Relaxation of Quasiconvex Functionals in B V (Ω, p) for Integrands f(x,u,Δu)

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RELAXATION OF QUASICONVEX FUNCTIONALS IN $BV(\Omega,\mathbb{R}^p)$ FOR INTEGRANDS $f(x,u,\nabla u)$

IRENE FONSECA† AND STEFAN MÜLLER††

Abstract. In this paper it is shown that if $f(x,u,.)$ is a quasiconvex function with linear growth then the relaxed functional in $BV(\Omega,\mathbb{R}^p)$ of

$$u \to \int_\Omega f(x,u(x),\nabla u(x))\, dx$$

with respect to the L^1 topology has an integral representation of the form

$$\mathcal{F}(u) = \int_\Omega f(x,u(x),\nabla u(x))\, dx + \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\, dH_{N-1}(x) + \int_\Omega f^\infty(x,u(x), dC(u))$$

where $Du = \nabla u\, dx + (u^+ - u^-)\otimes\nu\, dH_{N-1}\lfloor\Sigma(u) + C(u)$. The proof relies on a blow up argument introduced by the authors in the case where $u \in W^{1,1}$ and on a recent result by Alberti showing that the Cantor part $C(u)$ is rank-one valued.

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1. Introduction

In this paper we study the relaxation $\mathcal{F}(.)$ in $BV(\Omega, \mathbb{R}^p)$ of the functional

$$u \to \int_\Omega f(x,u(x),\nabla u(x))\, dx,$$

where $f(x,u,.)$ is quasiconvex, grows at most linearly with possibly degenerate bounds and satisfies some technical continuity conditions. We obtain the integral representation

$$\mathcal{F}(u) = \int_\Omega f(x,u(x),\nabla u(x))\, dx$$
$$+ \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\, dH_{N-1}(x) + \int_\Omega f^\infty(x,u(x),dC(u)) \quad (1.1)$$

where the distributional derivative Du is represented by[1]

$$Du = \nabla u\, dx + (u^+ - u^-)\otimes\nu\, dH_{N-1}\llcorner\Sigma(u) + C(u).$$

Here ∇u is the density of the absolutely continuous part of Du with respect to the Lebesgue measure $\mathcal{L}_N$, H_{N-1} is the N-1 dimensional Hausdorff measure, $(u^+ - u^-)$ is the jump of u across the interface $\Sigma(u)$ and $C(u)$ is the Cantor part of Du. As usual, the relaxation is defined by

$$\mathcal{F}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\, dx \mid u_n \in W^{1,1} \text{ and } u_n \to u \text{ in } L^1 \right\}$$

and f^∞ denotes the recession function

$$f^\infty(x,u,A) := \limsup_{t\to+\infty} \frac{f(x,u,tA)}{t}.$$

As opposed to the convex case, it may not be possible to replace in the latter definition limsup by lim, as illustrated by Müller [Mü]. However, it turns out that f^∞ is positively homogeneous of degree one and quasiconvex (see Remark 2.2 (ii)).

The proof of (1.1) is divided into two parts. In the first part, carried out on Sections 2, 3 and 4, we show that the representation in (1.1) is a lower bound for $\mathcal{F}(.)$, i. e. if $u_n \in W^{1,1}(\Omega; \mathbb{R}^p)$ are such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^p)$, $u \in BV(\Omega, \mathbb{R}^p)$, then

$$\liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\, dx \geq \int_\Omega f(x,u(x),\nabla u(x))\, dx \;+$$

[1]Rank-one matrices in $M^{p\times N}$ are represented by tensor products of the form $a\otimes b$, with $a \in \mathbb{R}^p$, $b \in \mathbb{R}^N$, where

$$(a\otimes b)_{ij} := a_i\, b_j$$

for $1 \leq i \leq p$, $1 \leq j \leq N$.

$$\int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\, dH_{N-1}(x) + \int_{C(u)} f^\infty(x,u(x),dC(u)). \qquad (1.2)$$

Here we use the blow up argument introduced in [FM], and the characterization of the surface energy density $K(x,a,b,\nu)$ is based on the work on relaxation of multiple convex integrals in $BV(\Omega; \mathbb{R}^p)$ by Fonseca and Rybka [FR]. The latter was derived from a conjecture by Fonseca and Tartar [FT2] concerning the isotropic singular perturbation problem for solid materials (see also Ambrosio and Pallara [AP]). The analysis of the Cantor part relies on a rank-one property of $C(u)$ obtained recently by Alberti [Al] (see Theorem 2.11).

In the second part of this paper (Section 5) we assert equality in (1.2). We use the same arguments exploited by Ambrosio, Mortola and Tortorelli [AMT].

Under the convexity assumption on $f(x,u,.)$ the integral representation (1.1) was obtained first by Dal Maso [DM] in the scalar case $p = 1$, by Goffman and Serrin [GS] and Reshetnyak [R] when f depends only on ∇u, while Giaquinta, Modica and Soucek [GMS] treated the case where $f = f(x,\nabla u)$. In the vectorial case $p > 1$, the result was proven by Ball and Murat [BM] and by Reshetnyak [R] provided $f = f(A)$ is convex and $u \in W^{1,1}(\Omega; \mathbb{R}^p)$. Fonseca [Fo] and Kinderlehrer [K] extended the latter to the class of functions $f = f(x,\nabla u)$, with $f(x,.)$ quasiconvex. When $f = f(x,u,.)$ is convex, Aviles and Giga [AG] obtained lower semicontinuity results in the BV setting (see also Ambrosio, Mortola and Tortorelli [AMT], Ambrosio and Pallara [AP] and Fonseca and Rybka [FR]) .

Here we generalize our previous work in [FM], where, under the same assumptions, we identified the absolutely continuous part of $\mathcal{F}(u)$, proving that

$$u \to \int_\Omega f(x,u(x),\nabla u(x))\, dx$$

is lower semicontinuous in $W^{1,1}$ with respect to the L^1 topology. As mentioned in [FM], the study of this problem was motivated by the analysis of variational problems for phase transitions. Equilibria of materials are often associated to minima of a bulk energy

$$I(u) := \int_\Omega f(x,u(x),\nabla u(x))\, dx$$

where the possibility of a phase transformation is related to the nonconvexity of $f(x,u,.)$. Here the function spaces involved should allow discontinuous vector-valued u, and a linear growth condition on $f(x,u,.)$ suggests, naturally, the need to relax $I(.)$ in BV. In addition, singular perturbation problems derived from phase transitions such as the one considered by Modica [Mo] (see also Baldo [B], Fonseca and Tartar [FT1], [Gu1], [Gu2], Kohn and Sternberg [KS], Owen and Sternberg [OS]) lead us to energy densities of the type

$$f(x,u,A) = 2\sqrt{W(u)}\; h(A)$$

where W vanishes at more than one point, thus preventing the coerciveness of f(x,u,.) and suggesting the need to consider degenerate bounds for f.

We remark that the convexity techniques employed on previous related works cannot be used in our context. Also, Sverak [S] and Zhang [Z] provided examples of quasiconvex functions with linear growth that are not convex. More recently, Müller [Mü] constructed functions that are positively homogeneous of degree one and quasiconvex but not convex.

As we were writing this paper we became aware of a result by Ambrosio and Dal Maso [ADM] providing the relaxation of

$$u \to \int_\Omega f(\nabla u(x))\, dx$$

in the L^1 topology, where f is a quasiconvex function such that

$$0 \le f(A) \le C(1 + \|A\|).$$

The remainder of this article is organized as follows. In Section 2 we introduce the notion of recession function for quasiconvex integrands and we recall some results on measure theory and on the theory of functions of bounded variation. We state the relaxation theorem, Theorem 2.16, and we introduce the surface energy density K. Sections 3 and 4 are dedicated to showing that

$$\mathcal{F}(u) \ge \int_\Omega f(x,u(x),\nabla u(x))\, dx + \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\, dH_{N-1}(x) + \int_\Omega f^\infty(x,u(x),dC(u)).$$

The absolutely continuous part of this inequality is treated at the end of Section 2 and we follow the argument in [FM], while the jump and the Cantor parts are studied in Sections 3 and 4, respectively. In Section 5, and using the same reasoning as in Ambrosio, Mortola and Tortorelli [AMT], we show that there is equality in (1.1). Finally, in the Appendix we provide the proofs for three results on measure theory and functions of bounded variation which we could not find in the literature although they are well-known to experts in the field.

2. Preliminaries

In what follows $\Omega \subset \mathbb{R}^N$ is an open, bounded set, p, $N \ge 1$, M^{pxN} is the vector space of all pxN real matrices and $S^{N-1} := \{x \in \mathbb{R}^N \mid \|x\| = 1\}$. Given $\nu \in S^{N-1}$, Q_ν is the open unit cube centered at the origin with two of its faces normal to ν, i. e. if $\{\nu_1, \nu_2, ..., \nu_{N-1}, \nu\}$ is an orthonormal basis of $\mathbb{R}^N$ then

$$Q_\nu := \{ x \in \mathbb{R}^N \mid |x.\nu_i| < 1/2, |x.\nu| < 1/2, i = 1, ..., N-1\}.$$

Definition 2.1([Mr]). A function $f : M^{pxN} \to \mathbb{R}$ is said to be *quasiconvex* if

$$f(A) \le \frac{1}{\text{meas}(D)} \int_D f(A+\nabla\varphi(x))\, dx \tag{2.1}$$

for all $A \in M^{p\times N}$, for every domain $D \subset \mathbb{R}^N$ and for all $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^p)$.

We define the *recession function of f* by

$$f^\infty(A) := \limsup_{t\to+\infty} \frac{f(tA)}{t}.$$

Remark 2.2. **(i)** If

$$|f(A)| \leq C(1 + \|A\|) \tag{2.2}$$

then, using a density argument, one shows easily that the inequality (2.1) holds for all $\varphi \in W_0^{1,1}(D; \mathbb{R}^p)$. Also, (2.1) can be extended to all Q - periodic functions $\varphi \in W^{1,1}(Q; \mathbb{R}^p)$, where Q is a cube in $\mathbb{R}^N$ (see Theorem 3.1 in [BM], Kohn [Ko]).

(ii) f^∞ is a quasiconvex function and is positively homogeneous of degree one. This class of functions was studied by Müller in [Mü], where he shows that these hypotheses do not imply convexity of f. To prove that f^∞ is quasiconvex, let $A \in M^{p\times N}$ and let $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^p)$ for some domain $D \subset \mathbb{R}^N$. Choosing $t_k \to +\infty$ with

$$f^\infty(A) = \lim_{k\to+\infty} \frac{f(t_kA)}{t_k},$$

by (2.1) we have

$$\frac{f(t_kA)}{t_k} \leq \frac{1}{(\text{meas}D)t_k} \int_D f(t_kA + \nabla(t_k\varphi(x)))\, dx$$
$$= \frac{1}{(\text{meas}D)t_k} \int_D f(t_k(A + \nabla\varphi(x)))\, dx.$$

Defining

$$H(x) := C(1 + \|A + \nabla\varphi(x)\|),$$

by (2.2) we deduce that

$$f^\infty(A) \leq \limsup_{k\to+\infty} \frac{1}{(\text{meas}D)t_k} \int_D f(t_k(A + \nabla\varphi(x)))\, dx$$
$$= \frac{1}{\text{meas}D} \int_D H(x)\, dx -$$
$$- \liminf_{k\to+\infty} \frac{1}{\text{meas}D} \int_\Omega \left[H(x) - \frac{1}{t_k} f(t_k(A + \nabla\varphi(x)))\right] dx$$

which, by Fatou's Lemma, yields

$$f^\infty(A) \leq \frac{1}{\text{meas}D} \int_D \limsup_{k\to+\infty} \frac{1}{t_k} f(t_k(A + \nabla\varphi(x)))\, dx$$
$$\leq \frac{1}{\text{meas}D} \int_D f^\infty(A + \nabla\varphi(x))\, dx. \qquad \blacksquare$$

Definition 2.3. A function $u \in L^1(\Omega; \mathbb{R}^p)$ is said to be of *bounded variation*, $u \in BV(\Omega; \mathbb{R}^p)$, if for all $i \in \{1, ..., p\}$, $j \in \{1, ..., N\}$ there exists a Radon measure μ_{ij} such that

$$\int_\Omega u_i(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx = - \int_\Omega \varphi(x)\, d\mu_{ij}$$

for every $\varphi \in C^1_0(\Omega)$. The distributional derivative Du is the matrix-valued measure with components μ_{ij}.

We briefly recall some facts on functions of bounded variation. For more details we refer the reader to Ambrosio, Mortola and Tortorelli [AMT], Evans and Gariepy [EG], Federer [Fe], Giusti [Gi], Ziemer [Zi].

The *approximate upper and lower limit* of each component u_i, for all $i \in \{1, ..., p\}$, are given by

$$u_i^+(x) := \inf \{t \in \mathbb{R} \mid \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \mathcal{L}_N[\{u_i > t\} \cap B(x,\varepsilon)] = 0\}$$

and

$$u_i^-(x) := \sup \{t \in \mathbb{R} \mid \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \mathcal{L}_N[\{u_i < t\} \cap B(x,\varepsilon)] = 0\}.$$

The set $\Sigma(u)$ is called the *singular set of u* or *jump set* and is defined by

$$\Sigma(u) = \bigcup_{i=1}^{p} \{x \in \Omega \mid u_i^-(x) < u_i^+(x)\}.$$

It is well known that $\Sigma(u)$ is N-1 rectifiable, i. e.

$$\Sigma(u) = \bigcup_{n=1}^{\infty} K_n \cup E$$

where $H_{N-1}(E) = 0$ and K_n is a compact subset of a C^1 hypersurface. If $x \in \Omega \setminus \Sigma(u)$ then u(x) is understood as the common value of $(u_1^+(x),...,u_p^+(x))$ and $(u_1^-(x),...,u_p^-(x))$, which may be $+\infty$ or $-\infty$ in some components. It can be shown that $u(x) \in \mathbb{R}^p$ for H_{N-1} a. e. $x \in \Omega \setminus \Sigma(u)$ (see [Fe], 4.5.9 (3)).

Theorem 2.4. If $u \in BV(\Omega; \mathbb{R}^p)$ then

(i) for $\mathcal{L}_N$ a. e. $x \in \Omega$

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left\{\frac{1}{meas(B(x,\varepsilon))} \int_{B(x,\varepsilon)} |u(y) - u(x) - \nabla u(x).(x - y)|^{N/(N-1)}\, dy \right)\Big\}^{N-1)/N} = 0;$$

(ii) for H_{N-1} a. e. $x \in \Sigma(u)$ there exists a unit vector $\nu(x) \in S^{N-1}$, normal to $\Sigma(u)$ at x, and there exist vectors $u^-(x), u^+(x) \in \mathbb{R}^p$ such that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in B(x,\varepsilon) | (y-x).\nu(x)>0\}} |u(y) - u^+(x)|^{N/(N-1)}\, dy = 0,$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in B(x,\varepsilon) | (y-x).\nu(x)<0\}} |u(y) - u^-(x)|^{N/(N-1)}\, dy = 0;$$

(iii) for H_{N-1} a. e. $x_0 \in \Omega\backslash\Sigma(u)$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\text{meas}(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |u(x) - u(x_0)|\, dx = 0$$

and for H_{N-1} a. e. $x_0 \in \Sigma(u)$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\text{meas}(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} u(x)\, dx = \frac{u^+(x_0) + u^-(x_0)}{2}.$$

We remark that in general $(u_i)^\pm \neq (u^\pm)_i$. If $u \in BV(\Omega; \mathbb{R}^p)$ then Du may be represented as

$$Du = \nabla u\, dx + (u^+ - u^-)\otimes\nu\, dH_{N-1}\lfloor\Sigma(u) + C(u) \tag{2.3}$$

where ∇u is the density of the absolutely continuous part of Du with respect to the N-dimensional Lebesgue measure $\mathcal{L}_N$ and H_{N-1} is the N-1 dimensional Hausdorff measure. The three measures in (2.3) are mutually singular ; if $H_{N-1}(B) < +\infty$ then $|C(u)|(B) = 0$ and there exists a Borel set E such that

$$\mathcal{L}_N(E) = 0 \text{ and } |C(u)|(X) = |C(u)|(X\cap E)$$

for all Borel sets $X \subset \Omega$.

Lemma 2.5. Let $u \in BV(\Omega; \mathbb{R}^p)$, and let $\rho \in C_0^\infty(\mathbb{R}^N)$ be a nonnegative function such that

$$\int_{\mathbb{R}^N} \rho(x)\, dx = 1, \quad \text{supp}\, \rho = \overline{B}(0,1)\ , \rho(x) = \rho(-x) \quad \text{for every } x \in \mathbb{R}^N.$$

Let $\rho_n(x) := n^N\rho(nx)$ and

$$u_n(x) := (u*\rho_n)(x) = \int_\Omega u(y)\, \rho_n(x-y)\, dy.$$

Then

(i) $$\int_{B(x_0,\varepsilon)} h(x)\, |\nabla u_n(x)|\, dx \leq \int_{B(x_0,\varepsilon+1/n)} (h*\rho_n)(x)\, |Du(x)|$$

whenever $\text{dist}(x_0,\partial\Omega) > \varepsilon+1/n$ and h is a nonnegative Borel function ;

(ii) $$\lim_{n\to+\infty} \int_{B(x_0,\varepsilon)} \theta(\nabla u_n(x))\, dx = \int_{B(x_0,\varepsilon)} \theta(Du(x))$$

for every function θ positively homogeneous of degree one and for every $\varepsilon \in (0, \mathrm{dist}(x_0,\partial\Omega))$ such that $|Du|(\partial B(x_0,\varepsilon)) = 0$;

(iii) if, in addition, $u \in L^\infty(\Omega ; \mathbb{R}^p)$ then for every $x_0 \in \Omega \setminus \Sigma(u)$

$$u_n(x_0) \to u(x_0) \quad \text{and} \quad (|u_n - u| * \rho_n)(x_0) \to 0$$

as $n \to +\infty$.

