NAMT
92-016

Jensen's Inequality in the Calculus of Variations

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Research Report No. 92-NA-016

May 1992

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

JENSEN'S INEQUALITY IN THE CALCULUS OF VARIATIONS

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1. Introduction

Jensen's inequality has lately received some attention for it appears to be closely connected not only to the usual notion of convexity but also to much more general kinds of convexity. Since these are the basic constitutive assumptions for lower semicontinuity results, we somehow expect to relate both. This is the goal of this paper. As a matter of fact, this standpoint opens the gate to a different way of understanding weak lower semicontinuity based on Jensen's inequality, which might be very useful in more general situations than the ones described here (see [14]). The principal ingredient in all this is the concept of parametrized measure or Young measure. And when we talk about Jensen's inequality we mean Jensen's inequality with respect to this parametrized measure. These were originally introduced as a tool to deal with non-convex problems in the Calculus of Variations. We show that they are very helpful in working with regular variational principles as well. Indeed, the structure of Young measures is deeply connected to weak lower semicontinuity. Some basic references are [5], [3], [7], [19], [22]. Previous work which greatly motivated this point of view is Kinderlehrer, Pedregal [15], [16], [17].

Assume we have a sequence of measurable functions, $\{z_j\}$, bounded in some space $L^p(\Omega)$ for some regular open, bounded set Ω. Then the sequence of pth powers, $\{|z_j|^p\}$, is bounded in $L^1(\Omega)$. Under these circumstances, there is a subsequence of the z_j's, not relabelled, and a family of probability measures, $\{\nu_x\}_{x\in\Omega}$, the corresponding parametrized measure, such that whenever the composites $\varphi(z_j)$ converge weakly in $L^1(\Omega)$ they do it towards the function

$$\overline{\varphi}(x) = \int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda).$$

This means

$$\lim_{j\to\infty} \int_E \varphi(z_j)\, dx = \int_E \overline{\varphi}\, dx, \tag{1.1}$$

for any measurable $E \subset \Omega$. We assume that the functions z_j take values in $\mathbf{R}^m$. But usually the main difficulty to be overcome is to make sure that for a particular φ the composites $\{\varphi(z_j)\}$ converge weakly to something so that we have equality in (1.1). This might be a tremendous job. And yet we claim that for weak lower semicontinuity we just need inequality in the right direction

$$\liminf_{j\to\infty} \int_E \varphi(z_j)\, dx \geq \int_E \int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda)\, dx. \tag{1.2}$$

Theorem 3.1 gives a general condition under which, even though the Young measure representation (1.1) may not be valid, we still have inequality (1.2). If now Jensen's inequality holds for φ and almost any individual ν_x, then

$$\liminf_{j\to\infty} \int_E \varphi(z_j)\, dx \geq \int_E \int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda)\, dx \geq \int_E \varphi\left(\int_{\mathbf{R}^m} \lambda\, d\nu_x(\lambda)\right) dx.$$

where

$$z(x) = \int_{\mathbf{R}^m} \lambda\, d\nu_x(\lambda), \quad \text{a.e. } x \in \Omega,$$

is the weak limit of the z_j's. We are also able to avoid the usual growth hypothesis in (1.2), so that as a consecuence we introduce a concept of quasiconvexity, stronger than $H^{1,p}$-quasiconvexity (see [5]), but suited for continuous functions without any growth requirement.

In section 2, we collet some basic facts concerning existence of parametrized measures and weak and biting convergence; we also remind the reader of the different notions of quasiconvexity and review the basic facts about $H^{1,p}$-Young measures as introduced in [17]. Next, we prove inequality (1.2). This essentially requires to understand the difference between weak and biting convergence. And in section 4, 5 and 6, we deal succesively with weak lower semicontinuity in three different situations but always through parametrized measures and Jensen's inequality. Those are superlinear growth, no growth assumptions and linear growth.

2. Notation and preliminaries

The existence theorem for Young measures suited for our purposes is the following. It can be found in Ball [3] and Matos [19].

THEOREM 2.1 *Let $\Omega \subset \mathbf{R}^n$ be Lebesgue measurable and let $z_j : \Omega \to \mathbf{R}^m$ be a sequence of measurable functions such that*

$$\sup_j \int_\Omega g(|z_j(x)|)\, dx < \infty, \tag{2.1}$$

for a function $g : [0,\infty) \to \mathbf{R}$ with $\lim_{t\to\infty} g(t) = \infty$. Then there is a subsequence, again $\{z_j\}$, and a family of probability measures $\{\nu_x\}_{x\in\Omega}$, depending measurably on x, in such a way that given any measurable $E \subset \Omega$,

$$f(z_j) \rightharpoonup \langle \nu_x, f\rangle \text{ in } L^1(E) \tag{2.2}$$

for any continuous $f : \mathbf{R}^m \to \mathbf{R}$ such that $\{f(z_j)\}$ is sequentially weakly relatively compact in $L^1(E)$.

The measurable dependence on x means that for any continuous f as in the theorem, the function of x, $\langle \nu_x, f\rangle$ is measurable in Ω. We will take $g(t) = t^p$ most of the time, so that for a sequence of functions uniformly bounded in $L^p(\Omega)$, $1 \le p < \infty$, we will have associated a parametrized measure with the property that whenever a subsequence converges weakly in $L^1(E)$ for $E \subset \Omega$, the weak limit can be represented through its parametrized measure. We call then (2.2) the integral representation for f in terms of the parametrized measure.

