\{ \int_Q \left[L\eta W(\xi(x)) + \frac{1}{L\eta} (h^\infty)^2(\nabla\xi(x))\right] dx : \xi \in \mathcal{A}(a, b, e_N),\ L > 0 \right\} =$$

$$= \frac{1}{\eta}K(a,b,e_N). \tag{4.10}$$

For $x \in a_0+\eta Q$ let

$$u_k(x) := v_k\left(\frac{x-a_0}{\eta}\right).$$

Clearly $u_k \in \mathcal{A}(a_0,a,b,e_N,\eta)$,

$$\int |u_k(x) - u(x)|\, dx = \int_{a_0+\eta Q} \left|v_k\left(\frac{x-a_0}{\eta}\right) - u(x)\right| dx = \eta^N \int_Q |v_k(x) - u(a_0+\eta x)|\, dx =$$

$$= \eta^N \int_Q |v_k(x) - u_0(x)|\, dx \to 0 \text{ as } k \to +\infty,$$

and, by (4.10),

$$\lim_{k\to+\infty} \int_{a_0+\eta Q} \left[\frac{1}{\varepsilon_{n_k}} W(u_k(x)) + \varepsilon_{n_k} h^2(\nabla u_k(x))\right] dx =$$

$$\lim_{k\to+\infty} \int_{a_0+\eta Q} \left[\frac{1}{\varepsilon_{n_k}} W\left(v_k\left(\frac{x-a_0}{\eta}\right)\right) + \varepsilon_{n_k} h^2\left(\frac{1}{\eta}\nabla v_k\left(\frac{x-a_0}{\eta}\right)\right)\right] dx =$$

$$= \lim_{k\to+\infty}\eta^N \int_Q \left[\frac{1}{\varepsilon_{n_k}} W(v_k(x)) + \varepsilon_{n_k} h_\eta^2(\nabla v_k(x))\right] dx = \eta^N K_\eta(a,b,e_N) = \eta^{N-1} K(a,b,e_N). \qquad \blacksquare$$

Proof of Proposition 4.1.

Step 1. Assume first that u has planar interface i.e.

$$u(x) = \begin{cases} b & \text{if } (x-a_0)\cdot\nu > 0 \\ a & \text{if } (x-a_0)\cdot\nu < 0. \end{cases}$$

Without loss of generality, assume that $a_0 = 0$ and $\nu = e_N$. Let $\Omega' = \{x \in \Omega : x_N = 0\}$. In order to ensure that property (H5) and Proposition 2.8 ii) are satisfied uniformly we will work on compact subsets of Ω. Fix $\varepsilon > 0$ and let $\Omega_\varepsilon' \subset\subset \Omega'$ be such that

$$H_{N-1}(\Omega' \setminus \Omega_\varepsilon') = O(\varepsilon). \tag{4.11}$$

Since Ω_ε' is compact we can find $\delta > 0$ such that $\Omega_\varepsilon' \times [-\delta/2, \delta/2] \subset\subset \Omega$ and (H5) and Proposition 2.8 ii) are satisfied uniformly in $\Omega_\varepsilon' \times [-\delta/2, \delta/2]$ i.e.

$$x,y \in \Omega_\varepsilon' \times [-\delta/2,\delta/2],\ |x-y| < \delta \Rightarrow |h^2(x,A) - h^2(y,A)| \le \varepsilon C(1+\|A\|^2), \text{ for all } A \in M^{p\times N} \tag{4.12}$$

and

$$x,y \in \Omega_\varepsilon' \times [-\delta/2,\delta/2],\ |x-y| < \delta \Rightarrow |K(x,a,b,\nu) - K(y,a,b,\nu)| < \varepsilon C\,(1+\|a\|^q+\|b\|^q+\|b-a\|^2). \tag{4.13}$$

We may write

$$\Omega_\varepsilon' = \bigcup_{i=1}^{p}(a_i + \eta Q') \cup \omega \tag{4.14}$$

where $H_{N-1}(\omega) = O(\varepsilon)$, $Q_i' := a_i + \eta Q'$ are cubes with disjoint interiors, $0 < \eta < \delta$ and

$$\bigcup_{i=1}^{p}(a_i + \eta Q) =: \bigcup_{i=1}^{p} Q_i \subset\subset \Omega.$$

Since $\partial\Omega$ is Lipschitz it is possible to pick η as above and so that

$$H_{N-1}\left(\mathrm{proj}_{\{x_N=0\}}\left(\Omega \cap \left\{|x_N| < \frac{\eta}{2}\right\}\right) \setminus \bigcup_{i=1}^{p} Q_i'\right) =: H_{N-1}(P) = O(\varepsilon). \tag{4.15}$$

We claim that given any sequence $\alpha_n \to 0^+$ there exists a subsequence $\{\alpha_{n_k}\}$ and a sequence $\{u_k\}$ in $H^1(\Omega;\mathbb{R}^p)$ such that $u_k \to u$ in $L^1(\Omega;\mathbb{R}^p)$ and

$$\lim_{k\to+\infty} \int_\Omega \frac{1}{\alpha_{n_k}} W(u_k(x)) + \alpha_{n_k} h^2(x,\nabla u_k(x))\, dx = \int_{\Sigma(u)} K(x,a,b,e_N)\, dH_{N-1}(x). \tag{4.16}$$

By Lemma 4.3 given a sequence $\alpha_n \to 0^+$ there exist a subsequence $\{\alpha_k^{(1)}\}$ and a sequence $\{u_k^{(1)}\}$ in $\mathcal{B}(a_1,a,b,e_N,\eta)$ such that $u_k^{(1)} \to u$ in $L^1(Q_1;\mathbb{R}^p)$ and

$$\lim_{k\to+\infty} \int_{Q_1} \left[\frac{1}{\alpha_k^{(1)}} W(u_k^{(1)}(x)) + \alpha_k^{(1)} h^2(a_1,\nabla u_k^{(1)}(x))\right] dx = \eta^{N-1} K(a_1,a,b,e_N). \tag{4.17}$$

By Lemma 3.2 there exists a subsequence $\{\beta_k^{(1)}\}$ of $\{\alpha_k^{(1)}\}$ and a sequence $\{w_k^{(1)}\}$ in $H^1(Q_1;\mathbb{R}^p)$ such that $w_k^{(1)} \to u$ in $L^1(Q_1;\mathbb{R}^p)$, $w_k^{(1)}(x) = v_{\beta_k}^{(1)}(\frac{x-a_1}{\eta})$ for $x \in \partial Q_1$ (the v_j are mollifications of u) and

$$\limsup_{k\to+\infty} \int_{Q_1} \left[\frac{1}{\beta_k^{(1)}} W(w_k^{(1)}(x)) + \beta_k^{(1)} h^2(a_1,\nabla w_k^{(1)}(x))\right] dx \le$$

$$\leq \liminf_{k\to+\infty} \int_{Q_1} [\frac{1}{\alpha_k^{(1)}} W(u_k^{(1)}(x)) + \alpha_k^{(1)} h^2(a_1, \nabla u_k^{(1)}(x))]\, dx = \eta^{N-1} K(a_1,a,b,e_N). \quad (4.18)$$