The proof of this lemma can be found in [AMT], Lemma 4.5. The next result will be used in Section 3.

Lemma 2.6. For H_{N-1} a. e. $x_0 \in \Sigma(u)$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \int_{\Sigma(u)\cap(x_0+\varepsilon Q_{\nu(x_0)})} |u^+(x) - u^-(x)|\, dH_{N-1}(x) = |u^+(x_0) - u^-(x_0)|.$$

In order to prove this lemma we need to recall some auxilliary theorems on measure theory that can be found in [EG]. The following version of Besicovitch Differentiation Theorem was proven by Ambrosio and Dal Maso, [ADM] Proposition 2.2.

Theorem 2.7. If λ and μ are Radon measures in Ω, $\mu \geq 0$, then there exists a Borel set E $\subset \Omega$ such that $\mu(E) = 0$ and for every $x \in \mathrm{supp}\,\mu \setminus E$

$$\frac{d\lambda}{d\mu}(x) := \lim_{\varepsilon\to 0} \frac{\lambda(x+\varepsilon C)}{\mu(x+\varepsilon C)}$$

exists and is finite whenever C is a bounded, convex, open set containing the origin.

We remark that in the above result the exceptional set E does not depend on C. An immediate corollary is the generalization of Lebesgue - Besicovitch Differentiation Theorem given below.

Theorem 2.8. If μ is a nonnegative Radon measure and if $f \in L^1_{loc}(\mathbb{R}^N,\mu)$ then

$$\lim_{\varepsilon\to 0} \frac{1}{\mu(x+\varepsilon C)} \int_{x+\varepsilon C} |f(y) - f(x)|\, d\mu(y) = 0$$

for μ a. e. $x \in \mathbb{R}^N$ and for every bounded, convex, open set C containing the origin.

Proposition 2.9. If μ is a Borel regular measure in $\mathbb{R}^N$ and if $A \subset \mathbb{R}^N$ is μ-measurable with $\mu(A) < +\infty$ then

$$\lim_{\varepsilon \to 0} \frac{\mu(A \cap B(x,\varepsilon))}{\varepsilon^m} = 0$$

for H_m a. e. $x \notin A$.

2.10. Change of Variables Formula. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, $n \leq m$, and if $u:\mathbb{R}^n \to \mathbb{R}^n$ is $\mathscr{L}_n$ summable then

$$\int_{\mathbb{R}^n} u(x) Jf(x)\, dx = \int_{\mathbb{R}^m} \left[\Sigma_{x \in f^{-1}\{y\}} u(x)\right] dH_n(y).$$

Proof of Lemma 2.6. Since $\Sigma(u)$ is rectifiable we can write

$$\Sigma(u) = \bigcup_{n=1}^{\infty} K_n \cup E$$

where $H_{N-1}(E) = 0$ and $K_n = \Phi_n(A_n)$, Φ_n is C^1 and $A_n \subset \mathbb{R}^{N-1}$ is compact. Let A_n' be the set of points of density 1 in A_n, i. e.

$$A_n' := \left\{ x' \in A_n \mid \lim_{\varepsilon \to 0} \frac{H_{N-1}(A_n \cap B(x',\varepsilon))}{H_{N-1}(B(x',\varepsilon))} = 1 \right\}. \tag{2.4}$$

It is well known that

$$H_{N-1}(A_n \setminus A_n') = 0. \tag{2.5}$$

Hence

$$\Sigma(u) = \bigcup_{n=1}^{\infty} \Phi_n(A_n') \cup E'$$

where

$$E' := \bigcup_{n=1}^{\infty} \Phi_n(A_n \setminus A_n') \cup E, \qquad H_{N-1}(E') = 0.$$

Indeed, by (2.5) and by the change of variables formula 2.10 we have

$$H_{N-1}(\Phi_n(A_n \setminus A_n')) \leq \int_{\mathbb{R}^N} \left[\Sigma_{x' \in \Phi_n^{-1}\{y\}} \chi_{A_n \setminus A_n'}(x')\right] dH_{N-1}(y)$$

$$= \int_{\mathbb{R}^{N-1}} \chi_{A_n \setminus A_n'}(x')\, J\Phi_n(x')\, dH_{N-1}(x') = 0.$$

Setting

$$K_n' := \Phi_n(A_n')$$

and as $H_{N-1}\lfloor K_n'$ is a Radon measure, by Theorem 2.8 there exists a set

$$E_n \subset K_n' \text{ with } H_{N-1}(E_n) = 0$$

such that if $x \in K_n' \setminus E_n$ then

$$\lim_{\varepsilon\to 0} \frac{1}{H_{N-1}(K_n' \cap B(x,\varepsilon))} \int_{K_n' \cap B(x,\varepsilon)} |u^+(y) - u^-(y) - [u^+(x) - u^-(x)]| \, dH_{N-1}(y) = 0. \qquad (2.6)$$

On the other hand, as

$$\mu := |u^+ - u^-| H_{N-1}\lfloor\Sigma(u)$$

has finite total variation, by Proposition 2.9 there exists

$$F_n \subset K_n' \text{ with } H_{N-1}(F_n) = 0$$

such that

$$\lim_{\varepsilon\to 0} \frac{\mu((x_0 + \varepsilon Q_{v(x_0)}) \cap [\Sigma(u) \setminus K_n'])}{\varepsilon^{N-1}} = 0 \qquad (2.7)$$

for every $x \in K_n' \setminus F_n$. Defining

$$E^* := \bigcup_{n=1}^{\infty} (E_n \cup F_n) \cup E'$$

then $H_{N-1}(E^*) = 0$ and if $x_0 \in \Sigma(u) \setminus E^*$, with $x_0 \in K_n' \setminus E^*$, for $\varepsilon > 0$ is small enough and after a rotation of the coordinate axes, we may write

$$K_n' \cap (x_0 + \varepsilon Q_{v(x_0)}) = \{x \in \mathbb{R}^N \mid x = (x', g(x')), x' \in A_n' \cap (x_0' + \varepsilon Q')\}$$

where Q' is the unit cube in $\mathbb{R}^{N-1}$ centered at the origin and g is a C^1 function, $\nabla g(x_0') = 0$. By (2.7) we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \int_{\Sigma(u)\cap(x_0+\varepsilon Q_{v(x_0)})} |u^+(x) - u^-(x)| \, dH_{N-1}(x) = \lim_{\varepsilon\to 0} \frac{\mu((x_0 + \varepsilon Q_{v(x_0)}) \cap K_n')}{\varepsilon^{N-1}}$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \int_{A_n' \cap (x_0' + \varepsilon Q')} |u^+(x', g(x')) - u^-(x', g(x'))| \sqrt{1 + |\nabla g(x')|^2} \, dx'$$

$$= |u^+(x_0) - u^-(x_0)|$$

due to (2.4) and (2.6). ■

Recently, Alberti [Al] showed that the density of the Cantor part C(u) is a rank-one matrix. Taking into consideration Theorem 2.7 we have the following property.

Theorem 2.11. If $u \in BV(\Omega; \mathbb{R}^p)$ then for $|C(u)|$ a. e. $x \in \Omega$

$$A(x) := \lim_{\varepsilon\to 0^+} \frac{D(u)(x+\varepsilon X)}{|D(u)|(x+\varepsilon X)} = \lim_{\varepsilon\to 0} \frac{C(u)(x+\varepsilon X)}{|C(u)|(x+\varepsilon X)}$$

exists and is a rank-one matrix of norm one, for every convex, open set X containing the origin.

The following lemma provides an estimate on the H_{N-1} measure of the level sets of Lipschitz functions. A minor variation was obtained in [FM], Lemma 2.7, and for the convenience of the reader its proof is presented in the Appendix.

Lemma 2.12. Let $K \subset \mathbb{R}^N$ be a compact set, let v be a Lipschitz function on K and let $A \subset K$ be a measurable set. If $0 < a < b$ then

$$\text{ess inf}_{t \in [a,b]}\, t\, H_{N-1}(\{x \in A \mid v(x) = t\}) \leq \frac{1}{\ln(b/a)} \int_{A \cap \{a \leq v \leq b\}} |\nabla v(x)|\, dx.$$

In Section 4 we will treat the density of $\mathcal{F}(.)$ with respect to the Cantor part of the derivative Du, and for this purpose we will need an uniform estimate on the measure $|Du|(B(x_0,t\varepsilon))$ with respect to $|Du|(B(x_0,\varepsilon))$.

Lemma 2.13. Let μ be a nonnegative Radon measure on $\mathbb{R}^N$. For μ a. e. $x_0 \in \mathbb{R}^N$ and for every $0<t<1$ one has

$$\limsup_{\varepsilon\to 0} \frac{\mu(B(x_0,t\varepsilon))}{\mu(B(x_0,\varepsilon))} \geq t^N.$$

The proof of this lemma can be found in the Appendix.

In the sequel

$$f : \Omega \times \mathbb{R}^p \times M^{p\times N} \to [0, +\infty)$$

satisfies the hypotheses:

(H1) f is continuous ;

(H2) f(x,u,.) is quasiconvex ;

(H3) there exists a nonnegative, bounded, continuous function $g : \Omega x \mathbb{R}^p \rightarrow [0, +\infty)$, c, C > 0 such that

$$cg(x, u)\|A\| \leq f(x, u, A) \leq Cg(x, u) (1 + \|A\|)$$

for all $(x,u,A) \in \Omega x \mathbb{R}^p x M^{pxN}$;

(H4) for every compact $K \subset\subset \Omega x \mathbb{R}^p$ there exists a continuous function ω with $\omega(0) = 0$ such that

$$|f(x,u,A) - f(x',u',A)| \leq \omega(|x - x'| + |u - u'|) (1 + \|A\|)$$

for all $(x,u,A), (x',u',A) \in K x M^{pxN}$. In addition, for every $x_0 \in \Omega$ and for all $\delta > 0$ there exists $\varepsilon > 0$ such that if $|x - x_0| \leq \varepsilon$ then

$$f(x, u, A) - f(x_0, u, A) \geq -\delta\, g(x, u) (1 + \|A\|)$$

for every $(u,A) \in \mathbb{R}^p x M^{pxN}$.

(H5) there exist C', L > 0, $0 \leq m < 1$, such that

$$\left|f^\infty(x, u, A) - \frac{f(x, u, tA)}{t}\right| \leq C' g(x, u) \frac{\|A\|^{1-m}}{t^m}$$

for every $(x,u,A) \in \Omega x \mathbb{R}^p x M^{pxN}$ and for all $t > 0$ such that $t\|A\| > L$.

The latter hypothesis will be used only to obtain a lower bound for the density of the jump term on Section 3. (H5) is equivalent to the condition

$$|f^\infty(x, u, A) - f(x, u, A| \leq C' g(x, u) (1 + \|A\|^{1-m})$$

for every $(x,u,A) \in \Omega x \mathbb{R}^p x M^{pxN}$.

The following properties are an easy consequence of the definition of recession function.

Proposition 2.14. If (H3) holds then

$$cg(x, u)\|A\| \leq f^\infty(x, u, A) \leq Cg(x, u) \|A\| \qquad (H3')$$

for all $(x, u, A) \in \Omega x \mathbb{R}^p x M^{pxN}$. If (H3) and (H4) hold then for every compact $K \subset\subset \Omega x \mathbb{R}^p$ there exists a continuous function ω with $\omega(0) = 0$ such that

$$|f^\infty(x,u,A) - f^\infty(x',u',A)| \leq \omega(|x - x'| + |u - u'|) \|A\| \qquad (H4_1')$$

for all $(x,u,A), (x',u',A) \in K x M^{pxN}$. Also, for every $x_0 \in \Omega$ and for all $\delta > 0$ there exists $\varepsilon > 0$ such that if $|x - x_0| \leq \varepsilon$ then

$$f^\infty(x, u, A) - f^\infty(x_0, u, A) \geq -\delta\, g(x, u) \|A\| \qquad (H4_2')$$

for every $(u,A) \in \mathbb{R}^p x M^{pxN}$.

The goal in this paper is to obtain an integral representation for the relaxation $\mathcal{F}(.)$ in $BV(\Omega;\mathbb{R}^p)$ of

$$u \to \int_\Omega f(x,u(x),\nabla u(x))\,dx$$

with respect to the L^1 topology, namely

$$\mathcal{F}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx \mid u_n \in W^{1,1} \text{ and } u_n \to u \text{ in } L^1 \right\}.$$

We introduce the surface energy density $K(x,a,b,\nu)$ whose characterization is based on the work of Fonseca and Rybka [FR] on the relaxation of multiple convex integrals in $BV(\Omega; \mathbb{R}^p)$ (see also Ambrosio and Pallara [AP]) which, in turn, was inspired by a conjecture of Fonseca and Tartar [FT2] for the integral representation of the Γ-limit of a sequence of rescaled singular perturbations for the bulk energy of an elastic material that changes phase.

If $(a, b, \nu) \in \mathbb{R}^p \times \mathbb{R}^p \times S^{N-1}$, let $\{\nu_1, ..., \nu_{N-1}, \nu\}$ form an orthonormal basis of $\mathbb{R}^N$ and define

$$\mathcal{A}(a, b, \nu) := \{\xi \in W^{1,1}(Q_\nu;\mathbb{R}^p) \mid \xi(y) = a \text{ if } y.\nu = -1/2,\ \xi(y) = b \text{ if } y.\nu = 1/2, \text{ and } \xi \text{ is periodic with period one in the } \nu_1, \nu_2, ..., \nu_{N-1} \text{ directions}\}.$$

As usual, ξ is periodic with period one in the ν_i direction if

$$\xi(y) = \xi(y + k\nu_i)$$

for all $k \in \mathbb{Z}$, $y \in Q_\nu$. The surface energy density $K : \Omega \times \mathbb{R}^p \times \mathbb{R}^p \times S^{N-1} \to [0, +\infty)$ is defined by

$$K(x, a, b, \nu) := \inf \left\{ \int_{Q_\nu} f^\infty(x,\xi(y),\nabla\xi(y))\,dy \mid \xi \in \mathcal{A}(a, b, \nu) \right\}.$$

For a detailed study of this function we refer the reader to [FR].

Lemma 2.15. If (H1)-(H4) hold then

(a) $|K(x,a,b,\nu) - K(x,a',b',\nu)| \leq C(|a - a'| + |b - b'|)$ for every (x,a,b,ν), $(x,a',b',\nu) \in \Omega\times\mathbb{R}^p\times\mathbb{R}^p\times S^{N-1}$;

(b) $(x,\nu) \to K(x,a,b,\nu)$ is upper semicontinuous for every $(a,b) \in \mathbb{R}^p\times\mathbb{R}^p$;

(c) K is upper semicontinuous in $\Omega\times\mathbb{R}^p\times\mathbb{R}^p\times S^{N-1}$;

(d) $K(x,a,b,\nu) \leq C\,|a - b|$ for every $(x,\nu) \in \Omega\times S^{N-1}$, $a, b \in \mathbb{R}^p$.

Proof. First note that (c) is an immediate consequence of (a) and (b).