For a bounded sequence in $L^1(\Omega)$ we may not have compactness in the weak * topology. The best one can have is Chacon's biting lemma. Let us recall what is meant by biting convergence ([23]). The sequence $\{f^k\} \subset L^1(\Omega)$ converges in the biting sense to $f \in L^1(\Omega)$

and is denoted

$$f^k \xrightarrow{b} f \text{ in } L^1(\Omega),$$

if there is a non-increasing sequence of measurable sets $\{E^j\}$ such that $|E^j| \to 0$ and

$$f^k \rightharpoonup f \text{ in } L^1(\Omega - E^j), \quad \forall j.$$

We may restate Chacon's biting lemma by saying that a uniformly bounded sequence in $L^1(\Omega)$ contains a subsequence converging in the biting sense to a function in $L^1(\Omega)$ ([6], [8]).

This lemma yields a sufficient and necessary condition for biting convergence to become weak convergence. Its proof is elementary and can be found in [17], but for the convenience of the reader we also include it here.

LEMMA 2.2 *Let $f^k : \Omega \to \mathbf{R}^+$ ($f^k \geq 0$) be a sequence of measurable functions in $L^1(\Omega)$, converging in the biting sense to $f \in L^1(\Omega)$.*

A subsequence converges weakly in $L^1(\Omega)$ if and only if

$$\liminf_{k\to\infty} \int_\Omega f^k(x)\,dx \leq \int_\Omega f(x)\,dx. \tag{2.3}$$

And, $\{f^k\}$ is weakly relatively compact in $L^1(\Omega)$ if and only if

$$\limsup_{k\to\infty} \int_\Omega f^k(x)\,dx \leq \int_\Omega f(x)\,dx. \tag{2.4}$$

In reality, inequalities (2.3) and (2.4) can be changed to equalities, but we will keep this formulation for it seems less restrictive.

Proof. The only non-trivial thing is to show that inequality (2.3) implies weak convergence.

Let $\{E_j\} \subset \Omega$ be the sequence of subsets associated with the biting convergence so that $|E_j| \to 0$, $E_{j+1} \subset E_j$ and

$$f^k \rightharpoonup f \text{ in } L^1(\Omega - E_j), \quad \forall j.$$

We may assume

$$\lim_{k\to\infty} \int_\Omega f^k\,dx \leq \int_\Omega f\,dx < \infty. \tag{2.5}$$

Suppose that no subsequence converges weakly in $L^1(\Omega)$. By Dunford-Pettis, there is an $\epsilon > 0$ and a subsequence $\{k_j\}$ such that

$$0 < \epsilon \leq \int_{E_j} f^{k_j}\, dx, \quad \forall j \text{ large},$$

for outside E_j we do have weak convergence. In particular, if $i > j$, since $f^{k_i} \geq 0$,

$$0 < \epsilon \leq \int_{E_i} f^{k_i}\, dx \leq \int_{E_j} f^{k_i}\, dx,$$

and for fixed j, and $i > j$,

$$\begin{aligned} \int_\Omega f^{k_i}\, dx &= \int_{E_j} f^{k_i}\, dx + \int_{\Omega - E_j} f^{k_i}\, dx \\ &\geq \epsilon + \int_{\Omega - E_j} f^{k_i}\, dx. \end{aligned}$$

Finally, letting $i \to \infty$,

$$\lim_{i\to\infty} \int_\Omega f^{k_i}\, dx \geq \epsilon + \int_{\Omega - E_j} f\, dx.$$

This is true for every j, and consequently

$$\lim_{i\to\infty} \int_\Omega f^{k_i}\, dx \geq \epsilon + \int_\Omega f\, dx,$$

against (2.5). ∎

After Ball and Zhang [7], we identify biting limits with the help of Young measures.

LEMMA 2.3 *Let $w^k : \Omega \subset \mathbf{R}^n \to \mathbf{R}^m$ be a sequence of vector-valued functions with associated Young measure $\{\nu_x\}_{x\in\Omega}$. If $\varphi : \mathbf{R}^m \to \mathbf{R}$ is continuous and the sequence $\{\varphi(w^k)\}$ uniformly bounded in $L^1(\Omega)$, then (possibly for a subsequence)*

$$\varphi(w^k) \xrightarrow{b} \overline{\varphi}(x) = \langle \nu_x, \varphi \rangle = \int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda) \tag{2.6}$$

The proof is nothing more than the fact that whenever $\varphi(w^k)$ converges weakly in $L^1(E)$, $E \subset \Omega$, the limit has to be $\overline{\varphi}(x)$ by theorem 2.1. This weak convergence holds in $L^1(\Omega - E_j)$ and $|E_j| \to 0$, so that the biting limit is equal to $\overline{\varphi}(x)$ a.e. $x \in \Omega$. Note that in particular $\overline{\varphi} \in L^1(\Omega)$.

We say that a function $\varphi : \mathbf{M} \to \overline{\mathbf{R}}$ ($\mathbf{M}$ is the space of matrices) is quasiconvex or $H^{1,\infty}$-quasiconvex if

$$\varphi(A) \leq \frac{1}{|\Omega|} \int_\Omega \varphi(A + \nabla u)\, dx$$

for any $u \in H_0^{1,\infty}(\Omega)$, and for any matrix A. If this inequality is true for any $u \in H_0^{1,p}(\Omega)$ then we say that φ is $H^{1,p}$-quasiconvex. Both definitions are independent of the set Ω and they are the same when φ does not grow faster than the pth power ([5]).