By Proposition 3.1,

$$\liminf_{k\to+\infty} \int_{Q_1} [\frac{1}{\beta_k^{(1)}} W(w_k^{(1)}(x)) + \beta_k^{(1)} h^2(a_1, \nabla w_k^{(1)}(x))]\, dx \geq \eta^{N-1} K(a_1,a,b,e_N)$$

which, together with (4.18), implies

$$\lim_{k\to+\infty} \int_{Q_1} [\frac{1}{\beta_k^{(1)}} W(w_k^{(1)}(x)) + \beta_k^{(1)} h^2(a_1, \nabla w_k^{(1)}(x))]\, dx = \eta^{N-1} K(a_1,a,b,e_N). \quad (4.19)$$

By Lemma 4.3 there exists a subsequence $\{\alpha_k^{(2)}\}$ of $\{\beta_k^{(1)}\}$ and a sequence $\{u_k^{(2)}\}$ in $\mathcal{B}(a_2,a,b,e_N,\eta)$ such that $u_k^{(2)} \to u$ in $L^1(Q_2;\mathbb{R}^p)$ and

$$\lim_{k\to+\infty} \int_{Q_2} [\frac{1}{\alpha_k^{(2)}} W(u_k^{(2)}(x)) + \alpha_k^{(2)} h^2(a_2, \nabla u_k^{(2)}(x))]\, dx = \eta^{N-1} K(a_2,a,b,e_N).$$

Once again, by applying Lemma 3.2, we conclude that there is a subsequence $\{\beta_k^{(2)}\}$ of $\{\alpha_k^{(2)}\}$ and a sequence $\{w_k^{(2)}\}$ in $H^1(Q_2;\mathbb{R}^p)$ such that $w_k^{(2)} \to u$ in $L^1(Q_2;\mathbb{R}^p)$, $w_k^{(2)}(x) = v_{\beta_k^{(2)}}(\frac{x-a_2}{\eta})$ for $x \in \partial Q_2$ and

$$\lim_{k\to+\infty} \int_{Q_2} [\frac{1}{\beta_k^{(2)}} W(w_k^{(2)}(x)) + \beta_k^{(2)} h^2(a_2, \nabla w_k^{(2)}(x))]\, dx = \eta^{N-1} K(a_2,a,b,e_N).$$

By induction we repeat the above argument in order to obtain subsequences $\{\beta_k^{(p)}\} \subset \{\beta_k^{(p-1)}\} \subset \ldots \subset \{\beta_k^{(2)}\} \subset \{\beta_k^{(1)}\}$ and sequences $\{w_k^{(j)}\}$ in $H^1(Q_j;\mathbb{R}^p)$ such that $w_k^{(j)} \to u$ in $L^1(Q_j;\mathbb{R}^p)$, $w_k^{(j)}(x) = v_{\beta_k^{(j)}}(\frac{x-a_j}{\eta})$ for $x \in \partial Q_j$ and

$$\lim_{k\to+\infty} \int_{Q_j} [\frac{1}{\beta_k^{(j)}} W(w_k^{(j)}(x)) + \beta_k^{(j)} h^2(a_j, \nabla w_k^{(j)}(x))]\, dx = \eta^{N-1} K(a_j,a,b,e_N)$$

for $j = 1,\ldots,p$. Consider the sequence $\{\beta_k^{(p)}\}$ and for all $j = 1,\ldots,p$ let $\{\xi_k^{(j)}\}$ be the corresponding subsequence of $\{w_k^{(j)}\}$ such that

$$\lim_{k\to+\infty}\int_{Q_j}\Big[\frac{1}{\beta_k^{(p)}}W(\xi_k^{(j)}(x)) + \beta_k^{(p)}h^2(a_j,\nabla\xi_k^{(j)}(x))\Big]\,dx = \eta^{N-1}\,K(a_j,a,b,e_N). \qquad (4.20)$$

Define the sequence $u_{k,\varepsilon}(x)$ as follows,

$$u_{k,\varepsilon}(x) = \begin{cases} \xi_k^{(j)}(x) & \text{if } x \in Q_j \\ b & \text{if } x_N > \eta/2 \\ a & \text{if } x_N < -\eta/2 \end{cases}$$

and in $\left(\Omega \cap \left\{|x_N| < \frac{\eta}{2}\right\}\right)\setminus(\bigcup_{i=1}^{p} Q_i)$ we define $u_{k,\varepsilon}(x)$ using the periodicity of $\{v_{\beta_k^{(p)}}\}$. Clearly $u_{k,\varepsilon} \in H^1(\Omega;\mathbb{R}^p)$. As $\|v_{\beta_k^{(p)}}\|_\infty \leq$ const. and since

$$\operatorname{meas}\left(\left(\Omega \cap \left\{|x_N| < \frac{\eta}{2}\right\}\right)\setminus(\bigcup_{i=1}^{p} Q_i)\right) = O(\varepsilon)$$

we have

$$\|u_{k,\varepsilon} - u\|_{L^1(\Omega,\mathbb{R}^p)} = O(\varepsilon) + \sum_{i=1}^{p}\|u_{k,\varepsilon} - u\|_{L^1(Q_i,\mathbb{R}^p)}$$

and so,

$$\lim_{\varepsilon\to 0^+}\big[\lim_{k\to+\infty}\|u_{k,\varepsilon} - u\|_{L^1(\Omega,\mathbb{R}^p)}\big] = 0.$$

Also,

$$\int_{\Omega}\Big[\frac{1}{\beta_k^{(p)}}W(u_{k,\varepsilon}(x)) + \beta_k^{(p)}h^2(x,\nabla u_{k,\varepsilon}(x))\Big]\,dx =$$

$$= \sum_{i=1}^{p}\int_{Q_i}\Big[\frac{1}{\beta_k^{(p)}}W(\xi_k^{(i)}(x)) + \beta_k^{(p)}h^2(a_i,\nabla\xi_k^{(i)}(x))\Big]\,dx +$$

$$+ \sum_{i=1}^{p}\int_{Q_i}\beta_k^{(p)}\,[h^2(x,\nabla\xi_k^{(i)}(x)) - h^2(a_i,\nabla\xi_k^{(i)}(x))]\,dx +$$

$$+ \int_{(\Omega\cap\{|x_N|<\eta/2\})\setminus(\cup Q_i)}\Big[\frac{1}{\beta_k^{(p)}}W(u_{k,\varepsilon}(x)) + \beta_k^{(p)}h^2(x,\nabla u_{k,\varepsilon}(x))\Big]\,dx +$$