(a) Let $\xi \in \mathcal{A}(a,b,\nu)$, let θ be a smooth cut-off function with $0 \le \theta \le 1$, $\theta(t) = 0$ if $t \ge 1/2$ and $\theta(t) = 1$ if $t \le 1/4$, and define

$$\xi^*(x) = \begin{cases} \xi(2y) & \text{if } |y.\nu| < 1/4 \\ \theta(y.\nu)b + (1 - \theta(y.\nu))b' & \text{if } 1/4 < y.\nu < 1/2 \\ \theta(-y.\nu)a + (1 - \theta(-y.\nu))a' & \text{if } -1/2 < y.\nu < -1/4 \end{cases} .$$

Then $\xi^* \in \mathcal{A}(a',b',\nu)$ and

$$K(x,a',b',\nu) \le \int_{Q_\nu} f^\infty(x,\xi^*(y),\nabla\xi^*(y))\,dy$$

$$= \int_{Q'} \int_{|y.\nu|<1/4} f^\infty(x,\xi(2y),2\nabla\xi(2y))\,dy \; +$$

$$+ \int_{Q'} \int_{1/4<y.\nu<1/2} f^\infty(x,\theta(y.\nu)b+[1-\theta(y.\nu)]b', (b-b')\otimes\theta'(y.\nu)\nu)\,dy$$

$$+ \int_{Q'} \int_{-1/2<y.\nu<-1/4} f^\infty(x,\theta(-y.\nu)a+[1-\theta(-y.\nu)]a', (a'-a)\otimes\theta'(-y.\nu)\nu)\,dy$$

hence, by (H3') (Proposition 2.14) and by the periodicity of ξ

$$K(x,a',b',\nu) \le \frac{1}{2^{N-1}} \int_{2Q'} \int_{|y.\nu|<1/2} f^\infty(x,\xi(y),\nabla\xi(y))\,dy + C(|a - a'| + |b - b'|)$$

$$= \int_{Q_\nu} f^\infty(x,\xi(y),\nabla\xi(y))\,dy + C(|a - a'| + |b - b'|).$$

Taking the infimum in all $\xi \in \mathcal{A}(a,b,\nu)$ we conclude that

$$K(x,a',b',\nu) \le K(x,a,b,\nu) + C(|a - a'| + |b - b'|).$$

(b) It is clear that

$$K(x,a,b,\nu) = \inf \left\{ \int_Q f^\infty(x,\xi(Ry),\nabla\xi(Ry)R^T)\,dy \mid R \text{ is a rotation, } Re_N = \nu \text{ and } \xi \in \mathcal{A}(a,b,e_N) \right\}$$

where $Q = (-1/2,1/2)^N$. Also, due to (H3') it suffices to consider smooth functions ξ. Let $(x_n,\nu_n) \to (x,\nu)$ and given $\varepsilon > 0$ choose a rotation R such that $Re_N = \nu$, and let $\xi \in \mathcal{A}(a,b,e_N)$ be a smooth function such that

$$\left| K(x,a,b,\nu) - \int_Q f^\infty(x,\xi(Ry),\nabla\xi(Ry)R^T)\,dy \right| < \varepsilon.$$

Let X be a compact subset of $\Omega \times \mathbb{R}^p$ containing a neighborhood of $\{(x, \xi(Ry)) \mid y \in Q\}$. By $(H4_1')$

there exists a function ω with $\omega(0) = 0$ such that

$$|f^\infty(y,u,A) - f^\infty(y',u',A)| \leq \omega(|y - y'| + |u - u'|)\ \|A\| \tag{2.8}$$

for all $(y,u,A), (y',u',A) \in K \times M^{p \times N}$. As $f^\infty(x,u,.)$ is quasiconvex (see Remark 2.2) by (H3') we obtain a Lipschitz condition for $f^\infty(x,u,.)$, precisely

$$|f^\infty(x,u,A) - f^\infty(x,u,B)| \leq C\ \|A - B\|. \tag{2.9}$$

Choosing rotations R_n such that $R_n e_N = \nu_n$, by (2.8) and (2.9) and for n large enough we have

$$\left| \int_Q f^\infty(x,\xi(Ry),\nabla\xi(Ry)R^T)\, dy - \int_Q f^\infty(x_n,\xi(R_n y),\nabla\xi(R_n y)R_n^T)\, dy \right| < \varepsilon.$$

Hence

$$\begin{aligned} K(x_n,a,b,\nu_n) &\leq \int_Q f^\infty(x_n,\xi(R_n y),\nabla\xi(R_n y)R_n^T)\, dy \\ &\leq \varepsilon + \int_Q f^\infty(x,\xi(Ry),\nabla\xi(Ry)R^T)\, dy \\ &\leq K(x,a,b,\nu) + 2\varepsilon \end{aligned}$$

and letting $\varepsilon \to 0$ we conclude that

$$\limsup_{n\to+\infty} K(x_n,a,b,\nu_n) \leq K(x,a,b,\nu).$$

(d) Setting

$$\xi_0(y) := (b - a)(y.\nu) + \frac{a + b}{2}$$

by (H3') we have

$$\begin{aligned} K(x,a,b,\nu) &\leq \int_{Q_\nu} f^\infty(x,\xi_0(y),\nabla\xi_0(y))\, dy \\ &= \int_{Q_\nu} f^\infty(x,\xi_0(y),(b - a)\otimes\nu)\, dy \leq C\,|a - b|. \end{aligned}$$

■

In what follows, if g is a positively homogeneous function of degree one and if μ is a $\mathbb{R}^m$-valued measure we use the notation

$$\int_\Omega g(d\mu)$$

to designate

$$\int_\Omega g(\alpha(x))\, d|\mu(x)|,$$

where $|\mu|$ is the nonnegative total variation measure and $\alpha : \Omega \to S^{m-1}$ is the Radon-Nikodym derivative of μ with respect to $|\mu|$ (see Goffman and Serrin [GS], Fonseca [Fo], Reshetnyak [R]).

Theorem 2.16. If (H1)-(H6) hold and if $u \in BV(\Omega, \mathbb{R}^p)$ then

$$\mathcal{F}(u) = \int_\Omega f(x,u(x),\nabla u(x))\,dx + \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\,dH_{N-1}(x) + \int_\Omega f^\infty(x,u(x),dC(u)).$$

Remark 2.17. In the case where

$$f = f(x,\nabla u),$$

the surface energy density becomes

$$K(x, a, b, \nu) = f^\infty(x, (b - a)\otimes\nu)$$

and the jump term in $\mathcal{F}(u)$ reduces to

$$\int_{\Sigma(u)} f^\infty(x, (u^+(x) - u^-(x))\otimes\nu(x))\,dH_{N-1}(x).$$

Indeed, setting

$$\xi_0(x) := (b - a)(x.\nu) + \frac{a + b}{2}$$

and

$\mathfrak{P} := \{\varphi \in W^{1,1}(Q_\nu;\mathbb{R}^p) \mid \varphi(y) = 0$ if $y.\nu = \pm 1/2$, φ is periodic with period one in the $\nu_1, \nu_2, ..., \nu_{N-1}$ directions$\}$,

we have

$$K(x, a, b, \nu) = \inf\left\{ \int_{Q_\nu} f^\infty(x,\nabla(\xi_0(y) + \varphi(y)))\,dy \;\middle|\; \varphi \in \mathfrak{P} \right\}$$

$$= \inf\left\{ \int_{Q_\nu} f^\infty(x, (b - a)\otimes\nu + \nabla\varphi(y))\,dy \mid \varphi \in \mathfrak{P} \right\}.$$

Thus, by (H3') and Remark 2.2 (i) and (ii) we conclude that

$$K(x, a, b, \nu) = f^\infty(x, (b - a)\otimes\nu)$$ ■

We divide the proof of Theorem 2.16 into two parts. In the first one we show that

$$\mathcal{F}(u) \geq \int_\Omega f(x,u(x),\nabla u(x))\,dx + \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\,dH_{N-1}(x) + \int_\Omega f^\infty(x,u(x),dC(u))$$

and the second part asserts the reverse inequality. It is clear that the above inequality is equivalent to proving

Theorem 2.18. Let (H1)-(H5) hold, let $u \in BV(\Omega, \mathbb{R}^p)$, $u_n \in W^{1,1}(\Omega; \mathbb{R}^p)$ and suppose that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^p)$. Then

$$\liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx \geq \int_\Omega f(x,u(x),\nabla u(x))\,dx$$
$$+ \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\,dH_{N-1}(x) + \int_\Omega f^\infty(x,u(x),\,dC(u)).$$

Proof of Theorem 2.18. Due to (H3) we may assume without loss of generality that

$$\liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx = \lim_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx < +\infty$$

and $u_n \in C_0^\infty(\mathbb{R}^N;\mathbb{R}^p)$ (see Proposition 2.6 in [FM] and also Acerbi and Fusco [AF]). Using the blow-up argument as introduced in [FM], we reduce the problem to verifying the pointwise inequalities (2.10), (2.11) and (2.12) below. As f is nonnegative there exists a subsequence, which for convenience of notation is still labelled $\{u_n\}$, and a nonnegative finite Radon measure μ such that

$$f(.,u_n(.),\nabla u_n(.)) \to \mu \text{ weakly * in the sense of measures,}$$

i. e. for all $\varphi \in C_0(\Omega)$

$$\int_\Omega \varphi(x)\, f(x,u_n(x),\nabla u_n(x))\,dx \to \int_\Omega \varphi(x)\,d\mu(x).$$

Using the Radon-Nikodym Theorem, we can write μ as a sum of four mutually singular nonnegative measures

$$\mu = \mu_a\,\mathcal{L}_N + \zeta\,|u^+ - u^-|H_{N-1}\lfloor\Sigma(u) + \eta\,|C(u)| + \mu_s.$$

We claim that

$$\mu_a(x_0) \geq f(x_0,u(x_0),\nabla u(x_0)) \quad \text{for } \mathcal{L}_N \text{ a. e. } x_0 \in \Omega, \tag{2.10}$$

$$\zeta(x_0) \geq \frac{K(x_0,u^-(x_0),u^+(x_0),\nu(x_0))}{|u^+(x_0) - u^-(x_0)|} \quad \text{for } |u^+ - u^-|\,H_{N-1}\lfloor\Sigma(u) \text{ a. e. } x_0 \in \Sigma(u) \tag{2.11}$$

and, using the same notation as in Theorem 2.11,

$$\eta(x_0) \geq f^\infty(x_0,u(x_0),A(x_0)) \quad \text{for } |C(u)| \text{ a. e. } x_0 \in \Omega. \tag{2.12}$$

Then, considering an increasing sequence of smooth cut-off functions φ_k, with $0 \leq \varphi_k \leq 1$ and $\sup_k \varphi_k(x) = 1$ in Ω, we conclude that

$$\lim_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx \geq \liminf_{n\to+\infty} \int_\Omega \varphi_k(x)\, f(x,u_n(x),\nabla u_n(x))\,dx$$

$$= \int_\Omega \varphi_k(x)\, d\mu(x)$$

$$\geq \int_\Omega \varphi_k(x)\mu_a(x)\, dx + \int_{\Sigma(u)} \varphi_k(x)\zeta(x)\ |u^+(x) - u^-(x)|\, dH_{N-1}(x) +$$

$$+ \int_\Omega \varphi_k(x)\eta(x)\, d\, |C(u)|(x)$$

$$\geq \int_\Omega \varphi_k(x)\ f(x,u(x),\nabla u(x))\, dx +$$

$$+ \int_{\Sigma(u)} \varphi_k(x)\ K(x,u^-(x),u^+(x),\nu(x))\, dH_{N-1}(x) + \int_\Omega \varphi_k(x) f^\infty(x,u(x),dC(u)).$$

Letting $k \to +\infty$, the result follows from the Monotone Convergence Theorem. ■

The next two sections are dedicated to proving claims (2.11) and (2.12). The inequality (2.10) concerning the absolutely continuous part is easily obtained. Indeed, by the Besicovitch Differentiation Theorem (Theorem 2.7) for $\mathscr{L}_N$ a. e. $x_0 \in \Omega$ the limit

$$\mu_a(x_0) := \lim_{\varepsilon \to 0^+} \frac{\mu(B(x_0,\varepsilon))}{\text{meas}(B(x_0,\varepsilon))}$$

exists and is finite and Theorem 2.4 (i) holds. Here, and in what follows, we denote the $\mathscr{L}_N$ measure of a Borel set B by meas(B). Choosing one such x_0, (2.10) now follows from Steps 1, 2 and 3 in the proof of Theorem 2.3 in [FM].

3. The density of the jump term.

Here we prove inequality (2.11). By Lemma 2.6, Theorem 2.4 (ii) and by Theorem 2.7, for H_{N-1} a. e. $x_0 \in \Sigma(u)$ we have

(i) $$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{N-1}} \int_{\Sigma(u)\cap(x_0+\varepsilon Q_{\nu(x_0)})} |u^+(x) - u^-(x)|\, dH_{N-1}(x) = |u^+(x_0) - u^-(x_0)|,$$

(ii) $$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in B(x,\varepsilon) | (y-x).\nu(x)>0\}} |u(y) - u^+(x)|^{N/(N-1)}\, dy = 0,$$

and

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in B(x,\varepsilon) | (y-x).\nu(x)<0\}} |u(y) - u^-(x)|^{N/(N-1)}\, dy = 0,$$

(iii) $$\zeta(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(x_0+\varepsilon Q_{\nu(x_0)})}{|u^+ - u^-|\, H_{N-1}\lfloor\Sigma(u)(x_0+\varepsilon Q_{\nu(x_0)})}$$

exists and is finite.

Writting $Q := Q_{\nu(x_0)}$, $Q^* := \frac{1}{1+\delta} Q$, with $0 < \delta < 1$, let $\varphi \in C_0^\infty(Q)$ be such that $0 \le \varphi \le 1$, $\varphi = 1$ on Q^*. By (i) and (iii)

$$\zeta(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(x_0+\varepsilon Q_{\nu(x_0)})}{|u^+ - u^-| \, H_{N-1}\lfloor\Sigma(u)(x_0+\varepsilon Q_{\nu(x_0)})}$$

$$= \frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N-1}} \int_{x_0+\varepsilon Q} d\mu(x)$$

$$\ge \frac{1}{|u^+(x_0) - u^-(x_0)|} \limsup_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{\varepsilon^{N-1}} \int_{x_0+\varepsilon Q} \varphi\left(\frac{x - x_0}{\varepsilon}\right) f(x,u_n(x),\nabla u_n(x))\, dx$$

$$= \frac{1}{|u^+(x_0) - u^-(x_0)|} \limsup_{\varepsilon \to 0^+} \lim_{n \to \infty} \int_Q \varepsilon\, \varphi(y)\, f(x_0+\varepsilon y,u_n(x_0+\varepsilon y),\nabla u_n(x_0+\varepsilon y))\, dy$$

$$\ge \frac{1}{|u^+(x_0) - u^-(x_0)|} \limsup_{\varepsilon \to 0^+} \limsup_{n \to \infty} \int_{Q^*} \varepsilon f(x_0+\varepsilon y,u_n(x_0+\varepsilon y),\nabla u_n(x_0+\varepsilon y))\, dy. \quad (3.1)$$

Let

$$u_{n,\varepsilon}(y) := u_n(x_0+\varepsilon y),$$

$$u_0(y) := \begin{cases} u^+(x_0) & \text{if } y.\nu(x_0) > 0 \\ u^-(x_0) & \text{if } y.\nu(x_0) \le 0 \end{cases}.$$

As $u_n \to u$ in L^1, by (ii) we obtain

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int_Q |u_{n\varepsilon}(y) - u_0(y)|\, dy = \lim_{\varepsilon \to 0^+} \int_{Q^+} |u(x_0+\varepsilon y) - u^+(x_0)|\, dy$$
$$+ \lim_{\varepsilon \to 0^+} \int_{Q^-} |u(x_0+\varepsilon y) - u^-(x_0)| dy = 0. \quad (3.2)$$

On the other hand

$$\zeta(x_0) \ge \frac{1}{|u^+(x_0) - u^-(x_0)|} \limsup_{\varepsilon \to 0^+} \limsup_{n \to \infty} \Big[\int_{Q^*} f^\infty(x_0+\varepsilon y,u_{n,\varepsilon}(y),\nabla u_{n,\varepsilon}(y))\, dy$$
$$+ \int_{Q^*} \Big(\varepsilon\, f(x_0+\varepsilon y,u_{n,\varepsilon}(y),\frac{1}{\varepsilon} \nabla u_{n,\varepsilon}(y)) - f^\infty(x_0+\varepsilon y,u_{n,\varepsilon}(y),\nabla u_{n,\varepsilon}(y)) \Big)\, dy \Big] \quad (3.3)$$

where, by (H3), (H3') and (H5)

$$\int_{Q^*} \Big| \varepsilon\, f(x_0+\varepsilon y,u_{n,\varepsilon}(y),\frac{1}{\varepsilon} \nabla u_{n,\varepsilon}(y)) - f^\infty(x_0+\varepsilon y,u_{n,\varepsilon}(y),\nabla u_{n,\varepsilon}(y))\Big|\, dy \le$$

$$\le \text{meas}(Q^* \cap \{ \|\nabla u_{n,\varepsilon}\| \le \varepsilon L\})\, \varepsilon\, C(1 + 2L) +$$

$$+ \int_{Q^* \cap \{\|\nabla u_{n\varepsilon}\| > \varepsilon L\}} C' \, g(x_0+\varepsilon y, u_{n,\varepsilon}(y)) \, \|\nabla u_{n\varepsilon}(y)\|^{1-m} \, \varepsilon^m \, dy$$

and so, as g is a bounded function, by Hölder's inequality, (H3) and (3.1) we conclude that

$$\int_{Q^*} \left| \varepsilon \, f(x_0+\varepsilon y, u_{n,\varepsilon}(y), \frac{1}{\varepsilon} \nabla u_{n,\varepsilon}(y)) - f^{\infty}(x_0+\varepsilon y, u_{n,\varepsilon}(y), \nabla u_{n,\varepsilon}(y)) \right| dy$$

$$\leq O(\varepsilon) + C \varepsilon^m \Big[\int_{Q^*} g(x_0+\varepsilon y, u_{n,\varepsilon}(y)) \, \|\nabla u_{n\varepsilon}(y)\| \, dy \Big]^{1-m}$$

$$= O(\varepsilon) + C \varepsilon^m \Big[\int_{Q^*} \varepsilon \, g(x_0+\varepsilon y, u_n(x_0+\varepsilon y)) \, \|\nabla u_n(x_0+\varepsilon y)\| \, dy \Big]^{1-m}$$

$$\leq O(\varepsilon) + C \varepsilon^m \Big[\int_{Q^*} \varepsilon \, f(x_0+\varepsilon y, u_n(x_0+\varepsilon y), \nabla u_n(x_0+\varepsilon y)) \, dy \Big]^{1-m}$$

$$\leq O(\varepsilon^m).$$

Thus (3.3) reduces to

$$\zeta(x_0) \geq \frac{1}{|u^+(x_0) - u^-(x_0)|} \limsup_{\varepsilon \to 0^+} \limsup_{n \to \infty} \Big\{ \int_{Q^*} f^{\infty}(x_0, u_{n,\varepsilon}(y), \nabla u_{n,\varepsilon}(y)) \, dy$$

$$+ \int_{Q^*} \big[f^{\infty}(x_0+\varepsilon y, u_{n,\varepsilon}(y), \nabla u_{n,\varepsilon}(y)) - f^{\infty}(x_0, u_{n,\varepsilon}(y), \nabla u_{n,\varepsilon}(y)) \big] \, dy \Big\} \qquad (3.4)$$

and $(H4_2')$, (H3) imply that

$$\int_{Q^*} [f^{\infty}(x_0+\varepsilon y, u_{n,\varepsilon}(y), \nabla u_{n,\varepsilon}(y)) - f^{\infty}(x_0, u_{n,\varepsilon}(y), \nabla u_{n,\varepsilon}(y))] \, dy$$

$$\geq - \delta \int_{Q^*} \varepsilon \, g(x_0+\varepsilon y, u_n(x_0+\varepsilon y)) \, \|\nabla u_n(x_0+\varepsilon y)\| \, dy$$

$$\geq - \delta C \int_{Q^*} \varepsilon \, f(x_0+\varepsilon y, u_n(x_0+\varepsilon y), \nabla u_n(x_0+\varepsilon y)) \, dy$$

where, by (3.1), the set

$$\Big\{ \int_{Q^*} \varepsilon f(x_0+\varepsilon y, u_n(x_0+\varepsilon y), \nabla u_n(x_0+\varepsilon y)) \, dy \ \Big| \ \varepsilon > 0, \text{ n positive integer} \Big\}$$

is bounded. Thus

$$\int_{Q^*} [f^{\infty}(x_0+\varepsilon y, u_{n,\varepsilon}(y), \nabla u_{n,\varepsilon}(y)) - f^{\infty}(x_0, u_{n,\varepsilon}(y), \nabla u_{n,\varepsilon}(y))] \, dy \geq O(\delta). \qquad (3.5)$$

