CONSTRUCTION OF A CLASS OF INTEGRAL MODELS FOR HEAT FLOW IN MATERIALS WITH MEMORY

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Research Report No. 88-1 2

February 1988

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1. Introduction

In this paper we construct a simple class of models for heat conduction in materials with memory. We concentrate on situations in which the heat flux depends on the temporal history of the temperature gradient (and possibly on the present value and the history of the temperature), but is independent of the present value of the temperature gradient. Our models are based on Gurtin and Pipkin's theory of heat conduction [12]. An important feature of this theory - one that is relevant to experimental studies of heat flow in certain materials at very low temperatures - is that it predicts finite speed of propagation for thermal disturbances. There are other theories of heat conduction in materials with memory[1] (cf., e.g. [6], [15]). However, the framework of Gurtin and Pipkin seems best suited for our purposes.

We restrict our attention to the one - dimensional case in which the only nonzero component of the heat flux is its' x - component, q; moreover, q and the absolute temperature $\theta > 0$ are functions of x and the time t. In addition, we assume that the material is homogeneous and has unit density. In the absence of deformation the law of balance of energy reduces to

$$\dot{e} + q_x = r, \tag{1.1}$$

where $e = e(x,t)$ is the (specific) internal energy and $r = r(x,t)$ is the external heat supply. A superposed dot indicates differentiation with respect to time, while a subscript x indicates spatial differentiation. Equation (1.1) must be supplemented with constitutive assumptions that characterize the particular type of material. Since we consider only materials whose thermodynamical properties at a point x are determined by the state

(1) The results of [6], [15] permit (but do not require) the heat flux to depend on the present value of the temperature gradient. If the heat flux is sensitive to small changes in the present value of the temperature gradient then the speed of propagation is not finite.

at that same point x, we suppress the spatial variable in our constitutive equations. We also omit the time argument when there is no likelihood of confusion.

In Fourier's classical theory of heat conduction the heat flux and the internal energy are assumed to be functions of the present values of the temperature and the temperature gradient. More precisely, the constitutive relations are

$$q = -\kappa(\theta)g,$$
$$e = e_0(\theta), \tag{1.2}$$

where $g := \theta_x$ is the temperature gradient and κ and e_0 are smooth functions with $\kappa, e_0' > 0$. The relations (1.2), together with (1.1), yield a parabolic equation for θ that predicts infinite speed of propagation for thermal disturbances. Despite this prediction Fourier's theory provides a description of heat conduction that is useful under an extremely wide range of conditions. However, there are situations in which departures from Fourier's law are observed experimentally; in particular, "wave - like" pulses of heat that propagate with finite speed have been observed in certain dielectrics at very low temperatures (cf. the references cited in [2], [5], and [7]).

There have been numerous attempts to develop theories of heat conduction that yield finite speed of propagation (cf. the review article [2]). The first such theory was apparently given by Cattaneo [1] who suggested that $(1.2)_1$ be replaced by

$$\tau(\theta)\dot{q} + q = -\kappa(\theta)g \tag{1.3}$$

with $\tau, \kappa > 0$. Coleman, Fabrizio, and Owen [5] discuss compatibility of (1.3) with thermodynamics. They use the second law to show that the classical equation $(1.2)_2$ for e is not appropriate for materials obeying (1.3); they also give a modified equation for e that is compatible with the second law. An existence theorem for the resulting (hyperbolic) system of partial differential equations for q and θ was established by Coleman, Hrusa, and Owen [7].

In Gurtin and Pipkin's theory a state is described by $(\theta(t),\bar{\theta}^t(\cdot),\bar{g}^t(\cdot))$, where $\bar{\theta}^t$

and $\bar{g}^t$ are the summed histories up to time t of the temperature and the temperature gradient. For $h : \mathbb{R} \to \mathbb{R}$ the summed history up to time t of h is the function $\bar{h}^t : [0,\infty) \to \mathbb{R}$ defined by

(1.4) $$\bar{h}^t(s) := \int_{t-s}^{t} h(z)\,dz \qquad \forall\ s \geq 0.$$

Following a procedure introduced by Coleman and Noll [8], Gurtin and Pipkin use the second law of thermodynamics to obtain restrictions on their constitutive relations. The procedure of Coleman and Noll is based on the following postulate: *For every part of the body under consideration the rate of production of entropy must be nonnegative for all smooth fields consistent with the constitutive equations and the law of balance of energy.* This postulate is equivalent to the requirement that the Clausius - Duhem inequality[2]

(1.5) $$\dot{\psi} + \dot{\theta}\eta + \frac{gq}{\theta} \leq 0,$$

hold for all smooth fields consistent with the constitutive equations; here η is the (specific) entropy and

(1.6) $$\psi := e - \theta\eta$$

is the (specific) free energy. It follows from (1.5) that the inequality

(1.7) $$\int_0^T \left(\frac{\dot{e}}{\theta} + \frac{gq}{\theta^2}\right)(t)\,dt \leq 0$$

is satisfied for every smooth and cyclic process, i.e. for every smooth process whose state at some time $T > 0$ coincides with its' initial state. The requirement that (1.7) hold for cyclic processes is sometimes taken as a statement of the second law (cf. [9] and [16] for more information).

(2) For deformable bodies this inequality contains an additional term involving the stress and the rate of strain. A general theory of thermodynamics for deformable media with memory, which is based on the Clausius - Duhem inequality, was developed by Coleman [4].

Gurtin and Pipkin consider a very general class of constitutive relations in which ψ, η and q are functionals of $(\theta(t), \bar{\theta}^t(\cdot), \bar{g}^t(\cdot))$ and without specifying the form of these functionals they derive conditions that are both necessary and sufficient for compatibility with thermodynamics. These conditions can be summarized roughly as follows:

(i) ***The entropy is minus the derivative of the free energy with respect to the present value of the temperature.***

(ii) ***The heat flux is determined from the free energy through a differential equation called the heat flux relation.***

(iii) ***A functional differential inequality, called the dissipation inequality, holds for all smooth processes.***

It is easy to construct constitutive functionals that satisfy (i) and (ii). However, it is not clear how to construct the functionals so that (iii) will be satisfied, and no examples are given in [12]. An example in which q is a linear functional of $\bar{g}^t$ is discussed in a subsequent paper of Chen and Gurtin [3]. To our knowledge there are no previous examples (consistent with the theory of [12]) in which q is a nonlinear functional. It is important to note that condition (ii) implies a relation between q and e; in particular, e will generally depend on $\bar{g}^t$.

In Gurtin and Pipkin's theory the linearized constitutive equations for q and e are[3]

$$
(1.8)\qquad
\begin{aligned}
q(t) &= \int_0^\infty \alpha'(s)\bar{g}^t(s)\,ds = -\int_0^\infty \alpha(s)g(t-s)\,ds,\\
e(t) &= b + c\theta(t) - \int_0^\infty \beta'(s)\bar{\theta}^t(s)\,ds = b + c\theta(t) + \int_0^\infty \beta(s)\theta(t-s)\,ds,
\end{aligned}
$$

where b and c are constants, and α and β are smooth kernels that decay sufficiently

(3) As one would expect, the linear equations (1.8) are not compatible with thermodynamics. They are, however, compatible to within terms that can be neglected in the linearized theory.

rapidly at infinity. We note that if τ and κ are constant then Cattaneo's relation (1.3) is equivalent to $(1.8)_1$ with $a(s) = (\kappa/\tau)\exp(-s/\tau)$.