$$+ \int_{\Omega\cap\{|x_N|>\eta/2\}}\beta_k^{(p)}\,h^2(x,0)\,dx =: I_1 + I_2 + I_3 + I_4,$$

where, by (4.20),

$$\lim_{k\to+\infty} I_1 = \eta^{N-1} \sum_{i=1}^{p} K(a_i,a,b,e_N)$$

and, since $\beta_k^{(p)} \to 0^+$,

$$\lim_{k\to+\infty} I_4 = 0.$$

Also, as $u_{k,\varepsilon}$ is the periodic extension of $v_{\beta_k^{(p)}}$ on $\left(\Omega \cap \left\{ |x_N| < \frac{\eta}{2} \right\}\right) \setminus (\bigcup_{i=1}^{p} Q_i)$ and

$$v_{\beta_k^{(p)}}(x) = \begin{cases} b & \text{if } x_N > \beta_k^{(p)} \\ a & \text{if } x_N < \beta_k^{(p)} \end{cases} \quad \text{with } \|\nabla v_{\beta_k^{(p)}}\|_\infty = O(\frac{1}{\beta_k^{(p)}}) \text{ if } |x_N| < \beta_k^{(p)}$$

we have, by (4.15),

$$I_3 \leq \int \int_{-\beta_k^{(p)}}^{\beta_k^{(p)}} C(\frac{1}{\beta_k^{(p)}} + \beta_k^{(p)})\, dx = O(\varepsilon).$$

As $\eta < \delta$, by (4.12) and (H3),

$$\limsup_{k\to+\infty} I_2 \leq \sum_{i=1}^{p} \int_{Q_i} \beta_k^{(p)}\, \varepsilon C\, (1+\|\nabla \xi_k^{(i)}(x)\|^2)\, dx \leq$$

$$\leq \limsup_{k\to+\infty} \sum_{i=1}^{p} \int_{Q_i} \beta_k^{(p)}\, \varepsilon C\, [1+h^2(a_i,\nabla \xi_k^{(i)}(x))]\, dx = O(\varepsilon)$$

since by (4.20) $\left\{ \int_{Q_i} \beta_k^{(p)}\, h^2(a_i,\nabla \xi_k^{(i)}(x))\, dx \right\}$ remains bounded. Finally, we note that, by (4.11), (4.13) and (4.14),

$$\left| \int_{\Omega'} K(x,a,b,e_N)\, dH_{N-1}(x) - \eta^{N-1} \sum_{i=1}^{p} K(a_i,a,b,e_N) \right| \leq$$

$$\leq \int_{\Omega' \setminus \bigcup_{i=1}^{p} Q_i'} K(x,a,b,e_N)\, dH_{N-1}(x) + \sum_{i=1}^{p} \int_{Q_i'} |K(x,a,b,e_N) - K(a_i,a,b,e_N)|\, dH_{N-1}(x) = O(\varepsilon),$$

so to obtain the desired approximating sequence it suffices to let $\varepsilon \to 0^+$ and use a diagonalization procedure.

Step 2. Now suppose that u has polygonal interface i.e. $u = \chi_A a + (1-\chi_A) b$ where $A \subset \Omega$ is of the form $A = A' \cap \Omega$, $\partial^* A \cap \Omega = \partial^* A' \cap \Omega$ with A' a polyhedral set (i.e. A' is a bounded, strongly Lipschitz domain and ∂A^f = Hi u ... u HM • Hi are closed subsets of hyperplanes of the type {x e IR^N: x.Vi = oq}). Notice that Step 1 corresponds to the case where A* is a large cube. We claim that for any sequence ^ -» 0+ there exists a subsequence {En^1} and a sequence {u_n} in HK^IRP) such that u_n -+ u in LH^;IR^P) and

$$\lim_{n\to 0} \int [\text{^WCunCx)} + \varepsilon_n V(x,\nabla u_n(x))]\, dx = \int K(x,a,b,\nu(x))\, dH_{N-1}(x). \quad (4.21)$$

Recall that 3*A n Q = 3*A^1 n *Cl* = $\bigcup_{i=1}^{M}$ (Hi n ft). Let I = { i € {1,...,M} : *$U^\wedge_v$ (H, nCl)*> 0}. If card 1 = 0 then u (x) = a a.e. in £2 or u(x) = b a.e. in *Q* so it suffices to take e^ = £n and u_n = u, for all n. If card 1 = 1 then 9*A n *£1* reduces to one planar interface and we are back to Step 1. Using an induction procedure, assume that the result is true if card I = k, k ^ M - l and we prove it is still true if card I = k. Assume that

$$\partial^*A \cap Q = (H_1 \cap \Omega) \cup \ldots \cup (H_k \cap Q).$$

Consider S := {x € $[R^N$: dist (x, Hi) = dist (x, H2 u... u HM)}. Then S is locally the graph of a Lipschitz function and for every xo € S there exists e > 0 such that

$$B(x_0,e) \cap \{x : \text{dist}(x, H_1) > \text{dist}(x, H_2 \cup \ldots \cup H_M)\}$$

is connected. Also

$$\partial^*A \cap \Omega \cap S = \{x \in (\bigcup_{i=1}^{k} H_i) \cap \Omega : \text{dist}(x, H_1) = \text{dist}(x, H_2 \cup \ldots \cup H_M)\} = \bigcup_{i=2}^{k} (H_j \cap H_1) \cap \Omega$$

and so HN-I(9*A n f i n S) = 0 because HN-i(Hi n Hj) = 0 for i * j. Let

$$\Omega_1 = \{x \in Q : \text{dist}(x,H_1) < \text{dist}(x,H_2 \cup \ldots \cup H_M)\}.$$

Clearly *Cl* is open and *Cl* n (H_2 u ... u H_K) = 0. Since *Cl* is the intersection of a strongly Lipschitz domain with *Q* and *dCl* is locally Lipschitz it follows that *Cl* is also a strongly Lipschitz domain. We would like to apply the induction hypothesis to *Cl* and to *Cl \ Q* := Q_2- But, although Ω_2 n d*A = (H_2 u... u HK) n *Q* consists of p flat interfaces and dQ_2 is locally Lipschitz, it may happen that Q_2 is no longer connected. We write

$$\Omega_2 = \bigcup_{i=1}^{} \omega_i$$

where a* are open, connected, strongly Lipschitz domains with ©i n ©j = 0 for i # j. It is easy to verify that if i * j then ©j n ©j *r\ O.* = 0. Thus we only need to match the deformations across the interfaces do)i n *dCl\ n Cl.* Fix 8 > 0 and let

$$U_\delta = \{x \in IR^N : \text{dist}(x,S) < \delta\},$$

$$U'_\delta = \{x \in IR^N : \text{dist}(x,S) < \delta,\ \text{dist}(x,H_1) < \text{dist}(x, H_2 \cup \ldots \cup H_k)\},$$

$$U''_\delta = \{x \in IR^N : \text{dist}(x,S) < \delta,\ \text{dist}(x,H_1) > \text{dist}(x,H_2 \cup \ldots \cup H_k)\}.$$

