EDGE DISJOINT SPANNING TREES IN RANDOM GRAPHS

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Abstract

We show that almost every G_{m-out} contains m edge disjoint spanning trees.

Introduction

In this note we consider the maximum number of edge disjoint spanning trees contained in the random graph G_{m-out}. Let a graph $G = (V,E)$ have property $\mathcal{A}_k$ if it contains spanning trees $T_1, T_2, \ldots, T_k$ which are pair-wise edge disjoint.

We consider the random graph $G_m = G_{m-out}$. This has vertex set $V_n = \{1,2,\ldots,n\}$. Each $v \in V_n$ independently chooses a set $out(v)$ of distinct vertices as neighbours, where each m-subset of $V_n - \{v\}$ is equally likely to be chosen. This produces a random m out-regular diagraph D_m which has been selected uniformly from $\binom{n-1}{m}^n$ distinct possibilities. G_m is obtained by ignoring orientation but without coalescing edges. (See [1], [2], [3] for properties of this model.)

Probability statements refer to the probability space of D_m and graph theoretic statements refer to G_m.

Theorem 1

Let $m \geq 2$ be a fixed constant. Then

$$\lim_{n\to\infty} \Pr(G_m \in \mathcal{A}_m) = 1. \qquad \square$$

[This is clearly best possible.]

The major graph theoretic result underpinning our proof is as follows.

Theorem 2 (Nash-Williams [5], Tutte [6])

A graph $G = (V,E)$ has property $\mathcal{A}_k$ if and only if for every partition $S_1, S_2, \ldots, S_t$ of V, $2 \leq t \leq |V|$, there at least $k(t-1)$ edges of G joining vertices in different subsets of the partition. □

(The necessity of the condition is obvious. The "meat" is in the sufficiency.)

Proof of main result

For $S \subseteq V_n$ let $\gamma(S) = |\{vw \in E(D_m): v \in S, w \notin S\}|$.

Lemma 1

The following events occur with probability tending to 1 (as $n \to \infty$).

(i) $S \subseteq V_n$, $1 \leq |S| \leq .49n$ implies $\gamma(S) \geq m$

(ii) $S,T \subseteq V_n$, $S \cap T = \phi$, $|S|, |T| \geq .49n$, implies $\gamma(S) + \gamma(T) \geq m$.

Proof

Observe that $\gamma(S) \geq |\{v \in S: \text{out}(v) \not\subseteq S\}|$. Hence $\gamma(S) \geq m$ for $|S| \leq m$ and

$$P(\exists S \subseteq V_n : m < |S| \leq .49n \text{ and } \gamma(S) < m) \leq \sum_{s=m+1}^{\lfloor .49n \rfloor} \binom{n}{s}\binom{s}{s-m+1} \left(\frac{\binom{s-1}{m}}{\binom{n-1}{m}} \right)^{s-m+1}$$

$$\leq \sum_{s=m+1}^{\lfloor .49n \rfloor} \binom{n}{s} s^{n-1} \left(\frac{s}{n}\right)^{m(s-m+1)}$$

$$= \sum_s u_s \text{ , say.}$$

Now $\sum_{s=m+1}^{\lfloor n/3 \rfloor} u_s \le \sum_{s=m+1}^{\lfloor n/3 \rfloor} \left(\frac{ne}{s}\right)^s s^{m-1} \left(\frac{s}{n}\right)^{m(s-m+1)}$

$$= \sum_{s=m+1}^{\lfloor n/3 \rfloor} \ldots s^{m-1} \left(\frac{s}{n}\right)^{(m-1)(s-m)}$$

$$= O(n^{-(m-1)}).$$

Next let $H(a) = \ldots (1-a)^{1-a}$, then

$$\sum_{s=\lceil n/3 \rceil}^{\lfloor .49n \rfloor} u_s \le \sum_{s=\lceil n/3 \rceil}^{\lfloor .49n \rfloor} e^{o(n)} H\left(\frac{s}{n}\right)^n \left(\frac{s}{n}\right)^{ms}$$

$$\le e^{o(n)} \sum_{s=\lceil n/3 \rceil}^{\lfloor .49n \rfloor} \ldots (1 - \ldots)^n$$

$$= o(1),$$

and (i) follows.

(ii)

$\Pr(\exists S,T \subseteq V_n.\ |S|, |T| \ge .49n,\ S \cap T = \emptyset \text{ and } T(S) + T(T) < m)$

$$\le \sum_{s=\lceil .49n \rceil}^{\lfloor .51n \rfloor} \sum_{t=\lceil .49n \rceil}^{n-s} \binom{n}{s}\binom{n-s}{t} \ldots m(s+t-m+1)$$

$$\le n^2 2^n 2^{.51n} n^{m-1} (.51)^{.98mn - m+1}$$

$$= o(1).$$ □

Proof of Theorem 1

Let $S_1 S_2, \ldots, S_t$ be a partition of V_n where $|S_1| \geq |S_2| \geq \ldots \geq |S_t|$. Now in the graph G_m there precisely $\gamma(S_1) + \gamma(S_2) + \ldots + \gamma(S_t)$ edges joining different subsets of the partition. But Lemma 1 implies

$$\gamma(S_1) + \gamma(S_2) \geq m \qquad \text{(ii)}$$

and

$$\gamma(S_3) + \ldots + \gamma(S_t) \geq (t-2)m \qquad \text{(i)}$$

and so we can apply Theorem 3. □

We note the following interesting consequence Theorem 1: $G_{2\text{-out}}$ is super-eulerian with probability tending to one. (A graph is super eulerian if it contains a trail which includes every vertex). This is because every graph in $\mathcal{A}_2$ has this property, Jaegar [4].

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