MacCamy [13] considered a model that was motivated by the linearized equations (1.8). He replaced $(1.8)_1$ with the nonlinear equation

$$(1.9) \qquad q(t) = -\int_0^\infty a(s)\, f(g(t-s))\, ds,$$

but retained the linear equation $(1.8)_2$ for e. He proved global existence and asymptotic stability for a corresponding initial - boundary value problem. Similar existence theorems for MacCamy's model were established by Dafermos and Nohel [10] and Staffans [17].

Gurtin and Pipkin's theory does not apply to $(1.8)_2$, (1.9) since q cannot be expressed as a suitable functional of $\bar{g}^t$ when f is nonlinear. The issue of compatibility with the second law is not addressed in [13]. However, by adapting an argument of Coleman, Fabrizio, and Owen [5], one can show that for the constitutive equations $(1.8)_2$, (1.9) (under very mild assumptions on a and f) there are smooth T - periodic functions θ and g for which (1.7) is violated[4]. Within the context of [10], [13] and [17] this probably is not a serious difficulty because the solutions obtained there remain close to an equilibrium state, and under reasonable assumptions on a, f, and β the inequality (1.7) is satisfied for a suitable class of smooth cyclic processes that are close to equilibrium.

In this paper we consider the constitutive relations

(4) The existence of such functions is a consequence of the structure of $(1.8)_2$, (1.9) rather than the particular assumptions made on c, β, a, and f. The sign of the product q(t)g(t) plays a crucial role both in the Clausius - Duhem inequality (1.5) and in (1.7). If the constitutive relation (1.9) is adopted then q(t)g(t) is generally not of fixed sign. Note that for Fourier's law $(1.2)_1$ we have $q(t)g(t) = -\kappa(\theta(t))g(t)^2 \le 0$.

$$\psi(t) = \hat{\psi}(\theta(t)) + \int_0^\infty \hat{\Psi}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\,ds,$$

(1.10)
$$\eta(t) = \hat{\eta}(\theta(t)) + \int_0^\infty \hat{H}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\,ds,$$

$$q(t) = \hat{q}(\theta(t)) + \int_0^\infty \hat{Q}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\,ds,$$

where $\hat{\Psi}$, $\hat{H}$, and $\hat{Q}$ are normalized so that

(1.11)
$$\hat{\Psi}(s,\upsilon,\upsilon s,0) = \hat{H}(s,\upsilon,\upsilon s,0) = \hat{Q}(s,\upsilon,\upsilon s,0) = 0 \qquad \forall\ s, \upsilon > 0$$

and satisfy hypotheses which ensure that the integrals in (1.10) will be well behaved for a reasonable class of functions θ, g. (A similar constitutive equation for e is implied by (1.6).) The normalization (1.11) means that the integrands in (1.10) vanish identically at equilibrium, i.e. for processes in which θ is constant and $g \equiv 0$. For equations of the form (1.10) this normalization can always be achieved by an appropriate modification of $\hat{\psi}$, $\hat{\Psi}$, $\hat{\eta}$, $\hat{H}$, $\hat{q}$, and $\hat{Q}$.

Functionals of similar nature are used for models in viscoelasticity[5]. We note that for rectilinear shearing motions the relative deformation gradient can be expressed in terms of the summed history of the velocity gradient; for shearing motions of K-BKZ fluids the constitutive relation can be written in the form

(1.12)
$$\sigma(t) = \int_0^\infty \hat{\sigma}(s,\bar{v}_x^t(s))\,ds,$$

where σ is the relevant stress component and v_x is the velocity gradient.

Our choice of the equations (1.10) is not made with the intention of describing a

(5) Gurtin and Hrusa [11] discuss compatibility with the Clausius - Planck inequality for a class of integral models in viscoelasticity. There are many similarities between the results of [11] and those obtained here.

particular physical substance. Our objective here is to obtain constitutive equations that incorporate the effects of memory in a qualitatively reasonable manner, that are compatible with thermodynamics, and that lead to initial value problems whose analysis is relatively clean[6].

We show that if[7]

(1.13) $$\hat{\eta}(\upsilon) = -\,\hat{\psi}'(\upsilon), \quad \hat{H}(s,\upsilon,\alpha,\gamma) = -\,\hat{\Psi}_{,2}(s,\upsilon,\alpha,\gamma),$$

(1.14) $$\hat{q}(\upsilon) = 0, \quad \hat{Q}(s,\upsilon,\alpha,\gamma) = -\,\upsilon\hat{\Psi}_{,4}(s,\upsilon,\alpha,\gamma),$$

(1.15) $$\hat{\Psi}_{,1}(s,\upsilon,\alpha,\gamma) + \upsilon\hat{\Psi}_{,3}(s,\upsilon,\alpha,\gamma) \le 0$$

$$\forall\ s,\upsilon > 0,\ \alpha \ge 0,\ \gamma \in \mathbb{R},$$

then Gurtin and Pipkin's conditions for compatibility with thermodynamics are satisfied. The important feature of this result is that a *pointwise* inequality for $\hat{\Psi}$, namely (1.15), ensures that the dissipation inequality holds for all smooth processes. If $\hat{\Psi}$ satisfies (1.15) (and the normalization $\hat{\Psi}(s,\upsilon,\upsilon s,0) = 0$ for all $s, \upsilon > 0$) then

(1.16) $$\hat{\Psi}_{,j}(s,\upsilon,\upsilon s,0) = 0 \quad j = 1, 2, 3, 4 \quad \forall\ s, \upsilon > 0.$$

Thus it is straightforward to produce constitutive equations of the form (1.10) that are compatible with thermodynamics: choose a function $\hat{\psi}$, construct a function $\hat{\Psi}$ satisfying (1.15), and then use (1.13), (1.14) to determine the entropy and the heat flux. (The functions $\hat{H}$ and $\hat{Q}$ constructed through (1.13) and (1.14) will automatically have the desired normalization by virtue of (1.16).). It is clear that there is a large class of functions $\hat{\Psi}$ obeying (1.15); in particular, (1.15) holds if $\hat{\Psi}_{,1} \le 0$ and $\hat{\Psi}_{,3} \le 0$.

(6) Existence of solutions to initial value problems will be discussed in future work.

(7) We use $F_{,j}$ to denote the partial derivative of a function F with respect to its' jth argument.

Conditions (1.13), (1.14), and (1.16) are necessary for compatibility. However, (1.15) is not necessary unless $\hat{\Psi}$ satisfies some additional assumptions. We show that if

(1.17) $$\hat{\Psi}(s,\upsilon,\alpha,0) = 0 \qquad \forall\ s, \upsilon > 0,\ \alpha \geq 0$$

then (1.15) is necessary for compatibility with thermodynamics. The physical interpretation of (1.17) is that for processes with $g \equiv 0$ the free energy depends only on the present value of θ.

