EXISTENCE AND UNIQUENESS OF MULTI-AGENT EQUILIBRIUM IN A STOCHASTIC, DYNAMIC CONSUMPTION/INVESTMENT MODEL

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Ioannis Karatzas
Department of Statistics
Columbia University
New York, NY 10027

John P. Lehoczky
Department of Statistics
Carnegie Mellon University
Pittsburgh, PA 15213

and

Steven E. Shreve
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

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Ioannis Karatzas*
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John P. Lehoczky**
Department of Statistics
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Steven E. Shreve**
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

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Abstract

We consider an economy in which a set of agents own productive assets which provide a commodity dividend stream, and the agents also receive individual commodity income streams over a finite time horizon. The agents can buy and sell this commodity at a certain spot price and buy and sell their shares of the productive assets. The proceeds can be invested in financial assets whose prices are modelled as semimartingales. Each agent's objective is to choose a commodity consumption process and to manage his portfolio so as to maximize the expected utility of his consumption, subject to having nonnegative wealth at the terminal time. We derive the optimal agent consumption and investment decision processes when the prices of the productive assets and commodity spot prices are specified. We prove the existence and uniqueness of an "equilibrium" commodity spot price process and productive asset prices. When the agents solve their individual optimization problems using the equilibrium prices, all of the commodity is exactly consumed as it is received, all of the productive assets are exactly owned and the financial markets are in zero net supply.

1. Introduction.

Over the last two decades, substantial progress has been made on the development of a mathematical theory for capital asset pricing. There has been a progressive depth of insight into the optimal actions of single agents and the way in which the aggregation of these actions leads to prices for capital assets. A major initial contribution was made by Merton [14,15], who studied the single agent optimal control problem. He produced closed form solutions for the consumption and investment policies and the agent's indirect utility, or value function, when the utility function for consumption was of the HARA class and satisfied the condition $U'(0) = \infty$. In the models for these solutions, stock prices are treated as geometric Brownian motion with constant coefficients. To address equilibrium, Merton proposed that the interest rate, the mean rates of return, and the diffusion coefficients of the stock price processes should not be constant, but should themselves be Itô processes with constant drift and diffusion coefficients. This generalized model is far more complex, and no comparable explicit solution was produced. A generalization of Merton's approach is to postulate an underlying Markov state process describing the economy and to allow the stock price coefficients to be functions of this state process; see Cox, Ingersoll and Ross [2], for example.

The present work builds on two previous papers concerning the single agent consumption/investment decision problem. The first paper, by Karatzas, Lehoczky, Sethi and Shreve [11], determined the optimal single agent consumption and investment policies and the value (indirect utility) function for wealth for arbitrary, smooth, concave utility functions of consumption which were assumed only to satisfy conditions required for the finiteness of the value function. This paper also removed the restriction $U'(0) = \infty$, carefully treated the consumption constraint $c \geq 0$, and addressed the possibility of bankruptcy. Stock prices were again modelled by constant coefficient geometric Brownian motion processes, and explicit optimal consumption and investment formulas were obtained.

The second paper, by Karatzas, Lehoczky and Shreve [12], developed a martingale-based characterization of the optimal decisions for a single agent. This approach is applicable to a much more general class of stock price processes, including non-Markovian models. An explicit characterization of single agent optimal consumption policies was provided for general utility functions and semimartingale stock price processes. As a consequence of these results, the optimal behavior of a single agent is now well understood. The present paper is the first to apply this explicit characterization to prove the existence and uniqueness of equilibrium in a multi-agent problem.

Our model of equilibrium was inspired by Duffie [3], Duffie and Huang [4,5], and Huang [4]. The multi-agent equilibrium problem arises when J agents (where J is some positive integer) have individual commodity earnings streams, and each agent is also endowed with a set of productive assets which produce commodity dividend streams. In this paper, there is a single, infinitely-divisible commodity, and each agent wishes to maximize his expected total utility from consumption of this commodity over time. The agents can trade the productive assets in order to hedge the risks associated with the commodity endowments and with the returns from the productive assets. Prices must be established endogenously for the trading of productive assets (stock prices) and for buying and selling the commodity (spot price). Because the market consisting only of productive assets may not be "complete", i.e., may not allow for hedging of all risk, we introduce financial assets whose prices are exogenous. We show that the price structure of these financial assets will influence the equilibrium prices of the productive assets and the equilibrium spot price, but it will not affect the equilibrium allocation of the commodity among agents.

Equilibrium prices (of the productive assets and the commodity) are those prices which, when accepted by the agents during the determination of their optimal consumption and portfolio policies, call for all the commodity to be exactly consumed, all productive assets to be exactly owned, and all pure hedging instruments (financial assets) to be in zero net supply. The goal of an equilibrium analysis is to establish the existence and uniqueness of equilibrium prices, and to characterize these prices as well as the consumption and investment decisions made by the individual agents.

Equilibrium in dynamic, stochastic, multi-agent problems has been studied by several authors. The usual approach (followed by, e.g., Duffie [3] and Duffie and Huang [4,5]) is to reduce the dynamic problem to a static one by considering agent consumption processes to be points in a suitable abstract space. Each agent has a preference structure defining a partial order over consumption plans. Under certain conditions, a deep fixed point theorem (e.g., the Kakutani fixed point theorem) can be invoked to prove the existence of a solution to the static equilibrium problem. A martingale representation theorem can then be employed to create a solution to the dynamic equilibrium problem. There are two drawbacks to this approach. First, the basic work by Mas-Colell [13] required "uniform properness", a strong restriction on the preference ordering. In particular, this property does not allow utility functions satisfying $U'(0) = \infty$, and so such utility functions are not allowed in [3,4,5]. On the other hand, Duffie and Zame [6] report an equilibrium analysis in which each agent has a utility function satisfying $U'(0) = \infty$. The analysis underlying [6] is quite involved, and it still does not include the case that the utility functions for some agents satisfy $U'(0) = \infty$, while the utility functions for other agents do not satisfy this condition. The second difficulty with the usual approach is that it gives little insight into the nature of equilibrium. The optimal consumption plans and spot price processes cannot be exhibited, nor can uniqueness of the equilibrium be established.

This paper is the first to bring the explicit characterization of optimal single agent behavior for general stock price processes to bear on the multi-agent equilibrium problem. The result is a major increase in knowledge about not only the existence, but also about the uniqueness and the structure of equilibrium. The use of the optimal single agent behavior allows a simple fixed point argument (specifically, the Tarski-Knaster lattice fixed point theorem; see Theorem 12.4) to be applied. The questions of existence and uniqueness are completely resolved under quite weak conditions on the agents' utility functions. In particular, all HARA functions, whether $U'(0)$ is finite or not, are included. The method also may provide tools for study of how economies which are not in equilibrium might converge to equilibrium, although that is beyond our present scope.

One important step in the search for equilibrium is to introduce a "representative agent", that is, to replace the many agents with distinct utility functions and incomes by a single agent who represents their individual interests and has their aggregate income. In Cox, Ingersoll and Ross [2], this step is simply eliminated by the assumption that all agents have the same utility functions and the same incomes. Under such an assumption, attention is immediately focussed on a single, generic agent, and questions of existence and uniqueness of equilibrium are trivialized. Huang [9], on the other hand, selects a set of positive weights $(\lambda_1,\ldots,\lambda_J)$ and defines a new utility function by

$$(1.1) \qquad U(c) = \max_{\substack{c_1 \geq 0 \ldots c_J \geq 0 \\ c_1 + \ldots + c_J = c}} [\lambda_1 U_1(c_1) + \ldots + \lambda_J U_J(c_J)].$$

The weights $(\lambda_1, \ldots, \lambda_J)$ characterize the representative agent. Huang's goal is to study the nature of equilibrium, and he is content to assume rather than prove its existence. In contrast, we wish to construct equilibrium, and we reduce that construction to the problem of finding an appropriate representative agent, i.e., a vector $(\lambda_1, \ldots, \lambda_J)$. This allows for the equilibrium to be constructed in $\mathbb{R}^J$, and not in some infinite-dimensional function space. In our setting, the optimal consumption strategies of the individual agents can be found explicitly in terms of the equilibrium values of $\lambda_1, \ldots, \lambda_J$.

This paper can be read independently of all previous work on capital asset pricing and equilibrium theory. It is organized as follows. Section 2 sets out the basic idea of equilibrium in a simple, two-stage model. The model of interest in this paper is considerably more complex than that of Section 2, but many of the essential features of the complex model are already present in the simpler setting. Consequently, the simpler model is a useful aid to understanding the more complicated one, which is presented in Section 3-7, culminating with the definitions of existence and uniqueness of equilibrium in Section 7. In Section 8 we show how the absence of arbitrage opportunities, a necessary ingredient in equilibrium, determines the prices of the productive assets. This allows us to effectively eliminate these assets and their price processes from the model, leaving the spot price process of the commodity as the only endogenous process. If a spot price process for the commodity is given, each individual agent then faces the problem of the maximization of his expected utility from consumption; in Section 9 we solve this stochastic control problem. It remains to determine the (equilibrium) spot price process which causes the markets to clear. We embark on that task in Section 10 with the introduction of the utility function for a "representative agent" and the explanation of how the representative agent relates to equilibrium (Theorems 10.2, 10.3). In Theorem 11.1 we state the existence and uniqueness of a fixed point for a certain operator from $(0,\infty)^J$ into itself, and in the remainder of Section 11 we show how all the properties we desire for equilibrium flow from this theorem. Section 12 proves the existence assertion of Theorem 11.1, Section 13 establishes uniqueness in the simplest case, and Section 15, the appendix, treats uniqueness in the more difficult case. These proofs are entirely self-contained. Section 14 gives three examples in which the equilibrium can be computed explicitly, and one example that shows how uniqueness can fail when our assumptions are violated.

2. The Idea of Equilibrium

In this section, we present an extremely simple model whose purpose is to illustrate the idea of equilibrium and to foreshadow the results that we shall obtain for the more elaborate model whose development begins in the next section.

Suppose there are J agents and each agent j will receive a positive income $\hat{c}_j(1)$ of units of a certain commodity in period one and a second positive income $\hat{c}_j(2)$ of units of the same commodity in period two. The agent wishes to maximize his utility from consumption of the commodity over these two periods. If he sets his period t consumption to be $c_j(t)$, $t = 1,2$, then the utility is defined to be

$$\log c_j(1) + \log c_j(2).$$

We shall always require that $c_j(t) \geq 0$, $t = 1,2$, and we define $\log 0 = -\infty$.