As Pern(A) < +«>, choose k = k(6) such that

$$\sum_{i\geq k} \mathrm{Per}_\Omega(\partial^*A \cap \omega_i) < \delta \tag{4.22}$$

and due to Proposition 2.8 i), we can also request that

$$\sum_{i\geq k} \int_{\partial^*A\cap\omega_i} K(x,a,b,\nu(x))\, dH_{N-1}(x) < \delta. \tag{4.23}$$

Since Ω_1 contains only one interface, by Step 1, given any sequence $\varepsilon_n \to 0^+$ there exists a subsequence $\{\varepsilon_n^{(1)}\}$ and a sequence $\{v_n\}$ in $H^1(\Omega_1;\mathbb{R}^p)$ such that $v_n \to u$ in $L^1(\Omega_1;\mathbb{R}^p)$ and

$$\lim_{n\to+\infty} \int_{\Omega_1} \left[\frac{1}{\varepsilon_n^{(1)}} W(v_n(x)) + \varepsilon_n^{(1)} h^2(x,\nabla v_n(x))\right] dx = \int_{\partial^*A\cap\Omega_1} K(x,a,b,\nu(x))\, dH_{N-1}(x).$$

$\partial^*A \cap \omega_1$ contains at most M-1 flat interfaces so we can use the induction hypothesis to obtain a subsequence $\{\varepsilon_n^{(2)}\}$ of $\{\varepsilon_n^{(1)}\}$ and a sequence $\{u_n^{(1)}\}$ in $H^1(\omega_1;\mathbb{R}^p)$ such that $u_n^{(1)} \to u$ in $L^1(\omega_1;\mathbb{R}^p)$

and

$$\lim_{n\to+\infty} \int_{\omega_1} \left[\frac{1}{\varepsilon_n^{(2)}} W(u_n^{(1)}(x)) + \varepsilon_n^{(2)} h^2(x,\nabla u_n^{(1)}(x))\right] dx = \int_{\partial^*A\cap\omega_1} K(x,a,b,\nu(x))\, dH_{N-1}(x).$$

We continue this process inductively in order to obtain subsequences $\{\varepsilon_n^{(k+1)}\} \subset \ldots \subset \{\varepsilon_n^{(1)}\} \subset \{\varepsilon_n\}$ and sequences $\{u_n^{(i)}\}$ in $H^1(\omega_i;\mathbb{R}^p)$ such that $u_n^{(i)} \to u$ in $L^1(\omega_i;\mathbb{R}^p)$ and

$$\lim_{n\to+\infty} \int_{\omega_i} \left[\frac{1}{\varepsilon_n^{(i+1)}} W(u_n^{(i)}(x)) + \varepsilon_n^{(i+1)} h^2(x,\nabla u_n^{(i)}(x))\right] dx = \int_{\partial^*A\cap\omega_i} K(x,a,b,\nu(x))\, dH_{N-1}(x)$$

for all $i = 1,\ldots k$. Consider the sequence $\{\varepsilon_n^{(k+1)}\} =: \{\alpha_n\}$ and for all $i = 1,\ldots,k$ let $\{\xi_n^{(i)}\}$ be the corresponding subsequence of $\{u_n^{(i)}\}$ such that

$$\lim_{n\to+\infty} \int_{\omega_i} \left[\frac{1}{\alpha_n} W(\xi_n^{(i)}(x)) + \alpha_n h^2(x,\nabla \xi_n^{(i)}(x))\right] dx = \int_{\partial^*A\cap\omega_i} K(x,a,b,\nu(x))\, dH_{N-1}(x) \tag{4.24}$$

and let $\{v_n^{(1)}\}$ be the corresponding subsequence of $\{v_n\}$ such that

$$\lim_{n\to+\infty} \int_{\Omega_1} \left[\frac{1}{\alpha_n} W(v_n^{(1)}(x)) + \alpha_n h^2(x,\nabla v_n^{(1)}(x))\right] dx = \int_{\partial^*A\cap\Omega_1} K(x,a,b,\nu(x))\, dH_{N-1}(x). \tag{4.25}$$

Define $\bar{u}(x) = \chi_{A'}(x)\, a + (1-\chi_{A'}(x))\, b$ so that the restriction of $\bar{u}$ to Ω is u and let

$$w_n(.) = \frac{1}{(\alpha_n)^N}\rho(\frac{\cdot}{\alpha_n}) * \bar{u}.$$

Notice that w_n is bounded in L^∞ and

$$\|\nabla w_n\|_\infty = O(\frac{1}{\alpha_n}); \text{ supp } \nabla w_n \subset \{x \in \mathbb{R}^N : \text{dist }(x,\Sigma(\bar{u})) < \alpha_n\}. \tag{4.26}$$

Since $w_n \to u$ and $v_n^{(1)} \to u$ in $L^1(U_\delta^- \cap \Omega_1;\mathbb{R}^P)$ we may use the slicing method to connect $v_n^{(1)}$ and w_n across $U_\delta^- \cap \Omega_1$ (as in the proof of Lemma 3.2 we let $M = C(1 + 2\|u\|_q^q)$, $C = \max\{c, c \text{ meas}\Omega\}$ and c is the constant appearing in (H2) and we divide U_δ^- into k_n slices of the form $S_i = \{x \in U_\delta^- : \alpha_i < \text{dist }(x,S) < \alpha_{i+1}\}$ where $\alpha_1 = 0$, $\alpha_{k_n+1} = \delta$ and k_n is given by $[\frac{2M}{\alpha_n(\sqrt{\sigma_n})}] + 1$, $\sigma_n = \|v_n^{(1)} - w_n\|_{L^2(U_\delta^- \cap \Omega_1;\mathbb{R}^P)}$). We obtain a sequence $\{\eta_n\}$ in $H^1(\Omega_1;\mathbb{R}^P)$ such that $\eta_n \to u$ in $L^1(\Omega_1;\mathbb{R}^P)$, $\eta_n = w_n$ on $\partial\Omega_1 \cap \Omega$ and, by (4.26), (H1), (H2), (H3) and (4.25)

$$\limsup_{n\to+\infty} \int_{\Omega_1} [\frac{1}{\alpha_n}W(\eta_n(x)) + \alpha_n h^2(x,\nabla\eta_n(x))]\, dx \le$$

$$\le \limsup_{n\to+\infty}\Big[\int_{\Omega_1} [\frac{1}{\alpha_n}W(v_n^{(1)}(x)) + \alpha_n h^2(x,\nabla v_n^{(1)}(x))]\, dx +$$

$$+ \int_{U_\delta^- \cap \Omega_1} [\frac{1}{\alpha_n}W(w_n(x)) + \alpha_n h^2(x,\nabla w_n(x))]\, dx\Big] \le$$

$$\le \int_{\partial^* A \cap \Omega_1} K(x,a,b,\nu(x))\, dH_{N-1}(x) + \limsup_{n\to+\infty}\frac{C}{\alpha_n}\text{meas}\{x \in U_\delta^- \cap \Omega_1 : \text{dist }(x,\Sigma(u)) < \alpha_n\} =$$

$$= \int_{\partial^* A \cap \Omega_1} K(x,a,b,\nu(x))\, dH_{N-1}(x) + C\, \text{Per}_\Omega(\partial^* A \cap U_\delta^- \cap \Omega_1). \tag{4.27}$$