Using a standard diagonalization procedure, by (3.2), (3.4) and (3.5) we construct a sequence $\{v_k\}$ such that

$$v_k \to u_0 \text{ in } L^1(Q)$$

and

$$\zeta(x_0) \geq \frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{k \to \infty} \int_{Q^*} f^{\infty}(x_0, v_k(y), \nabla v_k(y)) \, dy + O(\delta).$$

Making the change of variables

$$y = \frac{x}{1+\delta},$$

setting

$$w_k(x) := v_k\left(\frac{x}{1+\delta}\right)$$

and using the invariance of u_0 under the above change of variables we have

$$w_k \to u_0 \text{ in } L^1(Q) \tag{3.6}$$

with

$$\zeta(x_0) \geq \frac{(1+\delta)^{1-N}}{|u^+(x_0) - u^-(x_0)|} \lim_{k\to\infty} \int_Q f^\infty(x_0, w_k(y), \nabla w_k(y))\, dy + O(\delta). \tag{3.7}$$

In order to conclude (2.11) and taking into account the definition of K, we must modify w_k on the boundary of Q in such a way that the new sequence is in $\mathcal{A}(u^-(x_0), u^+(x_0), \nu(x_0))$ and the total energy does not increase. This is accomplished with the help of the following lemma, well known to experts in Gamma convergence. This result uses the idea of multiple cut-off functions which appears frequently in connection with certain convexity hypotheses (see [DG], [DD]) . We are grateful to G. Dal Maso for pointing out to us that these are not needed in the case of linear growth conditions. We include a proof for the convenience of the reader.

Lemma 3.1. Let $Q = [0, 1]^N$, and let $f : \Omega \times \mathbb{R}^p \times M^{p\times N} \to [0, +\infty)$ be a Carathéodory function such that

$$0 \leq f(x,u,A) \leq C\,(1 + \|A\|)$$

for some $C > 0$ and for all $(x,u,A) \in \Omega \times \mathbb{R}^p \times M^{p\times N}$. Let

$$u_0(y) := \begin{cases} b & \text{if } x_N > 0 \\ a & \text{if } x_N \leq 0 \end{cases}$$

and suppose that $w_n \to u_0$ in $L^1(Q; \mathbb{R}^p)$, where $w_n \in W^{1,1}(Q; \mathbb{R}^p)$. If ρ is a mollifier, $\rho_n(x) := n^N\rho(nx)$, then there exists a sequence of functions $\xi_n \in W^{1,1}(Q; \mathbb{R}^p) \cap \mathcal{A}(a,b,e_N)$ such that

$$\xi_n = \rho_n * u_0 \text{ on } \partial Q, \quad \xi_n \to u_0 \text{ in } L^1(Q; \mathbb{R}^p)$$

and

$$\liminf_{n\to+\infty} \int_Q f(x, w_n(x), \nabla w_n(x))\, dx \geq \limsup_{n\to+\infty} \int_Q f(x, \xi_n(x), \nabla \xi_n(x))\, dx.$$

Proof. Without loss of generality, assume that

$$\liminf_{n\to+\infty}\int_Q f(x,w_n(x),\nabla w_n(x))\,dx = \lim_{n\to+\infty}\int_Q f(x,w_n(x),\nabla w_n(x))\,dx < +\infty.$$

Define

$$v_n(x) := (\rho_n * u_0)(x) = \int_{B(x,1/n)} \rho_n(x-y)\, u_0(y)\, dy.$$

As $\rho \geq 0$, supp $\rho = \overline{B}(0, 1)$ and

$$\int_{B(0,1)} \rho(x)\, dx = 1,$$

we have for $i = 1, ..., N-1$

$$v_n(x + e_i) = \int_{\mathbb{R}^N} \rho_n(x+e_i-y)\, u_0(y)\, dy =$$

$$= \int_{\mathbb{R}^N} \rho_n(x-z)\, u_0(z+e_i)\, dz$$

$$= (\rho_n * u_0)(x)$$

and so

$$v_n(y) := \begin{cases} b & \text{if } x_N > 1/n \\ a & \text{if } x_N < -1/n \end{cases}, \quad \|\nabla v_n\|_\infty = O(n), \quad v_n \in \mathcal{A}(a,b,e_N). \tag{3.8}$$

Let

$$\alpha_n := \sqrt{\|w_n - v_n\|_{L^1(Q)}}, \quad k_n := n\,[1 + \|w_n\|_{1,1} + \|v_n\|_{1,1}], \quad s_n := \frac{\alpha_n}{k_n}$$

where $[k]$ denotes the largest integer less than or equal to k. As $\alpha_n \to 0^+$ we may assume that $0 \leq \alpha_n < 1$ and we set

$$Q_0 := (1 - \alpha_n)\, Q, \quad Q_i := (1 - \alpha_n + i\, s_n)\, Q,\ i = 1, ..., k_n.$$

Consider a family of cut-off functions

$$\varphi_i \in C_0^\infty(Q_i),\ 0 \leq \varphi_i \leq 1,\ \varphi_i = 1 \text{ in } Q_{i-1},\ \|\nabla\varphi_i\|_\infty = O(\tfrac{1}{s_n})$$

for $i = 1, ..., k_n$, and define

$$w_n^{(i)}(x) := (1 - \varphi_i(x))v_n(x) + \varphi_i(x)w_n(x).$$

As $w_n^{(i)} = v_n$ on ∂Q, by (3.8) we conclude that

$$w_n^{(i)} \in \mathcal{A}(a, b, e_N). \tag{3.9}$$

Clearly

$$\nabla w_n^{(i)} = \nabla w_n \text{ in } Q_{i-1}, \quad \nabla w_n^{(i)} = \nabla v_n \text{ on } Q\setminus Q_i$$

and in $Q_i \setminus Q_{i-1}$

$$\nabla w_n^{(i)} = \nabla v_n + \varphi_i(\nabla w_n - \nabla v_n) + (w_n - v_n) \otimes \nabla\varphi_i.$$

Due to the growth condition on f we deduce that

$$\int_Q f(x,w_n^{(i)}(x),\nabla w_n^{(i)}(x))\,dx \le \int_Q f(x,w_n(x),\nabla w_n(x))\,dx +$$

$$C \int_{Q_i\setminus Q_{i-1}} (1+ |w_n(x) - v_n(x)|^{\frac{1}{s_n}} + \|\nabla w_n(x)\| + \|\nabla v_n(x)\|)\,dx + C \int_{Q\setminus Q_i} (1+ \|\nabla v_n(x)\|)\,dx$$

and averaging this inequality among all the layers $Q_i \setminus Q_{i-1}$ we obtain

$$\frac{1}{k_n}\sum_{i=1}^{k_n}\int_Q f(x, w_n^{(i)}(x), \nabla w_n^{(i)}(x))\,dx \le \int_Q f(x, w_n(x), \nabla w_n(x))\,dx +$$

$$+\frac{C}{k_n}\int_Q (1+ \|\nabla w_n(x)\| + \|\nabla v_n(x)\|)\,dx + \frac{C}{k_n}\int_Q |w_n(x) - v_n(x)|^{\frac{1}{s_n}}\,dx + C\int_{Q\setminus Q_0}(1 + \|\nabla v_n(x)\|)\,dx$$

i. e.

$$\frac{1}{k_n}\sum_{i=1}^{k_n}\int_Q f(x, w_n^{(i)}(x), \nabla w_n^{(i)}(x))\,dx \le \int_Q f(x, w_n(x), \nabla w_n(x))\,dx +$$

$$+ O(1/n) + C\sqrt{\|w_n - v_n\|_{L^1(Q)}} + C\int_{Q\setminus Q_0}(1 + \|\nabla v_n(x)\|)\,dx.$$

By (3.7), as $\text{meas}(Q \setminus Q_0) = O(\alpha_n)$ and

$$\nabla v_n(x) = 0 \text{ if } |x_N| > 1/n,$$

we estimate

$$\int_{Q\setminus Q_0}(1 + \|\nabla v_n(x)\|)\,dx \le O(\alpha_n) + H_{N-1}(Q\setminus Q_0 \cap \{x_N = 0\}) \int_{-1/n}^{1/n} O(n)\,dx_N = O(\alpha_n).$$

Thus, setting

$$\varepsilon_n := O(1/n) + C\sqrt{\|w_n - v_n\|_{L^1(Q)}} + O(\alpha_n),$$

it is clear that

$$\varepsilon_n \to 0^+$$

with

$$\frac{1}{k_n}\sum_{i=1}^{k_n}\int_Q f(x, w_n^{(i)}(x), \nabla w_n^{(i)}(x))\,dx \le \int_Q f(x, w_n(x), \nabla w_n(x))\,dx + \varepsilon_n,$$

and so there must exist an index $i(n) \in \{1, ..., k_n\}$ for which

$$\int_Q f(x,w_n^{(i(n))}(x),\nabla w_n^{(i(n))}(x))\,dx \le \int_Q f(x,w_n(x),\nabla w_n(x))\,dx + \varepsilon_n.$$

By (3.9) it suffices to define

$$\xi_n := w_n^{(i(n))}.$$

■

End of the proof of (2.11). By (H3'), (3.6) and Lemma 3.1 there exists a sequence $\{\xi_k\}$ in $\mathcal{A}(u^-(x_0), u^+(x_0), \nu(x_0))$ such that

$$\lim_{k\to\infty} \int_Q f^\infty(x_0,w_k(y),\nabla w_k(y))\,dy \geq \limsup_{k\to\infty} \int_Q f^\infty(x_0,\xi_k(y),\nabla\xi_k(y))\,dy$$

which, together with (3.7), yields

$$\zeta(x_0) \geq \frac{(1+\delta)^{1-N}}{|u^+(x_0) - u^-(x_0)|} K(x_0, u^-(x_0), u^+(x_0), \nu(x_0)) + O(\delta).$$

Letting $\delta \to 0^+$ we conclude (2.11). ∎

4. The density of the Cantor part.

Here we want to show (2.12), i. e. for $|C(u)|$ a. e. $x_0 \in \Omega$

$$\eta(x_0) \geq f^\infty(x_0, u(x_0), A(x_0)).$$

Let $Q = (-1/2,1/2)^N$ and $Q(x_0,\varepsilon) := x_0 + \varepsilon Q$. For $|C(u)|$ a. e. $x_0 \in \Omega$ we have

$$\lim_{\varepsilon\to 0} \frac{|Du|(Q(x_0,\varepsilon))}{|C(u)|(Q(x_0,\varepsilon))} = 1$$

and so, by Theorem 2.4 (iii), Theorem 2.7 and Theorem 2.11, for $|C(u)|$ a. e. $x_0 \in \Omega$ the following hold :

$$\eta(x_0) = \lim_{\varepsilon\to 0} \frac{\mu(Q(x_0,\varepsilon))}{|Du|(Q(x_0,\varepsilon))}, \tag{4.1}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\text{meas}(Q(x_0,\varepsilon))} \int_{Q(x_0,\varepsilon)} |u(x) - u(x_0)|\,dx = 0, \tag{4.2}$$

$$A(x_0) = \lim_{\varepsilon\to 0} \frac{Du(Q(x_0,\varepsilon))}{|D(u)|(Q(x_0,\varepsilon))},\quad \|A(x_0)\| = 1,\ A(x_0) = a\otimes\nu, \tag{4.3}$$

$$\lim_{\varepsilon\to 0} \frac{|Du|(Q(x_0,\varepsilon))}{\varepsilon^{N-1}} = 0 \text{ and } \lim_{\varepsilon\to 0} \frac{|Du|(Q(x_0,\varepsilon))}{\varepsilon^N} = +\infty. \tag{4.4}$$

Also, by Lemma 2.13 we may assume that

$$\lim_{t\to 1^-} \liminf_{\varepsilon\to 0} \frac{|Du|(Q(x_0,\varepsilon)\setminus Q(x_0,t\varepsilon))}{|Du|(Q(x_0,\varepsilon))} = 0. \tag{4.5}$$

If $g(x_0,u(x_0)) = 0$ then $f^\infty(x_0,u(x_0),A(x_0)) = 0$ and (2.12) holds trivially. Without loss of generality we may assume that

$$u(x_0) = 0,\ g(x_0,u(x_0)) > 0,\ A_0 := A(x_0) = a\otimes e_N \text{ where } |a| = 1.$$

Step 1 [Diagonalization]. There exists a continuous function ω such that $\omega(t) \to 0^+$ when $t \to 0^+$ and for each $0 < t < 1$ and each $\gamma \in (t, 1)$ there exists a sequence $\{r_k\}$ such that, with $Q_k := Q(x_0,r_k)$,

$$r_k \to 0, \ \frac{1}{\text{meas}(Q_k)} \int_{Q_k} |u_k(x)|\, dx \to 0, \tag{4.6}$$

$$\frac{1}{r_k|Du|(Q_k)} \int_{Q_k} \left| u_k(x) - \frac{1}{\text{meas}(Q_k)} \int_{Q_k} u_k(y)dy - \left[u(x) - \frac{1}{\text{meas}(Q_k)} \int_{Q_k} u(y)dy\right] \right| dx \to 0, \tag{4.7}$$

$$\limsup_{k\to+\infty} \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x,u_k(x),\nabla u_k(x))\, dx \leq \eta(x_0), \tag{4.8}$$

$$\frac{Du(Q_k)}{|D(u)|(Q_k)} \to A_0 \tag{4.9}$$

and

$$\limsup_{k\to+\infty} \frac{|Du|(Q_k \setminus Q(x_0,r_k t))}{|Du|(Q_k)} \leq \omega(1 - t). \tag{4.10}$$

Indeed, by (4.2)

$$\lim_{\varepsilon\to 0} \lim_{n\to+\infty} \frac{1}{\text{meas}(Q_\varepsilon)} \int_{Q_\varepsilon} |u_n(x)|\, dx = 0,$$

$$\lim_{\varepsilon\to 0} \lim_{n\to+\infty} \frac{1}{\varepsilon\, |Du|(Q_\varepsilon)} \int_{Q_\varepsilon} \left| u_n(x) - u(x) - \frac{1}{\text{meas}(Q_\varepsilon)} \int_{Q_\varepsilon} [u_n(y) - u(y)]dy \right| dx = 0,$$

by (4.1)

$$\limsup_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{|Du|(Q_\varepsilon)} \int_{\gamma Q_\varepsilon} f(x,u_n(x),\nabla u_n(x))\, dx$$

$$\leq \lim_{\varepsilon\to 0} \frac{\mu(Q(x_0,\varepsilon))}{|Du|(Q(x_0,\varepsilon))} = \eta(x_0)$$

and by (4.5)

$$\liminf_{\varepsilon\to 0} \frac{|Du|(Q(x_0,\varepsilon) \setminus Q(x_0,t\varepsilon))}{|Du|(Q(x_0,\varepsilon))} \leq \omega(1-t).$$

Hence, a standard diagonalization procedure yields (4.6), (4.7), (4.8) and (4.10) while (4.9) follows from (4.3). ■

Step 2 [Truncation]. For every $0 < \varepsilon < \varepsilon_0$ there exist sequences $v_k \in W^{1,1}(\Omega;\mathbb{R}^p)$ and $a_k \to 0$ such that

$$\|v_k\|_\infty \le \varepsilon,$$

$$\limsup_{k\to\infty} \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x_0,0,\nabla v_k(x))\, dx \le (1 + \omega(\varepsilon))\, \eta(x_0) \tag{4.11}$$

and

$$\frac{1}{r_k\, |Du|(Q_k)} \int_{Q_k} \left| v_k(x) - a_k - \left[u(x) - \frac{1}{\text{meas}(Q_k)} \int_{Q_k} u(y)\, dy\right]\right| dx \to 0, \tag{4.12}$$

where $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. The proof of (4.11) is very similar to the argument used in proving Step 3 of Theorem 2.2 in [FM]. Firstly, by (4.2) and (4.6) we may assume that

$$\frac{1}{\text{meas}(Q_k)} \int_{Q_k} |u_k(x)|\, dx \le \varepsilon^2 \ \text{ and } \frac{1}{\text{meas}(Q_k)} \int_{Q_k} |u(x)|\, dx \le \varepsilon^2. \tag{4.13}$$