If the only commodity available to the agent is his income $\hat{c}_j(1)$, $\hat{c}_j(2)$, and we assume that the commodity is perishable (so that commodity not consumed in period one is not available in period two), then the agent must choose $c_j(1) \in [0, \hat{c}_j(1)]$, $c_j(2) \in [0, \hat{c}_j(2)]$, and his optimal choices are

$$c_j(1) = \hat{c}_j(1), \quad c_j(2) = \hat{c}_j(2). \tag{2.1}$$

However, if agent j is allowed to trade with the other agents, his lot in life will be no worse and can probably be improved. To facilitate this trading, we postulate a <u>spot price</u> $\psi(t) > 0$ for the commodity in period t, $t = 1,2$. Thus, agent j can turn his endowment into

(2.2) $$\xi_j \triangleq \psi(1)\hat{c}_j(1) + \psi(2)\hat{c}_j(2)$$

dollars, and he can finance any consumption plan $c_j(1)$, $c_j(2)$ as long as

(2.3) $$\psi(1)c_j(1) + \psi(2)c_j(2) \leq \xi_j.$$

Note that we are allowing agent j to "borrow" against period two income in order to finance period one consumption. We thus have the following optimization problem for agent j:

$$\begin{aligned} \text{To maximize} \quad & \log c_j(1) + \log c_j(2) \quad , \\ \text{subject to} \quad & \psi(1)c_j(1) + \psi(2)c_j(2) \leq \xi_j, \\ & c_j(1) \geq 0,\ c_j\ (2) \geq 0. \end{aligned}$$

The unique solution to this problem is easily determined to be

(2.4) $$c_j^*(1) \triangleq \frac{\xi_j}{2\psi(1)}, \quad c_j^*(2) \triangleq \frac{\xi_j}{2\psi(2)},$$

and a bit of algebra gives: $\sum_{t=1}^{2} \log \hat{c}_j(t) \leq \sum_{t=1}^{2} \log c_j^*(t)$, with equality holding if and only if $\psi(1)\hat{c}_j(1) = \psi(2)\hat{c}_j(2)$. In other words, trading will strictly improve the lot of the j^{th} agent, unless the value ξ_j of his endowment is equally divided over the two periods.

The optimization problem for agent j can be stated and solved irrespectively of the choice of $\psi(1) > 0$, $\psi(2) > 0$. However, the commodity in question is perishable, and its only source in each period is the aggregate income of the agents in that period. Define the supply in period t to be

$$\hat{c}(t) = \sum_{j=1}^{J} \hat{c}_j(t); \qquad t = 1,2.$$

According to (2.4), the demand in period t is $\frac{1}{2\psi(t)} \sum_{j=1}^{J} \xi_j = \frac{1}{2\psi(t)} [\psi(1)\hat{c}(1) + \psi(2)\hat{c}(2)]$. An equilibrium spot price pair $(\psi(1), \psi(2))$ is one which causes supply to equal demand in each period, i.e.

$$\hat{c}(1) = \frac{1}{2\psi(1)} [\psi(1)\hat{c}(1) + \psi(2)\hat{c}(2)] , \quad \hat{c}(2) = \frac{1}{2\psi(2)} [\psi(1)\hat{c}(1) + \psi(2)\hat{c}(2)].$$

It is easily verified that these equilibrium conditions reduce to

$$\psi(1)\hat{c}(1) = \psi(2)\hat{c}(2). \tag{2.5}$$

Thus, the equilibrium prices are determined only up to a multiplicative constant, and are inversely proportional to supply. Substitution of (2.5) into (2.2), (2.4) results in

$$c_j^*(1) = \lambda_j\hat{c}(1), \quad c_j^*(2) = \lambda_j\hat{c}(2), \tag{2.6}$$

where

$$\lambda_j \triangleq \frac{1}{2} \left[\frac{\hat{c}_j(1)}{\hat{c}(1)} + \frac{\hat{c}_j(2)}{\hat{c}(2)} \right]. \tag{2.7}$$

Even though the equilibrium prices are not completely determined, the equilibrium optimal consumption plan of each agent is unique. Moreover, the consumption of agent j in each period is a fixed fraction λ_j of supply, and λ_j is directly related to agent j's relative importance in the economy.

We have given a complete analysis of this simple, two-stage, deterministic equilibrium model. We list here four ingredients of a more realistic model.

(1) Agents should not perfectly know their future incomes, nor the future spot prices. In this paper, these will be modelled by stochastic processes.

(2) Money which is borrowed or held between periods should incur an interest charge or could be invested, respectively. In this paper, we shall create stochastically priced financial instruments to model borrowing and investing.

(3) Not all agents should have the same utility from consumption. In this paper, each agent will have his own utility function, in constrast to the above model in which each agent had the logarithmic utility function.

(4) Trading opportunities and consumption decisions should be allowed to occur more than twice. The model of this paper is in continuous time with a finite planning horizon.

The principal results we obtain for the model of this paper are essentially those obtained for the simplified model of this section. They can be formulated as follows:

(I) An equilibrium spot price process exists, and is unique up to a multiplicative constant.

(II) The equilibrium optimal consumption processes of the individual agents are unique.

3. The Model Primitives

We begin with an exogenous N-dimensional Brownian motion $W = \{W(t) = (W_1(t),\ldots,W_N(t))^{tr}, \mathcal{F}(t); 0 \leq t \leq T\}$ on a probability space $(\Omega,\mathcal{F},P)$. Here and elsewhere, the superscript tr denotes transposition. The filtration $\{\mathcal{F}(t)\}$ is the augmentation under P of the filtration generated by W; it represents the information available to the agents at time t, and all processes which follow are assumed to be $\{\mathcal{F}(t)\}$-adapted.

Our model has M productive assets, and associated with each one of them is an $\{\mathcal{F}(t)\}$-adapted, bounded, measurable, nonnegative, exogenous dividend process $\{\delta_m(t); 0 \leq t \leq T\}$. Ownership of one share of asset m entitles one to receive the dividend process δ_m, which is denominated in units of the single commodity in our economy, not in cash. We denote by $\delta(t)$ the M-dimensional column vector whose m-th component is $\delta_m(t)$.

There are J agents in the economy, and each agent j has an initial endowment of $\epsilon_{j,m}$ shares of productive asset m. We assume that $\epsilon_{j,m} \geq 0$; $\forall\ 1 \leq j \leq J,\ 1 \leq m \leq M$, and

$$(3.1) \qquad \sum_{j=1}^{J} \epsilon_{j,m} = 1, \quad m = 1,\ldots,M;$$

in other words, exactly one share of each asset is owned. We denote by ϵ_j the M-dimensional row vector $(\epsilon_{j,1},\ldots,\epsilon_{j,M})$ of agent j's endowments. In addition to his endowment, each agent j is entitled to a bounded, measurable, $\{\mathcal{F}(t)\}$-adapted, nonnegative, exogenous earnings process $\{e_j(t); 0 \leq t \leq T\}$, measured in units of commodity. Thus, if he takes no action, agent j will receive the income process, measured in units of commodity,

$$\hat{c}_j(t) \triangleq e_j(t) + \epsilon_j\delta(t); \quad 0 \leq t \leq T. \tag{3.2}$$

We assume that the nonnegative process $\hat{c}_j(t,\omega)$ is positive on a set of positive product (i.e., Lebesgue × P) - measure; otherwise, agent j would have no role to play in the equilibrium model. The <u>aggregate income process</u> is

$$\hat{c}(t) \triangleq \sum_{j=1}^{J} \hat{c}_j(t) = \sum_{j=1}^{J} e_j(t) + \sum_{m=1}^{M} \delta_m(t); \quad 0 \leq t \leq T, \tag{3.3}$$

which we assume to satisfy

$$0 < k \leq \hat{c}(t) \leq K; \quad \forall\ (t,\omega) \in [0,T] \times \Omega \tag{3.4}$$

for some constants k and K.

Each agent j has a measurable <u>utility function</u> $U_j(t,c)\colon [0,T] \times (0,\infty) \to \mathbb{R}$, which quantifies the "utility" that he derives by consuming his wealth at the rate $c > 0$ at time t. For every $t \in [0,T]$, the function $U_j(t,\cdot)\colon (0,\infty) \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, strictly concave, and satisfies

$$U_j(t,c) \leq k_1 + k_2 c^{\rho}; \ \forall\ c > 0, \tag{3.5}$$

$$\lim_{c\to\infty} U_j'(t,c) = 0, \tag{3.6}$$

$$\frac{d}{dc}(c\, U_j'(t,c)) \geq 0; \ \forall\ c > 0, \tag{3.7}$$

where $k_1, k_2 \in (0,\infty)$ and $\rho \in (0,1)$ are independent of t. Here and throughout the paper, prime denotes differentiation with respect to the second (consumption) argument. An immediate consequence of (3.7) is $U_j'(t,c) \geq \frac{1}{c} U_j'(t,1);\ \forall\ c \geq 1$, and integrating this inequality we see that

(3.8) $$\lim_{c\to\infty} U_j(t,c) = \infty.$$

We set $U_j(t,0) \triangleq \lim_{c\downarrow 0} U_j(t,c)$, $U_j'(t,0) \triangleq \lim_{c\downarrow 0} U_j'(t,c)$. Note that $-\infty \leq U_j(t,0) < \infty$ and $0 < U_j'(t,0) \leq \infty$.

If $U_j'(t,0) < \infty$ for some $t \in [0,T]$ and $j \in \{1,\dots,J\}$, then the assumptions made thus far are not sufficient to guarantee the uniqueness of equilibrium (Example 14.4). Consequently, we also assume that

(3.9) $$U_j'(t,0) = \infty,\ \forall\ t \in [0,T],\ \forall\ j \in \{1,\dots,J\}$$
$$\text{or } \hat{c}_j(t) > 0 \text{ a.s.},\ \forall\ t \in [0,T],\ \forall\ j \in \{1,\dots,J\}.$$

4. The Financial Assets

The J agents in Section 2 will be buying and selling among themselves the commodity and ownerships of the productive assets, but these instruments alone may not be sufficient to allow agents to hedge against all the risk inherent in the information pattern represented by $\{\mathcal{F}(t)\}$. This hedging occurs when agents finance their consumption strategies, and it finds its mathematical expression in the representation of $\{\mathcal{F}(t)\}$-martingales as stochastic integrals with respect to the underlying Brownian motion. To aid in this hedging, we introduce $N + 1$ <u>financial assets</u> with prices per share $\{f_n(t); 0 \leq t \leq T\}$ governed by the differential equations

(4.1) $$df_0(t) = r(t)f_0(t)dt; \quad 0 \leq t \leq T,$$

(4.2) $$df_n(t) = f_n(t)[b_n(t)dt + a_n(t)dW(t)]; \quad 0 \leq t \leq T, \quad n = 1,\dots,N.$$

We take these equations to have the initial condition

(4.3) $$f_n(0) = 1; \quad n = 1,\dots,N.$$

Equations (4.1), (4.2) with initial conditions (4.3) have the unique solutions

(4.4) $$f_0(t) = \exp\{\int_0^t r(s)ds\},$$

(4.5) $$f_n(t) = \exp\{\int_0^t [b_n(s) - \frac{1}{2}\|a_n(s)\|^2]ds + \int_0^t a_n(s)dW(s)\}; \quad n = 1,\dots,N.$$

Note that these solutions are always strictly positive.

We denote by $F(t)$ the $(N + 1)$-dimensional column vector of financial asset prices $F(t) = (f_0(t),\dots,f_N(t))^{tr}$ and by $f(t)$ the N-dimensional vector $(f_1(t),\dots,f_N(t))^{tr}$. The <u>interest rate</u> process $\{r(t);\ 0 \leq t \leq T\}$ as well as the vector of <u>mean rates of return</u> $\{b(t) = (b_1(t),\dots,b_n(t))^{tr};\ 0 \leq t \leq T\}$ and the $N \times N$ <u>dispersion matrix</u> $a(t)$, whose n-th row is $a_n(t) = (a_{n,1}(t),\dots,a_{n,N}(t))$, are assumed to be measurable, $\{\mathcal{F}(t)\}$-adapted, and bounded uniformly in $(t,\omega) \in [0,T] \times \Omega$. These processes are exogenous.