It follows from (1.6) and (1.10) that the internal energy is given by

(1.18) $$e(t) = \hat{e}(\theta(t)) + \int_0^\infty \hat{E}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds,$$

where

(1.19) $$\hat{e}(\upsilon) := \hat{\psi}(\upsilon) + \upsilon\hat{\eta}(\upsilon),$$
$$\hat{E}(s,\upsilon,\alpha,\gamma) := \hat{\Psi}(s,\upsilon,\alpha,\gamma) + \upsilon\hat{H}(s,\upsilon,\alpha,\gamma)$$
$$\forall\ s, \upsilon > 0,\ \alpha \geq 0,\ \gamma \in \mathbb{R}.$$

Observe that

(1.20) $$\hat{E}(s,\upsilon,\upsilon s,0) = 0 \qquad \forall\ s, \upsilon > 0$$

by virtue of (1.11). The quantity $\hat{e}(\upsilon)$ is called the equilibrium internal energy at the temperature υ; its' derivative

(1.21) $$\hat{c}_E(\upsilon) := \hat{e}'(\upsilon)$$

is called the equilibrium specific heat or equilibrium heat capacity at the temperature υ. The instantaneous specific heat or instantaneous heat capacity $\hat{c}_I(\theta(t),\bar{\theta}^t,\bar{g}^t)$ at the state $(\theta(t),\bar{\theta}^t,\bar{g}^t)$ is given by

$$(1.22) \qquad \hat{c}_l(\theta(t), \bar{\theta}^t, \bar{g}^t) := \hat{e}'(\theta(t)) + \int_0^{\infty} \text{^(smeVs),!^))}\, ds.$$

We note that the second law of thermodynamics places no restrictions on the sign of $\hat{c}_£$ or of $\hat{c}_r$ Of course, it is generally assumed in practice that the heat capacities are positive.

The paper is organized as follows. Section 2 contains some preliminary material concerning integral functionals of the type appearing in (1.10). In Section 3 we establish conditions for compatibility of (1.10) with thermodynamics. Section 4 is devoted to examples. Finally, in Section 5, we discuss various modifications including dependence of q on the present value of g.

2. Preliminaries

For a function $h : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$, the summed history up to time t of h is the function $\bar{h}^t : [0,\infty) \to \mathbb{R}$ defined by

$$(2.1) \qquad \bar{h}^t(s) := \int_{t-s}^{t} h(z)\, dz \qquad \forall\ s \geq 0.$$

If h is continuous on $\mathbb{R}$ then

$$(2.2) \qquad \begin{aligned} &\bar{h}^t(0) = 0, \\ &\frac{d}{ds}\bar{h}^t(s) = h(t-s), \\ &\frac{d}{dt}\bar{h}^t(s) = h(t) - h(t-s) \end{aligned} \qquad \forall\ s \geq 0,\ t \in \mathbb{R}.$$

A pair (θ,g) of real-valued functions on $\mathbb{R}$ will be called an **admissible pair** if[8] $\theta \in C_b^1(\mathbb{R})$, $g \in C_b(\mathbb{R})$, and $\theta > 0$. We note that if (θ,g) is an admissible pair then for each $t \in \mathbb{R}$ the function $s \mapsto \bar{\theta}^t(s)$ is strictly increasing.

The following definition is designed to ensure that

$$(2.3) \qquad \int_0^\infty F(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds$$

is well behaved for admissible pairs (θ,g): A function F will be called an **admissible kernel** if $F \in C((0,\infty)\times(0,\infty)\times[0,\infty)\times\mathbb{R})$ and for each compact set $K \subset (0,\infty)\times[0,\infty)\times\mathbb{R}$ there is a function $v \in L^1(0,\infty)$ such that

$$(2.4) \qquad |F(s,\upsilon,ys,zs)| \leq v(s) \qquad \forall\ s > 0,\ (\upsilon,y,z) \in K.$$

(8) Here C_b is the set of bounded continuous functions and C_b^1 is the set of all functions in C_b whose derivatives belong to C_b.

In several of our proofs we construct sequences of admissible pairs and pass to the limit in integrals of the form (2.3). For this purpose we say that a sequence $\{u_n\}_{n=1}^{\infty}$ of functions on $[0,\infty)$ is **linearly dominated** if there is a constant L such that

$$|u_n(s)| \le Ls \qquad \forall\ s \ge 0,\ n = 1, 2, \cdots . \tag{2.5}$$

The remarks below follow easily from (2.1), (2.2), the definitions of an admissible pair and an admissible kernel, the dominated convergence theorem, and the fact that if $w \in C^1(0,\infty)$ with $w, \dot{w} \in L^1(0,\infty)$ then $w(s) \to 0$ as $s \to \infty$.

Remark 2.1: *If $f \in C(0,\infty)$, F is an admissible kernel, and (θ,g) is an admissible pair then the mapping*

$$t \mapsto f(\theta(t)) + \int_0^\infty F(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds \tag{2.6}$$

is continuous on $\mathbb{R}$.

Remark 2.2: *Assume that $f \in C^1(0,\infty)$, $F \in C^1((0,\infty)\times(0,\infty)\times[0,\infty)\times\mathbb{R})$, F and $F_{,j}$, $j = 1, 2, 3, 4$ are admissible kernels, and (θ,g) is an admissible pair. Then the mapping (2.6) is continuously differentiable on $\mathbb{R}$ and*

$$\begin{aligned} \frac{d}{dt}\int_0^\infty F(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds = {} & \dot{\theta}(t)\int_0^\infty F_{,2}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds \\ & + \int_0^\infty F_{,3}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))[\theta(t) - \theta(t-s)]\, ds \\ & + \int_0^\infty F_{,4}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))[g(t) - g(t-s)]\, ds. \end{aligned} \tag{2.7}$$

Moreover, the mapping

$$s \mapsto \frac{d}{ds} F(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s)) \tag{2.8}$$

is (absolutely) integrable on $(0,\infty)$ and consequently

(2.9) $$F(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s)) \longrightarrow 0 \quad \text{as } s \to \infty.$$

The following two lemmas will be used to infer pointwise relations from relations involving integrals.