Set

$$a_k := \frac{1}{\text{meas}(Q_k)} \int_{Q_k} u_k(x)\, dx$$

and define

$$v_k := a_k + \varphi_k(|u_k - a_k|)\,(u_k - a_k)$$

where φ_k is a smooth cut-off function, $0 \le \varphi_k \le 1$, $2\varepsilon^2 \le s_k \le t_k \le \frac{\varepsilon}{2}$,

$$\varphi_k(t) = 1 \text{ if } t \le s_k,\ \ \varphi_k(t) = 0 \ \text{ if } t \ge t_k,\ \ \|\varphi_k'\|_\infty \le \frac{C}{t_k - s_k}.$$

Clearly, $\|v_k - a_k\|_\infty \le t_k \le \varepsilon/2$ which implies that

$$\|v_k\|_\infty \le \varepsilon,$$

$$\nabla v_k = \varphi_k \nabla u_k + (u_k - a_k) \otimes \varphi'_k(|u_k - a_k|)\, \nabla |u_k - a_k|$$

and

$$|\nabla v_k| \le |\nabla u_k| + \frac{C}{t_k - s_k} |u_k - a_k|\ |\nabla |u_k - a_k||.$$

Thus, by (H4)

$$\frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x_0,0,\nabla v_k(x))\, dx = \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} [f(x_0,0,\nabla v_k(x)) - f(x,v_k,\nabla v_k(x))]\, dx$$

$$+ \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x,v_k,\nabla v_k(x))\, dx$$

$$\le \frac{1}{|Du|(Q_k)} \omega(O(r_k) + \varepsilon) \int_{\gamma Q_k} (1 + \|\nabla v_k(x)\|)\, dx + \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x,v_k,\nabla v_k(x))\, dx$$

and so, by (H3) and as for k sufficiently large and ε small

$$g(x,v_k(x)) \ge \frac{g(x_0,0)}{2} > 0,$$

we have

$$\frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x_0,0,\nabla v_k(x))\,dx \le \frac{1}{|Du|(Q_k)} \omega(O(r_k)+\varepsilon)\, C\, r_k^N +$$

$$+ [C\omega(O(r_k)+\varepsilon)+1] \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x,v_k,\nabla v_k(x))\,dx. \qquad (4.14)$$

On the other hand, by (4.13) we have

$$\begin{aligned} meas(\gamma Q_k \cap \{ |u_k - a_k| > t_k\}) &\le meas(\{x \in Q_k \mid |u_k(x)| \ge \varepsilon^2\}) \\ &\le \frac{C}{\varepsilon^2} r_k^N \frac{1}{meas(Q_k)} \int_{Q_k} |u_k(x)|\,dx \\ &\le C\, r_k^N \end{aligned}$$

which, together with (4.8) and (H3) yields

$$\frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x,v_k,\nabla v_k(x))\,dx \le \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x,u_k(x),\nabla u_k(x))\,dx$$

$$+ \frac{C}{|Du|(Q_k)} \frac{1}{t_k - s_k} \int_{\gamma Q_k \cap \{s_k \le |u_k - a_k| \le t_k\}} |u_k - a_k|\, |\nabla |u_k - a_k||\,dx + \frac{C}{|Du|(Q_k)} \int_{\gamma Q_k \cap \{s_k \le |u_k - a_k| \le t_k\}} |\nabla u_k(x)|\,dx \quad +$$

$$+ \frac{C r_k^N}{|Du|(Q_k)} + \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k \cap \{|u_k - a_k| > t_k\}} f(x,a_k,0)\,dx$$

$$\le \eta(x_0) + o(1) + \frac{C r_k^N}{|Du|(Q_k)} + \frac{C}{|Du|(Q_k)} \frac{1}{t_k - s_k} \int_{s_k}^{t_k} \rho H_{N-1}(|u_k - a_k| = \rho \cap \gamma Q_k)\,d\rho$$

$$+ \frac{C}{|Du|(Q_k)} \int_{\gamma Q_k \cap \{s_k \le |u_k - a_k| \le t_k\}} |\nabla u_k(x)|\,dx. \quad (4.15)$$

For fixed k and a. e. t_k one has

$$\lim_{s \to t_k^-} \frac{1}{t_k - s} \int_s^{t_k} \rho H_{N-1}(|u_k - a_k| = \rho \cap \gamma Q_k)\,d\rho = t_k H_{N-1}(|u_k - a_k| = t_k \cap \gamma Q_k) \quad (4.16)_1$$

and

$$\lim_{s \to t_k^-} \int_{\gamma Q_k \cap \{s \le |u_k - a_k| \le t_k\}} |\nabla u_k(x)|\,dx = 0. \qquad (4.16)_2$$

In view of Lemma 2.12 and by (4.4) and (4.8) we can choose $t_k \in (2\varepsilon^2, \varepsilon/2)$ and $s_k \in (2\varepsilon^2, t_k)$ such that

$$\frac{C}{|Du|(Q_k)} \frac{1}{t_k - s_k} \int_{s_k}^{t_k} \rho H_{N-1}(|u_k - a_k| = \rho \cap \gamma Q_k)\,d\rho + \frac{C}{|Du|(Q_k)} \int_{\gamma Q_k \cap \{s_k \le |u_k - a_k| \le t_k\}} |\nabla u_k(x)|\,dx$$

$$\leq 1/k + \frac{C}{-\ln\varepsilon}\ \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k \cap \{ |u_k - a_k| < \varepsilon/2\}} |\nabla u_k(x)|\, dx$$

$$\leq 1/k + \omega(\varepsilon)\ \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k \cap \{ |u_k| \leq \varepsilon\}} f(x,u_k(x),\nabla u_k(x))\, dx + C\omega(\varepsilon)\, \frac{r_k^N}{|Du|(Q_k)}$$

$$\leq 1/k + \overline{w}(\varepsilon)\, (\eta(x_0) + O\,(1)), \tag{4.17}$$

where we used (H3) and the fact that $g(x_0,0) > 0$. By (4.4), (4.14), (4.15), (4.16) and (4.17) we conclude (4.11). To prove (4.12) we consider in $Q = (-1/2, 1/2)^N$ the rescaled functions

$$\bar{u}_k(z) := \frac{r_k^{N-1}}{|Du|(Q_k)} \Big[u(x_0 + r_k z) - \frac{1}{meas(Q_k)} \int_{Q_k} u(x)\, dx\Big],$$

$$w_k(z) := \frac{r_k^{N-1}}{|Du|(Q_k)}\ [u_k(x_0 + r_k z) - a_k],$$

$$w_k^*(z) := \frac{r_k^{N-1}}{|Du|(Q_k)}\, [v_k(x_0 + r_k z) - a_k].$$

Then (4.12) becomes

$$\|\bar{u}_k - w_k^*\|_{L^1(Q)} \to 0 \ \text{ as } k \to +\infty \tag{4.18}$$

and

$$\int_Q \bar{u}_k(z)\, dz = 0 = \int_Q w_k(z)\, dz,\ |D\bar{u}_k|\,(Q) = 1.$$

As BV is compactly imbedded in L^1 we deduce that

$$\{\bar{u}_k\} \text{ is equi-integrable} \tag{4.19}$$

and by (4.7) we have

$$\|\bar{u}_k - w_k\|_{L^1(Q)} \to 0 \ \text{ as } k \to +\infty. \tag{4.20}$$

Moreover, by (4.4)

$$\lambda_k := \frac{r_k^{N-1}}{|Du|(Q_k)} \to +\infty,$$

$$w_k^*(z) = \varphi_k\Big(\frac{|w_k(z)|}{\lambda_k}\Big)\ w_k(z).$$

and

$$\|w_k^* - w_k\|_{L^1(Q)} \leq \int_{Q \cap \{ |w_k| \geq \lambda_k \varepsilon^2\}} |w_k(x)|\, dx. \tag{4.21}$$

Since $\{w_k\}$ is equi-integrable by (4.19) and (4.20) and as

$$\text{meas}(\{x \in Q \mid |w_k(x)| \geq \lambda_k \varepsilon^2\}) = \text{meas}(\{x \in Q \mid |u_k(x_0 + r_k z) - a_k| \geq 2\varepsilon^2\})$$

$$\leq \text{meas}(\{x \in Q \mid |u_k(x_0 + r_k z)| \geq \varepsilon^2\})$$

$$\leq \frac{C}{\varepsilon^2} r_k^N \frac{1}{\text{meas}(Q_k)} \int_{Q_k} |u_k(x)|\, dx \leq C\ r_k^N \to 0,$$

(4.18) follows from (4.21).

Step 3 [Main Estimate]. Let $\{v_k\}$ be as defined in Step 2. We claim that for all $\delta > 0$

$$\liminf_{k\to+\infty} \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x_0,0,\nabla v_k(x))\, dx \geq f^\infty(x_0,0,A_0) - C\,\omega(1-t). \tag{4.22}$$

After extracting a subsequence we may replace liminf by lim. Also, without loss of generality we assume that $x_0 = 0$ and we use the notation

$$tQ_k := Q(x_0, tr_k).$$

Defining

$$f^*(A) := f(x_0, 0, A)$$

then

$$\lim_{k\to+\infty} \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x_0,0,\nabla v_k(x))\, dx =$$

$$= \lim_{k\to+\infty} \frac{1}{\mu_k} \int_{\gamma Q} f^*(\mu_k \nabla w_k^*(x))\, dx \tag{4.23}$$

where by (4.4)

$$\mu_k := \frac{|Du|(Q_k)}{r_k^N} \to +\infty.$$

By (4.3)

$$D\bar{u}_k(Q) = \frac{Du(Q(x_0,r_k))}{|D(u)|(Q(x_0,r_k))} \to A_0 = a \otimes e_N \text{ as } k \to +\infty$$

and by Proposition A.1 (see the Appendix)

$$|D\bar{u}_k - (D\bar{u}_k.A_0)A_0|(Q) \to 0 \text{ as } k \to +\infty \tag{4.24}$$

from which we conclude that

$$|D\bar{u}_k.e_i|(Q) \to 0 \quad \text{for all } i = 1, \ldots, N-1.$$

Thus, it is possible (e. g. by averaging in $x_1,\ldots,x_{N-1}$ and smoothing in x_N) to find a sequence of smooth functions $\xi_k(x) = \bar{\xi}_k(x_N)$ such that

$$\|\xi_k - \bar{u}_k\|_{L^1(Q)} \to 0, \tag{4.25$_1$}$$

and for a. e. $\tau \in (0, 1)$

$$\nabla\xi_k(\tau Q) - D\bar{u}_k(\tau Q) \to 0. \tag{4.25$_2$}$$

Fix $\tau \in (t, \gamma)$ for which $(4.25)_2$ holds. Then we may choose $\delta > 0$ such that $(1 - \delta)\tau > t$ and we may assume that

$$|D\xi_k|(\tau Q\setminus \tau(1-\delta)Q) \leq |D\bar{u}_k|(Q\setminus tQ)$$
$$= \frac{|Du|(Q_k\setminus tQ_k)}{|Du|(Q_k)}. \tag{4.26}$$

We remark that, as ξ_k depends only on x_N, its trace on $\partial(\tau Q)$ agrees with the trace of

$$A_k x + p(x),\ A_k := \frac{\xi_k(\tau/2) - \xi_k(-\tau/2)}{\tau} \otimes e_N = \tau^{-N}\, \nabla\xi_k(\tau Q)$$

where p is τQ periodic. Hence, taking into account (4.23), by (4.20) and using the construction introduced in Lemma 3.1 we will modify w_k^* on the layer $\tau Q \setminus \tau(1-\delta)Q$ so that it coincides with ξ_k on the boundary of τQ, and then we will apply the quasiconvexity property of f* (see Remark 2.2). Let

$$\alpha_k := \sqrt{\|\xi_k - w_k^*\|_{L^1(Q)}},\quad \Delta_k := k\,[\,|D\xi_k|(\gamma Q) + |D\, w_k^*|(\gamma Q) + 1],\ s_k := \frac{\alpha_k}{\Delta_k}.$$

By (4.18), (4.25), $\alpha_k \to 0^+$. We assume that $0 \leq \alpha_k < 1$ and we set

$$Q_i := \tau(1 - \delta)(1 - \alpha_k + i\, s_k)\, Q,\ i = 1, \ldots, \Delta_k.$$

Clearly

$$tQ \subset Q_i \subset Q_{i+1} \subset \tau Q.$$

Consider a family of cut-off functions

$$\varphi_i \in C_0^\infty(Q_i),\ 0 \leq \varphi_i \leq 1,\ \varphi_i = 1 \text{ in } Q_{i-1},\ \|\nabla\varphi_i\|_\infty = O(\frac{1}{s_k})$$

for $i = 1, \ldots, \Delta_k$, and define

$$w_k^{(i)}(x) := (1 - \varphi_i(x))\xi_k(x) + \varphi_i(x) w_k^*(x).$$

We have

$$\frac{1}{\mu_k}\int_{\tau Q} f^*(\mu_k \nabla w_k^{(i)}(x))\, dx \leq \frac{1}{\mu_k}\int_{\tau Q} f^*(\mu_k \nabla w_k^*(x))\, dx \ +$$

$$C \int_{Q_i \setminus Q_{i-1}} \left[|\xi_k(x) - w^*_k(x)| \frac{1}{s_k} + \frac{1}{\mu_k} \right] dx + C \left[|D\xi_k|(Q_i \setminus Q_{i-1}) + |D\bar{w}^*_k|(Q_i \setminus Q_{i-1}) \right]$$

$$+ C \int_{\tau Q \setminus Q_i} \left[\|\nabla \xi_k(x)\| + \frac{1}{\mu_k} \right] dx$$

and averaging this inequality among all the layers $Q_i \setminus Q_{i-1}$, by (4.26) we obtain

$$\frac{1}{\Delta_k} \Sigma \frac{1}{\mu_k} \int_{\tau Q} f^*(\mu_k \nabla w^{(i)}_k(x))\, dx \leq \frac{1}{\mu_k} \int_{\tau Q} f^*(\mu_k \nabla w^*_k(x))\, dx + C \frac{|Du|(Q_k \setminus tQ_k)}{|Du|(Q_k)} + O(\alpha_k + 1/\mu_k).$$

Hence there must exist an index $i = i(k) \in \{1, ..., \Delta_k\}$ for which

$$\frac{1}{\mu_k} \int_{\tau Q} f^*(\mu_k \nabla w^{(i)}_k(x))\, dx \leq \frac{1}{\mu_k} \int_{\tau Q} f^*(\mu_k \nabla w^*_k(x))\, dx + C \frac{|Du|(Q_k \setminus tQ_k)}{|Du|(Q_k)} + O(1/k)$$

and by (4.10), (4.23), $(4.25)_2$, by the quasiconvexity of f* and since f* is a Lipschitz function we conclude that

$$\lim_{k \to +\infty} \frac{1}{|Du|(Q_k)} \int_{\tau Q_k} f(x_0, 0, \nabla v_k(x))\, dx \geq$$

$$\geq \limsup_{k \to +\infty} \left[\frac{1}{\mu_k} \int_{\tau Q} f^*(\mu_k \nabla w^{(i)}_k(x))\, dx - C \frac{|Du|(Q_k \setminus tQ_k)}{|Du|(Q_k)} \right]$$

$$= \limsup_{k \to +\infty} \frac{1}{\mu_k} f^*(\mu_k \nabla \xi_k(\tau Q)) - C\, \omega(1 - t).$$

In view of (4.9), (4.10) and $(4.25)_2$

$$\limsup_{k \to +\infty} |A_0 - \nabla \xi_k(\tau Q)| = \limsup_{k \to +\infty} |A_0 - D\bar{u}_k(\tau Q)| = \limsup_{k \to +\infty} \left| A_0 - \frac{Du(\tau Q_k)}{|Du|(Q_k)} \right|$$

$$\leq \limsup_{k \to +\infty} \left| \frac{Du(Q_k \setminus tQ_k)}{|Du|(Q_k)} \right| \leq \omega(1 - t)$$

therefore, due to the lipschitz continuity of f*

$$\limsup_{k \to +\infty} \left| \frac{1}{\mu_k} f^*(\mu_k A_0) - \frac{1}{\mu_k} f^*(\mu_k \nabla \xi_k(\tau Q)) \right| \leq C\, \omega(1 - t).$$

Finally,

$$s \to f^*(sA_0)$$

is convex because rank $A_0 = 1$ and so

$$\limsup_{s\to+\infty} \frac{1}{s} f^*(sA_0) = \lim_{s\to+\infty} \frac{1}{s} f^*(sA_0) = f^\infty(x_0,0,A_0)$$

and we deduce that

$$\limsup_{k\to+\infty} \frac{1}{\mu_k} f^*(\mu_k \nabla\xi_k(\tau Q)) \geq f^\infty(x_0,0,A_0) - C\,\omega(1 - t),$$

and hence (4.22).