The financial assets represent contracts between agents and in equilibrium will be in zero net supply. Although they are rather arbitrarily chosen, we shall see that the particular choices of $r(\cdot)$, $b(\cdot)$ and $a(\cdot)$ have minimal effect on the equilibrium.

A market in which all risk can be hedged against is referred to as <u>complete</u>. It may be possible to obtain a complete market in our model by introducing fewer than $N + 1$ financial assets, but the feasibility of this depends on the nature of the equilibrium itself. We have, therefore, taken the convenient approach of making available enough financial assets to complete the market, regardless of the nature of the equilibrium we finally obtain.

We impose the <u>nondegeneracy assumption</u> that for some $\epsilon > 0$,

$$(4.6)\qquad \xi^{tr}a(t)a^{tr}(t)\xi \geq \epsilon\|\xi\|^2;\ \forall\ \xi \in \mathbb{R}^N,\ (t,\omega) \in [0,T] \times \Omega.$$

Under this assumption, the matrices $a(t)$ and $a^{tr}(t)$ are invertible, and we have, according to Lemma 2.1 of KLS:

$$(4.7)\qquad \|a^{tr}(t,\omega)^{-1}\xi\| \leq \frac{1}{\sqrt{\epsilon}}\|\xi\|,\ \|a(t,\omega)^{-1}\xi\| \leq \frac{1}{\sqrt{\epsilon}}\|\xi\|;\ \forall\ \xi \in \mathbb{R}^N, (t,\omega) \in [0,T] \times \Omega.$$

The financial asset prices f_0 and $f_1, \dots, f_N$ have mean rates of return r and $b_1, \dots, b_N$, respectively, and our first task is to change the probability measure so as to make them all have the same mean rate of return. Toward this end, let us introduce the "relative risk" process

$$\theta(t) = (a(t))^{-1}[b(t) - r(t)\underset{\sim}{1}_N]. \tag{4.8}$$

Condition (4.7) ensures that $\|\theta\|$ is bounded by some constant. Define also the "likelihood ratio" process

$$Z(t) \triangleq \exp\{\int_0^t \theta^{tr}(s)dW(s) - 1/2 \int_0^t \|\theta(s)\|^2 ds\}, \quad 0 \le t \le T. \tag{4.9}$$

Then $\{Z(t), \mathcal{F}(t); 0 \le t \le T\}$ is a martingale, and the new probability measure given by

$$\tilde{P}(A) = E[Z(T) \cdot 1_A]; \quad \forall A \in \mathcal{F}(T), \tag{4.10}$$

is such that P and $\tilde{P}$ are mutually absolutely continuous on $\mathcal{F}(T)$. When making statements which hold almost surely, we are thus not obliged to distinguish between these two probability measures. Furthermore,

$$\tilde{W}(t) \triangleq W(t) + \int_0^t \theta(s)ds; \quad 0 \le t \le T, \tag{4.11}$$

is a standard N-dimensional Brownian motion under $\tilde{P}$ (Girsanov [8] or Karatzas & Shreve [10], §3.5).

The financial assets in the model will dynamically affect the value of money. We shall see that the process

$$\zeta(t) \triangleq Z(t)e^{-\int_0^t r(u)du}; \quad 0 \le t \le T, \tag{4.12}$$

can serve as a "deflator", in the sense that multiplication by $\zeta(t)$ converts wealth held at time t to the equivalent amount of wealth at time zero.

5. The Endogenous Price Processes

As we mentioned earlier, there is a single commodity traded in the economy; agents may buy or sell it at each time t at a spot price of $\psi(t)$ dollars per unit. This process will be determined endogenously, by the equilibrium considerations to be specified later, in such a way as to satisfy

$$(5.1) \qquad 0 < k(\psi) \le \zeta(t)\psi(t) \le K(\psi); \quad \forall\ (t,\omega) \in [0,T] \times \Omega.$$

Here, $k(\psi)$ and $K(\psi)$ are constants which may depend on ψ but not on (t,ω). Agents may also buy or sell part or all of the productive assets at their nonnegative prices per share $P_1(t),\dots,P_M(t)$, which will likewise be endogenous. We denote by $P(t)$ the M-dimensional column vector of productive asset prices $P(t) = (P_1(t),\dots,P_M(t))^{tr}$. We require that for each m,

$$(5.2) \qquad \zeta P_m \text{ is bounded (uniformly in } t \text{ and } \omega),$$

and P_m is a nonnegative $\{\mathcal{F}(t)\}$-semimartingale of the form

$$(5.3) \qquad dP_m(t) = \beta_m(t)dt + \alpha_m(t)dW(t); \ 0 \le t \le T, \quad m = 1,\dots,M,$$

where $\{\beta(t) = (\beta_1(t),\dots,\beta_m(t))^{tr};\ 0 \le t \le T\}$ and $\{\alpha(t) = (\alpha_{m,n}(t));$ $m = 1,\dots,M;\ n = 1,\dots,N;\ 0 \le t \le T\}$ are processes to be determined endogenously so that

$$(5.4) \qquad \int_0^T [\|\beta(t)\| + \sum_{m=1}^{M} \|\alpha_m(t)\|^2]dt < \infty, \quad \text{a.s.}$$

At the terminal time T, the productive asset m has paid out all its dividends and has no further value. We thus require that

(5.5) $$P_m(T) = 0; \text{ a.s., } m = 1,\ldots,M.$$

6. The Optimization Problem for an Individual Agent

Each agent j will choose for himself a consumption process $\{c_j(t); 0 \le t \le T\}$, a productive asset portfolio process $\{\pi_j(t) = (\pi_{j,1}(t),\ldots,\pi_{j,m}(t));\ 0 \le t \le T\}$, and a financial asset portfolio process $\{\Phi_j(t) = (\phi_{j,0}(t),\ldots,\phi_{j,N}(t));\ 0 \le t \le T\}$, such that

$$(6.1) \qquad \inf_{0\le t\le T} c_j(t) \ge 0, \quad \sup_{0\le t\le T} c_j(t) < \infty, \quad \text{a.s.},$$

$$(6.2) \qquad \sup_{0\le t\le T} [\|\pi_j(t)\| + \|\Phi_j(t)\|] < \infty, \quad \text{a.s.}$$

We denote by $\phi_j(t)$ the N-dimensional process $(\phi_{j,1}(t),\ldots,\phi_{j,N}(t))$. The nonnegative consumption process represents the rate at which the agent consumes the commodity, and is thus denominated in units of the commodity. The components of the portfolio processes may be either positive or negative, and represent the j^{th} agent's positions, measured in numbers of shares, in the respective assets. Initially, we have $\pi_{j,m}(0) = \epsilon_{j,m}$; $m = 1,\ldots,M$, and $\phi_{j,n}(0) = 0$; $n = 0,\ldots,N$, and we require that

$$(6.3) \qquad \pi_j(t)P(t)+\Phi_j(t)F(t) = \epsilon_j P(0) - \int_0^t \psi(s)c_j(s)ds + \int_0^t \psi(s)e_j(s)ds$$

$$+ \int_0^t \psi(s)\pi_j(s)\delta(s)ds + \int_0^t \pi_j(s)dP(s) + \int_0^t \Phi_j(s)dF(s);$$

$$0 \le t \le T, \text{ a.s.}$$

The integrals on the right-hand side of (5.3) account, respectively, for:

(i) the decrease in wealth due to consumption, (ii) the increase in wealth due to earnings, (iii) the increase in wealth due to dividends paid by productive assets held, (iv) capital gains or losses from productive assets held, and (v) capital gains or losses from financial assets held. We call relation (6.3) the budget equation and refer to

$$(6.4) \qquad X_j(t) \triangleq \pi_j(t)P(t) + \Phi_j(t)F(t)$$

as the wealth of agent j at time t. Triples (c_j, π_j, Φ_j) which satisfy the budget equation are self-financing, in the sense that all changes in wealth are accounted for within the model.

We note that the definition of X_j leads to the formula

$$(6.5) \qquad \phi_{j,0}(t) = \frac{1}{f_0(t)} [X_j(t) - \pi_j(t)P(t) - \phi_j(t)f(t)].$$

Substitution of (6.4) and (6.5) into (6.3) gives the budget equation in revised form

$$(6.6) \qquad X_j(t) = \epsilon_j P(0) + \int_0^t \psi(s)[e_j(s) - c_j(s)]ds + \int_0^t r(s)X_j(s)ds$$

$$+ \int_0^t \phi_j(s)\mathrm{diag}(f(s))[b(s) - r(s)\underset{\sim}{1}_N]ds$$

$$+ \int_0^t \pi_j(s)[\psi(s)\delta(s) + \beta(s) - r(s)P(s)]ds$$

$$+ \int_0^t [\pi_j(s)\alpha(s) + \phi_j(s)\mathrm{diag}(f(s))a(s)]dW(s),$$

where diag(f(s)) is the $N \times N$ diagonal matrix whose diagonal entries are the components of $f(s)$, and $\underset{\sim}{1}_N$ is the N-dimensional column vector whose every component is 1. In terms of the process $\tilde{W}$ of (4.11) and the coefficient processes in (4.1), (4.2), and (5.3), the budget equation (6.6) becomes

(6.7) $$X_j(t) = \epsilon_j P(0) + \int_0^t \psi(s)[e_j(s) - c_j(s)]ds + \int_0^t r(s)X_j(s)ds$$

$$+ \int_0^t \pi_j(s)[\psi(s)\delta(s) + \beta(s) - r(s)P(s) - \alpha(s)\theta(s)]ds$$

$$+ \int_0^t [\pi_j(s)\alpha(s) + \phi_j(s)\text{diag}(f(s))a(s)]d\tilde{W}(s).$$

While agents may have short-term deficits, we require that they choose consumption and portfolio processes so that for some positive constant $K(c,\pi,\phi)$ depending on the indicated processes but not on (t,ω),

(6.8) $$\zeta(t)X_j(t) \geq - K(c,\pi,\phi); \ \forall\ t \in [0,T],\ \text{a.s.},$$

(6.9) $$X_j(T) \geq 0,\ \text{a.s.}$$

6.1 Definition. Let a spot price process ψ satisfying (5.1) and a vector of productive asset prices $(P_1,\ldots,P_M)$ of the form (5.3), (5.5) be given, where (5.2), (5.4) are also satisfied. Relative to these, a triple (c_j, π_j, ϕ_j) of consumption, productive asset portfolio, and financial asset portfolio processes is <u>feasible for agent j</u> if (6.1), (6.2) and (6.3) are satisfied, and $X_j(\cdot)$ defined by (6.4), or equivalently by (6.7), satisfies (6.8), (6.9). A triple $(c_j^*, \pi_j^*, \Phi_j^*)$ is <u>optimal for agent j</u> if it is feasible and maximizes the expected total utility from consumption

$$E\int_0^T U_j(t, c_j(t))dt, \tag{6.10}$$

over all feasible triples (c_j, π_j, Φ_j) for which

$$E\int_0^T \max\{0, -U_j(t, c_j(t))\}dt < \infty. \tag{6.11}$$

7. The Definition of Equilibrium

When agent j is attempting to solve his optimization problem, he acts as a price-taker. In particular, he has no influence over ψ and $(P_1,\ldots,P_M)$. In aggregate, however, the actions of the agents should determine the prices ψ and $(P_1,\ldots,P_M)$ through the law of supply and demand. This law dictates that all the commodity be consumed as it enters the economy, that the aggregate demand for each productive asset be one share (which is the initial supply, cf. (3.1)), and that the aggregate demand for each financial asset be zero.