Lemma 2.1: *Assume that $f \in C(0,\infty)$, F is an admissible kernel, and that*

(2.10) $$F(s,\upsilon,\upsilon s,0) = 0 \quad \forall\ s, \upsilon > 0.$$

If

(2.11) $$f(\theta(t)) + \int_0^\infty F(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds = 0 \quad \forall\ t \in \mathbb{R}$$

for every admissible pair (θ,g) then

(2.12) $$f(\upsilon) = 0, \quad F(s,\upsilon,\alpha,\gamma) = 0 \quad \forall\ s, \upsilon > 0,\ \alpha \geq 0,\ \gamma \in \mathbb{R}.$$

Proof: Let $\tau, \upsilon, \alpha > 0$ and $t, \gamma \in \mathbb{R}$ be given. That $f(\upsilon) = 0$ follows from (2.11) with $\theta \equiv \upsilon$ and $g \equiv 0$. We want to show that

(2.13) $$F(\tau,\upsilon,\alpha,\gamma) = 0.$$

Suppose first that

(2.14) $$\alpha < \tau\upsilon,$$

put

(2.15) $$\lambda := \frac{\alpha}{\tau},$$

fix $A > 0$, and construct a sequence $\{(\theta_n,g_n)\}_{n=1}^{\infty}$ of admissible pairs such that $\theta_n(t) = \upsilon$ for all n, $\{\bar{\theta}_n^t\}_{n=1}^{\infty}$ and $\{\bar{g}_n^t\}_{n=1}^{\infty}$ are linearly dominated, and

$$\bar{\theta}_n^t(s) \longrightarrow \begin{cases} \lambda s & \text{if } s \in [0,A) \\ \upsilon s & \text{if } s \in (A,\infty) \end{cases}$$

(2.16)

$$\bar{g}_n^t(s) \longrightarrow \begin{cases} \dfrac{\gamma s}{\tau} & \text{if } s \in [0,A) \\ 0 & \text{if } s \in (A,\infty), \end{cases}$$

as $n \to \infty$. (The limits of $\bar{\theta}_n^t(A)$ and $\bar{g}_n^t(A)$ are unimportant. We note that the construction of $\{\theta_n\}_{n=1}^{\infty}$ requires $\upsilon > \lambda$, which is implied by (2.14), (2.15).) If we apply (2.11) to this sequence, let $n \to \infty$, and recall that $f(\upsilon) = 0$ we obtain

(2.17) $$\int_0^A F(s,\upsilon,\lambda s,\frac{\gamma s}{\tau})\, ds = 0.$$

Since (2.17) holds for every $A > 0$ we have

(2.18) $$F(A,\upsilon,\lambda A,\frac{\gamma A}{\tau}) = 0 \quad \forall\ A > 0.$$

If we put $A = \tau$ then (2.18) yields (2.13).

Suppose now that

(2.19) $$\alpha \geq \tau\upsilon,$$

define λ through (2.15), and fix $B > 0$. Applying (2.11) to an appropriate sequence of admissible pairs and passing to the limit we find that

(2.20) $$\int_B^{\infty} F(s,\upsilon,\lambda s,\frac{\gamma s}{\tau})\, ds = 0 \quad \forall\ B > 0,$$

which yields

(2.21) $$F(B,\upsilon,\lambda B,\frac{\gamma B}{\tau}) = 0 \quad \forall\ B > 0,$$

and consequently (2.13). We have therefore shown that (2.13) holds for all $\tau, \upsilon, \alpha > 0$, $\gamma \in \mathbb{R}$; the continuity of F implies that (2.13) also holds for $\alpha = 0$. //

If F is an admissible kernel satisfying (2.10) and

(2.22) $$\int_0^{\infty} F(s,6(t),\bar{9}\backslash s),\bar{g}\backslash s))\, ds \wedge 0 \quad V\ t\ G\ IR$$

for every admissible pair (8,g), it does not follow that the pointwise inequality

(2.23) $$F(s,t>,a,Y) \lessdot 0 \quad V\ S,\ D > 0,\ a \wedge O,\ y\ e\ K,$$

holds. This is because if (8,g) is an admissible pair then the mapping $s \wedge \bar{S}^t(s)$ is strictly increasing, and consequently the class of admissible pairs is not large enough to ensure that (2.22) implies (2.23). However, we do have the following result.

Lemma 2.2: *Assume that F is an admissible kernel and*

(2.24) $$F(s,D,a,0) = 0 \quad V\ s, \backslash) > 0,\ a \wedge 0.$$

If (222) holds for every admissible pair (Q,g) then (223) is satisfied.

Proof: Let A, B > 0 with A < B, t, y G IR, and 6 G C_b^*(1R) with 6 > 0 be given. We construct a sequence $\{g_n\}_{n=1}^{\sim}$ of functions in C_b([R) such that $\{i^\wedge\}^\wedge_{=1}$ is linearly dominated and

(2.25) $$i^l_f(s) \to \begin{cases} 1^o & \text{if } s\ G\ [0,A)U(B,oo) \\ ILy & \text{if } S\ G\ (A,B). \end{cases}$$

If we apply (2.22) to the admissible pairs $(9,g_n)$ and let n-> oo we obtain

(2.26) $$\int_A^B F(s,6(t),\bar{G}^t(s),Y)\, ds \lessdot 0.$$

Since (2.26) holds for every A, B with 0 < A < B, it follows that

(2.27) $$F(s,0(t),\bar{6}^t(s),Y) \wedge 0 \quad V\ s > 0.$$

For given $s, \upsilon, \alpha > 0$ we can choose θ such that

$$(2.28) \qquad \theta(t) = \upsilon, \qquad \bar{\theta}^t(s) = \alpha,$$

and the proof is complete. //

3. Compatibility with Thermodynamics

We consider constitutive relations of the form

$$\psi(t) = \hat{\psi}(\theta(t)) + \int_0^\infty \hat{\Psi}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\,ds,$$

(3.1)
$$\eta(t) = \hat{\eta}(\theta(t)) + \int_0^\infty \hat{H}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\,ds,$$

$$q(t) = \hat{q}(\theta(t)) + \int_0^\infty \hat{Q}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\,ds$$

for the free energy ψ, the entropy η, and the heat flux q, where θ is the absolute temperature and g is the temperature gradient. A similar constitutive equation for the internal energy e is implied by the relation

(3.2)
$$e = \psi + \theta\eta.$$

Throughout this section we assume that (A1) through (A4) below hold:

(A1) $\hat{\psi} \in C^1(0,\infty)$, $\hat{\eta}$, $\hat{q} \in C(0,\infty)$, $\hat{\Psi} \in C^1((0,\infty)\times(0,\infty)\times[0,\infty)\times \mathbb{R})$;

(A2) $\hat{\Psi}$, $\hat{H}$, $\hat{Q}$ *and* $\hat{\Psi}_{,j}$, $j = 1, 2, 3, 4$ *are admissible kernels (in the sense of Section 2);*

(A3) $\hat{\Psi}(s,\upsilon,\upsilon s,0) = \hat{H}(s,\upsilon,\upsilon s,0) = \hat{Q}(s,\upsilon,\upsilon s,0) = 0 \qquad \forall\ s, \upsilon > 0;$

(A4) *For each* $\upsilon > 0$ *we have*

$$\hat{\Psi}(s,\upsilon,ys,zs) \to 0 \qquad as\ s \to 0^+,$$

uniformly for (y,z) in compact subsets of $[0,\infty)\times \mathbb{R}$.