Step 4 [Conclusion]. By (4.11) and (4.22) we conclude that

$$(1 + \omega(\varepsilon))\, \eta(x_0) \geq f^\infty(x_0,0,A_0) - C\,\omega(1 - t)$$

and so, letting $\varepsilon \to 0^+$, $t \to 1^-$ we obtain

$$\eta(x_0) \geq f^\infty(x_0,0,A_0).$$

■

Remark 4.1. If f(x,u,.) is convex then Alberti's result (Theorem 2.11) concerning the rank-one property of the Cantor set is not needed. Indeed, by (4.20)

$$\|\bar{u}_k - w_k^*\|_{L^1(\partial(\tau Q))} \to 0 \ \text{ as } k \to +\infty \ \text{ for } L_1 \text{ a. e. } \tau \in (0, 1).$$

Choosing $\tau \in (t, \gamma)$ and using Jensen's inequality and the Gauss-Green formula, from (4.23) we have

$$\lim_{k\to+\infty} \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x_0,0,\nabla v_k(x))\, dx = \lim_{k\to+\infty} \frac{1}{\mu_k} \int_{\tau Q} f^*(\mu_k \nabla w_k^*(x))\, dx$$

$$\geq \limsup_{k\to+\infty} \frac{1}{\mu_k} f^*\Big(\int_{\tau Q} \mu_k \nabla w_k^*(x)\, dx\Big)$$

$$= \limsup_{k\to+\infty} \frac{1}{\mu_k} f^*\Big(\int_{\partial(\tau Q)} \mu_k w_k^*(x)\otimes\nu(x)\, dH_{N-1}(x) \Big)$$

$$= \limsup_{k\to+\infty} \frac{1}{\mu_k} f^*\Big(\mu_k \int_{\partial(\tau Q)} \bar{u}_k(x)\otimes\nu(x)\, dH_{N-1}(x)\Big)$$

$$= \limsup_{k\to+\infty} \frac{1}{\mu_k} f^*(\mu_k D\bar{u}_k(\tau Q))$$

$$= \limsup_{k\to+\infty} \frac{1}{\mu_k} f^*(\mu_k \frac{Du\,(\tau Q_k)}{|Du|(\tau Q_k)})$$

$$\geq f^\infty(x_0,0,A_0) - C\omega(1 - t).$$

In the last step we used (4.9), (4.10) and the convexity of f. Taking (4.11) into account and letting $\varepsilon \to 0^+$, $t \to 1^-$ we conclude that

$$\eta(x_0) \geq f^\infty(x_0,0,A_0).$$

To make the above calculation rigorous one should mollify $\bar{u}_k$, but this process poses no additional difficulty. One thus obtains a proof of the lower bound needed in Ambrosio and Pallara [AP] without recourse to the sophisticated geometric measure theory tools used in [AG].

5. Relaxation.

Due to Theorem 2.18, the proof of Theorem 2.16 is complete once we show that

$$\mathcal{F}(u) \leq \int_\Omega f(x,u(x),\nabla u(x))\,dx + \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\,dH_{N-1}(x) + \int_\Omega f^\infty(x,u(x),dC(u)). \qquad (5.1)$$

The proof of (5.1) follows closely that of the analogous estimate in Ambrosio, Mortola and Tortorelli [AMT]. It is divided into four steps and we begin by considering

$$\mathcal{F}(u;A) := \inf_{\{u_n\}} \left\{ \liminf_{n \to +\infty} \int_A f(x,u_n(x),\nabla u_n(x))\,dx \mid u_n \in W^{1,1}(A;\mathbb{R}^p) \text{ and } u_n \to u \text{ in } L^1 \right\}$$

whenever $A \subset \Omega$ is an open set.

Step 1. We claim that

$\mathcal{F}(u;A)$ is a variational functional with respect to the L^1 topology and

$$\mathcal{F}(u;A) \leq C\,(\text{meas}(A) + |Du|(A)). \qquad (5.2)$$

We recall that $\mathcal{F}(u;A)$ is said to be a *variational functional with respect to the L^1 topology* if

(i) $\mathcal{F}(.;A)$ is local, i. e.

$$\mathcal{F}(u;A) = \mathcal{F}(v;A)$$

for every $u, v \in BV(A;\mathbb{R}^p)$ verifying $u = v$ a. e. in A;

(ii) $\mathcal{F}(.;A)$ is sequentially lower semicontinuous, i. e. if $u_n, u \in BV(A;\mathbb{R}^p)$ and $u_n \to u$ in $L^1(A;\mathbb{R}^p)$ then

$$\mathcal{F}(u;A) \leq \liminf_{n\to+\infty} \mathcal{F}(u_n;A).$$

(iii) $\mathcal{F}(.;A)$ is the trace on $\{A \subset \Omega \mid A \text{ is open}\}$ of a Borel measure on the set $\mathcal{B}(\Omega)$ of all Borel subsets of Ω.

De Giorgi and Letta [DGL] introduced the following criterion to assert (iii) :

A set function $\alpha : \{A \subset \Omega \mid A \text{ is open}\} \to [0, +\infty]$ is the trace of a Borel measure if

(a) $\alpha(B) \leq \alpha(A)$ for all $A, B \in X := \{U \subset \Omega \mid U \text{ is open}\}$ with $B \subset A$;

(b) $\alpha(A \cup B) \geq \alpha(A) + \alpha(B)$ for all $A, B \in X$ such that $A \cap B = \emptyset$;

(c) $\alpha(A \cup B) \le \alpha(A) + \alpha(B)$ for all $A, B \in X$;

(d) $\alpha(A) = \sup\{\alpha(B) \mid B \subset\subset A\}$ for all $A \in X$.

The argument used to show that $\mathcal{F}(u;A)$ is a variational functional with respect to the L^1 topology is exactly the same as in Theorem 4.3 of [AMT], where the only assumptions on f are continuity and the bounds

$$0 \le f(x,u,A) \le C\,(1 + \|A\|).$$

Also, due to (H3), given $u \in BV(A; \mathbb{R}^p)$ and considering a sequence of smooth functions u_n such that

$$u_n \to u \text{ in } L^1(A; \mathbb{R}^p) \text{ and } \int_A |\nabla u_n(x)|\,dx \to |Du|(A),$$

we conclude that

$$\mathcal{F}(u;A) \le \liminf_{n\to+\infty} \int_A f(x,u_n(x),\nabla u_n(x))\,dx$$

$$\le C\,(\text{meas}(A) + \liminf_{n\to+\infty} \int_A |\nabla u_n(x)|\,dx\,) = C\,(\text{meas}(A) + |Du|(A)).$$

Step 2. We claim that if $u \in BV(\Omega; \mathbb{R}^p) \cap L^\infty(\Omega; \mathbb{R}^p)$ then

$$\mathcal{F}(u; \Omega\setminus\Sigma(u)) \le \int_{\Omega\setminus\Sigma(u)} f(x,u(x),\nabla u(x))\,dx + \int_{\Omega\setminus\Sigma(u)} f^\infty(x,u(x),dC(u)). \tag{5.3}$$

By Step 1 $\mathcal{F}(u; .)$ is a Radon measure, absolutely continuous with respect to $\mathcal{L}_N + |Du|$. Thus (5.3) holds if and only if for $\mathcal{L}_N$ a. e. $x_0 \in \Omega$

$$\frac{d\mathcal{F}(u; .)}{d\mathcal{L}_N}(x_0) \le f(x_0,u(x_0),\nabla u(x_0)) \tag{5.4}$$

and for $|C(u)|$ a. e. $x_0 \in \Omega$

$$\frac{d\mathcal{F}(u; .)}{d|C(u)|}(x_0) \le f^\infty(x_0,u(x_0),A(x_0)). \tag{5.5}$$

We start by showing (5.4). Let $\{u_n\}$ be the regularized sequence defined in Lemma 2.5. Writing

$$Du = \nabla u\,dx + D_s u,$$

by Theorem 2.4, Theorem 2.7 and Theorem 2.8 for $\mathcal{L}_N$ a. e. $x_0 \in \Omega$ we have

$$\lim_{\varepsilon\to 0^+} \frac{1}{\text{meas}(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |u(x) - u(x_0)|\,(1+ |\nabla u(x)|)\,dx = 0, \tag{5.6}$$

$$\lim_{\varepsilon\to 0^+} \frac{|D_s u|(B(x_0,\varepsilon))}{\mathcal{L}_N(B(x_0,\varepsilon))} = 0,\ \lim_{\varepsilon\to 0^+} \frac{|Du|(B(x_0,\varepsilon))}{\mathcal{L}_N(B(x_0,\varepsilon))} \text{ exists and is finite,} \tag{5.7}$$

$$\frac{1}{\text{meas}(B(x_0,\varepsilon))} \int_\Omega f(x_0,u(x_0),\nabla u(x))\,dx \to f(x_0,u(x_0),\nabla u(x_0)) \tag{5.8}$$

and

$$\frac{d\mathcal{F}(u; .)}{d\mathcal{L}_N}(x_0) \text{ exists and is finite.}$$

Choose a sequence of numbers $\varepsilon \in (0, \text{dist}(x_0,\partial\Omega))$. Then

$$\frac{d\mathcal{F}(u; .)}{d\mathcal{L}_N}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u; B(x_0,\varepsilon))}{\mathcal{L}_N(B(x_0,\varepsilon))}$$

$$\leq \liminf_{\varepsilon \to 0^+} \liminf_{n \to +\infty} \frac{1}{\text{meas}(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} f(x,u_n(x),\nabla u_n(x))\, dx. \qquad (5.9)$$

Following [AMT], Proposition 4.6, we introduce the Yosida transforms of f, given by

$$f_\lambda(x,u) := \sup\{f(x',u',A) - \lambda[\,|x - x'| + |u - u'|]\,(1 + \|A\|) : \ (x',u') \in \Omega x\mathbb{R}^p\}$$

for every $\lambda > 0$. Then

(i)$f_\lambda(x,u,A) \geq f(x,u,A)$ and $f_\lambda(x,u,A)$ decreases to $f(x,u,A)$ as $\lambda \to +\infty$;

(ii) $f_\lambda(x,u,A) \geq f_\eta(x,u,A)$ if $\lambda \leq \eta$, for every $(x,u,A) \in \Omega x\mathbb{R}^p x M^{pxN}$;

(iii) $|f_\lambda(x,u,A) - f_\lambda(x',u',A)| \leq \lambda(|x - x'| + |u - u'|)\,(1 + \|A\|)$

for every (x,u,A), $(x',u',A) \in \Omega x\mathbb{R}^p x M^{pxN}$;

(iv) the approximation is uniform on compact sets. Precisely, let K be a compact subset of $\Omega x\mathbb{R}^p$ and let $\delta > 0$. There exists $\lambda > 0$ such that

$$f(x,u,A) \leq f_\lambda(x,u,A) \leq f(x,u,A) + \delta\,(1 + \|A\|)$$

for every $(x,u,A) \in K x M^{pxN}$.

Fix $\delta > 0$ and let

$$K := \overline{B}(x_0,\frac{\text{dist}(x_0,\partial\Omega)}{2}) \times \overline{B}(0, \|u\|_\infty).$$

By (i), (ii) and (iv)

$$f(x,u_n(x),\nabla u_n(x)) \leq f_\lambda(x,u_n(x),\nabla u_n(x))$$

$$\leq f_\lambda(x_0,u(x_0),\nabla u_n(x)) + \lambda(\,|x - x_0| + |u_n(x) - u(x_0)|)\,(1 + \|\nabla u_n(x)\|)$$

$$\leq f(x_0,u(x_0),\nabla u_n(x)) + \delta(1 + \|\nabla u_n(x)\|) + \lambda(\,|x - x_0| + |u_n(x) - u(x_0)|)\,(1 + \|\nabla u_n(x)\|). \qquad (5.10)$$

Taking into account that $\nabla u_n = \rho_n * \nabla u + \rho_n * D_s u$ and that $f(x_0,u(x_0), .)$ is a Lipschitz function, by (H3), Lemma 2.5 and (5.9) we have

$$\frac{d\mathcal{F}(u;\,.)}{d\mathcal{L}_N}(x_0) \le \liminf_{\varepsilon\to 0^+} \liminf_{n\to+\infty} \frac{1}{meas(B(x_0,\varepsilon))}\Big[\int_{B(x_0,\varepsilon)} f(x_0,u(x_0),(\rho_n*\nabla u)(x))\,dx +$$
$$+ C\ |D_s u|(B(x_0,\varepsilon+1/n)) + (\lambda\varepsilon+\delta)\ meas\,(B(x_0,\varepsilon)) + (\lambda\varepsilon+\delta)\ |Du|(B(x_0,\varepsilon+1/n))$$
$$+\lambda \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|\,(1 + |\nabla u_n(x)|)\,dx\ \Big].$$

Since

$$\lim_{n\to+\infty} \int_{B(x_0,\varepsilon)} f(x_0,u(x_0),(\rho_n*\nabla u)(x))\,dx = \int_{B(x_0,\varepsilon)} f(x_0,u(x_0),\nabla u(x))\,dx,$$
$$|Du|\,(B(x_0,\varepsilon+1/n)) \to |Du|\,(\overline{B}(x_0,\varepsilon)) = |Du|\,(B(x_0,\varepsilon))$$

for a. e. ε, invoking (5.7) and (5.8) one deduces

$$\frac{d\mathcal{F}(u;\,.)}{d\mathcal{L}_N}(x_0) \le f(x_0,u(x_0),\nabla u(x_0)) + C\delta$$
$$+\ \lambda\ \liminf\,ess_{\,\varepsilon\to 0^+} \liminf_{n\to+\infty} \frac{1}{meas(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|\,(1 + |\nabla u_n(x)|)\,dx. \quad (5.11)$$

To prove (5.4) it remains to show that the last term converges to zero. By (5.6)

$$\lim_{\varepsilon\to 0^+} \lim_{n\to+\infty} \frac{1}{meas(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|\,dx =$$
$$= \lim_{\varepsilon\to 0^+} \frac{1}{meas(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |u(x) - u(x_0)| = 0$$

and by Lemma 2.5 and the dominated convergence theorem (with respect to the measure $|Du|$)

$$\limsup_{n\to+\infty} \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|\,|\nabla u_n(x)|\,dx \le$$

$$\le \limsup_{n\to+\infty} \Big[\int_{B(x_0,\varepsilon)} |u_n(x) - u(x)|\,|\nabla u_n(x)|\,dx + \int_{B(x_0,\varepsilon)} |u(x) - u(x_0)|\,|\nabla u_n(x)|\,dx \Big]$$

$$\le \limsup_{n\to+\infty} \Big[\int_{B(x_0,\varepsilon+1/n)} (|u_n - u|*\rho_n)(x)\,|Du|(x) + \int_{B(x_0,\varepsilon+1/n)} (|u - u(x_0)|*\rho_n)(x)\,|Du|(x) \Big]$$

$$\le \limsup_{n\to+\infty} \Big[\int_{B(x_0,\varepsilon+1/n)\setminus\Sigma(u)} (|u_n - u|*\rho_n)(x)\,|Du|(x) + \int_{B(x_0,\varepsilon+1/n)\setminus\Sigma(u)} (|u - u(x_0)|*\rho_n)(x)\,|Du|(x)$$
$$+ 4\|u\|_\infty\,|Du|(B(x_0,\varepsilon+1/n)\cap\Sigma(u)) \Big]$$

$$\le \int_{\overline{B}(x_0,\varepsilon)\setminus\Sigma(u)} |u(x) - u(x_0)|\,|Du|(x) + 4\ \|u\|_\infty |Du|\ (\overline{B}(x_0,\varepsilon)\cap\Sigma(u))$$

$$\leq \int_{\overline{B}(x_0,\varepsilon)\setminus\Sigma(u)} |u(x) - u(x_0)|\, |Du|(x) + \quad 4\ \|u\|_\infty\, |D_s u|(B(x_0,\varepsilon)). \tag{5.12}$$

Taking into account that $|Du|(\partial B(x_0,\varepsilon)) = 0$ for a. e. ε and that

$$\int_{B(x_0,\varepsilon)} |u(x) - u(x_0)|\, |Du|(x) \leq \int_{B(x_0,\varepsilon)} |u(x) - u(x_0)|\, |\nabla u(x)|\, dx + 2\ \|u\|_\infty\, |D_s u|(B(x_0,\varepsilon)),$$

we obtain from (5.6) and (5.7) that

$$\text{limsup ess}_{\varepsilon\to 0^+}\ \text{limsup}_{n\to+\infty} \frac{1}{\text{meas}(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|\, |\nabla u_n(x)|\, dx \ = 0$$

and (5.4) follows from (5.11).

Next we prove (5.5), where using Radon-Nikodym Theorem we write

$|Du| = |C(u)| + \mu$, where μ and $|C(u)|$ are mutually singular Radon measures.