7.1 Definition. An equilibrium consists of a spot price process ψ satisfying (5.1), a vector of productive asset prices $(P_1,\ldots,P_m)$ of the form (5.3), (5.5) for which (5.2) and (5.4) are also satisfied, and a collection of consumption, productive asset portfolio, and financial asset portfolio triples (c_j^*,π_j^*,Φ_j^*), $j = 1,\ldots,J$. Each (c_j^*,π_j^*,Φ_j^*) must be optimal for agent j relative to ψ and $(P_1,\ldots,P_M)$, and for Lebesgue-almost every $t \in [0,T]$ and P-a.e. $\omega \in \Omega$, the market clearing conditions must hold:

$$\sum_{j=1}^{J} c_j^*(t) = \hat{c}(t), \tag{7.1}$$

$$\sum_{j=1}^{J} \pi_j^*(t) = \underset{\sim}{1}_M^{tr}, \tag{7.2}$$

$$\sum_{j=1}^{J} \Phi_j^*(t) = \underset{\sim}{0}_{N+1}^{tr}. \tag{7.3}$$

Here, $\underset{\sim}{1}_M$ is the M-dimensional column vector with all components equal to 1, and $\underset{\sim}{0}_{N+1}$ is the (N+1)-dimensional column vector with all components equal to 0. □

<u>7.2 Remark</u>. If $[\psi, (P_1,\dots,P_M), \{(c_j^*,\pi_j^*,\Phi_j^*); j = 1,\dots,J\}]$ is an equilibrium, then for every j, we have $c_j^*(t) \le \hat{c}(t)$. From condition (3.4), it follows then that

$$(7.4) \qquad E\int_0^T U_j(t,c_j^*(t))dt < \infty; \quad j = 1,\dots,J. \qquad \square$$

It would not be correct to think of the collective actions of agents as determining the prices, unless equilibrium is essentially unique. Of course, prices cannot be entirely unique, since the currency can always be revalued, which would have the effect of scaling ψ and $(P_1,\dots,P_M)$. We are thus led to the following concept.

<u>7.3 Definition</u>. Suppose that for any two equilibria $[\psi, (P_1,\dots,P_M), \{(c_j^*,\pi_j^*,\Phi_j^*); j = 1,\dots,J\}]$ and $[\tilde{\psi}, (\tilde{P}_1,\dots,\tilde{P}_M), \{(\tilde{c}_j^*,\tilde{\pi}_j^*,\tilde{\Phi}_j^*); j = 1,\dots,J\}]$, there exists a positive constant γ for which

$$(7.5) \qquad \psi(t) = \gamma\tilde{\psi}(t), \; P_1(t) = \gamma\tilde{P}_1(t),\dots,P_M(t) = \gamma\tilde{P}_M(t),$$

for Lebesgue-almost every $t \in [0,T]$ and P-a.e. $\omega \in \Omega$. Then we say that <u>equilibrium is unique</u>. $\square$

If (7.5) holds, then agent j faces the same optimization problem relative to $\tilde{\psi}$ and $(\tilde{P}_1,\dots,\tilde{P}_M)$ as he does relative to ψ and $(P_1,\dots,P_M)$. The optimal productive asset and financial asset portfolios for this problem may not be unique, but we shall see that the optimal consumption process is (Theorem 9.4). Thus, for the two equilibria in Definition 7.3, relations (7.5) will imply that for Lebesgue-almost every $t \in [0,T]$ and P-a.e. $\omega \in \Omega$,

$$c_j^*(t) = \tilde{c}_j^*(t). \qquad (7.6)$$

It will also follow from our analysis that if $[\psi, (P_1,\ldots,P_M), \{(c_j^*,\pi_j^*,\Phi_j^*); j = 1,\ldots,J\}]$ is an equilibrium and $[\tilde{\psi}, (\tilde{P}_1,\ldots,\tilde{P}_M), \{(\tilde{c}_j^*,\tilde{\pi}_j^*,\tilde{\Phi}_j^*); j = 1,\ldots,J\}]$ is an equilibrium for another model which differs from the first only in the choice of $r(\cdot)$, $b(\cdot)$ and $a(\cdot)$, then (7.6) holds (Corollary 11.3), although (7.5) may not. The conclusion we draw is that the exogenously selected financial assets can affect the value of money by more than a multiplicative factor, but they cannot affect the way in which real wealth, measured in units of commodity, is ultimately distributed among the agents.

8. The Equilibrium Prices of Productive Assets

In this section we shall assume the existence of equilibrium and draw conclusions about the prices of the productive assets. Our principal result is that their associated gains processes must be martingales under the probability measure $\tilde{P}$, a fact which imposes the particular form (8.6) on the productive asset prices.

8.1 Lemma. Let $[\psi, (P_1,\dots,P_M), \{(c_j^*,\pi_j^*,\Phi_j^*);\ j = 1,\dots,J\}]$ be an equilibrium. Then

$$(8.1) \qquad \psi(t)\delta(t) + \beta(t) - r(t)P(t) - \alpha(t)\theta(t) = 0$$

can fail only on a subset of $[0,T] \times \Omega$ with zero Lebesgue $\times$ P - measure.

Proof: Let j be a given integer between 1 and J, and let

$$(8.2) \qquad X_j^*(t) = \epsilon_j P(0) + \int_0^t \psi(s)[e_j(s) - c_j^*(s)]ds + \int_0^t r(s)X_j^*(s)ds$$

$$+ \int_0^t \pi_j^*(s)[\psi(s)\delta(s) + \beta(s) - r(s)P(s) - \alpha(s)\theta(s)]ds$$

$$+ \int_0^t [\pi_j^*(s)\alpha(s) + \phi_i^*(s)\mathrm{diag}(f(s))a(s)]d\tilde{W}(s)$$

be the wealth process corresponding to (c_j^*,π_j^*,Φ_j^*). The feasibility of (c_j^*,π_j^*,Φ_j^*) implies that ζX_j^* is bounded below and $X_j^*(T) \ge 0$, a.s. Define

$$\pi_j(t) \triangleq \operatorname{sgn} [\psi(t)\delta(t) + \beta(t) - r(t)P(t) - \alpha(t)\theta(t)]^{tr},$$

where the signum function is applied separately to each component of the above vector. Because $a(\cdot)$ is invertible, there exists a unique, N-dimensional process $\phi_j(t) = (\phi_{j,1}(t),\ldots,\phi_{j,N}(t))$ such that

$$\pi_j(t)\alpha(t) + \phi_j(t)\mathrm{diag}(f(t))a(t) = 0;\ 0 \le t \le T, \text{ a.s.}$$

Set $\Phi_j(t) = (0,\phi_{j,1}(t),\ldots,\phi_{j,N}(t))$, $0 \le t \le T$. We define also

$$c_j(t) = \frac{1}{\psi(t)}\pi_j(t)\ [\psi(t)\delta(t) + \beta(t) + r(t)P(t) - \alpha(t)\theta(t)],$$

$$\tilde{c}_j(t) = c_j^*(t) + c_j(t),\ \tilde{\pi}_j(t) = \pi_j^*(t) + \pi_j(t),\ \tilde{\Phi}_j(t) = \Phi_j^*(t) + \Phi_j(t);\ 0 \le t \le T.$$

Note that X_j^* given by (8.2) is also the wealth process corresponding to $(\tilde{c}_j,\tilde{\pi}_j,\tilde{\Phi}_j)$; it follows from the feasibility of (c_j^*,π_j^*,Φ_j^*) that $(\tilde{c}_j,\tilde{\pi}_j,\tilde{\Phi}_j)$ is feasible. By construction, $\tilde{c}_j(t) \ge c_j^*(t)$, and if (8.1) failed on a subset of $[0,T] \times \Omega$ with positive Lebesgue $\times$ P - measure, then this inequality would be strict on this set. According to Remark 7.2 and the strict monotonicity of $U_j(t,\cdot)$, we would then have

$$E\int_0^T U_j(t,c_j^*(t))dt < E\int_0^T U_j(t,\tilde{c}_j(t))dt,$$

which would be a violation of the optimality of (c_j^*,π_j^*,Φ_j^*). □

We may solve (8.1) for the drift in the productive assets $\beta(\cdot)$ and substitute this into (5.3), to obtain

$$dP(t) = [r(t)P(t) - \psi(t)\delta(t)]dt + \alpha(t)d\tilde{W}(t).$$

This linear stochastic differential equation has a unique solution, which leads to the expression

$$(8.3) \qquad G(t) = P(0) + \int_0^t e^{-\int_0^s r(u)du} \alpha(s)d\tilde{W}(s)$$

for the <u>gains process</u>

$$(8.4) \qquad G(t) \triangleq e^{-\int_0^t r(s)ds} P(t) + \int_0^t e^{-\int_0^s r(u)du} \psi(s)\delta(s)ds.$$

We see from (8.3) that under the $\tilde{P}$-measure, with respect to which $\tilde{W}$ is a Brownian motion, the gains process $G(\cdot)$ is a vector of local martingales. In particular, for each positive integer n, we may define

$$\sigma_n \triangleq T \wedge \inf\{t \in [0,T]: \int_0^t \|\alpha(s)\|^2 ds = n\},$$

and then we will have for $0 \leq t \leq T$,

$$G(t \wedge \sigma_n) = \tilde{E}[G(\sigma_n)|\mathcal{F}(t)]$$

$$= 1_{\{\sigma_n < t\}} G(\sigma_n) + 1_{\{\sigma_n \geq t\}} \frac{1}{Z(t)} E[Z(\sigma_n)G(\sigma_n)|\mathcal{F}(t)].$$

Here we have used Lemma 3.5.3 of Karatzas & Shreve [10] to change from the conditional expectation under $\tilde{P}$ to the conditional expectation under P. Letting $n \to \infty$ and recalling (5.4), we obtain

$$G(t) = \lim_{n\to\infty} 1_{\{\sigma_n \geq t\}} \frac{1}{Z(t)} E[Z(\sigma_n)G(\sigma_n)|\mathcal{F}(t)], \quad \text{a.s.} \tag{8.5}$$

But on the event $\{\sigma_n > t\}$,

$$\frac{1}{Z(t)} E[Z(\sigma_n)G(\sigma_n)|\mathcal{F}(t)] = \frac{1}{Z(t)} E[\zeta(\sigma_n)P(\sigma_n)|\mathcal{F}(t)]$$

$$+ \frac{1}{Z(t)} \cdot E[Z(\sigma_n)|\mathcal{F}(t)] \cdot \int_0^t e^{-\int_0^s r(u)du} \psi(s)\delta(s)ds$$

$$+ \frac{1}{Z(t)} \cdot E[Z(\sigma_n) \int_t^{\sigma_n} e^{-\int_0^s r(u)du} \psi(s)\delta(s)ds|\mathcal{F}(t)]$$

$$= \frac{1}{Z(t)} E[\zeta(\sigma_n)P(\sigma_n)|\mathcal{F}(t)] + \int_0^t e^{-\int_0^s r(u)du} \psi(s)\delta(s)ds$$

$$+ \frac{1}{Z(t)} \cdot E[\int_t^{\sigma_n} \zeta(s)\psi(s)\delta(s)ds|\mathcal{F}(t)].$$

We have assumed that ζP and $\zeta\psi$ are bounded (see (5.1), (5.2)), and so the bounded convergence theorem asserts that the limit in (8.5) is

$$\frac{1}{Z(t)} E[\zeta(T)P(T)|\mathcal{F}(t)] + \int_0^t e^{-\int_0^s r(u)du} \psi(s)\delta(s)ds$$

$$+ \frac{1}{Z(t)} \cdot E[\int_t^T \zeta(s)\psi(s)\delta(s)ds \ |\mathcal{F}(t)] = \tilde{E}[G(T)|\mathcal{F}(t)].$$

We conclude that G is a martingale under $\tilde{P}$. The process $G(\cdot)$ at time t records the current values of the productive assets plus the values of the dividends paid out during $[0,t]$, converted to dollars. All these values are discounted back to the initial time via the interest rate process $r(\cdot)$. The essence of the proof of Lemma 8.1 is that, if G is not a martingale under $\tilde{P}$, then arbitrage opportunities exist in the trading of productive assets against financial assets. This observation allows us to express the productive asset prices in terms of the other data of the model, a fact which we now state as a theorem.