An $H^{1,p}$-Young measure is a parametrized measure in the sense of theorem 2.1 associated to a sequence of gradients, $\{\nabla u^j\}$, such that $\{u^j\}$ is bounded in $H^{1,p}(\Omega)$. The key fact proved in [17] is that we can always assume that $\{|\nabla u^j|^p\}$ is weakly convergent in $L^1(\Omega)$ and therefore it is an equiintegrable family of functions. In other words, if $\{\nabla u^j\}$ generates $\{\nu_x\}_{x\in\Omega}$ then there is another sequence $\{\nabla v^j\}$, such that $\{|\nabla v^j|^p\}$ is weakly convergent in $L^1(\Omega)$ and whose parametrized measure is the same $\{\nu_x\}_{x\in\Omega}$. Another important fact is that in this situation each individual ν_x can be understood as a homogeneous (i.e. independent of the point in Ω) $H^{1,p}$-Young measure for a.e. $x \in \Omega$ (see [17]).

We formally say that Jensen's inequality holds for a continuous function $\varphi : \mathbf{R}^m \to \overline{\mathbf{R}}$ and a probability measure ν supported in $\mathbf{R}^m$ if

$$\int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu(\lambda) \geq \varphi\left(\int_{\mathbf{R}^m} \lambda\, d\nu(\lambda)\right).$$

3. A previous theorem

One important fact for a general approach to weak lower semicontinuity is the following. It has two main advantages. On the one hand it will enable us to understand weak lower semicontinuity on arbitrary measurable sets. And on the other, it will allow us to avoid the usual growth conditions on the integrand.

THEOREM 3.1 *Let $g : \mathbf{R}^+ \to \overline{\mathbf{R}^+}$ be a continuous function with $\lim_{t\to\infty} g(t) = \infty$, and $z^j : \Omega \to \mathbf{R}^m$, a sequence of vector valued functions defined in a regular open bounded set $\Omega \subset \mathbf{R}^n$, such that*

$$\sup_j \int_\Omega g\left(|z^j|\right)\, dx < \infty. \tag{3.1}$$

If $\{\nu_x\}_{x\in\Omega}$ is the parametrized measure associated to the z^j's according to theorem 2.1, then

$$\liminf_{j\to\infty}\int_E \varphi\left(z^j\right)\, dx \geq \int_E\int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda)\, dx, \tag{3.2}$$

for any measurable $E \subset \Omega$ and for every continuous φ such that

$$\liminf_{\lambda\to\infty} \frac{\varphi(\lambda)}{g(|\lambda|)} \geq 0. \tag{3.3}$$

Notice that we are assuming that any subsequence out of the $\{z^j\}$ give rise to the same parametrized measure and we are not saying anything about the weak limit of $\{z^j\}$. On the other hand, φ is also allowed to take on the value $+\infty$.

If the function g takes the value $+\infty$ from some value R on, then the condition on the sup implies that the sequence is uniformly bounded in the L^∞-norm, and condition (3.3) is fulfilled for any continuous function.

Under the hypothesis of the lemma the sequence $\{\varphi(z^j)\}$ may fail to be weakly relatively compact in $L^1(\Omega)$, so that the integral representation in (3.2) might give the wrong answer. And yet, under the further restriction expressed in (3.3), we have inequality in the right direction in order to prove weak lower semicontinuity.

A previous step in the proof of the theorem is the following lemma in the spirit of de la Vallée Poussin's compactness criterion in L^1.

LEMMA 3.2 *Assume that φ, g and $\{z^j\}$ are as in the theorem, i.e., conditions (3.1) and (3.3) hold. Then*

$$\lim_{\alpha\to-\infty}\int_{\{\varphi(z^j)\leq\alpha\}} \left|\varphi\left(z^j\right)\right|\, dx = 0, \tag{3.4}$$

uniformly in j.

Proof. Let $\epsilon > 0$. There is a $C_\epsilon > 0$ such that

$$\varphi(\lambda) \geq -\epsilon g(|\lambda|), \quad |\lambda| \geq C_\epsilon,$$

by (3.3). Let $D_\epsilon = \min\{\varphi(\lambda) : |\lambda| \leq C_\epsilon\}$. If $\alpha < \min\{0, D_\epsilon\}$ then clearly the set

$$\{\varphi(z^j) \leq \alpha, |z^j| \leq C_\epsilon\}$$

is empty and

$$\int_{\{\varphi(z^j)\leq\alpha\}} \left|\varphi\left(z^j\right)\right| \, dx = \int_{\{\varphi(z^j)\leq\alpha\}} -\varphi\left(z^j\right) \, dx$$
$$= \int_{\{\varphi(z^j)\leq\alpha, |z^j|\geq C_\epsilon\}} -\varphi\left(z^j\right) \, dx$$
$$\leq \epsilon \int_{\{\varphi(z^j)\leq\alpha, |z^j|\geq C_\epsilon\}} g\left(\left|z^j\right|\right) \, dx$$
$$\leq \epsilon M,$$

uniformly in j, where M is a uniform bound for the L^1-norms of $\{g(|z^j|)\}$. ∎

We now divide the proof of the theorem in two easy steps.