By Lemma 2.5 (iii)

$$\rho_n * u\ (x) \to u(x) \quad \text{for } |C(u)| \text{ a. e. } x \in \Omega$$

hence u is $|C(u)|$ measurable and by Theorem 2.7, Theorem 2.8 and Theorem 2.11, for $|C(u)|$ a. e. $x_0 \in \Omega$ we have

$$\lim_{\varepsilon\to 0^+} \frac{\mu(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))} = 0,\ \lim_{\varepsilon\to 0^+} \frac{|Du|(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))} \text{ exists and is finite,} \tag{5.13}$$

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon^N}{|C(u)|(B(x_0,\varepsilon))} = 0, \tag{5.14}$$

$$\frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |u(x) - u(x_0)|\, |C(u)|(x) \to 0, \tag{5.15}$$

$$A(x) := \lim_{\varepsilon\to 0^+} \frac{C(u)(B(x,\varepsilon))}{|C(u)|(B(x,\varepsilon)\ )} \text{ exists and is a rank-one matrix of norm one,} \tag{5.16}$$

$$\text{liminf}_{\varepsilon\to 0^+} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} f^\infty(x_0,u(x_0),A(x))\, d|C(u)| = f^\infty(x_0,u(x_0),A(x_0)), \tag{5.17}$$

and

$$\frac{d\mathcal{F}(u;\, .)}{d\ |C(u)|}(x_0) \text{ exists and is finite.}$$

As before, using (5.10) and (5.12) one sees that

$$\frac{d\mathcal{F}(u;\, .)}{d\ |C(u)|}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mathcal{F}(u;\, B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))}$$

$$\leq \liminf_{\varepsilon \to 0^+} \liminf_{n \to +\infty} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} f(x,u_n(x),\nabla u_n(x))\, dx$$

$$\leq \liminf_{\varepsilon \to 0^+} \liminf_{n \to +} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \Big[\int_{B(x_0,\varepsilon)} f(x_0,u(x_0),\nabla u_n(x))\, dx \, +$$

$$+ (\delta + \lambda\varepsilon)) \int_{B(x_0,\varepsilon)} |\nabla u_n(x)|\, dx + (\delta + \lambda\varepsilon)\, \mathrm{meas}(B(x_0,\varepsilon)) + \lambda \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|\, (1 + |\nabla u_n(x)|) dx \Big]$$

$$\leq \liminf_{\varepsilon \to 0^+} \liminf_{n \to +\infty} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} f(x_0,u(x_0),\nabla u_n(x))\, dx \, +$$

$$+ \limsup_{\varepsilon \to 0} \frac{1}{|C(u)|(B(x_0,\varepsilon))} (\delta + \lambda\varepsilon) \left[|Du|(B(x_0,\varepsilon)) + \mathrm{meas}\,(B(x_0,\varepsilon))\right]$$

$$+ \lambda \limsup_{\varepsilon \to 0^+} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \Big[\int_{\overline{B}(x_0,\varepsilon)\setminus\Sigma(u)} |u(x) - u(x_0)|\, |Du|(x) +$$

$$+ \int_{B(x_0,\varepsilon)} |u(x) - u(x_0)|\, dx + 4\, \|u\|_\infty |Du|\, (\overline{B}(x_0,\varepsilon) \cap \Sigma(u)).\Big].$$

By (5.13) - (5.15) and, due to the rectifiability of the jump set, as $C(B(x_0,\varepsilon)\cap\Sigma(u)) = 0$ we conclude that

$$\frac{d\mathcal{F}(u;\,.)}{d\,|C(u)|}(x_0) \leq \liminf_{\varepsilon \to 0^+} \liminf_{n \to +\infty} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \Big[\int_{B(x_0,\varepsilon)} f(x_0,u(x_0),\nabla u_n(x))\, dx \, +$$

$$+ \lambda \int_{B(x_0,\varepsilon)} |u(x) - u(x_0)|\, |C(u)|(x) + 2\lambda\, \|u\|_\infty\, \mu(B(x_0,\varepsilon)))$$

$$+ 4\lambda\, \|u\|_\infty\, |Du|(B(x_0,\varepsilon) \cap \Sigma(u)) \Big] + C\delta$$

$$\leq \liminf_{\varepsilon \to 0^+} \liminf_{n \to +\infty} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} f(x_0,u(x_0),\nabla u_n(x))\, dx + C\delta. \qquad (5.18)$$

Now we use Ambrosio and DalMaso's argument in [ADM], Proposition 4.2. Define

$$g(A) := \sup_{t>0} \frac{f(x_0,u(x_0),tA) - f(x_0,u(x_0),0)}{t}.$$

Then g is Lipschitz continuous, positively homogeneous of degree one and the rank-one convexity of $f(x_0,u(x_0),.)$ implies that

$$g(A) = f^\infty(x_0,u(x_0),A) \text{ whenever rank } A \le 1.$$

Thus, by (5.18), (5.14) and Lemma 2.5 we have

$$\frac{d\mathcal{F}(u;.)}{d\,|C(u)|}(x_0) \le \liminf_{\varepsilon\to 0^+} \liminf_{n\to+\infty} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} [f(x_0,u(x_0),0)+g(\nabla u_n(x))]dx + C\delta$$

$$= \liminf_{\varepsilon\to 0^+} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} g(Du) + C\delta$$

$$= \liminf_{\varepsilon\to 0^+} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} [g(A(x))\, d|C(u)| + g(d\mu)] + C\delta$$

and so, by (5.13), (5.16), (5.17) and Alberti's Theorem 2.11 we conclude that

$$\frac{d\mathcal{F}(u;.)}{d|C(u)|}(x_0) \le \liminf_{\varepsilon\to 0^+} \frac{1}{|C(u)|(B(x_0,\varepsilon))}\Big[\int_{B(x_0,\varepsilon)} f^\infty(x_0,u(x_0),A(x))\, d|C(u)| + C\mu(B(x_0,\varepsilon))\Big] + C\delta$$

$$= f^\infty(x_0,u(x_0),A(x_0)) + C\delta.$$

It suffices to let $\delta \to 0^+$.

Step 3. We show that

$$\mathcal{F}(u;\Sigma(u)) \le \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\, dH_{N-1}(x) \qquad (5.19)$$

for every $u \in BV(\Omega; \mathbb{R}^p) \cap L^\infty(\Omega; \mathbb{R}^p)$. The proof is divided into three parts :

1. $u(x) = a\chi_E(x) + b(1 - \chi_E(x))$ with $Per_\Omega(E) < +\infty$;
2. $u(x) = \sum a_i\chi_{E_i}(x)$ where $\{E_i\}_{i=1}^{+\infty}$ forms a partition of Ω into sets of finite perimeter ;
3. General case, $u \in BV(\Omega; \mathbb{R}^p) \cap L^\infty(\Omega; \mathbb{R}^p)$.

1. Let $u(x) = a\chi_E(x) + b(1 - \chi_E(x))$, $Per_\Omega(E) < +\infty$. We start by proving that for every open set $A \subset \Omega$

$$\mathcal{F}(u;A) \le \int_A f(x,u(x),0)\, dx + \int_{\Sigma(u)\cap A} K(x,a,b,\nu(x))\, dH_{N-1}(x). \qquad (5.20)$$

a) Suppose first that

$$u(x) = \begin{cases} b & \text{if } x.\nu > 0 \\ a & \text{if } x.\nu < 0 \end{cases}.$$

In Fonseca and Rybka [FR] (Proposition 4.1 and Lemma 4.2) it was shown that if $A = \alpha + \lambda Q_\nu$ is an open cube with two faces orthogonal to ν then there exists a sequence $u_n \in W^{1,1}(A; \mathbb{R}^p)$ such that

$$\int_A f(x,u_n(x),\nabla u_n(x))\, dx \to \int_A f(x,u(x),0)\, dx + \int_{\Sigma(u)\cap A} K(x,a,b,\nu(x))\, dH_{N-1}(x)$$

and so (5.20) holds.

b) Consider u as in a) and let $A \subset \Omega$ be an arbitrary open set in $\mathbb{R}^N$. Let π be the plane

$$\pi := \{x \in \mathbb{R}^N \mid x.\nu = 0\}.$$

It is clear that

$$A = \bigcup_{n=1}^{+\infty} (\cup A_n)$$

where A_n is an increasing finite collection of non-overlapping (i. e. with disjoint interiors) cubes $\overline{Q}$ of the form $a_i + \varepsilon\overline{Q}_\nu$ with edge length bigger than or equal to 1/n and such that

$$H_{N-1}(\partial Q \cap \pi) = 0. \tag{5.21}$$

Thus, by Step 1 (iii) and applying a) to a decreasing sequence of open cubes whose intersection is the closed cube $\overline{Q}$ one has

$$\mathcal{F}(u; \overline{Q}) \leq \int_{\overline{Q}} f(x,u(x),0)\, dx + \int_{\Sigma(u)\cap\overline{Q}} K(x,a,b,\nu(x))\, dH_{N-1}(x)$$

and so

$$\mathcal{F}(u;A) \leq \lim_{n \to +\infty} \mathcal{F}(u; \cup A_n)$$

$$\leq \lim_{n \to +\infty} \sum_{\overline{Q} \in A_n} \mathcal{F}(u; \overline{Q})$$

$$\leq \liminf_{n \to +\infty} \sum_{\overline{Q} \in A_n} \left[\int_{\overline{Q}} f(x,u(x),0)\, dx + \int_{\Sigma(u)\cap\overline{Q}} K(x,a,b,\nu(x))\, dH_{N-1}(x) \right].$$

By (5.21) and Lebesgue's Monotone Convergence Theorem we conclude that

$$\mathcal{F}(u;A) \leq \liminf_{n \to +\infty} \left[\int_{\cup A_n} f(x,u(x),0)\, dx + \int_{\Sigma(u)\cap(\cup A_n)} K(x,a,b,\nu(x))\, dH_{N-1}(x) \right]$$

$$= \int_A f(x,u(x),0)\, dx + \int_{\Sigma(u)\cap A} K(x,a,b,\nu(x))\, dH_{N-1}(x).$$

c) Now suppose that u has polygonal interface i.e. $u = \chi_E a + (1-\chi_E)b$ where E is a polyhedral set i.e. E is a bounded, strongly Lipschitz domain and $\partial E = H_1 \cup \ldots \cup H_M$, H_i are closed subsets of hyperplanes of the type $\{x \in \mathbb{R}^N : x.\nu_i = \alpha_i\}$. Let A be an open set contained in Ω and let $I = \{i \in \{1,\ldots,M\} \mid H_{N-1}(H_i \cap A) > 0\}$. If $A \cap \Sigma(u) = \emptyset$, i. e. if card I = 0 then $u \in W^{1,1}(A; \mathbb{R}^p)$ and it suffices to consider $u_n = u \in W^{1,1}(A; \mathbb{R}^p)$, with (5.20) reducing to

$$\mathcal{F}(u;A) \leq \int_A f(x,u(x),0)\, dx.$$

The case card I = 1 was studied in part b) where E is a large cube so that $\Sigma(u)\cap\Omega$ reduces to the flat interface $\{x \in \Omega \mid x.\nu = 0\}$. Using an induction procedure, assume that (5.20) is true if card I = k, $k \leq M - 1$ and we prove it is still true if card I = k. Assume that

$$\partial E \cap A = (H_1 \cap \Omega) \cup \ldots \cup (H_k \cap \Omega).$$

and consider $S := \{x \in \mathbb{R}^N : \text{dist}(x, H_1) = \text{dist}(x, H_2 \cup ... \cup H_M)\}$. Note that $H_{N-1}(S \cap \Sigma(u)) = 0$ because $H_{N-1}(H_i \cap H_j) = 0$ for $i \neq j$. Fix $\delta > 0$ and let

$$U_\delta = \{x \in \mathbb{R}^N : \text{dist}(x,S) < \delta\},$$

$$U_\delta^- = \{x \in \mathbb{R}^N : \text{dist}(x,S) < \delta, \text{dist}(x,H_1) < \text{dist}(x,H_2 \cup ... \cup H_k)\},$$

$$U_\delta^+ = \{x \in \mathbb{R}^N : \text{dist}(x,S) < \delta, \text{dist}(x,H_1) > \text{dist}(x,H_2 \cup ... \cup H_k)\}.$$

Let

$$A_1 = \{x \in A : \text{dist}(x,H_1) < \text{dist}(x,H_2 \cup ... \cup H_M)\}.$$

Clearly A_1 is open and $A_1 \cap (H_2 \cup ... \cup H_k) = \emptyset$. We apply the induction hypothesis to A_1 and to $A \setminus \overline{A}_1 := A_2$ to obtain sequences $u_n \in W^{1,1}(A_1; \mathbb{R}^p)$, $v_n \in W^{1,1}(A_2; \mathbb{R}^p)$ such that

$$u_n \to u \text{ in } L^1(A_1; \mathbb{R}^p), \quad v_n \to u \text{ in } L^1(A_2; \mathbb{R}^p)$$

and

$$\lim_{n\to+\infty} \int_{A_1} f(x,u_n(x),\nabla u_n(x))\, dx \leq \int_{A_1} f(x,u(x),0)\, dx + \int_{\Sigma(u)\cap A_1} K(x,a,b,\nu(x))\, dH_{N-1}(x) + \delta/2,$$

$$\lim_{n\to+\infty} \int_{A_2} f(x,v_n(x),\nabla v_n(x))\, dx \leq \int_{A_2} f(x,u(x),0)\, dx + \int_{\Sigma(u)\cap A_2} K(x,a,b,\nu(x))\, dH_{N-1}(x) + \delta/2.$$

As in Lemma 3.1 we will use the slicing method to connect u_n to v_n. Let ρ be a mollifier, $\rho_n(x) := n^N\rho(nx)$ and define

$$w_n(x) := (\rho_n * u)(x) = \int_{B(x,1/n)} \rho_n(x-y)\, u(y)\, dy.$$

As $\rho \geq 0$, $\text{supp}\, \rho = \overline{B}(0, 1)$ and

$$\int_{B(0,1)} \rho(x)\, dx = 1,$$

we have

$$\|\nabla w_n\|_\infty \leq Cn, \quad \text{supp}\, \nabla w_n \subset \{x \in \mathbb{R}^N \mid \text{dist}(x,\Sigma(u)) \leq 1/n\}. \tag{5.23}$$

Let

$$\alpha_n := \sqrt{\|w_n - v_n\|_{L^1(A_1)}}, \quad k_n := n\,[1 + \|w_n\|_{1,1} + \|v_n\|_{1,1}], \quad s_n := \frac{\alpha_n}{k_n}$$

where $[k]$ denotes the largest integer less than or equal to k, set

$$U_i^- := U_{\delta_i}^-, \text{ where } \delta_i = (1 - \alpha_n + i\, s_n)\, U_\delta^-, \; i = 1, ..., k_n,$$

and consider a family of cut-off functions

$$\varphi_i \in W_0^{1,\infty}(U_i^-), \; 0 \leq \varphi_i \leq 1, \; \varphi_i = 1 \text{ in } U_{i-1}^-, \; \|\nabla\varphi_i\|_\infty = O(\tfrac{1}{s_n})$$

for $i = 1, ..., k_n$. Define

$$u_n^{(i)}(x) := (1 - \varphi_i(x))w_n(x) + \varphi_i(x)u_n(x), \; x \in A_1.$$

Then

$$u_n^{(i)} = w_n \text{ on } \partial A_1 \cap S,$$
$$\nabla u_n^{(i)} = \nabla u_n \text{ in } U_{i-1}^-, \quad \nabla u_n^{(i)} = \nabla w_n \quad \text{in } A_1 \setminus U_i^-$$

and in $U_i^- \setminus U_{i-1}^-$

$$\nabla u_n^{(i)} = \nabla w_n + \varphi_i(\nabla u_n - \nabla w_n) + (u_n - w_n) \otimes \nabla \varphi_i.$$

Due to the growth condition (H3) on f we deduce that

$$\int_{A_1} f(x,u_n^{(i)}(x),\nabla u_n^{(i)}(x))\, dx \le \int_{A_1} f(x,u_n(x),\nabla u_n(x))\, dx +$$

$$C \int_{U_i^- \setminus U_{i-1}^-} (1+ |w_n(x) - u_n(x)|\frac{1}{s_n} + \|\nabla w_n(x)\| + \|\nabla u_n(x)\|)\, dx + C \int_{A_1 \setminus U_i^-} (1+ \|\nabla w_n(x)\|)\, dx$$

and averaging this inequality among all the layers $U_i^- \setminus U_{i-1}^-$ and by (5.23) we obtain

$$\frac{1}{k_n}\sum_{i=1}^{k_n} \int_{A_1} f(x, u_n^{(i)}(x), \nabla u_n^{(i)}(x))\, dx \le \int_{A_1} f(x, u_n(x), \nabla u_n(x))\, dx,$$
$$+ \frac{C}{k_n} \int_Q (1+ \|\nabla w_n(x)\| + \|\nabla v_n(x)\|)\, dx + \frac{C}{k_n} \int_Q |w_n(x) - v_n(x)|\frac{1}{s_n}\, dx$$
$$+ C\,(1 + n) \text{ meas } \{x \in U_\delta^- \cap A_1 |\, \text{dist}(x,\Sigma(u)) \le 1/n\}.$$

Thus, there must exist an index $i(n) \in \{1, ..., k_n\}$ for which

$$\overline{u}_n := u_n^{(i(n))} \to u \text{ in } L^1(A_1; \mathbb{R}^p),$$

and taking into account that $\Sigma(u)$ is a union of finitely many closed subsets of hyperplanes

$$\limsup_{n\to\infty} \int_{A_1} f(x,\overline{u}_n(x),\nabla \overline{u}_n(x))\, dx \le$$

$$\int_{A_1} f(x,u(x),0)\, dx + \int_{\Sigma(u)\cap A_1} K(x,a,b,\nu(x))\, dH_{N-1}(x) + \delta/2 + C H_{N-1}\,(U_\delta^- \cap A_1 \cap \Sigma(u)).$$

Similarly, we may construct a sequence $\overline{v}_n$ such that

$$\overline{v}_n = w_n \text{ on } \partial A_2 \cap S, \qquad \overline{v}_n \to u \text{ in } L^1(A_2; \mathbb{R}^p),$$

$$\limsup_{n\to\infty} \int_{A_2} f(x,\overline{v}_n(x),\nabla \overline{v}_n(x))\, dx \le$$

$$\int_{A_2} f(x,u(x),0)\, dx + \int_{\Sigma(u)\cap A_2} K(x,a,b,\nu(x))\, dH_{N-1}(x) + \delta/2 + C H_{N-1}\,(U_\delta^- \cap A_2 \cap \Sigma(u)).$$

We set

$$\xi_n(x) := \chi_{A_1}(x)\, \overline{u}_n(x) + \chi_{A_2}(x)\, \overline{v}_n(x).$$