Similarly, for each $i = 1,\dots,k$, we connect $\xi_n^{(i)}$ to w_n across $U_\delta^+ \cap \omega_i$ and we obtain sequences $\varphi_n^{(i)}$ in $H^1(\omega_i;\mathbb{R}^P)$ such that $\varphi_n^{(i)} \to u$ in $L^1(\omega_i;\mathbb{R}^P)$, $\varphi_n^{(i)} = w_n$ on $\partial\omega_i \cap \Omega$ and

$$\limsup_{n\to+\infty} \int_{\omega_i} [\frac{1}{\alpha_n} W(\varphi_n^{(i)}(x)) + \alpha_n h^2(x,\nabla\varphi_n^{(i)}(x))]\, dx \le$$

$$\le \int_{\partial^*A\cap\omega_i} K(x,a,b,\nu(x))\, dH_{N-1}(x) + C\, Per_\Omega(\partial^*A \cap U_\delta^+ \cap \omega_i). \tag{4.28}$$

Define

$$u_n(x) := \sum_{i=1}^{k} \chi_{\omega_i}(x)\, \varphi_n^{(i)}(x) + \chi_{\overline{\Omega}_1\cap\Omega}(x)\, \eta_n(x) + \sum_{i=k+1}^{\infty} \chi_{\omega_i}(x)\, w_n(x).$$

Then,

$$\|u_n - u\|_{L^1(\Omega;\mathbb{R}^p)} \le$$

$$\le \sum_{i=1}^{k} \int_{\omega_i} |\varphi_n^{(i)}(x) - u(x)|\, dx + \int_{\Omega_1} |\eta_n(x) - u(x)|\, dx + C \sum_{i=k+1}^{\infty} meas\{x \in \omega_i : dist(x,\Sigma(u)) < \alpha_n\} \le$$

$$\le \sum_{i=1}^{k} \int_{\omega_i} |\varphi_n^{(i)}(x) - u(x)|\, dx + \int_{\Omega_1} |\eta_n(x) - u(x)|\, dx + C\, meas\{x \in \Omega : dist(x,\Sigma(u)) < \alpha_n\}.$$

Hence,

$$\lim_{n\to+\infty} \|u_n - u\|_{L^1(\Omega;\mathbb{R}^p)} \le \sum_{i=1}^{k} \lim_{n\to+\infty} \int_{\omega_i} |\varphi_n^{(i)}(x) - u(x)|\, dx + \lim_{n\to+\infty} \int_{\Omega_1} |\eta_n(x) - u(x)|\, dx +$$

$$+ \lim_{n\to+\infty} C\, meas\{x \in \Omega : dist(x,\Sigma(u)) < \alpha_n\} = 0$$

since $\Sigma(u)$ is polyhedral. On the other hand, by (4.26), (H1), (H2) and (H3)

$$\int_{\Omega} [\frac{1}{\alpha_n} W(u_n(x)) + \alpha_n h^2(x,\nabla u_n(x))]\, dx \le \sum_{i=1}^{k} \int_{\omega_i} [\frac{1}{\alpha_n} W(\varphi_n^{(i)}(x)) + \alpha_n h^2(x,\nabla\varphi_n^{(i)}(x))]\, dx +$$

$$+ \int_{\Omega_1} [\frac{1}{\alpha_n} W(\eta_n(x)) + \alpha_n h^2(x,\nabla\eta_n(x))]\, dx + \sum_{i=k+1}^{\infty} \frac{C}{\alpha_n} meas\{x \in \omega_i : dist(x,\Sigma(u)) < \alpha_n\} =$$

$$= \sum_{i=1}^{k} \int_{\omega_i} [\frac{1}{\alpha_n} W(\varphi_n^{(i)}(x)) + \alpha_n h^2(x,\nabla\varphi_n^{(i)}(x))]\, dx + \int_{\Omega_1} [\frac{1}{\alpha_n} W(\eta_n(x)) + \alpha_n h^2(x,\nabla\eta_n(x))]\, dx +$$

$$+ \frac{C}{\alpha_n} meas\{x \in \bigcup_{i=k+1}^{\infty} \omega_i : dist\,(x,\Sigma(u)) < \alpha_n\}.$$

Thus, by (4.22), (4.23), (4.27) and (4.28),

$$\limsup_{n\to+\infty}\int_\Omega [\frac{1}{\alpha_n}W(u_n(x)) + \alpha_n h^2(x,\nabla u_n(x))]\,dx \le \sum_{i=1}^{k}\int_{\omega_i\cap\partial^*A} K(x,a,b,\nu(x))\,dH_{N-1}(x) +$$

$$+ C\sum_{i=1}^{k} Per_\Omega(\partial^*A \cap U_\delta^+ \cap \omega_i) + \int_{\Omega_1\cap\partial^*A} K(x,a,b,\nu(x))\,dH_{N-1}(x) + C\,Per_\Omega(\partial^*A \cap U_\delta^- \cap \Omega_1) +$$

$$+ C\,Per_\Omega(\partial^*A \cap (\bigcup_{i=k+1}^{\infty}\omega_i)) \le$$

$$\le \int_{\Omega\cap\partial^*A} K(x,a,b,\nu(x))\,dH_{N-1}(x) - \sum_{i=k+1}^{\infty}\int_{\omega_i\cap\partial^*A} K(x,a,b,\nu(x))\,dH_{N-1}(x) + CPer_\Omega(\partial^*A \cap U_\delta) + O(\delta)$$

$$= \int_{\Omega\cap\partial^*A} K(x,a,b,\nu(x))\,dH_{N-1}(x) + O(\delta).$$

Hence we proved that for all $\delta > 0$ there exists a subsequence $\{\alpha_n(\delta)\}$ of $\{\varepsilon_n\}$ and there exists a sequence $\{u_n(\delta)\}$ such that

i) $\lim_{n\to+\infty}\|u_n(\delta) - u\|_{L^1(\Omega;\mathbb{R}^P)} = 0$;

ii) $$\limsup_{n\to+\infty}\int_\Omega [\frac{1}{\alpha_n(\delta)}W(u_n(\delta)(x)) + \alpha_n(\delta)h^2(x,\nabla u_n(\delta)(x))]\,dx \le$$

$$\le \int_{\Omega\cap\partial^*A} K(x,a,b,\nu(x))\,dH_{N-1}(x) + O(\delta).$$

Let $\delta = 1$ and choose n_1 such that

$$\|u_{n_1}(1) - u\|_{L^1(\Omega;\mathbb{R}^P)} \le 1$$

and

$$\int_\Omega [\frac{1}{\alpha_{n_1}(1)}W(u_{n_1}(1)(x)) + \alpha_{n_1}(1)h^2(x,\nabla u_{n_1}(1)(x))]\,dx \le \int_{\Omega\cap\partial^*A} K(x,a,b,\nu(x))\,dH_{N-1}(x) + 2O(1).$$