<u>8.2 Theorem</u>. Let $[\psi, (P_1,\dots,P_M), \{(c_j^*,\pi_j^*,\Phi_j^*); \ j = 1,\dots,J\}]$ be an equilibrium. Then

$$(8.6)\qquad P(t) = \tilde{E}\left[\int_t^T e^{-\int_t^s r(u)du}\psi(s)\delta(s)ds \,\middle|\, \mathcal{F}(t)\right]$$

$$= \frac{1}{\zeta(t)} E\left[\int_t^T \zeta(s)\psi(s)\delta(s)ds \,\middle|\, \mathcal{F}(t)\right], \qquad 0 \leq t \leq T, \text{ a.s.}$$

Proof: From (5.5) and (8.4), we have $G(T) = \int_0^T e^{-\int_0^s r(u)du}\psi(s)\delta(s)ds$, from which follows for $0 \leq t \leq T$,

$$P(t) = e^{\int_0^t r(s)ds} G(t) - \int_0^t e^{-\int_t^s r(u)du}\psi(s)\delta(s)ds$$

$$= e^{\int_0^t r(s)ds} \tilde{E}[G(T)\,|\,\mathcal{F}(t)] - \int_0^t e^{-\int_t^s r(u)du}\psi(s)\delta(s)ds$$

$$= \tilde{E}\left[\int_t^T e^{-\int_t^s r(u)du}\psi(s)\delta(s)ds \,\middle|\, \mathcal{F}(t)\right], \quad \text{a.s.}$$

The second equality in (8.6) is the result of changing from the $\tilde{P}$-measure to the P-measure. □

Theorem 8.2 assumes the existence of an equilibrium, which includes the assumption of the existence of a vector of productive asset prices. However, that theorem provides the formula (8.6) for this vector (in terms of the dividend and spot price processes), a fact which suggests that the existence of productive asset prices could be a conclusion rather than a hypothesis of the model. In our eventual construction of equilibrium, we will in fact obtain these prices via formula (8.6), so we must show that the prices so obtained satisfy the conditions imposed on them in Definition 7.1. For this and later purposes, it will be necessary to represent martingales (under $\tilde{P}$) as stochastic integrals with respect to $\tilde{W}$. We first state this representation result, and then verify that formula (8.6) can be used to construct productive asset prices.

8.3 Lemma. Let $\{Y(t), \mathcal{F}(t); 0 \leq t \leq T\}$ be a martingale under $\tilde{P}$. Then there exists an N-dimensional process $\{H(t) = (H_1(t),\ldots,H_N(t)), \mathcal{F}(t); 0 \leq t \leq T\}$ such that

(8.7) $$\int_0^T \|H(t)\|^2 dt < \infty, \quad \text{a.s.},$$

$$Y(t) = Y(0) + \int_0^t H(s)d\tilde{W}(s), \quad 0 \leq t \leq T, \text{ a.s.}$$

Proof: We note first of all that for $0 \leq s \leq t \leq T$,

$$E[Z(t)Y(t)|\mathcal{F}(s)] = Z(s)\tilde{E}[Y(t)|\mathcal{F}(s)] = Z(s)Y(s), \quad \text{a.s.},$$

so ZY is a martingale under P. Because $\{\mathcal{F}(t)\}$ is the augmentation of the filtration generated by the Brownian motion W (under P), there exists a process $L = (L_1,\ldots,L_N)$ such that

$$E \int_0^T \|L(t)\|^2 dt < \infty$$

$$Z(t)Y(t) = Y(0) + \int_0^t L(s)dW(s), \quad 0 \le t \le T, \text{ a.s.}$$

(Karatzas & Shreve [10], Theorem 3.4.15, Problem 3.4.16). Defining $u(v,z) = \frac{v}{z}$ and applying Itô's rule, we obtain

$$Y(t) = u(Z(t)Y(t),Z(t))$$

$$= Y(0) + \int_0^t [\frac{1}{Z(t)} L(s) - Y(s)\theta^{tr}(s)]\theta(s)ds$$

$$+ \int_0^t [\frac{1}{Z(s)} L(s) - Y(s)\theta^{tr}(s)]dW(s)$$

$$= Y(0) + \int_0^t H(s)d\tilde{W}(s), \quad 0 \le t \le T, \text{ a.s.},$$

where

$$H(t) \triangleq \frac{1}{Z(t)} L(t) - Y(t)\theta^{tr}(t), \; 0 \le t \le T, \text{ a.s.} \qquad \square$$

<u>8.4 Theorem</u>. Let a spot price ψ satisfying (5.1) be given, and define $P(t) = (P_1(t),\ldots,P_M(t))tr$ by (8.6). Then, for each $m = 1,\ldots,M$, P_m is a nonnegative Itô process satisfying the conditions (5.2) and (5.5), as well as

a differential equation of the form (5.3), where the coefficient processes β_m and α_m satisfy (5.4) and (8.1).

Proof: From (8.4) and (8.6) written componentwise, we obtain

$$G_m(t) = \tilde{E}\left[\int_0^T e^{-\int_0^s r(u)du} \psi(s)\delta_m(s)ds \,\middle|\, \mathcal{F}(t)\right]; \quad m = 1,\dots,M,$$

which is a martingale under $\tilde{P}$. According to Lemma 8.3, there exists an N-dimensional process $H = (H_1,\dots,H_N)$ satisfying (8.7) and for which

$$G_m(t) = G_m(0) + \int_0^t H(s)d\tilde{W}(s)$$

$$= G_m(0) + \int_0^t H(s)\theta(s)ds + \int_0^t H(s)dW(s).$$

Equation (8.4) now leads to the formula

$$P_m(t) = e^{\int_0^t r(s)ds}\left[G_m(0) + \int_0^t H(s)\theta(s)ds + \int_0^t H(s)dW(s)\right.$$

$$\left. - \int_0^t e^{-\int_0^s r(u)du} \psi(s)\delta(s)ds\right].$$

It follows from Itô's rule that P_m has the form indicated by (5.3) with

$$\beta_m(t) = r(t)P(t) - \psi(t)\delta(t) + e^{\int_0^t r(s)ds} H(t)\theta(t)$$

$$\alpha_m(t) = e^{\int_0^t r(s)ds} H(t). \qquad \square$$

9. The Solution of the Optimization Problem for an Individual Agent

Throughout this section, we have a fixed spot price process ψ satisfying (5.1), in terms of which the productive asset price process vector $P(t) = (P_1(t),\dots,P_M(t))$ is given by (8.6). We also fix an element $j \in \{1,\dots,J\}$. Agent j is unaware of any equilibrium considerations used to obtain ψ and P; he simply takes these as given and is not bound by any market clearing conditions. He also takes as given the model primitives of Section 3 and the exogenous processes of Section 4. We show in this section how agent j maximizes his expected utility of consumption.

9.1 Lemma. Let (c_j, π_j, ψ_j) be a feasible triple as described in Definition 6.1. Then

$$(9.1) \qquad E \int_0^T \zeta(s)\psi(s)c_j(s)ds \le E \int_0^T \zeta(s)\psi(s)\hat{c}_j(s)ds.$$

Proof: Under our assumptions, Theorem 8.4 implies the validity of (8.1), and so the budget equation (6.7) for the wealth of the j^{th} agent becomes

$$X_j(t) = \epsilon_j P(0) + \int_0^t \psi(s)[e_j(s) - c_j(s)]ds + \int_0^t r(s)X_j(s)ds$$

$$+ \int_0^t [\pi_j(s)\alpha(s) + \phi_j(s)\mathrm{diag}(f(s))a(s)]d\tilde{W}(s),$$

for which the unique solution is

$$(9.2) \quad X_j(t) = e^{\int_0^t r(u)du}\{\epsilon_j P(0) + \int_0^t e^{-\int_0^s r(u)du}\psi(s)[e_j(s) - c_j(s)]ds$$

$$+ \int_0^t e^{-\int_0^s r(u)du}[\pi_j(s)\alpha(s) + \phi_j(s)\mathrm{diag}(f(s))a(s)]d\tilde{W}(s)\}.$$

For each positive integer n, let

$$\tau_n = T \wedge \inf\{t \geq 0: \int_0^t \|\pi_j(s)\alpha(s) + \phi_j(s)\mathrm{diag}(f(s))a(s)\|^2 ds = n\}.$$

Because of (5.4), (6.2), and the boundedness of the coefficient processes $r(\cdot)$, $b(\cdot)$, and $a(\cdot)$ appearing in (4.4), (4.5), we have $\lim_{n\to\infty} \tau_n = T$, a.s. From (9.2) we obtain

$$E\,[\zeta(\tau_n)X_j(\tau_n)] + E \int_0^{\tau_n} \zeta(s)\psi(s)c_j(s)ds$$

$$= \tilde{E}\,[e^{-\int_0^{\tau_n} r(u)du} X_j(\tau_n)] + \tilde{E} \int_0^{\tau_n} e^{-\int_0^s r(u)du}\psi(s)c_j(s)ds$$

$$= \epsilon_j P(0) + \tilde{E} \int_0^{\tau_n} e^{-\int_0^s r(u)du}\psi(s)e_j(s)ds$$

$$= \epsilon_j P(0) + E \int_0^{\tau_n} \zeta(s)\psi(s)e_j(s)ds.$$

Letting $n \to \infty$ and using (6.8), (6.9), Fatou's lemma, and the monotone convergence theorem, we obtain

$$E \int_0^T \zeta(s)\psi(s)c_j(s)ds \leq \epsilon_i P(0) + E \int_0^T \zeta(s)\psi(s)e_j(s)ds.$$

Relation (9.1) now follows from

$$P(0) = E \int_0^T \zeta(s)\psi(s)\delta(s)ds, \tag{9.3}$$

a consequence of (8.6). □

<u>9.2 Theorem</u>. Consider a consumption process c_j (i.e., a nonnegative, measurable, $\{\mathcal{F}(t)\}$-adapted process satisfying (6.1)), for which (9.1) is valid. Then there exist a productive asset portfolio π_j and a financial asset portfolio Φ_j such that (c_j, π_j, Φ_j) is feasible. Furthermore, we can take $\pi_j \equiv \epsilon_j$, i.e., agent j need not change his initial position in the productive assets.