We note that the normalization (A3) is used in an essential way. For functionals of the form appearing in (3.1) this normalization can always be achieved by an appropriate modification of $\hat{\psi}$, $\hat{\Psi}$, $\hat{\eta}$, $\hat{H}$, $\hat{q}$ and $\hat{Q}$. (Indeed, one can replace $\hat{\Psi}(s,\upsilon,\alpha,\gamma)$ with

$\hat{\Psi}(s,\upsilon,\alpha,\gamma)$ - $\hat{\Psi}(s,\upsilon,\upsilon s,0)$ and add a correction term to $\hat{\psi}$.) It follows from (A1) and (A2) that ψ is continuously differentiable and η and q are continuous on $\mathbb{R}$ if (θ,g) is an admissible pair, i.e. if $\theta \in C_b^1(\mathbb{R})$, $g \in C_b(\mathbb{R})$, and $\theta > 0$. The purpose of (A4) is to ensure that

$$\hat{\Psi}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s)) \to 0 \qquad \text{as } s \to 0^+ \tag{3.3}$$

whenever (θ,g) is an admissible pair. We note that (A4) is implied by (A3) if $\hat{\Psi} \in C([0,\infty)\times(0,\infty)\times[0,\infty)\times\mathbb{R})$.

Our assumptions are certainly not the weakest possible. However, we feel that they are reasonable from the viewpoint of applications.

By an **admissible thermodynamical process** we mean an ordered quintuplet (θ,g,ψ,η,q) of functions on $\mathbb{R}$ where (θ,g) is an admissible pair and ψ, η, q are computed through (3.1). We say that the constitutive equations (3.1) are **compatible with thermodynamics** if the Clausius - Duhem inequality

$$\dot{\psi} + \eta\dot{\theta} + \frac{qg}{\theta} \leq 0 \tag{3.4}$$

holds for every admissible thermodynamical process (θ,g,ψ,η,q).

The following result is an immediate consequence[9] of the work of Gurtin and Pipkin [12].

Proposition 3.1: *The constitutive equations (3.1) are compatible with thermodynamics if and only if*

(9) Gurtin and Pipkin use a slightly different definition of admissible pair and their technical assumptions are formulated accordingly. However, it is clear that the arguments used in [12] can be applied in the present setting.

$$(3.5)\quad \hat{\eta}(\theta(t)) + \int_0^\infty \hat{H}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds = -\ \hat{\psi}'(\theta(t)) - \int_0^\infty \hat{\Psi},_2(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds,$$

$$(3.6)\quad \hat{q}(\theta(t)) + \int_0^\infty \hat{Q}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds = -\ \theta(t)\int_0^\infty \hat{\Psi},_4(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds,$$

and

$$(3.7)\quad \int_0^\infty \hat{\Psi},_3(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))[\theta(t) - \theta(t-s)]\, ds - \int_0^\infty \hat{\Psi},_4(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, g(t-s)\, ds \le 0$$

for every admissible pair (θ,g).

It is obvious that if

$$(3.8)\quad \hat{\eta}(\upsilon) = -\ \hat{\psi}'(\upsilon),\quad \hat{H}(s,\upsilon,\alpha,\gamma) = -\ \hat{\Psi},_2(s,\upsilon,\alpha,\gamma),$$

$$(3.9)\quad \hat{q}(\upsilon) = 0,\quad \hat{Q}(s,\upsilon,\alpha,\gamma) = -\ \upsilon\hat{\Psi},_4(s,\upsilon,\alpha,\gamma)$$

$$\forall\ s, \upsilon > 0,\ \alpha \ge 0,\ \gamma \in \mathbb{R}$$

then (3.5) and (3.6) hold for all admissible pairs (θ,g). The next proposition gives a pointwise condition on $\hat{\Psi}$ which is sufficient to ensure that (3.7) holds for every admissible pair (θ,g).

Proposition 3.2: *If*

$$(3.10)\quad \hat{\Psi},_1(s,\upsilon,\alpha,\gamma) + \upsilon\hat{\Psi},_3(s,\upsilon,\alpha,\gamma) \le 0 \quad \forall\ s, \upsilon > 0,\ \alpha \ge 0,\ \gamma \in \mathbb{R}$$

then (3.7) holds for every admissible pair (θ,g). Consequently, if (3.8), (3.9), (3.10) hold then the constitutive relations (3.1) are compatible with thermodynamics.

Proof: Let (θ,g) be an admissible pair. To see that (3.10) implies (3.7) we first observe that

$$(3.11)\quad \hat{\Psi}_{,3}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))[\theta(t) - \theta(t\text{-}s)] - \hat{\Psi}_{,4}(s,e(t),\bar{e}^t(s),l^t(s))g(t\text{-}s)$$

$$= \hat{\Psi}_{,1}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s)) + \theta(t)\hat{\Psi}_{,3}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s)) - \frac{d}{ds}\hat{\Psi}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s)).$$

It follows from (A4) and Remark 2.2 that

(3.12) ∞ J^^meWlVs)) ds = 0 0

and consequently integration of (3.11) shows that (3.7) is equivalent to

(3.i3) oo J^(smeWlto) ds + e(t)J^MaxeVs),!^)) ds ≤o. 0 oo 0

Clearly, (3.10) implies (3.13) which completes the proof. //

We now discuss the necessity of various pointwise conditions, including (3.8), (3.9), and (3.10), for compatibility with thermodynamics[10].

Proposition 3.3: *If the inequality (3,7) holds for every admissible pair (0,g) then*

(3.14) £,.(s/D,\)s,0) = 0 j = 1, 2, 3,4 V s, x> > 0.

If the constitutive equations (3.1) are compatible with thermodynamics then (3,8), (3,9), and (3.14) hold.

Proof: Let t, ye IR and D, *X*, A, B > 0 with A < B be given. We first construct a sequence {(Q^gn)}^! of admissible pairs such that g_n=0 for all n, {GJJ}^ is uniformly bounded, and

(10) The necessity of (3.8)j and (3.9)j can be obtained directly from the results in Section 3 of [12]. A necessary condition that is closely related to (3.14) is also established in Section 3 of [12].

$$\text{(3.15)}\qquad \theta_n(t\text{-}s) \longrightarrow \begin{cases} \upsilon & \text{if } s \in [0,A) \cup (B,\infty) \\ \lambda & \text{if } s \in (A,B) \end{cases}$$

$$\bar{\theta}^t_n(s) \longrightarrow \begin{cases} \upsilon s & \text{if } s \in [0,A) \\ \upsilon A + \lambda(s - A) & \text{if } s \in [A,B] \\ \upsilon A + \lambda(B - A) + \upsilon(s - B) & \text{if } s \in (B,\infty). \end{cases}$$

as $n \to \infty$. Applying (3.7) to this sequence and letting $n \to \infty$, we obtain

$$\text{(3.16)}\qquad \int_A^B \hat{\Psi}_{,3}(s,\upsilon,\upsilon A+\lambda(s\text{-}A),0)(\upsilon - \lambda)\, ds \le 0.$$