Suppose that $\alpha_{n_1}(1) = \alpha_{j_1}$. Now let $\delta = \frac{1}{2}$ and choose n_2 large enough so that

$$\alpha_{n_2}(2) = \alpha_{j_2} \text{ with } j_2 > j_1,$$

$$\|u_{n_2}(2) - u\|_{L^1(\Omega;\mathbb{R}^P)} \le \frac{1}{2}$$

and

$$\int_\Omega \left[\frac{1}{\alpha_{n_j}(2)} W(u_{n_2}(2)(x)) + \alpha_{n_2}(2) h^2(x, \nabla u_{n_2}(2)(x))\right] dx \le \int_{\Omega\cap\partial^* A} K(x,a,b,\nu(x))\, dH_{N-1}(x) + 2O(\tfrac{1}{2}).$$

Continuing this process, **we choose n^* large enough** so that

$$\alpha_{n_k}(k) = \varepsilon_{j_k} \text{ with } j_k > j_{k-1} > \dots > j_2 > j_1.$$

$$\|u_{n_k}(k) - u\|_{L^1(\Omega;\mathbb{R}^p)} \le \frac{1}{k}$$

and

$$\int_\Omega \left[\frac{1}{\alpha_{n_k}(k)} W(u_{n_k}(k)(x)) + \alpha_{n_k}(k) h^2(x, \nabla u_{n_k}(k)(x))\right] dx \le \int_{\Omega\cap\partial^* A} K(x,a,b,\nu(x))\, dH_{N-1}(x) + 2O(\tfrac{1}{j}).$$

Then $\{\alpha_{n_k}(k)\}$ is a subsequence of $\{\alpha_n\}$ and defining $v_k := u_{n_k}(k)$ we have $v_k \to u$ in $L^1(\Omega;\mathbb{R}^p)$

and

$$\limsup_{k\to\infty} \int_\Omega \left[\frac{1}{\alpha_{n_k}} W(v_k(x)) + \alpha_{n_k}(k) h^2(x, \nabla v_k(x))\right] dx \le \int_{\Omega\cap\partial^* A} K(x,a,b,\nu(x))\, dH_{N-1}(x).$$

This, together with Proposition 3.1, gives the result.

Step 3. Finally consider an arbitrary $u = \chi_A a + (1-\chi_A)b$ with $Per_\Omega(A) < +\infty$. By Theorem 2.6 there exist polyhedral sets A_k such that $\chi_{A_k} \to \chi_A$ in $L^1(\Omega)$, $Per_\Omega(A_k) \to Per_\Omega(A)$ and meas(A_k) = meas(A). By Step 2 for every **k** there exist sequences $\varepsilon_n^{(k)} \to 0^+$ as $n \to +\infty$, and $u_n^{(k)} \to \chi_{A_k} a + (1-\chi_{A_k})b$ as $n \to +\infty$ in $L^1(\Omega;\mathbb{R}^p)$ such that

$$\lim_{n\to+\infty} \int_\Omega \left[\frac{1}{\varepsilon_n^{(k)}} W(u_n^{(k)}(x)) + \varepsilon_n^{(k)} h^2(x, \nabla u_n^{(k)}(x))\right] dx = \int_{\partial^* A_k\cap\Omega} K(x,a,b,\nu(x))\, dH_{N-1}(x).$$

Consider $n(k)$ such that

$$\|u_{n(k)}^{(k)} - (\chi_{A_k} a + (1-\chi_{A_k})b)\|_{L^1(\Omega;\mathbb{R}^p)} \le 1/k$$

and

$$\left| \int_{\partial^* A_k\cap\Omega} K(x,a,b,\nu(x))\, dH_{N-1}(x) - \int_\Omega \left[\frac{1}{\varepsilon_{n(k)}^{(k)}} W(u_{n(k)}^{(k)}(x)) + \varepsilon_{n(k)}^{(k)} h^2(x, \nabla u_{n(k)}^{(k)}(x))\right] dx \right| < 1/k$$

with $0 \le \varepsilon_{n(k)}^{(k)} < 1/k$. Set $v_k = u_{n(k)}^{(k)}$ and $\alpha_k = \varepsilon_{n(k)}^{(k)}$; *then*

$$v_k \to \chi_A a + (1-\chi_A) b \text{ in } L^1(\Omega;\mathbb{R}^p)$$

and for every continuous function $g : \Omega \times \mathbb{R}^p \to [0,+\infty)$ we have

$$\int_{\partial^* A_k \cap \Omega} g(x,\nu(x))\, dH_{N-1}(x) \to \int_{\partial^* A \cap \Omega} g(x,\nu(x))\, dH_{N-1}(x).$$

As $K(.,a,b,.)$ is upper semicontinuous there exist continuous functions $g_m : \Omega \times \mathbb{R}^p \to [0,+\infty)$ such that

$$K(x,a,b,\xi) \le g(x,\xi) \le C|\xi|$$

and

$$K(x,a,b,\xi) = \inf_m g_m(x,\xi)$$

for every $(x,\xi) \in \Omega \times \mathbb{R}^p$, where we have extended $K(x,a,b,.)$ as a homogeneous function of degree one (see [FM2], Lemma 2.15 and Step 3, Section 5). Thus for all m

$$\limsup_{k\to+\infty} \int_\Omega \left[\frac{1}{\alpha_k} W(v_k(x)) + \alpha_k h^2(x,\nabla v_k(x))\right] dx$$

$$= \limsup_{k\to+\infty} \int_{\partial^* A_k \cap \Omega} K(x,a,b,\nu(x))\, dH_{N-1}(x) \le \limsup_{k\to+\infty} \int_{\partial^* A_k \cap \Omega} g_m(x,\nu(x))\, dH_{N-1}(x)$$

$$= \int_{\partial^* A \cap \Omega} g_m(x,\nu(x))\, dH_{N-1}(x).$$

Taking the limit when $m \to +\infty$ and using Lebesgue's Monotone Convergence Theorem we deduce that

$$\limsup_{k\to+\infty} \int_\Omega \left[\frac{1}{\alpha_k} W(v_k(x)) + \alpha_k h^2(x,\nabla v_k(x))\right] dx \le \int_{\partial^* A \cap \Omega} K(x,a,b,\nu(x))\, dH_{N-1}(x)$$

which, by Proposition 3.1 concludes the proof. ■

5. A constrained penalized minimization problem.

In this section we assume the following additional hypotheses:

(H6) $W \in W^{1,\infty}_{loc}(\mathbb{R}^p)$;

(H7) there exist constants $\alpha, \delta > 0$ such that

$$\|u - a\| < \delta \Rightarrow \alpha \|u - a\|^q \le W(u) \le \frac{1}{\alpha} \|u - a\|^q$$

and

$$\|u - b\| < \delta \Rightarrow \alpha \|u - b\|^q \leq W(u) \leq \frac{1}{\alpha} \|u - b\|^q;$$

(H8) $h^2(x, .)$ is quasiconvex for all $x \in \Omega$.