Step 1. Assume $\varphi \geq 0$. If the liminf in (3.2) is not finite, there is nothing to do. If it is finite, then for any subsequence, which we do not relabel, giving the liminf we have that $\{\varphi(z^j)\}$ is uniformly bounded in $L^1(E)$. Therefore by lemma 3.3 the biting limit is given by

$$\overline{\varphi}(x) = \int_{\mathbf{R}^m} \varphi(\lambda) \, d\nu_x(\lambda).$$

If there is a subsequence converging weakly in $L^1(E)$ then we have equality in (3.2) according to theorem 2.1. And if there is some subsequence not converging weakly then we can certainly apply lemma 3.2 and conclude

$$\int_E \overline{\varphi}(x) \, dx \leq \liminf_{j\to\infty} \int_E \varphi(z^j) \, dx.$$

In any case we obtain the inequality.

Step 2. Changing the lower bound 0 by any other constant does not make any difference in step 1. Consider then for $\alpha < 0$ the function

$$\varphi_\alpha = \max\{\alpha, \varphi\} \geq \alpha.$$

In this way, we may conclude through step 1,

$$\liminf_{j\to\infty} \int_E \varphi_\alpha\left(z^j\right) \, dx \geq \int_E \int_{\mathbf{R}^m} \varphi_\alpha(\lambda) \, d\nu_x(\lambda) \, dx.$$

And since the ν_x's are probability measures

$$\liminf_{j\to\infty} \int_E \varphi_\alpha\left(z^j\right) \, dx \geq \int_E \int_{\mathbf{R}^m} \varphi(\lambda) \, d\nu_x(\lambda) \, dx,$$

for any $\alpha < 0$. But now

$$\left|\int_E (\varphi_\alpha(z^j) - \varphi(z^j))\ dx\right| = \int_{E\cap\{\varphi(z^j)\le\alpha\}} (\alpha - \varphi(z^j))\ dx$$
$$\le \int_{\{\varphi(z^j)\le\alpha\}} |\varphi(z^j)|\ dx.$$

And this last quantity may be made small uniformly in j by lemma 3.2. Therefore we get (3.2). ■

When we deal with a bounded sequence in $L^p(\Omega)$, $1 \le p < \infty$ a lower bound such as

$$\varphi(\lambda) \ge -C\,(1 + |\lambda|^q), \quad q < p,$$

is sufficient for having the inequality (3.2). And for $p = \infty$ the inequality holds for any continuous φ.

Condition (3.3) is sharp in the sense that we cannot allow functions φ with

$$\liminf_{\lambda\infty} \frac{\varphi(\lambda)}{g(|\lambda|)} < 0,$$

and still have inequality (3.2). The now classical example is due to Tartar (see Ball and Murat [5] and Dacorogna [9]).

Take $\varphi : \mathbf{M}^{2\times 2} \to \mathbf{R}$, $\varphi(A) = \det A$, and $g(t) = t^2$. Then it is not hard to see that

$$\liminf_{A\to\infty} \frac{\det A}{|A|^2} = -\frac{1}{2} < 0.$$

If $u^n : \Omega = (0,a)^2 \subset \mathbf{R}^2 \to \mathbf{R}^2$, $0 < a < 1$, are defined by

$$u^n(x,y) = \frac{1}{\sqrt{n}}(1-y)^n\,(\sin(nx), \cos(nx)),$$

so that

$$\nabla u^n = \sqrt{n}\begin{pmatrix} (1-y)^n\cos(nx) & -(1-y)^{n-1}\sin(nx) \\ -(1-y)^n\sin(nx) & -(1-y)^{n-1}\cos(nx) \end{pmatrix},$$

then $\nu_x = \delta_0$, for all $x \in \Omega$, and

$$\liminf_{n\to\infty} \int_\Omega \det(\nabla u^n)\, dx = -\frac{a}{2},$$

but

$$\int_\Omega \int_{\mathbf{M}^{2\times 2}} \det(A)\, d\nu_x(A)\, dx = 0.$$

COROLLARY 3.3 *Assume, under the same hypothesis of theorem 3.1, that*

$$\lim_{j\to\infty} \int_\Omega \varphi(z^j)\, dx = \int_\Omega \int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda)\, dx. \tag{3.5}$$

Then the whole sequence $\{\varphi(z^j)\}$ *converges weakly in* $L^1(\Omega)$.

The proof is nothing more than applying theorem 3.1 to a given measurable $E \subset \Omega$ and its complement $\Omega - E$, keeping in mind (3.5). This yields

$$\lim_{j\to\infty} \int_E \varphi(z^j)\, dx = \int_E \int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda)\, dx.$$

4. Weak lower semicontinuity

We may now prove that almost any weak lower semicontinuity result can be recast through a particular Jensen's inequality for appropriate functions and probability measures. In general and without any "a priori" distinction between the scalar case or the vectorial case in the calculus of variation or between the cases with or without derivatives in the context of [10], suppose we know that Jensen's inequality holds for φ and ν_x for a.e. $x \in \Omega$, for every possible Young measure coming from subsequences of the z^j's, and that theorem 3.1 holds for φ, then

$$\liminf_{j\to\infty} \int_E \varphi\left(z^j\right)\, dx \geq \int_E \int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda)\, dx \geq \int_E \varphi\left(\int_{\mathbf{R}^m} \lambda\, d\nu_x(\lambda)\right)\, dx, \tag{4.1}$$

and if the z^j's converge weakly in the appropriate space to z, then the integral representation

$$z(x) \doteq \int_{\mathbf{R}^m} \lambda\, d\nu_x(\lambda), \quad \text{a.e. } x \in \Omega,$$

is valid regardless of the subsequence we are considering. Thus we get the weak lower semicontinuity. Hence any time we have a Jensen type inequality for a function and a family of parametrized measures, we obtain a lower semicontinuity result for the sequence giving rise to the parametrized measure. In particular, we get immediately the following general fact. Notice we do not need any growth restriction on φ.