<u>Proof</u>: From (5.1), (9.1), (9.3) and the boundedness of e_j, the random variable

$$Q_j \triangleq \int_0^T e^{-\int_0^s r(u)du} \psi(s)[e_j(s) - c_j(s)]ds$$

is $\tilde{P}$-integrable, with $\epsilon_j P(0) + \tilde{E}Q_j \geq 0$. According to Lemma 8.3, the

$\tilde{P}$-martingale $\tilde{E}[Q_j|\mathcal{F}(t)]$ admits the stochastic representation

$$\tilde{E}[Q_j|\mathcal{F}(t)] = \tilde{E}Q_j + \int_0^t H(s)\tilde{W}(s), \quad 0 \leq t \leq T, \text{ a.s.},$$

where the N-dimensional, measurable process H is $\{\mathcal{F}(t)\}$-adapted and satisfies (8.7). We define $\pi_j \equiv \epsilon_j$ and

$$\phi_j(s) = -[e^{\int_0^s r(u)du} H(s) + \epsilon_j\alpha(s)](a(s))^{-1}(\mathrm{diag}(f(s))^{-1}, \quad 0 \leq s \leq T.$$

The corresponding wealth process (cf. (9.2)) is

$$(9.4) \quad X_j(t) = e^{\int_0^t r(u)du}\{\epsilon_j P(0) + \int_0^t e^{-\int_0^s r(u)du}\psi(s)[e_j(s) - c_j(s)]ds$$

$$- \int_0^t H(s)d\tilde{W}(s)\}$$

$$= e^{\int_0^t r(u)du}\{\epsilon_j P(0) + \tilde{E}Q_j + \tilde{E}[\int_t^T e^{-\int_0^s r(u)du}\psi(s)[c_j(s) - e_j(s)]ds|\mathcal{F}(t)]\}$$

$$= e^{\int_0^t r(u)du}\{\epsilon_j P(0) + \tilde{E}Q_j + \frac{1}{Z(t)} E[\int_t^T \zeta(s)\psi(s)[c_j(s) - e_j(s)]ds|\mathcal{F}(t)]\},$$

which satisfies (6.8) and (6.9). Moreover, with $\phi_{j,0}$ defined by (6.5), π_j and the (N+1)-dimension process $\Phi = (\phi_{j,0},\phi_j)$ satisfy (6.2) (recall (4.7), the strict positivity of f_n in (4.4), and (8.7)). □

Using Lemma 9.1 and Theorem 9.2, we see that the optimization problem for agent j reduces to maximizing

(6.10) $$E \int_0^T U_j(t, c_j(t))dt,$$

subject to the constraints

(6.1) $$\inf_{0 \leq t \leq T} c_j(t) \geq 0, \quad \sup_{0 \leq t \leq T} c_j(t) < \infty, \quad \text{a.s.},$$

(6.11) $$E \int_0^T \max\{0, -U_j(t, c_j(t))\}dt < \infty,$$

(9.1) $$E \int_0^T \zeta(s)\psi(s)c_j(s)ds \leq E \int_0^T \zeta(s)\psi(s)\hat{c}_j(s)ds,$$

where ζ is determined by (4.12). This is a problem involving the consumption process, but not the portfolio process. The productive asset prices do not enter this formulation of the problem; the financial asset prices enter only through ζ.

We now present the solution of this problem. Recall our assumption that, for each $t \in [0,T]$, the function $U_j'(t,\cdot)$ is strictly decreasing and satisfies (3.6); we may define $I_j(t,\cdot)$ to be the inverse of $U_j'(t,\cdot)$, i.e., a strictly decreasing, continuous mapping from $(0, U_j'(t,0))$ onto $(0,\infty)$. In other words, (9.5) $$I_j(t,y) > 0,\ U_j'(t,\ I_j(t,y)) = y;\quad \forall\ y \in (0,\ U_j'(t,0)).$$

We extend the domain of $I_j(t,\cdot)$ by setting

(9.6) $$I_j(t,y) = 0;\ \forall\ y \in [U_j'(t,0),\ \infty).$$

For $y \in (0,\infty)$, define

(9.7) $$\mathcal{X}_j(y) \triangleq E \int_0^T \zeta(s)\psi(s)I_j(s,y\zeta(s)\psi(s))ds.$$

<u>9.3 Lemma</u>. The function $\mathcal{X}_j$ maps $(0,\infty)$ into $[0,\infty)$, is continuous, nondecreasing, and satisfies

(9.8) $$\lim_{y\downarrow 0} \mathcal{X}_j(y) = \infty,\quad \lim_{y\uparrow \bar{y}_j} \mathcal{X}_j(y) = 0,$$

where

(9.9) $$\bar{y}_j \triangleq \sup\ \{y > 0;\ \mathcal{X}_j(y) > 0\}.$$

On $(0,\bar{y}_j)$, $\mathcal{X}_j$ is strictly decreasing.

<u>Proof</u>: It is apparent that $\mathcal{X}_j(y) \geq 0;\ \forall\ y \in (0,\infty)$. We show that $\mathcal{X}_j(y) < \infty$. If $0 < y < U_i'(t,2)$, then $c \triangleq I_j\ (t,y) > 2$, and from (3.5), (9.5), and the concavity of $U_j(t,\cdot)$, we have

$$y = U_j'(t,c) \le \frac{U_j(t,c) - U_j(t,1)}{c - 1}$$

$$\le \frac{2}{c}\,[k_1 + k_2\, c^{\rho} - U_j(t,1)]$$

$$\le k_3 c^{-1} + k_4 c^{\rho-1} \le k_5\, c^{\rho-1}$$

for some positive constants k_3, k_4 and k_5, which do not depend on c. Therefore,

$$(9.10) \qquad I_j(t,y) \le \left(\frac{k_5}{y}\right)^{\frac{1}{1-\rho}}; \quad \forall\, y \in (0,\ U_j'(t,2)).$$

We also have

$$I_j(t,y) \le 2; \quad \forall \quad y \in [U_j'(t,2),\ \infty),$$

and so the finiteness of $\mathfrak{X}_j$ for all $y \in (0,\infty)$ will follow from the finiteness of

$$E \int_0^T 1_{\{y\zeta(s)\psi(s) < U_j'(t,2)\}}\, \zeta(s)\psi(s) I_j(s,\ y\zeta(s)\psi(s))ds.$$

According to (9.10), this expression is bounded above by

$$\left(\frac{k_5}{y}\right)^{\frac{1}{1-\rho}} E \int_0^T [\zeta(s)\psi(s)]^{-\frac{\rho}{1-\rho}}\, ds,$$

which is finite because of (5.1).

Because $I_j(t,\cdot)$ is nondecreasing, $\mathcal{X}_j(\cdot)$ is also. The right-continuity of $\mathcal{X}_j$ and the first part of (9.8) are consequences of the monotone convergence theorem; the left-continuity of $\mathcal{X}_j$ follows from its finiteness and the dominated convergence theorem. If $\bar{y}_j < \infty$, the second part of (9.8) follows from the continuity of $\mathcal{X}_j$; if $\bar{y}_j = \infty$, we use the dominated convergence theorem to obtain this result.

For $y \in (0, \bar{y}_j)$, we have $\mathcal{X}_j(y) > 0$, which implies that on a set of positive Lebesgue x P-measure,

$$y\,\zeta(t)\psi(t) < U_j'(t,0).$$

But $I_j(t,\cdot)$ is <u>strictly</u> decreasing on $(0, U_j'(t,0))$, and so $\mathcal{X}_j(y+\eta) < \mathcal{X}_j(y), \quad \forall \quad \eta > 0.$ □

When restricted to $(0, \bar{y}_j)$, the function $\mathcal{X}_j$ has a continuous, strictly decreasing inverse $\mathcal{Y}_j\colon (0,\infty) \xrightarrow{\text{onto}} (0,\bar{y}_j)$. Let

$$(9.11) \qquad \xi_j \triangleq E\int_0^T \zeta(s)\psi(s)\hat{c}_j(s)ds, \qquad \eta_j \triangleq \mathcal{Y}(\xi_j).$$

(Note that $\xi_j > 0$ because of the assumption that $\hat{c}_j$ is not Lebesgue x P-almost everywhere zero.) We shall show that the optimal consumption process for the j^{th} agent is

$$(9.12) \qquad c_j^*(t) \triangleq I_j\,(t,\ \eta_j\zeta(t)\psi(t)),\quad 0 \le t \le T.$$

<u>9.4 Theorem</u>. The unique (up to Lebesgue x P-almost everywhere equivalence) optimal consumption policy for the j^{th} agent is given by (9.12).

Proof: According to our definitions,

$$E\int_0^T \zeta(s)\psi(s)c_j^*(s)ds = \mathcal{X}_j(\eta_j) = \xi_j = E\int_0^T \zeta(s)\psi(s)\hat{c}_j(s)ds,$$

so c_j^* satisfies (9.1) with equality. Let c_j be any process satisfying (6.1), (6.11) and (9.1), so

$$E\int_0^T \zeta(s)\psi(s)\ [c_j^*(s) - c_j(s)] \geq 0.$$

From elementary calculus, one can show that

(9.13) $U_j(t,\ I_j(t,y)) - y\ I_j(t,y) = \max_{c\geq 0}\{U_j(t,c)-yc\};\ \forall\ \ y \in (0,\infty),\ t \in [0,T],$

and thus

(9.14) $$E\int_0^T U_j(s,\ c_j(s))ds$$

$$\leq E\int_0^T U_j(s,c_j(s))ds + y_j\ E\int_0^T \zeta(s)\psi(s)[c_j^*(s) - c_j(s)]ds$$

$$\leq E\int_0^T U_j(s,c_j^*(s))ds.$$

Thus, if it is feasible, then c_j^* is optimal.

There is at least one feasible consumption process; namely

$$c_j \equiv \xi_j\left[E \int_0^T \zeta(s)\psi(s)ds\right]^{-1}.$$

This constant process satisfies (9.1) with equality, and (6.1), (6.11) are also clearly satisfied. With this choice of c_j in (9.14), we see that c_j^* satisfies (6.11).

Because the maximum in (9.13) is uniquely attained at $I_j(t,y)$, c_j^* is the unique optimal consumption policy for agent j. □

10. The Representative Agent

In order to facilitate the proof of the existence of equilibrium in the next section, we introduce here the notion of a representative agent. Given a vector $\Lambda = (\lambda_1,\ldots,\lambda_J) \in (0,\infty)^J$, we define the function

$$(10.1)\quad U(t,c;\ \Lambda) \triangleq \max_{\substack{c_1\geq 0,\ldots,c_J\geq 0 \\ c_1+\ldots+c_J=c}} \sum_{j=1}^{J} \lambda_j U_j(t,c_j);\ \forall\ (t,c) \in [0,T] \times (0,\infty).$$

As we show in Lemma 10.1, the function U inherits many of the properties of $U_1,\ldots,U_J$. It can thus be thought of as the utility function of a "representative" agent, who assigns the weights $\lambda_1,\ldots,\lambda_J$ to the utilities of the individual agents in the economy.