In (H8) we could just as well have taken $h(x, .)$ quasiconvex since by Jensen's inequality the quasiconvexity of $h^2(x, .)$ follows trivially.

We consider the following minimization problem:

(P_0) minimize

$$E(u) = \int_\Omega W(u(x))\, dx$$

subject to the constraint $\frac{1}{\text{meas}(\Omega)}\int_\Omega u(x)\, dx = m$, where $m = \theta a + (1-\theta)b$ for some $\theta \in (0,1)$. Clearly any piecewise constant function of bounded variation of the form $u = \chi_A a + (1 - \chi_A)b$ with $\text{meas}(A) = \theta\text{meas}(\Omega)$ is a solution of (P_0), so there exist an infinite number of solutions to this problem. In order to single out one of them, and keeping in mind that the Wulff set is the preferred shape for some types of materials for which the surface energy density is anisotropic (see [Fo], [FM3], [T1], [T2], [W]), we consider the family of anisotropic singular perturbations

$$E_\varepsilon(u) = \int_\Omega [W(u(x)) + \varepsilon^2 h^2(x,\nabla u(x))]\, dx$$

and the corresponding minimization problems

(P_ε) minimize $E_\varepsilon(u)$ on $\left\{u \in H^1(\Omega;\mathbb{R}^P) : \frac{1}{\text{meas}(\Omega)}\int_\Omega u(x)\, dx = m\right\}$.

Since h^2 is quasiconvex, the growth conditions on W and h guarantee the existence of a minimizer u_ε of E_ε, by direct methods of the Calculus of Variations. We show that the solutions u_ε of (P_ε) select the solution of (P_0) which minimizes the integral over the interface $\Omega \cap \partial^*\{u=a\}$ of the surface energy density, namely:

Theorem 5.1. Assume hypotheses (H1)-(H7) hold and let $\{u_n\}$ be a sequence of minimizers of E_{ε_n} converging to u_0 in $L^1(\Omega;\mathbb{R}^P)$. Then u_0 is a solution of the problem:

$$\text{(P) minimize } \int_{\Omega\cap\partial^*\{u=a\}} K(x,a,b,\nu(x))\, dH_{N-1}(x) =: J_0(u) \text{ on}$$

$$V := \left\{u \in BV(\Omega;\mathbb{R}^P),\ u \in \{a,b\} \text{ a.e.} : \frac{1}{\text{meas}(\Omega)}\int_\Omega u(x)\, dx = m\right\}.$$

Proof. We follow the proof of Fonseca and Tartar [FT]. We assume that $\Omega = Q$ and that h and K are independent of x. Since all subsequent constructions were based on this one it is easy to see that the theorem remains valid in the general case.

Step 1. We begin by showing that there exists a constant $C > 0$ such that $J_{\varepsilon_n}(u_n) \leq C$ for sufficiently large n. Indeed, let γ be a smooth function with compact support that satisfies $\gamma(-1) = 0$, $\gamma(1) = 1$ and $0 \leq \gamma \leq 1$. Define

$$w_n(x) = \begin{cases} a & \text{if } x_N > \eta_n + \varepsilon_n \\ \gamma(\frac{x_N - \eta_n}{\varepsilon_n})a + (1 - \gamma(\frac{x_N - \eta_n}{\varepsilon_n}))b & \text{if } |x_N - \eta_n| < \varepsilon_n \\ b & \text{if } x_N < \eta_n - \varepsilon_n \end{cases}$$

where η_n is chosen so that

$$\text{meas}(\{x \in Q : x_N > \eta_n + \varepsilon_n\}) + \int_{\{x \in Q: |x_N - \eta_n| < \varepsilon_n\}} \gamma(\frac{x_N - \eta_n}{\varepsilon_n})\, dx = \theta.$$

It follows that $w_n \in H^1(Q;\mathbb{R}^P)$ and $\int_Q w_n(x)\, dx = m$ so by (H3) we have

$$J_{\varepsilon_n}(u_n) \leq J_{\varepsilon_n}(w_n) \leq \frac{1}{\varepsilon_n}\int_{\{x \in Q: |x_N - \eta_n| < \varepsilon_n\}} W(w_n(x))\, dx + \varepsilon_n \int_Q C\,(1 + \|\nabla w_n(x)\|^2)\, dx \leq$$

$$\leq \text{const.} \max\{W(v) : v \in [a,b]\} + \varepsilon_n C + \frac{1}{\varepsilon_n} \int_{\{x \in Q: |x_N - \eta_n| < \varepsilon_n\}} |\gamma'(\frac{x_N - \eta_n}{\varepsilon_n})|^2\, \|a - b\|^2\, dx \leq$$

$$\leq \text{const.}\, [\max\{W(v) : v \in [a,b]\} + 1 + \|\gamma'\|_\infty^2\, \|a - b\|^2],$$

where $[a,b]$ denotes the line segment joining a and b.