THEOREM 4.1 *Let $\varphi : \mathbf{R}^m \to \overline{\mathbf{R}}$ be a real continuous function such that*

$$\liminf_{\lambda\to\infty} \frac{\varphi(\lambda)}{|\lambda|^p} \geq 0, \quad 1 < p < \infty,$$

no estimates for $p = \infty$.

Then the weak lower semicontinuity property

$$\liminf_{j\to\infty} \int_E \varphi(z^j)\, dx \geq \int_E \varphi(z)\, dx, \tag{4.2}$$

holds for any measurable E and any sequence $z^j : \Omega \to \mathbf{R}^m$ converging weakly in $L^p(\Omega)$ to z, if and only if φ is convex.

It is well-known that there is no restriction on the probability measures coming from sequences bounded in L^p. Therefore Jensen's inequality holds for any probability measure if and only if we have convexity.

In the context of compensated compactness, we have an example in which the lower semicontinuity can also be reinterpreted in this way. We deal with linear partial differential operators of the type

$$(Au)_i = \sum_{j,k} a_{ijk} \frac{\partial u^j}{\partial x_k}, \quad i = 1, \ldots, s,$$

with constant coefficients a_{ijk}. In [20], we even specialize to the kind of operators for which the decomposition

$$Au = \{\pi_i \cdot \nabla u \cdot p_i\}_{i=1}^N, \tag{4.3}$$

is valid, being π_i and p_i the canonical projections onto A_i and $\oplus_{i\neq j} E_j$ respectively, where

$$\mathbf{R}^p = \bigoplus_{i=1}^N E_i, \quad \mathbf{R}^q = \bigoplus_{i=1}^N A_i,$$

and $u : \Omega \subset \mathbf{R}^p \to \mathbf{R}^q$. Then theorem 1.7 in [20] unravels the structure of parametrized measures for sequences of functions uniformly bounded and verifying $Az^j = 0$: $\nu_x = \prod_{i=1}^N \nu_x^i$ where each ν_x^i is a probability measure supported over A_i. Therefore, if φ is a function separately convex over the A_i's, then Jensen's inequality holds for this φ and ν_x and we have the weak lower semicontinuity result. In fact, this is the approach tacitly assumed in [20].

The variational case, in which A =curl is much more delicate. We could begin declaring a function $\varphi : \mathbf{M} \to \mathbf{R}$ as quasiconvex if Jensen's inequality holds for any Young measure coming from a sequence of gradients bounded in $L^\infty(\Omega)$, so that weak lower semicontinuity results for quasiconvex functions would be immediate. But then the task would be to identify these quasiconvex functions with the classical ones. See also [21] for further discussion on rank-one convexity related to this setting. Likewise, we could say that a function is $H^{1,p}$-quasiconvex if Jensen's inequality holds for it and Young measures associated to sequences of gradients uniformly bounded in $H^{1,p}(\Omega)$ so that again we would automatically have weak lower semicontinuity results for this class of functions. Under growth assumptions, these are again the usual quasiconvex functions.

PROPOSITION 4.2 *Let $\varphi : \mathbf{M} \to \mathbf{R}$ be continuous and such that*

$$\varphi(A) \le C(1+|A|^p), \quad 1 < p < \infty,$$
$$\text{no restriction if } p = \infty.$$

Then φ is quasiconvex if and only if

$$\varphi\Big(\int_{\mathbf{M}} A\, d\nu(A)\Big) \le \int_{\mathbf{M}} \varphi(A)\, d\nu(A),$$

for any homogeneous $H^{1,p}$-Young measure.

Proof. If Jensen's inequality is valid for any such parametrized measure, then in particular, for any $u \in H^{1,p}(\Omega; \mathbf{R}^m)$ with affine boundary values $u(x) = Fx$, $x \in \partial\Omega$, we consider the homogeneous $H^{1,p}$-Young measure, ν, given by the average of $\delta_{\nabla u}$ as discussed in section 2. Then

$$\frac{1}{|\Omega|}\int_\Omega \varphi(\nabla u)\, dx = \int_{\mathbf{M}} \varphi(A)\, d\nu(A) \ge \varphi\Big(\int_{\mathbf{M}} A\, d\nu(A)\Big) = \varphi(F),$$

and φ is quasiconvex.

By truncation, we may well assume that φ is bounded below and quasiconvex. Then the important fact is that for any homogeneous $H^{1,p}$-Young measure, ν, we can find a sequence of functions, $u^j \in H^{1,p}(\Omega; \mathbf{R}^m)$, whose gradients give rise to ν and whose pth power is weakly convergent in $L^1(\Omega)$ as discussed in section 2. Hence the integral representation holds for such a φ, and

$$\int_{\mathbf{M}} \varphi(A)\, d\nu(A) = \lim_{j\to\infty} \frac{1}{|\Omega|}\int_\Omega \varphi(\nabla u^j)\, dx \ge \varphi(F).$$

For the case $p = \infty$, we always have weak convergence in $L^1(\Omega)$ for any continuous function and the Young measure representation is valid. ∎

We now have the following lower semicontinuity results.