10.1 Lemma. For fixed $\Lambda \in (0,\infty)$ and $t \in [0,T]$, the function $U(t,\cdot;\Lambda)$: $(0,\infty) \to \mathbb{R}$ is strictly increasing and continuously differentiable, $U'(t,\cdot;\ \Lambda)$ is strictly decreasing, and

$$\lim_{c\to\infty} U'(t,c;\ \Lambda) = 0.$$

Furthermore, with ρ as in (3.5) and for some constants $\bar{k}_1 > 0$, $\bar{k}_2 > 0$,

$$(10.2)\qquad U(t,c;\ \Lambda) \leq \bar{k}_1 + \bar{k}_2\, c^{\rho} \quad \forall\ \ c > 0.$$

Proof: Define

$$(10.3) \qquad I(t,y;\ \Lambda) \triangleq \sum_{j=1}^{J} I_j(t, \frac{y}{\lambda_j}), \quad \forall \ y \in (0,\infty).$$

The function $I(t,\cdot;\ \Lambda)$ is continuous and nonincreasing, is strictly decreasing on $(0, \max_{1\le j\le J} \lambda_j U_j'(t,0))$, and maps this interval onto $(0,\infty)$. Thus, for every $c \in (0,\infty)$, there is a unique positive number $H(t,c) = H(t,c;\ \Lambda)$ with $I(t,H(t,c);\ \Lambda) = c$, and the mapping

$$H(t,\cdot): \quad (0,\infty) \xrightarrow{\text{onto}} (0, \max_{1\le j\le J} \lambda_j U_j'(t,0))$$

is continuous and strictly decreasing.

Let $c \in (0,\infty)$ be given, and define

$$(10.4) \qquad \bar{c}_j \triangleq I_j(t, \frac{H(t,c)}{\lambda_j}); \quad j = 1,\dots,J.$$

Then $\sum_{j=1}^{J} \bar{c}_j = I(t, H(t,c);\ \Lambda) = c$, and for each j,

$$U_j'(t,\bar{c}_j) = \begin{cases} \dfrac{H(t,c)}{\lambda_j} & \text{if } H(t,c) < \lambda_j U_j'(t,0), \\ 0 & \text{if } H(t,c) \ge \lambda_j U_j'(t,0). \end{cases}$$

In either case,

$$\lambda_j U_j'(t,\bar{c}_j) \le H(t,c); \quad j = 1,\dots,J.$$

Let $c_1,\dots,c_J$ be any other nonnegative numbers with $\sum_{j=1}^{J} c_j = c$. The concavity of each $U_j(t,\cdot)$ allows us to write

$$\sum_{j=1}^{J} \lambda_j U_j(t,c_j) \leq \sum_{j=1}^{J} \lambda_j [U_j(t,\bar{c}_j) + (c_j - \bar{c}_j)U_j'(t,\bar{c}_j)]$$

$$\leq \sum_{j=1}^{J} \lambda_j U_j(t,\bar{c}_j) + H(t,c) \sum_{j=1}^{J} (c_j - \bar{c}_j)$$

$$= \sum_{j=1}^{J} \lambda_j U_j(t,\bar{c}_j).$$

It follows that the vector $(\bar{c}_1,\ldots,\bar{c}_J)$ attains the maximum in (10.1), i.e.,

$$\text{(10.5)} \qquad U(t,c;\Lambda) = \sum_{j=1}^{J} \lambda_j U_j(t, I_j(t, \frac{H(t,c)}{\lambda_j})); \quad \forall \quad c \in (0,\infty).$$

Now each $I_j(t,\cdot)$ is differentiable except possibly at $U_j'(t,0)$, so $I(t,\cdot;\Lambda)$ is differentiable off the set $\{\lambda_j U_j'(t,0);\ j = 1,\ldots,J\}$ and $H(t,\cdot)$ is differentiable off the set $A \triangleq \{I(t, \lambda_j U_j'(t,0); \Lambda);\ j = 1,\ldots,J\}$. Moreover, for $1 \leq j \leq J$ and $c \in (I(t, \lambda_j U_j'(t,0); \Lambda), \infty)\backslash A$, we have $0 < \frac{1}{\lambda_j} H(t,c) < U_j'(t,0)$, and (9.5) gives

$$\text{(10.6)} \qquad \frac{d}{dc} \lambda_j U_j(t, I_j(t, \frac{H(t,c)}{\lambda_j})) = U_j'(t, I_j(t, \frac{H(t,c)}{\lambda_j}))I_j'(t, \frac{H(t,c)}{\lambda_j})H'(t,c)$$

$$= \frac{1}{\lambda_j} H(t,c)\ I_j'(t, \frac{H(t,c)}{\lambda_j})H'(t,c).$$

For $1 \leq j \leq J$ and $c \in (0, I(t,\lambda_j U_j'(t,0)))\backslash A$, we have $\frac{1}{\lambda_j} H(t,c) > U_j'(t,0)$ and $I_j(t, \frac{H(t,c)}{\lambda_j}) = I_j'(t, \frac{H(t,c)}{\lambda_j}) = 0$, so again (10.6) holds, this time with

both sides equal to zero. The derivative of $U(t,c;\ \Lambda)$ in (10.5) is thus seen to be

$$(10.7)\qquad U'(t,c;\ \Lambda) = H(t,c)H'(t,c) \sum_{j=1}^{J} \frac{1}{\lambda_j} I_j'(t,\ \frac{H(t,c)}{\lambda_j})$$

$$= H(t,c)H'(t,c)\ I'(t,\ H(t,c)) = H(t,c)$$

for all $c \in (0,\infty)\backslash A$. The expression (10.7) also gives the correct one-sided derivatives of $U(t,\cdot;\ \Lambda)$ at points in A, and since $H(t,\cdot)$ is continuous, $U(t,\cdot;\ \Lambda)$ must in fact be continuously differentiable and (10.7) must be valid on all of $(0,\infty)$. The properties for $U'(t,\cdot;\ \Lambda)$ claimed in the lemma are consequences of the known properties of $H(t,\cdot)$.

As for the bound (10.2), it follows from (3.5) once we observe that

$$U(t,c;\ \Lambda) \leq \sum_{j=1}^{J} \lambda_j U_j(t,c). \qquad \square$$

The properties established for $U(t,\cdot;\ \Lambda)$ in Lemma 10.1 are exactly those properties, shared by each $U_j(t,\cdot)$, which were used in Section 9 in the derivation of the optimal consumption process for agent j.

Because of (10.7), the function $I(t,\cdot;\ \Lambda)$ of (10.3) satisfies

$$(10.8)\qquad I(t,y;\ \Lambda) > 0,\ U'(t,\ I(t,y;\ \Lambda);\ \Lambda) = y;\ \forall\ y \in (0,\ U'(t,0;\ \Lambda))$$

$$I(t,y;\ \Lambda) = 0;\quad \forall\quad y \in [U_i'(t,0;\ \Lambda),\ \infty).$$

If a spot price ψ satisfying (5.1) is given, then by analogy with (9.7) we can define

$$\mathcal{X}(y;\Lambda) \triangleq E\int_0^T \zeta(s)\psi(s)I(s,y\,\zeta(s)\psi(s);\,\Lambda)ds,$$

$$\bar{y}(\Lambda) \triangleq \sup\{y > 0:\ \mathcal{X}(y;\Lambda) > 0\}.$$

The assertions of Lemma 9.3 are valid for $\mathcal{X}(\cdot;\Lambda)$, and the inverse $\mathcal{Y}(\cdot;\Lambda)\colon (0,\infty) \xrightarrow{\text{onto}} (0,\bar{y}(\Lambda))$ is continuous and strictly decreasing.

We imagine that the representative agent receives the aggregate income process $\hat{c}(\cdot)$ defined in (3.3), and attempts to maximize his total expected utility $E\int_0^T U(t,c(t))dt$ from consumption, subject to $E\int_0^T \zeta(s)\psi(s)c(s)ds \le \xi$, where

$$\xi \triangleq E\int_0^T \zeta(s)\psi(s)\hat{c}(s)ds. \tag{10.9}$$

Now with

$$\eta(\Lambda) \triangleq \mathcal{Y}(\xi;\Lambda), \tag{10.10}$$

the optimal consumption process for the representative agent is given by the analogue of (9.12):

$$c^*(t;\Lambda) \triangleq I(t,\,\eta(\Lambda)\zeta(t)\psi(t);\,\Lambda),\ 0 \le t \le T. \tag{10.11}$$

We elaborate on the fiction of the representative agent by further imagining that after he computes $c^*(t;\Lambda)$, rather than consuming the commodity himself, the representative agent parcels out this consumption to the J individual agents, according to the formula (10.4):

$$(10.12)\qquad \bar{c}_j(t;\Lambda) \triangleq I_j\left(t, \frac{1}{\lambda_j} U'(t, c^*(t;\Lambda);\Lambda)\right), \quad 0 \le t \le T.$$

Each agent j will be happy with this arrangement if $\bar{c}_j(t;\Lambda)$ agrees with his optimal consumption process $c_j^*(t)$ defined by (9.12). This agreement will in fact occur, provided that

$$(10.13)\qquad \eta_j \zeta(t)\psi(t) = \frac{1}{\lambda_j} U'(t, c^*(t;\Lambda);\Lambda); \quad 0 \le t \le T,$$

and under this condition we shall have

$$\sum_{j=1}^{J} c_j^*(t) = \sum_{j=1}^{J} \bar{c}_j(t;\Lambda) = I(t, U'(t, c^*(t;\Lambda);\Lambda) = c^*(t;\Lambda); \quad 0 \le t \le T.$$

It follows from (7.1) that a necessary condition for the existence of equilibrium is

$$(10.14)\qquad \hat{c}(t) = c^*(t;\Lambda)\ ; \quad 0 \le t \le T$$

almost surely, in terms of which (10.13) becomes

$$(10.15)\qquad \zeta(t)\psi(t) = \frac{1}{\lambda_j \eta_j} U'(t, \hat{c}(t);\Lambda); \quad 0 \le t \le T.$$

This last equation does not provide a direct formula for the deflated spot price process $\zeta\psi$, because the number η_j on the right-hand side depends on $\zeta\psi$ (recall (9.11)) and because the vector Λ has not yet been determined. Nevertheless, the equation (10.13) provides a valuable framework for further discussion of equilibrium.

10.2 Theorem. Let $\Lambda = (\lambda_1,\ldots,\lambda_J) \in (0,\infty)^J$ be given, and define a spot price process $\psi(\cdot;\Lambda)$ by

$$(10.16) \qquad \psi(t;\Lambda) \triangleq \frac{1}{\zeta(t)} U'(t, \hat{c}(t); \Lambda) , \quad 0 \le t \le T.$$

Using this spot price process, for each j define $\eta_j(\Lambda)$ and $c_j^*(\cdot;\Lambda)$ by (9.11) and (9.12), respectively. .If the vector Λ satisfies

$$(10.17) \qquad \lambda_j\eta_j(\Lambda) = 1; \quad \forall \ j = 1,\ldots,J,$$

then the spot price process $\psi(\cdot;\Lambda)$, the corresponding vector of productive assets given by (8.6), the consumption processes given by

$$(10.18) \qquad c_j^*(t;\Lambda) \triangleq I_j(t; \eta_j(\Lambda)U'(t, \hat{c}(t); \Lambda)), \quad 0 \le t \le T, \quad j = 1,\ldots,J,$$

the productive asset portfolio processes $\pi_j^* \equiv \epsilon_j$, $j = 1,\ldots,J$, and the corresponding financial asset portfolio processes Φ_j^*, $j = 1,\ldots,J$, given as in Theorem 9.2, constitute an equilibrium.