We then construct a new sequence $\{(\theta_n,g_n)\}_{n=1}^{\infty}$ of admissible pairs with $\theta_n \equiv \upsilon$ for all n, $\{g_n\}_{n=1}^{\infty}$ uniformly bounded, and

$$\text{(3.17)}\qquad g_n(t\text{-}s) \longrightarrow \begin{cases} 0 & \text{if } s \in [0,A) \cup (B,\infty) \\ \gamma & \text{if } s \in (A,B) \end{cases}$$

$$\bar{g}^t_n(s) \longrightarrow \begin{cases} 0 & \text{if } s \in [0,A) \\ \gamma(s - A) & \text{if } s \in [A,B] \\ \gamma(B - A) & \text{if } s \in (B,\infty) \end{cases}$$

as $n \to \infty$. We apply (3.7) to the new sequence and let $n \to \infty$ to obtain

$$\text{(3.18)}\qquad \int_A^B \hat{\Psi}_{,4}(s,\upsilon,\upsilon s,\gamma(s\text{-}A))\gamma\, ds \ge 0.$$

Since (3.16) and (3.18) hold for every A, B with $0 < A < B$, we conclude that

$$\text{(3.19)}\qquad \begin{aligned} &\hat{\Psi}_{,3}(s,\upsilon,\upsilon s,0)(\upsilon - \lambda) \le 0 \\ &\hat{\Psi}_{,4}(s,\upsilon,\upsilon s,0)\gamma \ge 0 \qquad \forall\ s > 0. \end{aligned}$$

It follows from (3.19) that

$$\text{(3.20)}\qquad \hat{\Psi}_{,3}(s,\upsilon,\upsilon s,0) = \hat{\Psi}_{,4}(s,\upsilon,\upsilon s,0) = 0 \quad \forall\ s, \upsilon > 0$$

since γ and $\upsilon, \lambda > 0$ are arbitrary.

If we differentiate the relation

(3.21) $$\hat{\Psi}(s,\upsilon,\upsilon s,0) = 0$$

with respect to s, then with respect to υ, and appeal to (3.20) we obtain

(3.22) $$\hat{\Psi}_{,1}(s,\upsilon,\upsilon s,0) = \hat{\Psi}_{,2}(s,\upsilon,\upsilon s,0) = 0 \quad \forall\ s, \upsilon > 0.$$

If the constitutive equations (3.1) are compatible with thermodynamics then (3.7) is satisfied for every admissible pair (θ,g), and consequently (3.14) holds.
The necessity of (3.8) and (3.9) follows from (3.20), (3.22) and Lemma 2.1. //

Without additional assumptions on $\hat{\Psi}$ the pointwise inequality (3.10) is not necessary for compatibility[11]. However, we do have the following result.

Proposition 3.4: *If the constitutive relations (3.1) are compatible with thermodynamics and*

(3.23) $$\hat{\Psi}(s,\upsilon,\alpha,0) = 0 \quad \forall\ s, \upsilon > 0,\ \alpha \geq 0,$$

then (3.10) holds.

Proof: We note that (3.23) implies

(3.24) $$\hat{\Psi}_{,1}(s,\upsilon,\alpha,0) = \hat{\Psi}_{,3}(s,\upsilon,\alpha,0) = 0 \quad \forall\ s, \upsilon > 0,\ \alpha \geq 0.$$

The desired conclusion therefore follows from (3.24), Lemma 2.2, Proposition 3.1, and the fact that (3.7) is equivalent to (3.13). //

(11) We have constructed an example of constitutive relations of the form (3.1) that are compatible with thermodynamics, but for which the pointwise inequality (3.10) fails. We omit the presentation of this example because it is rather involved.

4. Examples

We now discuss several examples of constitutive equations satisfying the conditions of Section 3. The following result, which is implied by Propositions 3.1, 3.2, and 3.3, will be used: *If* $\hat{\psi} \in C^1(0,\infty)$, $\hat{\Psi} \in C^1((0,\infty)\times(0,\infty)\times[0,\infty)\times \mathbb{R})$,

$$\hat{\Psi}(s,\upsilon,\upsilon s,0) = 0 \qquad \forall\ s, \upsilon > 0, \tag{4.1}$$

$\hat{\Psi}$ *and* $\hat{\Psi}_{,j}$, $j = 1, 2, 3, 4$ *are admissible kernels, (A4) of Section 3 holds, and*

$$\hat{\Psi}_{,1}(s,\upsilon,\alpha,\gamma) + \upsilon\hat{\Psi}_{,3}(s,\upsilon,\alpha,\gamma) \leq 0 \quad \forall\ s, \upsilon > 0,\ \alpha \geq 0,\ \gamma \in \mathbb{R}, \tag{4.2}$$

then the constitutive equations

$$\begin{aligned} \psi(t) &= \hat{\psi}(\theta(t)) + \int_0^\infty \hat{\Psi}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds, \\ \eta(t) &= -\ \hat{\psi}'(\theta(t)) - \int_0^\infty \hat{\Psi}_{,2}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds, \\ q(t) &= -\ \theta(t)\int_0^\infty \hat{\Psi}_{,4}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\, ds \end{aligned} \tag{4.3}$$

are compatible with thermodynamics. In order to avoid repeated hypotheses we assume that $\hat{\psi} \in C^1(0,\infty)$ and that all functions appearing in the examples below are such that $\hat{\Psi} \in C^1((0,\infty)\times(0,\infty)\times[0,\infty)\times \mathbb{R})$, $\hat{\Psi}$ and $\hat{\Psi}_{,j}$, $j = 1, 2, 3, 4$ are admissible kernels, and (A4) holds.

Example 4.1: (Cattaneo's Relation) Let τ and κ be strictly positive constants. If we define $\hat{\Psi}$ by

(4.4) $$\hat{\Psi}(s,\upsilon,\alpha,\gamma) := \frac{\kappa}{2\upsilon\tau^2} e^{-s/\tau}\gamma^2 \quad \forall\ s, \upsilon > 0,\ \alpha \geq 0,\ \gamma \in \mathbb{R},$$

so that the free energy is given by

(4.5) $$\psi(t) = \hat{\psi}(\theta(t)) + \frac{\kappa}{2\theta(t)\tau^2} \int_0^\infty e^{-s/\tau} (\bar{g}^t(s))^2\, ds,$$

then it is easy to check that (4.1) and (4.2) are satisfied. The corresponding equations for the entropy, internal energy, and heat flux are

$$\eta(t) = -\ \hat{\psi}'(\theta(t)) + \frac{\kappa}{2\theta(t)^2\tau^2} \int_0^\infty e^{-s/\tau} (\bar{g}^t(s))^2\, ds,$$

(4.6) $$e(t) = \hat{\psi}(\theta(t)) -\ \theta(t)\hat{\psi}'(\theta(t)) + \frac{\kappa}{\theta(t)\tau^2} \int_0^\infty e^{-s/\tau} (\bar{g}^t(s))^2\, ds,$$

$$q(t) = \frac{-\kappa}{\tau^2} \int_0^\infty e^{-s/\tau}\, \bar{g}^t(s)\, ds.$$

Equation $(4.6)_3$ is equivalent to Cattaneo's relation

(4.7) $$\tau\dot{q} + q = -\ \kappa g.$$

It is interesting to note that Chen and Gurtin [3] give a different free energy which yields equation $(4.6)_3$ for the heat flux. Chen and Gurtin's free energy cannot be expressed in the form $(4.3)_1$. //