THEOREM 4.3 *Let the operator A in (4.3) be given, and let $\varphi : \mathbf{R}^q \to \mathbf{R}$ be continuous. Then the weak lower semicontinuity property*

$$\liminf_{j\to\infty} \int_E \varphi(z^j)\, dx \geq \int_E \varphi(z)\, dx, \tag{4.4}$$

holds for any sequence $z^j \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega)$ such that $Az^j = 0$ if and only if φ is convex on each A_i separately.

And finally, for the variational case, A =curl, we have for $p > 1$,

THEOREM 4.4 *Let $\varphi : \mathbf{R}^m \to \mathbf{R}$ be a real continuous function such that*

$$0 \leq \liminf_{\lambda\to\infty} \frac{\varphi(\lambda)}{|\lambda|^p} \leq \limsup_{\lambda\to\infty} \frac{\varphi(\lambda)}{|\lambda|^p} < \infty$$

no estimates for $p = \infty$.

Then the weak lower semicontinuity property

$$\liminf_{j\to\infty} \int_E \varphi(\nabla u^j)\, dx \geq \int_E \varphi(\nabla u)\, dx, \tag{4.5}$$

holds for any measurable E and for any sequence $u^j : \Omega \to \mathbf{R}^m$ converging weakly in $H^{1,p}(\Omega)$ to u, if and only if φ is quasiconvex.

If we use succesively theorems 3.1 and proposition 4.2, we arrive at (4.1) for $z^j = \nabla u^j$. This is possible since almost every individual ν_x is a homogeneous $H^{1,p}$-Young measure.

5. Weak lower semicontinuity without growth restrictions

In the context of non-linear elasticity, it is crucial to have weak lower semicontinuity results for stored energy functions which take the value $+\infty$ when the determinant of the deformation gradient is non-positive in order to rule out interpenetration of matter. Therefore, we can not keep any hypothesis regarding polynomial growth of the energy function. This is the reason why existence theorems assume polyconvexity as a constituve assumption rather than quasiconvexity ([4], [5]) since all weak lower semicontinuity results involving quasiconvexity require polynomial growth on the energy function. We again take $p > 1$ throughout this section.

In [5], Ball and Murat introduced the notion of $H^{1,p}$-quasiconvexity requiring that for any $u \in H_0^{1,p}(\Omega)$ and any $A \in \mathbf{M}$,

$$\varphi(A) \leq \frac{1}{|\Omega|} \int_\Omega \varphi(A + \nabla u)\, dx. \tag{5.1}$$

The point is that under the hypothesis

$$c \leq \varphi(A) \leq C(1 + |A|^p) \tag{5.2}$$

$H^{1,p}$-quasiconvexity reduces to the usual notion of quasiconvexity requiring (5.1) for smooth functions with compact support, or equivalently for Lipschitz functions. They also proved that $H^{1,p}$-quasiconvexity, without the hypothesis (5.2), is a necessary condition for weak lower semicontinuity and conjectured that it should also be sufficient.

We introduce a somewhat stronger condition, which we call closed $H^{1,p}$-quasiconvexity for reasons that will be clear soon, and prove that this is sufficient for weak lower semicontinuity. We have been unable to show that this is necessary.

Let us reformulate the notion of $H^{1,p}$-quasiconvexity. For $u \in H^{1,p}(\Omega)$ with affine boundary values $u(x) = Fx$, $x \in \partial\Omega$, we may consider the homogeneous $H^{1,p}$-Young measure, ν_u, given by the average of $\delta_{\nabla u}$ as in the proof of proposition 4.2. Then (5.1) translates into

$$\varphi(F) \leq \int_{\mathbf{M}} \varphi(A)\, d\nu_u(A),$$

which is Jensen's inequality for φ and ν_u. On the other hand, we know that, in the appropriate setting, the probability measures ν_u are dense in the set of all homogeneous $H^{1,p}$-Young measures with underlying deformation F. Therefore we say that a continuous function $\varphi : \mathbf{M} \to \overline{\mathbf{R}}$ is closed $H^{1,p}$-quasiconvex if for any matrix F and any homogeneous $H^{1,p}$-Young measure, ν, with underlying deformation F, Jensen's inequality holds,

$$\varphi(F) \leq \int_{\mathbf{M}} \varphi(A)\, d\nu(A).$$

One immediately obtains the following two theorems.

THEOREM 5.1 *Let $\varphi : \mathbf{M} \to \overline{\mathbf{R}}$ be a closed $H^{1,p}$-quasiconvex continuous function such that*

$$\liminf_{A\to\infty} \frac{\varphi(A)}{|A|^p} \geq 0.$$

If u^j converges weakly to u in $H^{1,p}(\Omega)$, then

$$\liminf_{j\to\infty} \int_E \varphi(\nabla u^j)\, dx \geq \int_E \varphi(\nabla u)\, dx,$$

for any measurable $E \subset \Omega$.

THEOREM 5.2 *Let φ be as above and such that*

$$c(|A|^p - 1) \leq \varphi(A), \quad A \in \mathbf{M}.$$

Let $\mathcal{A} = \left\{u \in H^{1,p}(\Omega) : u - u_0 \in H_0^{1,p}(\Omega)\right\}$ for $u_0 \in H^{1,p}(\Omega)$ given and

$$J(u) = \int_\Omega \varphi(\nabla u)\, dx.$$

Assume that J is not identically $+\infty$ in $\mathcal{A}$. Then J admits absolute minimizers in $\mathcal{A}$.

The proof of theorem 5.1 reduces to the chain of inequalities in (4.1).