Clearly $\xi_n \in W^{1,1}(A;\mathbb{R}^p)$, $\xi_n \to u$ in $L^1(A;\mathbb{R}^p)$ and so

$$\mathcal{F}(u;A) \le \liminf_{n\to+\infty} \int_A f(x,\xi_n(x),\nabla\xi_n(x))\,dx$$

$$\le \limsup_{n\to\infty} \int_{A_1} f(x,\bar{u}_n(x),\nabla\bar{u}_n(x))\,dx + \limsup_{n\to\infty} \int_{A_2} f(x,\bar{v}_n(x),\nabla\bar{v}_n(x))\,dx$$

$$\le \int_A f(x,u(x),0)\,dx + \int_{\Sigma(u)\cap A} K(x,a,b,\nu(x))\,dH_{N-1}(x) + \delta + H_{N-1}(U_\delta\cap A\cap\Sigma(u)).$$

As $H_{N-1}(S\cap\Sigma(u)) = 0$, letting $\delta \to 0$ we obtain (5.20)

f) Finally, if E is an arbitrary set of finite perimeter in Ω, by De Giorgi's approximating lemma there exists a sequence of polyhedral sets E_n such that

$$\mathrm{meas}(E_n\Delta E) \to 0,\ \ \mathrm{Per}_\Omega(E_n) \to \mathrm{Per}_\Omega(E).$$

On the other hand, by Lemma 2.15 a), b), there exists a sequence of continuous functions $g_m : \Omega\times\mathbb{R}^N \to [0,+\infty)$ such that

$$K(x,a,b,y) \le g_m(x,y) \le C|y| \ \text{ for all } (x,y) \in \Omega\times\mathbb{R}^N$$

and

$$K(x,a,b,y) = \inf_m g_m(x,y),$$

where we extended K(x,a,b,.) as a homogeneous of degree one function. Setting

$$u_n(x) := a\chi_{E_n}(x) + b(1 - \chi_{E_n}(x)),$$

by Step 1, (i), (iii)

$$\mathcal{F}(u;A) \le \liminf_{n\to+\infty} \mathcal{F}(u_n;A)$$

$$\le \liminf_{n\to+\infty} \Big[\int_A f(x,u_n(x),0)\,dx + \int_{\Sigma(u_n)\cap A} K(x,a,b,\nu(x))\,dH_{N-1}(x) \Big]$$

$$= \int_A f(x,u(x),0)\,dx + \lim_{n\to+\infty} \int_{\Sigma(u_n)\cap A} g_m(x,\nu(x))\,dH_{N-1}(x)$$

$$= \int_A f(x,u(x),0)\,dx + \int_{\Sigma(u)\cap A} g_m(x,\nu(x))\,dH_{N-1}(x).$$

Letting $m \to +\infty$ and using Lebesgue's Monotone Convergence Theorem we obtain (5.20). This inequality together with Step 1, (iii) yields

$$\mathcal{F}(u;\Sigma(u)) \le \inf\{\mathcal{F}(u;A) \mid A \subset \Omega, A \text{ is open}, \Sigma(u)\subset A\}$$

$$\le \inf\Big\{ \int_A f(x,u(x),0)\,dx + \int_{\Sigma(u)\cap A} K(x,a,b,\nu(x))\,dH_{N-1}(x) \mid A \subset \Omega, A \text{ is open } \Sigma(u)\subset A\Big\}$$

$$= \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\,dH_{N-1}(x)$$

and we conclude (5.19). The cases 2 and 3 are now obtained as in [AMT] Proposition 4.8, Steps 1 and 2, respectively, where the upper semicontinuity of K is needed (see Lemma 2.15).

Step 4. By Step 2 and Theorem 2.18 and as $\mathcal{F}(u;.)$ is a variational functional we have

$$\mathcal{F}(u;\Omega) = \int_{\Omega} f(x,u(x),\nabla u(x))\,dx + \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\,dH_{N-1}(x) + $$
$$+ \int_{\Omega} f^{\infty}(x,u(x),dC(u))$$

for every $u \in BV(\Omega;\mathbb{R}^p) \cap L^{\infty}(\Omega;\mathbb{R}^p)$. In order to extend this result to $BV(\Omega;\mathbb{R}^p)$, we use the same argument exploited in [AMT], Theorem 4.9. Let $\phi_n \in C_0^1(\mathbb{R}^p;\mathbb{R}^p)$ be such that

$$\phi_n(y) = y \quad \text{if } y \in B(0,n),\ \|\nabla\phi_n\|_{\infty} \le 1,$$

and fix $u \in BV(\Omega;\mathbb{R}^p)$. Because it is important to have $\|\nabla\phi_n\|_{\infty}$ bounded above by one and not just by an arbitrary constant C, we specify the norm we are using for matrices, namely

$$\|A\| = \sup\{|Ax| : |x| \le 1\}.$$

Then

$$\phi_n(u) \in BV(\Omega;\mathbb{R}^p) \cap L^{\infty}(\Omega;\mathbb{R}^p),$$
$$\Sigma(\phi_n(u)) \subset \Sigma(u),$$
$$(\phi_n(u)^-, \phi_n(u)^+, \nu_{\phi_n(u)}) = (\phi_n(u^-), \phi_n(u^+), \nu_{(u)}) \quad \text{if } x \in \Sigma(\phi_n(u))$$

and

$$\int_B |D\phi_n(u)| \le \int_B |D(u)| \quad \text{for every Borel set } B \subset \Omega. \tag{5.24}$$

As $\mathcal{F}(.;\Omega)$ is a variational functional we conclude that

$$\mathcal{F}(u;\Omega) \le \liminf_{n\to\infty} \mathcal{F}(\phi_n(u);\Omega)$$
$$= \liminf_{n\to\infty} \Big[\int_{\Omega} f(x,\phi_n(u),\nabla(\phi_n(u))(x))\,dx + \int_{\Sigma(\phi_n(u))} K(x,\phi_n(u)^-,\phi_n(u)^+,\nu_{\phi_n(u)})\,dH_{N-1}(x) + $$
$$+ \int_{\Omega} f^{\infty}(x,\phi_n(u),dC(\phi_n(u))).$$

By Lemma 2.15 (c), (d) K is upper-semicontinuous and

$$K(x,\phi_n(u^-)(x), \phi_n(u^+)(x), \nu(x)) \le C\,|u^-(x) - u^+(x)|$$

and so, by Fatou's Lemma we obtain

$$\limsup_{n\to\infty} \int_{\Sigma(\phi_n(u))} K(x,\phi_n(u)^-,\phi_n(u)^+,\nu_{\phi_n(u)})\,dH_{N-1}(x) \le \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\,dH_{N-1}(x).$$

On the other hand, setting

$$\Omega_n := \{x \in \Omega \setminus \Sigma(u) : |u(x)| \leq n\}$$

we have

$$\limsup_{n\to\infty} \int_\Omega f(x,\phi_n(u),\nabla(\phi_n(u))(x))\,dx =$$

$$= \limsup_{n\to\infty} \Big[\int_{\Omega_n\setminus\Sigma(u)} f(x,\phi_n(u),\nabla(\phi_n(u))(x))\,dx + \int_{(\Omega\setminus\Omega_n)\setminus\Sigma(u)} f(x,\phi_n(u),\nabla(\phi_n(u))(x))\,dx \Big]$$

$$\leq \int_\Omega f(x,u(x),\nabla u(x))\,dx + C \limsup_{n\to\infty} [\mathrm{meas}(\Omega\setminus\Omega_n) + |D(\phi_n(u))|((\Omega\setminus\Omega_n)\setminus\Sigma(u))].$$

By (5.24) we deduce that

$$\limsup_{n\to\infty} |D(\phi_n(u))|((\Omega\setminus\Omega_n)\setminus\Sigma(u)) \leq \limsup_{n\to\infty} |D(u)|(\Omega\setminus(\Omega_n\cup\Sigma(u))) = 0$$

and so

$$\limsup_{n\to\infty} \int_\Omega f(x,\phi_n(u),\nabla(\phi_n(u))(x))\,dx \leq \int_\Omega f(x,u(x),\nabla u(x))\,dx.$$

In a similar way

$$\limsup_{n\to\infty} \int_\Omega f^\infty(x,\phi_n(u),dC(\phi_n(u)))$$

$$= \limsup_{n\to\infty} \Big[\int_{\Omega_n\setminus\Sigma(u)} f^\infty(x,\phi_n(u),dC(\phi_n(u))) + \int_{(\Omega\setminus\Omega_n)\setminus\Sigma(u)} f^\infty(x,\phi_n(u),dC(\phi_n(u))) \Big]$$

$$\leq \int_\Omega f(x,u(x),dC(u)) + C \limsup_{n\to\infty} [\mathrm{meas}(\Omega\setminus\Omega_n) + |D(\phi_n(u))|((\Omega\setminus\Omega_n)\setminus\Sigma(u))]$$

$$= \int_\Omega f(x,u(x),dC(u)).$$

■

Appendix.

In this appendix we prove Lemmas 2.12, 2.13 as well the following proposition justifying (4.24).

Proposition A.1. Let $\{\mu_k\}$ be a sequence of $\mathbb{R}^p$-valued Radon measures on Ω such that

$$|\mu_k|(\Omega) \to 1, \text{ and } \mu_k(\Omega) \to a \text{ where } |a| = 1.$$

Then

$$|\mu_k - (\mu_k.a)\,a|(\Omega) \to 0.$$

Proof. By the Radon-Nikodym Theorem we may write

$$\mu_k = \lambda_k \, |\mu_k|$$

where $\lambda_k : \Omega \to \mathbb{R}^p$ is $|\mu_k|$-measurable and $|\lambda_k(x)| = 1$ for $|\mu_k|$ a. e. $x \in \Omega$. Define

$$\alpha_K := \lambda_k - (\lambda_k .a)\, a.$$

Then

$$|\alpha_k|^2 = 1 - (\lambda_k .a)^2$$
$$\leq 2(1 - \lambda_k .a). \qquad \text{(A.1)}$$

On the other hand

$$1 = \lim_{k\to+\infty} \mu_k(\Omega).a$$
$$= \lim_{k\to+\infty} \int_\Omega (\lambda_k.a)\, d|\mu_k|$$

and so

$$\int_\Omega [1 - (\lambda_k.a)]\, d|\mu_k| \to 0. \qquad \text{(A.2)}$$

By (A.1) we have

$$|\mu_k - (\mu_k.a)\, a|(\Omega) = \int_\Omega |\alpha_k(x)|\, d|\mu_k|$$
$$\leq C \Big(\int_\Omega |\alpha_k(x)|^2\, d|\mu_k| \Big)^{1/2}$$
$$\leq C \Big(\int_\Omega [1 - (\lambda_k.a)]\, d|\mu_k| \Big)^{1/2}$$

and the result now follows from (A.2). ■

We recall that if $w \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R})$ and $g \in L^1(\mathbb{R}^N; \mathbb{R})$ then the *change of variables formula* (or *coarea formula*) holds, namely

$$\int_{\mathbb{R}^N} g(x)\, |\nabla w(x)|\, dx = \int_{-\infty}^{+\infty} \Big(\int_{w^{-1}(t)} g(x)\, dH_{N-1}(x) \Big)\, dt. \qquad \text{(A.3)}$$

For details see Evans and Gariepy [EG] and Ziemer [Zi].

Lemma 2.12. Let $K \subset \mathbb{R}^N$ be a compact set, let $v \in W^{1,\infty}(K)$ and let $A \subset K$ be a measurable set. Then

$$\operatorname{ess\,inf}_{t \in [a,b]} t\, H_{N-1}(\{x \in A \mid v(x) = t\}) \leq \frac{1}{\ln(b/a)} \int_{A \cap \{a \leq v \leq b\}} |\nabla v(x)|\, dx.$$

Proof. Let $\bar{v}$ be a Lipschitz extension of v to $\mathbb{R}^N$ with compact support. Applying the co-area formula (A.3) to

$$w(x) := \bar{v}(x) \text{ and } g(x) := \chi_A(x)\, \chi_{[a,b]}(\bar{v}(x))$$

we have

$$\int_{A\cap\{a\le v\le b\}} |\nabla v(x)|\, dx = \int_{-\infty}^{+\infty} \Big(\int_{\bar{v}^{-1}(t)} g(x)\, dH_{N-1}(x) \Big)\, dt$$

$$= \int_a^b H_{N-1}(\{x \in A \mid \bar{v}(x) = t\})\, dt.$$

As v and $\bar{v}$ agree on A, we can replace $\bar{v}$ by v and defining

$$\alpha := \operatorname{ess\,inf}_{t\in(a,b)}\ t\, H_{N-1}(\{x \in A \mid v(x) = t\})$$

we conclude that

$$\int_{A\cap\{a\le v\le b\}} |\nabla v(x)|\, dx = \int_a^b \frac{1}{t}\, t\, H_{N-1}(\{x \in A \mid v(x) = t\})\, dt$$

$$\ge \alpha \int_a^b \frac{1}{t} dt = \alpha \ln(b/a).$$

■

Lemma 2.13. Let μ be a nonnegative Radon measure on $\mathbb{R}^N$. For μ a. e. $x_0 \in \mathbb{R}^N$ and for every $0 < t < 1$ one has

$$\limsup_{\varepsilon\to 0} \frac{\mu(B(x_0,t\varepsilon))}{\mu(B(x_0,\varepsilon))} \ge t^N. \tag{A.4}$$

Proof. Let

$$N := \{x \in \mathbb{R}^N \mid \lim_{\varepsilon\to 0} \frac{\mu(B(x,\varepsilon))}{\mathcal{L}_N(B(x,\varepsilon))} \text{ does not exist}\}.$$

By the Besicovitch Differentiation Theorem (Theorem 2.7)

$$\mathcal{L}_N(N) = 0,$$

and setting

$$E := \{x \in \mathbb{R}^N \mid \liminf_{\varepsilon\to 0} \frac{\mu(B(x,\varepsilon))}{\mathcal{L}_N(B(x,\varepsilon))} = 0\} \cup \{x \in \mathbb{R}^N \mid \mu(B(x,\varepsilon)) = 0 \text{ for some } \varepsilon > 0\}$$

we have

$$\mu(E) = 0.$$

Thus, if $x_0 \in \mathbb{R}^N \setminus (E \cup N)$ we deduce that

$$\lim_{\varepsilon\to 0} \frac{\mu(B(x_0,\varepsilon))}{\mathscr{L}_N(B(x_0,\varepsilon))} \neq 0$$

with

$$\lim_{\varepsilon\to 0} \frac{\mu(B(x_0,t\varepsilon))}{\mu(B(x_0,\varepsilon))} = \lim_{\varepsilon\to 0} \frac{\mu(B(x_0,t\varepsilon))}{\mathscr{L}_N(B(x_0,t\varepsilon))} \frac{\mathscr{L}_N(B(x_0,t\varepsilon))}{\mathscr{L}_N(B(x_0,\varepsilon))} \frac{\mathscr{L}_N(B(x_0,\varepsilon))}{\mu(B(x_0,\varepsilon))}$$

$$= t^N. \tag{A.5}$$

Now consider the set of points at which (A.4) fails, i. e.

$$A := \{x \in \mathbb{R}^N \mid \limsup_{\varepsilon\to 0} \frac{\mu(B(x_0,t\varepsilon))}{\mu(B(x_0,\varepsilon))} < t^N\}.$$

By (A.5) we have that

$$A \subset (E \cup N)$$

and so, it suffices to show that $\mu\llcorner A$ is absolutely continuous with respect to $\mathscr{L}_N$. Suppose that we prove that for every $x \in A$

$$\liminf_{\varepsilon\to 0} \frac{\mu(B(x,\varepsilon))}{\mathscr{L}_N(B(x,\varepsilon))} < +\infty. \tag{A.6}$$

Then

$$A \subset \cup_{k=1}^{+\infty} A_k$$

where

$$A_k := \{x \in A \mid \liminf_{\varepsilon\to 0} \frac{\mu(B(x,\varepsilon))}{\mathscr{L}_N(B(x,\varepsilon))} \leq k\},$$

and since $\mu\llcorner A_k$ is absolutely continuous with respect to $\mathscr{L}_N$, with $\mu\llcorner A_k \leq k\mathscr{L}_N$, by the monotone convergence theorem we conclude that

$$\mu(A \cap B) = 0 \text{ whenever } \mathscr{L}_N(B) = 0.$$

It remains to prove (A.6). If $x \in A$ then by the definition of the set A there exists $\delta(x) > 0$ such that

$$\mu(B(x,t\varepsilon)) \leq t^N \mu(B(x,\varepsilon))$$

for every $\varepsilon \in (0, \delta(x))$. Fixing $r_0 \in (t\delta(x), \delta(x))$ and setting $r_i := t^i r_0$ we conclude that

$$\liminf_{\varepsilon\to 0} \frac{\mu(B(x,\varepsilon))}{\mathscr{L}_N(B(x,\varepsilon))} \leq \lim_{i\to\infty} \frac{\mu(B(x,r_i))}{\mathscr{L}_N(B(x,r_i))}$$

$$\leq \limsup_{i\to\infty} \frac{\mu(B(x,r_0))}{t^{iN}\,\mathscr{L}_N(B(x,r_0))} \left[\frac{\mu(B(x,r_i))}{\mu(B(x,r_{i-1}))} \frac{\mu(B(x,r_{i-1}))}{\mu(B(x,r_{i-2}))} \cdots \frac{\mu(B(x,r_1))}{\mu(B(x,r_0))}\right]$$

$$\leq \lim_{i\to\infty} \frac{\mu(B(x,r_0))}{t^{iN}\,\mathcal{L}_N(B(x,r_0))}\,t^{iN}$$

$$= \frac{\mu(B(x,r_0))}{\mathcal{L}_N(B(x,r_0))}. \qquad \blacksquare$$

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