Example 4.2: An interesting generalization of $(4.6)_3$ is provided by

(4.8) $$q(t) = \int_0^\infty a'(s) f(\bar{g}^t(s))\, ds,$$

where $a : (0,\infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are smooth functions with $f(0) = 0$. If we integrate the relation (3.9) we find that the simplest candidate for $\hat{\Psi}$ is

$$(4.9) \qquad \hat{\Psi}(s,\upsilon,\alpha,\gamma) := \frac{-a'(s)}{\upsilon} F(\gamma) \qquad \forall\ s, \upsilon > 0,\ \alpha \geq 0,\ \gamma \in \mathbb{R},$$

where

$$(4.10) \qquad F(\gamma) := \int_0^\gamma f(\lambda)\, d\lambda.$$

The inequality (4.2) is equivalent to

$$(4.11) \qquad a''(s)F(\gamma) \geq 0 \quad \forall\ s > 0,\ \gamma \in \mathbb{R}.$$

Consequently, if

$$(4.12) \qquad a \text{ is convex}, \qquad F(\gamma) \geq 0 \quad \forall\ \gamma \in \mathbb{R}$$

then the constitutive equations

$$(4.13) \qquad \begin{aligned} \psi(t) &= \hat{\psi}(\theta(t)) - \frac{1}{\theta(t)} \int_0^\infty a'(s)F(\bar{g}^t(s))\, ds, \\ \eta(t) &= -\ \hat{\psi}'(\theta(t)) - \frac{1}{\theta(t)^2} \int_0^\infty a'(s)F(\bar{g}^t(s))\, ds \end{aligned}$$

together with (4.8) are compatible with thermodynamics. The corresponding equation for e is

$$(4.14) \qquad e(t) = \hat{\psi}(\theta(t)) - \theta(t)\hat{\psi}'(\theta(t)) - \frac{2}{\theta(t)} \int_0^\infty a'(s)F(\bar{g}^t(s))\, ds. \qquad //$$

Example 4.3: In the special case when the free energy is independent of the summed history of the temperature, i.e. when

$$(4.15) \qquad \hat{\Psi}(s,\upsilon,\alpha,\gamma) = F(s,\upsilon,\gamma) \qquad \forall\ s, \upsilon > 0,\ \alpha \geq 0,\ \gamma \in \mathbb{R}$$

with

$$(4.16) \qquad F(s,\upsilon,0) = 0 \qquad \forall\ s, \upsilon > 0$$

the inequality (4.2) becomes

(4.17) $$F_{,1}(s,\upsilon,\gamma) \leq 0 \quad \forall\ s, \upsilon > 0,\ \gamma \in \mathbb{R}.$$

If (4.16), (4.17) hold then the constitutive equations

$$\psi(t) = \hat{\psi}(\theta(t)) + \int_0^\infty F(s,\theta(t),\bar{g}^t(s))\, ds,$$

(4.18) $$\eta(t) = -\hat{\psi}'(\theta(t)) - \int_0^\infty F_{,2}(s,\theta(t),\bar{g}^t(s))\, ds,$$

$$q(t) = -\theta(t)\int_0^\infty F_{,3}(s,\theta(t),\bar{g}^t(s))\, ds$$

are compatible with thermodynamics.

One can also start with an expression for q of the form

(4.19) $$q(t) = \int_0^\infty G(s,\theta(t),\bar{g}^t(s))\, ds,$$

with $G(s,\upsilon,0) = 0$ for all $s, \upsilon > 0$, and then construct the remaining constitutive equations. It is easy to show that there is a function F satisfying (4.16), (4.17) and

(4.20) $$G(s,\upsilon,\gamma) = -\upsilon F_{,3}(s,\upsilon,\gamma) \quad \forall\ s, \upsilon > 0,\ \gamma \in \mathbb{R}$$

if and only if

(4.21) $$\int_0^\gamma G_{,1}(s,\upsilon,\lambda)\, d\lambda \geq 0 \quad \forall\ s, \upsilon > 0,\ \gamma \in \mathbb{R}.$$

It is important to note that there may be free energies which cannot be expressed in the form $(4.3)_1$ but satisfy the conditions of Gurtin and Pipkin's theory and lead to equations of the form (4.19) for the heat flux. (See the example in Section 6 of Chen and Gurtin [3].) //

Example 4.4: Suppose that

(4.22) $$\hat{\Psi}(s,\upsilon,\alpha,\gamma) = F(s,\upsilon,\alpha - \upsilon s,\gamma) \quad \forall\ s, \upsilon > 0,\ \alpha \geq 0,\ \gamma \in \mathbb{R}$$

with

(4.23) F(s,\),O,O) = 0 V s,\>>0.

In this case the inequality (4.2) reduces to

(4.24) F.jfowx--os,y) ≤ 0 V s,\)>0, cc ≥ 0, ye R.

If (4.23) and (4.24) are satisfied then the constitutive equations

$$V(t) = \hat{v}(G(t)) + \int_0 F(s,e(t),\bar{6}^t(s) - 9(t)s,?(s))\,ds,$$

(4.25)
$$Ti(t) = - \wedge(0(0) - \int_0 F\wedge s.eCD\wedge Cs) - \theta(t)s,\bar{g}^t(s))\,ds$$

$$+\int_0^\infty sF_{,3}(s, e(t),\bar{e}^t(s) - \theta(t)s,\bar{g}^t(s))\,ds,$$

$$q(t) = - 9(t)\int_0^\infty F_{,4}(s,9(t),\bar{e}^t(s) - e(t)s,i^l(s))\,ds.$$

are compatible with thermodynamics. //

5. Modifications

A. Dependence on the Present Value of g

Gurtin and Pipkin's theory is not applicable if the constitutive relations allow dependence on the present value of g. One can show that the free energy and the entropy must be independent of the present value of g and that the entropy is minus the derivative of the free energy with respect to the present value of θ. However, it is no longer true that the heat flux is determined by the free energy. A complete discussion of this situation is beyond the scope of the present paper. However, the following result can be obtained from the arguments of Section 3 and a straightforward calculation: *If (A1) through (A4), (3.8), (3.9), (3.10), and*

$$yF(\upsilon,y) \leq 0 \quad \forall\ \upsilon > 0,\ y \in \mathbb{R} \tag{5.1}$$

hold then the constitutive equations

$$\begin{aligned} \psi(t) &= \hat{\psi}(\theta(t)) + \int_0^\infty \hat{\Psi}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\,ds, \\ \eta(t) &= \hat{\eta}(\theta(t)) + \int_0^\infty \hat{H}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\,ds, \\ q(t) &= F(\theta(t),g(t)) + \int_0^\infty \hat{Q}(s,\theta(t),\bar{\theta}^t(s),\bar{g}^t(s))\,ds \end{aligned} \tag{5.2}$$

are compatible with thermodynamics in the sense that the Clausius - Duhem inequality (3.4) holds whenever (θ,g) is an admissible pair and ψ, η, q are computed through (5.2).