PROPOSITION 5.3 *i) Every polyconvex function $G : \mathbf{M} \to \overline{\mathbf{R}}$ is closed $H^{1,p}$-quasiconvexity for $p \geq N$ if $\mathbf{M}$ is the set of $N \times N$ matrices.*

ii) Let $g : \mathbf{R}^+ \to \overline{\mathbf{R}}$ be a convex increasing function, and $\varphi : \mathbf{M} \to \mathbf{R}^+$ a quasiconvex function with pth growth, then $G = g \cdot \varphi$ is closed $H^{1,p}$-quasiconvex.

Proof. Let $F \in \mathbf{M}$ and ν a homogeneous $H^{1,p}$-Young measure with underlying deformation F, generated by the sequence $\{\nabla u^k\}$ where $\{|\nabla u^k|^p\}$ is equiintegrable in Ω.

i) Let us write $G(A) = g(M(A))$ where $M(A)$ represents the set of all minors of $A \in \mathbf{M}$, and g is any convex function. Then, for any $p \geq N$, and possibly for an appropriate subsequence,

$$M(\nabla u^k) \rightharpoonup M(F) \quad \text{in} \quad L^1(\Omega).$$

Notice that this weak convergence is also true for $p = N$ because $\left\{|\nabla u^k|^N\right\}$ is equiintegrable. Then we have

$$\int_{\mathbf{M}} M(A)\, d\nu(A) = M(F),$$

and

$$\int_{\mathbf{M}} G(A)\, d\nu(A) = \int_{\mathbf{M}} g(M(A))\, d\nu(A) \geq g\left(\int_{\mathbf{M}} M(A)\, d\nu(A)\right) = g(M(F)) = G(F).$$

ii) In the same situation, apply g to both sides of the inequality

$$\int_{\mathbf{M}} \varphi(A)\, d\nu(A) \geq \varphi(F),$$

and then use Jensen's inequality. ∎

It is hard to grasp the nature of closed $H^{1,p}$-quasiconvexity. We do not even know how to understand quasiconvexity. The main trouble is in the nonlocal character of quasiconvexity. On the other hand, having the weak lower semicontinuity property for any measurable set $E \subset \Omega$ might be something too ambitious to have. There is however a more natural property for a function to be closed $H^{1,p}$-quasiconvex, which is sufficient. For a continuous function $\varphi : \mathbf{M} \to \overline{\mathbf{R}}$, let us define the p-quasiconvexification by

$$\varphi_p^\sharp = \sup\{\psi : \psi \text{ is quasiconvex}, |\psi| \leq C(1 + |\cdot|^p), \text{ some } C > 0, \psi \leq \varphi\}.$$

PROPOSITION 5.4 *Let $\varphi : \mathbf{M} \to \overline{\mathbf{R}}$ be such that it coincides with its p-quasiconvexification. Then φ is closed $H^{1,p}$-quasiconvex. And in particular, it is $H^{1,p}$-quasiconvex.*

Proof. Let ν be a homogeneous $H^{1,p}$-Young measure, with underlying deformation F. Since any quasiconvex $\psi \leq \varphi$ in the above sup is closed $H^{1,p}$-quasiconvex because of the bound on ψ, we have

$$\psi(F) \leq \int_{\mathbf{M}} \psi(A)\, d\nu(A) \leq \int_{\mathbf{M}} \varphi(A)\, d\nu(A).$$

And just notice that $\psi(F) \to \varphi(F)$. ∎

COROLLARY 5.5 *In the situation of theorem 5.2, if φ is such that it coincides with its p-quasiconvexification, then J admits absolute minimizers in $\mathcal{A}$.*

6. Weak lower semicontinuity under linear growth

It is interesting to look at the situation previously discussed and see what conclusions we can draw under linear growth conditions, i.e., $g(t) = t$ in theorem 3.1. Precisely the case $p = 1$ is always very special for well-known reasons and general existence theorems for this kind of variational problems do not seem to be available. While for $p > 1$ the uniform boundedness of a sequence of functions z^j or of a sequence of gradients ∇u^j in $L^p(\Omega)$ is enough to force weak convergence, this is not so for $p = 1$. And in variational principles this uniform boundedness is the best one can obtain under coercivity assumptions. So if we place ourselves in the situation of theorem 3.1 taking $g(t) = t$, what can we say about weak lower semicontinuity?

To begin with, theorem 3.1 is still true. If moveover we assume convexity on φ then what we get is

$$\liminf_{j\to\infty} \int_E \varphi\left(z^j\right)\,dx \geq \int_E \int_{\mathbf{R}^m} \varphi(\lambda)\,d\nu_x(\lambda)\,dx \geq \int_E \varphi\left(\int_{\mathbf{R}^m} \lambda\,d\nu_x(\lambda)\right)\,dx.$$

If,

$$z(x) = \int_{\mathbf{R}^m} \lambda\,d\nu_x(\lambda), \quad \text{a.e. } x \in \Omega,$$

then we have the weak lower semicontinuity result

$$\liminf_{j\to\infty} \int_E \varphi\left(z^j\right)\,dx \geq \int_E \varphi(z)\,dx.$$

How do the z^j's converge to z? According to lemma 2.3 this convergence is only in the biting sense. Thus we have proved the following weak lower semicontinuity result.