Step 2. By Step 1 and Proposition 3.1 it follows that $u_0 \in V$ and

$$\liminf_{n \to +\infty} J_{\varepsilon_n}(u_n) \geq J_0(u_0). \tag{5.1}$$

Let $u \in V$. It suffices to show that there exists a sequence $\varepsilon_n \to 0^+$ and a sequence $\{v_n\}$ in $H^1(Q;\mathbb{R}^P)$ such that $v_n \to u$ in $L^1(Q;\mathbb{R}^P)$, $\lim_{n \to +\infty} J_{\varepsilon_n}(v_n) = J_0(u)$ and $\int_Q v_n(x)\, dx = m$. Then, by (5.1) and since u_n is a minimizer of E_{ε_n}, we have

$$J_0(u) = \lim_{n \to +\infty} J_{\varepsilon_n}(v_n) \geq \limsup_{n \to +\infty} J_{\varepsilon_n}(u_n) \geq J_0(u_0)$$

and thus u_0 is a solution of (P). To prove the existence of $\{v_n\}$ we modify the sequence $\{w_n\}$ constructed in the proof of Step 1 of Lemma 4.2 to obtain a new sequence satisfying

$$\lim_{n \to +\infty} J_{\varepsilon_n}(v_n) = \lim_{n \to +\infty} J_{\varepsilon_n}(w_n) \text{ and } \int_Q v_n(x)\, dx = m.$$

Let $L_n > 0$ and $\xi_n \in \mathcal{A}(a,b,e_N)$ be such that

$$\lim_{n\to+\infty}\int_Q [L_n W(\xi_n(x)) + \frac{1}{L_n}(h^{\infty})^2(\nabla\xi_n(x))]\,dx = K(a,b,e_N).$$

For n fixed, we defined

$$w_\delta^n(x) = w_\delta^n(x',x_N) = \begin{cases} b & \text{if } x_N > \delta/2 \\ \xi_n(x',\frac{x_N}{\delta}) & \text{if } -\delta/2 \le x_N \le \delta/2 \\ a & \text{if } x_N < -\delta/2. \end{cases}$$

We showed in Lemma 4.2 that

$$\|w_\delta^n - u\|_{L^1(Q;\mathbb{R}^p)} = O(\delta) \tag{5.2}$$

and

$$\int_Q [\frac{L_n}{\delta}W(w_\delta^n(x)) + \frac{\delta}{L_n}h^2(\nabla w_\delta^n(x))]\,dx = \int_Q [L_n W(\xi_n(x)) + \frac{1}{L_n}(h^{\infty})^2(\nabla\xi_n(x))]\,dx +$$

$$+ \text{const.}\frac{\delta}{L_n} + \text{const.}(\frac{\delta}{L_n})^{m/2}.$$

Let $\bar{w}_\delta^n = w_\delta^n + m - \int_Q w_\delta^n(x)\,dx$. Then $\int_Q \bar{w}_\delta^n(x)\,dx = m$ and $\bar{w}_\delta^n \to u$ in $L^1(Q;\mathbb{R}^p)$ as $\delta \to 0^+$. We claim that, for n fixed, $J_{\delta/L_n}(\bar{w}_\delta^n) = J_{\delta/L_n}(w_\delta^n) + O(\delta)$. Indeed, since $\nabla\bar{w}_\delta^n(x) = \nabla w_\delta^n(x)$,

$$J_{\delta/L_n}(\bar{w}_\delta^n) = J_{\delta/L_n}(w_\delta^n) + \int_Q \frac{L_n}{\delta}[W(\bar{w}_\delta^n(x)) - W(w_\delta^n(x))]\,dx$$

where ,

$$\int_Q \frac{L_n}{\delta}[W(\bar{w}_\delta^n(x)) - W(w_\delta^n(x))]\,dx =$$

$$= \int_{Q\cap\{x_N<-\delta/2\}} \frac{L_n}{\delta}W(a + m - \int_Q w_\delta^n(x)\,dx)\,dx + \int_{Q\cap\{x_N>\delta/2\}} \frac{L_n}{\delta}W(b + m - \int_Q w_\delta^n(x)\,dx)\,dx +$$

$$+ \int_{Q\cap\{|x_N|<\delta/2\}} \frac{L_n}{\delta}[W(w_\delta^n(x) + m - \int_Q w_\delta^n(x)\,dx) - W(w_\delta^n(x))]\,dx.$$

By (H7) and (5.2),

$$\int_{Q\cap\{x_N<-\delta/2\}} \frac{L_n}{\delta} W\Big(a + m - \int_Q w_\delta^n(x)\,dx\Big)\,dx \le \wedge \Big| \int u(x) - w_\delta^n(x)\,dx\Big|^q = O(\delta),$$

$$\int_{Q\cap\{x_N>\delta/2\}} \wedge W\Big(b + m - \int_Q w_\delta^n(x)\,dx\Big)\,dx \le \wedge \Big|\int_Q u(x) - w_\delta^n(x)\,dx\Big|^q = O(\delta),$$

and by (H6) and (5.2),

$$\Big| \int_{Q\cap\{|x_N|<\delta/2\}} \frac{L_n}{\delta}\Big[W\Big(w_\delta^n(x) + m - \int_Q w_\delta^n(x)\,dx\Big) - W(w_\delta^n(x))\Big]\,dx\Big| \le$$

$$\le \text{const.}\frac{L_n}{\delta}\text{meas}(\{x \in Q : |x_N| < \delta/2\}) \Big|\int_Q u(x) - w_\delta^n(x)\,dx\Big| = O(\delta).$$

Choose $\delta(n)$ such that $\delta(n) \to 0^+$, $\frac{\varepsilon}{\cdot} \to 0^+$, $\|\bar{w}^n_{\delta(n)} - u\|_1 \to 0$ and

$$\Big|\int_Q \Big[L_n W(\xi_n(x)) + \frac{1}{L_n}(h^{**})^2(\nabla\xi_n(x))\Big]\,dx - \int_Q \Big[\frac{L_n}{\delta(n)} W(\bar{w}^n_{\delta(n)}(x)) + \frac{\delta(n)}{L_n} h^2(\nabla \bar{w}^n_{\delta(n)}(x))\Big]\,dx\Big| < \frac{1}{n}.$$

Let $v_n = \bar{w}^n_{\delta(n)}$ and $\varepsilon_n = \frac{\delta(n)}{L_n}$. Then $\varepsilon_n \to 0^+$, $v_n \to u$ in $L^1(Q;\mathbb{R}^p)$, $\int v_n(x)\,dx = m$ and

$$\lim_{n\to+\infty} J_{\varepsilon_n}(v_n) = \lim_{n\to+\infty} \int_Q \Big[\frac{1}{\varepsilon_n} W(v_n(x)) + \varepsilon_n h^2(\nabla v_n(x))\Big]\,dx =$$

$$= \lim_{n\to+\infty} \int_Q \Big[L_n W(\xi_n(x)) + \frac{1}{L_n}(h^{**})^2(\nabla\xi_n(x))\Big]\,dx$$

$$= K(a,b,e_N) = \int_{Q\cap\{x_N=0\}} K(a,b,e_N)\,dH_{N-1}(x) = J_0(u). \qquad \blacksquare$$

<u>Final Comments</u>: In the isotropic case with no explicit dependence on x, i.e. $h(\nabla u) = \|\nabla u\|$, it was shown in [FT] that

$$K(a,b,\nu) = \bar{K}$$

where

$$\overline{K} = 2 \inf \left\{ \int_{-1}^{1} \sqrt{W(g(s))}\, h^{\infty}(g'(s))\, ds : g \text{ is piecewise } C^1,\ g(-1) = a,\ g(1) = b \right\}.$$

The same result was proved by [OS] in the anisotropic scalar case. We conjecture that, in general, in the anisotropic case $K > \overline{K}$.

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