We note that models of the form

$$(5.3)\qquad \begin{aligned} q(t) &= -\,h(g(t)) - \int_0^\infty a(s)f(g(t-s))\,ds, \\ e(t) &= b + c\theta(t) + \int_0^\infty \beta(s)\theta(t-s)\,ds \end{aligned}$$

with

$$(5.4)\qquad yh(y) \geq 0 \quad \forall\ y \in \mathbb{R}$$

have been studied by several authors (cf. the review article [14]). Models of this structure are generally incompatible with thermodynamics in the sense that (under very mild assumptions on a and f) one can construct smooth T - periodic functions θ, g such that (1.7) is violated when e and q are given by (5.3).

B. Coldness

In applications involving very low temperatures it is often convenient to use the coldness $\phi := 1/\theta$ and the coldness gradient $p := \phi_x = -\,g/(\theta^2)$ in place of the temperature θ and the temperature gradient g. If we define χ through

$$(5.5)\qquad \chi := \eta - \phi e$$

then the Clausius - Duhem inequality becomes

$$(5.6)\qquad \dot{\chi} + e\dot{\phi} + qp \geq 0.$$

We now consider the constitutive equations

$$\chi(t) = \tilde{\chi}(\phi(t)) + \int_0^\infty \tilde{X}(s,\phi(t),\bar{\phi}^t(s),\bar{p}^t(s))\, ds,$$

(5.7)
$$e(t) = \tilde{e}(\phi(t)) + \int_0^\infty \tilde{E}(s,\phi(t),\bar{\phi}^t(s),\bar{p}^t(s))\, ds,$$

$$q(t) = \tilde{q}(\phi(t)) + \int_0^\infty \tilde{Q}(s,\phi(t),\bar{\phi}^t(s),\bar{p}^t(s))\, ds$$

under the following assumptions:

($\tilde{A}1$) $\tilde{\chi} \in C^1(0,\infty)$, $\tilde{e}$, $\tilde{q} \in C(0,\infty)$, $\tilde{X} \in C^1((0,\infty)\times(0,\infty)\times[0,\infty)\times \mathbb{R})$;

($\tilde{A}2$) *$\tilde{X}$, $\tilde{E}$, $\tilde{Q}$, and $\tilde{X}_{,j}$ $j = 1, 2, 3, 4$ are admissible kernels (in the sense of Section 2);*

($\tilde{A}3$) $\tilde{X}(s,\upsilon,\upsilon s,0) = \tilde{E}(s,\upsilon,\upsilon s,0) = \tilde{Q}(s,\upsilon,\upsilon s,0) = 0 \quad \forall\ s,\ \upsilon > 0;$

($\tilde{A}4$) *For each $\upsilon > 0$ we have*

$$\tilde{X}(s,\upsilon,ys,zs) \to 0 \quad as\ s \to 0^+,$$

uniformly for (y,z) in compact subsets of $[0,\infty)\times \mathbb{R}$.

The same definition of an admissible pair is still appropriate, i.e. a pair of functions (ϕ,p) on $\mathbb{R}$ is called admissible if $\phi \in C_b^1(\mathbb{R})$, $p \in C_b(\mathbb{R})$, and $\phi > 0$. By compatibility with thermodynamics we now mean that the inequality (5.6) holds whenever (ϕ,p) is an admissible pair and χ, e, q are computed through (5.7).

Using arguments of the type employed by Gurtin and Pipkin [12] one can establish the following analog of Proposition 3.1: *The constitutive equations (5.7) are compatible with thermodynamics if and only if*

(5.8) $\tilde{e}(<|)(t)) + \int_0^{\infty} \tilde{E}(s,<t>(t),y(s),\bar{p}^t(s))\,ds = -\ \%(<t>(t)) - \int_0^{\infty} \tilde{X}\wedge s \wedge CO.iWp^t e))\,ds,$

(5.9) $\tilde{q}((Kt)) + \int_0^{\infty} \tilde{Q}Cs.iKtXiWp\bar{V}s))\,ds = -\int_0^{\infty} \tilde{x}_{,4}(smiW?(s))\,ds,$

and

(5.10) $\int_0^{\infty} x\tilde{\wedge} s \wedge O.iWp\bar{W}Wt) - <t>(t-s)]\,ds - \int_0^{\infty} \tilde{x}_{,4}(s,«t),?(s),p^t(s))p(t-s)\,ds \geq 0$

for every admissible pair $(<f>,p)$.

If (ϕ,p) is an admissible pair one can argue as in the proof of Proposition 3.2 to show that: *The inequality (5.10) holds if and only if*

(5.11) $\int_0^{\infty} \tilde{x}_{,1}(s,(|)(t),?(s),5^t(s))\,ds + «t)\int_0^{\infty} \tilde{x}\wedge(smiWp\bar{t}o)\,ds \geq 0.$

Moreover, one can show that: *If (5.10) holds for every admissible pair $(<j>,p)$ then*

(5.12) $\tilde{X}_{,j}(s,a),i)s,0) = 0, \quad j = 1, 2, 3, 4 \quad \forall\ s, D > 0.$

Using Lemmas 2.1, 2.2, and the remarks above it is straightforward to establish analogs of all of the results of Section 3.

We close with a very simple example in which the internal energy e depends only on the present value of ϕ.

Example 5.1: Suppose that $\tilde{\%} \in C^1(0,\infty)$ and

(5.13) $\%(t) := \tilde{\%}(<(>(t)) + \int_0^{\infty} a^f(s)F\wedge(s))\,ds$

where $a:(0,\infty)\to\mathbb{R}$ and $F: \mathbb{R}\to\mathbb{R}$ are smooth functions with

(5.14) $F(0) = 0:$

We assume that a and F are such that $\tilde{X}$, $\tilde{X}_{,j}$, $j = 1, 4$ are admissable kernels, where

(5.15) $$\tilde{X}(s,\upsilon,\alpha,\gamma) := a'(s)F(\gamma) \quad \forall\ s, \upsilon > 0,\ \alpha \geq 0,\ \gamma \in \mathbb{R}.$$

If

(5.16) $$a \text{ is convex,} \qquad F(\gamma) \geq 0 \quad \forall\ \gamma \in \mathbb{R}$$

then the constitutive equations

(5.17)
$$e(t) = -\ \tilde{\chi}'(\phi(t)),$$
$$q(t) = -\int_0^\infty a'(s)F'(\bar{p}^t(s))\, ds$$

together with (5.13) are compatible with thermodynamics. //

Acknowledgement. We are grateful to B. D. Coleman, M. E. Gurtin, D. R. Owen, and H. M. Soner for valuable discussions and suggestions. This work was supported by the United States Air Force under grant AFOSR - 85 - 0307.

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