THEOREM 6.1 *If a sequence of $L^1(\Omega)$-functions, $\{z^j\}$, converges to z in the biting sense and φ is a convex function such that*

$$\liminf_{\lambda\to\infty} \frac{\varphi(\lambda)}{|\lambda|} \geq 0, \tag{6.1}$$

or in particular φ is bounded below, then

$$\liminf_{j\to\infty} \int_E \varphi\left(z^j\right)\,dx \geq \int_E \varphi(z)\,dx,$$

for any $E \subset \Omega$.

In the variational case in which $z^j = \nabla u^j$, $u^j \in H^{1,1}(\Omega)$, we have again the same trouble: uniform boundedness in $H^{1,1}(\Omega)$ does not imply weak convergence. However by the Sobolev embedding theorems we do have strong convergence in $L^1(\Omega)$ to $u \in L^1(\Omega)$. The function u may or may not be in $H^{1,1}(\Omega)$. In case it is, lower semicontinuity results in all of

Ω have been proved by I. Fonseca and S. Müller, [12] and [13]. In the worst possible case, u is a BV-function whose gradient is a measure, and it is a natural problem to extend the functional to BV-functions in such a way that it remains lower semicontinuous with respect to strong convergence in $L^1(\Omega)$. For this topic see also [2], [11].

This approach in terms of parametrized measures and Jensen's inequality yields a lower semicontinuity result with respect to biting convergence in the gradient case for it is also true that we can extract a biting converging subsequence to a gradient from every bounded sequence in $H^{1,1}(\Omega)$. Indeed, in [17] it was proved that if $\{u^j\}$ is a bounded sequence in $H^{1,1}(\Omega)$ then there is a weakly convergent sequence, $\{v^j\}$, in $H^{1,1}(\Omega)$ which shares the parametrized measure with $\{u^j\}$. Therefore, if $\{\nu_x\}_{x\in\Omega}$ is the parametrized measure for both $\{u^j\}$ and $\{v^j\}$ then there is a $u \in H^{1,1}(\Omega)$ such that

$$\nabla u(x) = \int_{\mathbf{M}} \lambda \, d\nu_x(\lambda), \quad \text{a.e. } x \in \Omega,$$

and by lemma 2.3, $\nabla u^j \stackrel{b}{\rightharpoonup} \nabla u$.

On the other hand, proposition 4.2 is still true for $p = 1$ with exactly the same proof. Hence,

THEOREM 6.2 *If a sequence of functions in $H^{1,1}(\Omega)$, $\{u^j\}$, is such that their gradients, $\{\nabla u^j\}$, converge in the biting sense to ∇u, for $u \in H^{1,1}(\Omega)$ and φ is a quasiconvex function with*

$$0 \le \liminf_{\lambda\to\infty} \frac{\varphi(\lambda)}{|\lambda|} \le \limsup_{\lambda\to\infty} \frac{\varphi(\lambda)}{|\lambda|} < \infty,$$

then

$$\liminf_{j\to\infty} \int_E \varphi\left(\nabla u^j\right) dx \ge \int_E \varphi(\nabla u) \, dx, \tag{6.2}$$

for any $E \subset \Omega$, measurable.

However, biting convergence is of no use in variational principles because of the following fact.

THEOREM 6.3 *Let $\varphi : \mathbf{M} \to \mathbf{R}$ be continuous, quasiconvex and such that*

$$c(|A|^p - 1) \le \varphi(A) \le C(|A|^p + 1), \quad 1 \le p < \infty,$$

and consider the variational principle

$$\inf\left\{\int_\Omega \varphi(\nabla v) \, dx : v \in \mathcal{A}\right\},$$

for some suitable class $\mathcal{A} \subset H^{1,p}(\Omega;\mathbf{R}^m)$. If $\{u^j\}$ is a minimizing sequence converging weakly for $p > 1$ or in the biting sense for $p = 1$ to a minimizer u, then $\{|\nabla u^j|^p\}$ is a weakly convergent sequence in $L^1(\Omega)$.

For the proof, just observe that

$$\liminf_{j\to\infty} \int_\Omega \varphi\left(\nabla u^j\right)\, dx \geq \int_\Omega \int_{\mathbf{M}} \varphi(A)\, d\nu_x(A)\, dx \geq \int_\Omega \varphi\left(\nabla u\right)\, dx,$$

but in this case we have equality throughout because we are dealing with a minimizing sequence and a minimizer. Through corollary 3.3 we conclude that $\{\varphi\left(\nabla u^j\right)\}$ is weakly convergent in $L^1(\Omega)$, and the lower bound on φ and the Dunford-Pettis compactness criterion yields the result.

As a consequence, for $p = 1$ and whenever we have minimizers, minimizing sequences should converge weakly to minimizers. The trouble with this resides in the fact that having the weak lower semicontinuity property (6.2) for any measurable set is too ambitious. For this reason, it is also plausible that if we restrict attention to all of Ω, lower semicontinuity results in the strong topology of $L^1(\Omega)$ are stronger than those with respect to biting convergence. For a bounded sequence in $H^{1,1}(\Omega)$, $\{u^j\}$, we have associated a strong limit in $L^1(\Omega)$, u, and a biting limit $v \in H^{1,1}(\Omega)$. If u is also an $H^{1,1}(\Omega)$-function then it is reasonable to expect

$$\int_\Omega \varphi(\nabla u)\, dx \geq \int_\Omega \varphi(\nabla v)\, dx,$$

for any quasiconvex function φ as in theorem 6.2, with equality only in the case $u = v$ when the given sequence converges weakly in $H^{1,1}(\Omega)$.

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