Proof: By assumption, $\eta_j(\Lambda)$ is the unique positive number η for which

$$(10.19)\qquad E\int_0^T U'(t,\ \hat{c}(t);\ \Lambda)\ I_j(t;\eta\ U'(t,\hat{c}(t);\ \Lambda))dt$$

$$= E\int_0^T U'(t,\ \hat{c}(t);\ \Lambda)\ \hat{c}_j(t)dt$$

holds. Because of (3.4) and Lemma 10.1, $\zeta(\cdot)\psi(\cdot;\Lambda)$ satisfies (5.1). For each j, the optimality of the (c_j^*,π_j^*,Φ_j^*) follows from Theorems 9.2 and 9.4.

It remains to verify the market clearing conditions (7.1) - (7.3). From (3.1) we have (7.2). As for (7.1), we note from (10.17), (10.18) that

$$(10.20)\qquad \sum_{j=1}^{J} c_j^*(t;\Lambda) = \sum_{j=1}^{J} I_j(t,\ \eta_j(\Lambda)\zeta(t)\ \psi(t;\Lambda))$$

$$= \sum_{j=1}^{J} I_j(t,\ \frac{1}{\lambda_j} U'(t,\ \hat{c}(t);\ \Lambda))$$

$$= I(t,\ U'(t,\ \hat{c}(t);\ \Lambda);\ \Lambda) = \hat{c}(t),\qquad 0 \le t \le T,\ \text{a.s.}$$

We turn now to (7.3). Because for each j, $\pi_j^* \equiv \epsilon_j$ and Φ_j^* is also given as in Theorem 9.2, the corresponding wealth process is given by (9.4), that is,

$$(10.21)\qquad X_j^*(t) = e^{\int_0^t r(u)du}\{\epsilon_j P(0) + \tilde{E}Q_j^*$$

$$+ \frac{1}{Z(t)} E[\int_t^T \zeta(s)\psi(s;\Lambda)[c_j^*(s;\ \Lambda) - e_j(s)]ds\,|\mathcal{F}(t)]\},$$

where

$$Q_j^* \triangleq \int_0^T e^{-\int_0^s r(u)du} \psi(s;\Lambda)[e_j(s) - c_j^*(s;\ \Lambda)]ds.$$

Using (3.3), (10.20) and (9.3), we see that

$$\sum_{j=1}^J \tilde{E}Q_j^* = E\int_0^T \zeta(s)\psi(s;\Lambda)[\sum_{j=1}^J e_j(s) - \hat{c}(s)]ds$$

$$= - E\int_0^T \zeta(s)\psi(s;\Lambda) \sum_{m=1}^M \delta_m(s)ds$$

$$= - \sum_{j=1}^J \epsilon_j\ P(0),$$

and so a summation over j in (10.21) yields

$$\zeta(t) \sum_{j=1}^J X_j^*(t) = E\left[\int_t^T \zeta(s)\psi(s;\Lambda) \sum_{m=1}^M \delta_m(s)ds \,|\mathcal{F}(t)\right]. \tag{10.22}$$

From (6.4) and (8.6) we have also

$$\zeta(t) \sum_{j=1}^J X_j^*(t) = \zeta(t) \sum_{m=1}^M P_m(t) + \zeta(t) \sum_{j=1}^J \Phi_j^*(t)F(t) \tag{10.23}$$

$$= E\left[\int_t^T \zeta(s)\psi(s;\Lambda) \sum_{m=1}^M \delta_m(s)ds \,|\mathcal{F}(t)\right] + \zeta(t) \sum_{j=1}^J \Phi_j^*(t)\ F(t).$$

Comparison of (10.22), (10.23) shows that $\sum_{j=1}^{J} \Phi_j^*(t)F(t) = 0,\ 0 \le t \le T$, a.s. Because $\pi_j^* \equiv \epsilon_j$, (6.3) reduces to

$$\Phi_j^*(t)\ F(t) = \int_0^t \psi(s;\Lambda)[\hat{c}_j(s) - c_j^*(s;\Lambda)]ds + \int_0^t \Phi_j^*(s)dF(s),$$

which yields, in conjunction with (10.20):

$$(10.24) \qquad 0 = \int_0^t \sum_{j=1}^{J} \Phi_j^*(s)dF(s)$$

$$= \int_0^t \sum_{j=1}^{J} [\phi_{j,0}^*(s)f_0(s)r(s) + \phi_j^*(s)\ \mathrm{diag}(f(s))b(s)]ds$$

$$+ \int_0^t \sum_{j=1}^{J} \phi_j^*(s)\ \mathrm{diag}(f(s))a(s)\ dW(s); \quad 0 \le t \le T, \text{ a.s.}$$

The local martingale part of the right-hand side of (10.24) and hence also its quadratic variation

$$\int_0^t \|(\sum_{j=1}^{J} \phi_j^*(s))\ \mathrm{diag}\ (f(s))\ a(s)\|^2 ds;\ 0 \le t \le T,$$

must be identically equal to zero. It follows from the nonsingularity of $\mathrm{diag}(f(s))a(s)$ that

$$\sum_{j=1}^{J} \phi_j^*(t) = \underset{\sim}{0}_N^{tr}, \quad \text{for a.e.} \quad t \in [0,T],$$

almost surely. From (10.24) we see now that also $\sum_{j=1}^{J} \phi_{j,0}^*(t) = 0$, for a.e. $t \in [0,T]$, a.s. □

<u>10.3 Theorem</u>: Let $[\psi, P = (P_1,\ldots,P_M), \{(c_j^*,\pi_j^*,\Phi_j^*); j = 1,\ldots,J\}]$ be an equilibrium as set forth in Definition 7.1. For each j, let η_j be defined by (9.11), and set $\Lambda = (\frac{1}{\eta_1},\ldots,\frac{1}{\eta_J})$. Then

$$\psi(t) = \frac{1}{\zeta(t)} U'(t, \hat{c}(t); \Lambda); \quad 0 \le t \le T.$$

<u>Proof</u>: From the equilibrium conditions, Theorem 9.4, and (10.3) we have

$$\hat{c}(t) = \sum_{j=1}^{J} c_j^*(t) = \sum_{j=1}^{J} I_j(t, \eta_j\zeta(t)\psi(t)) = I(t, \zeta(t)\psi(t); \Lambda),$$

and thus from (10.8) (recalling from (3.4) that $\hat{c}(t) > 0$), we conclude that

$$U'(t, \hat{c}(t); \Lambda) = \zeta(t)\psi(t); \quad 0 \le t \le T, \quad \text{a.s.}$$ □

11. Existence and Uniqueness of Equilibrium

11.1 Theorem. There exists $\Lambda \in (0,\infty)^J$ such that $\eta_j(\Lambda)$, $j = 1,\dots,J$, defined as in Theorem 10.2, satisfy (10.17). If $\tilde{\Lambda}$ is another element of $(0,\infty)^J$ with this property, then $\Lambda = \gamma\tilde{\Lambda}$ for some $\gamma > 0$.

We defer the proof of Theorem 11.1 to Sections 12, 13 and 15, and devote this section to a discussion of its consequences.

11.2 Corollary. There exists a unique equilibrium.

Proof: Existence follows from Theorems 10.2 and 11.1. For uniqueness, suppose $[\psi, (P_1,\dots,P_M), \{(c_j^*,\pi_j^*,\Phi_j^*);\ j = 1,\dots,J\}]$ and $[\tilde{\psi}, (\tilde{P}_1,\dots,\tilde{P}_M), \{\tilde{c}_j^*,\tilde{\pi}_j^*,\tilde{\Phi}_j^*);\ j = 1,\dots,J\}]$ are both equilibria. According to Theorem 10.3, there exist Λ, $\tilde{\Lambda} \in (0,\infty)^J$ such that $\eta_j(\Lambda)$ and $\eta_j(\tilde{\Lambda})$, $j = 1,\dots,J$, satisfy their respective versions of (10.17) and

$$\psi(t) = \frac{1}{\zeta(t)} U'(t, \hat{c}(t); \Lambda),\ \tilde{\psi}(t) = \frac{1}{\zeta(t)} U'(t, \hat{c}(t); \tilde{\Lambda}),\ 0 \le t \le T, \text{ a.s.}$$

Theorem 11.1 implies $\Lambda = \gamma\tilde{\Lambda}$ for some $\gamma > 0$, so $\psi = \gamma\tilde{\psi}$. Theorem 8.2 implies that $P_m = \gamma\tilde{P}_m$, $m = 1,\dots,M$. □

11.3 Corollary. Suppose $[\psi, (P_1,\dots,P_M), \{c_j^*,\pi_j^*,\Phi_j^*;\ j = 1,\dots,J\}]$ is an equilibrium and $[\tilde{\psi}, (\tilde{P}_1,\dots,\tilde{P}_M), \{\tilde{c}_j^*,\tilde{\pi}_j^*,\tilde{\Phi}_j^*;\ j = 1,\dots,J\}]$ is an equilibrium for another model which differs from the first only in the choice of the coefficients of the financial assets $r(\cdot)$, $b(\cdot)$ and $a(\cdot)$. Let ζ be the deflator defined by (4.12) for the first model, and let $\tilde{\zeta}$ be the analogously defined deflator for the second model. Then for Lebesgue $\times$ P almost every

(t,ω), we have

$$\zeta(t)\psi(t) = \gamma\tilde{\zeta}(t)\tilde{\psi}(t), \quad c_j^*(t) = \tilde{c}_j^*(t); \quad j = 1,\ldots,J,$$

for some $\gamma > 0$.

<u>Proof</u>: For $\Lambda \in (0,\infty)^J$ and $j \in \{1,\ldots,J\}$, let $\eta_j(\Lambda)$ be the unique positive number satisfying (10.19). The mapping $\eta_j\colon (0,\infty)^J \to (0,\infty)$ depends on the model primitives of Section 3, but not on the financial assets. According to Theorem 10.3, there exist $\Lambda, \tilde{\Lambda} \in (0,\infty)^J$ such that

$$(11.1)\ \zeta(t)\psi(t) = U'(t, \hat{c}(t); \Lambda), \quad \tilde{\zeta}(t)\tilde{\psi}(t) = U'(t, \hat{c}(t); \tilde{\Lambda}); \quad 0 \le t \le T.$$

Indeed, by comparing (11.1) and (10.15), we conclude that these particular vectors $\Lambda = (\lambda_1,\ldots,\lambda_J)$ and $\tilde{\Lambda} = (\tilde{\lambda}_1,\ldots,\tilde{\lambda}_J)$ satisfy

$$\lambda_j\eta_j(\Lambda) = \tilde{\lambda}_j\eta_j(\tilde{\Lambda}) = 1; \quad j = 1,\ldots,J,$$

and so Theorem 11.1 asserts the existence of $\gamma > 0$ such that $\Lambda = \gamma\tilde{\Lambda}$. It follows from (11.1) that $\zeta\psi = \gamma\tilde{\zeta}\tilde{\psi}$. Furthermore, the (unique by Theorem 9.4) optimal consumption processes are given by (10.18), and therefore satisfy

$$\begin{aligned} c_j^*(t) &= I_j(t;\ \eta_j(\Lambda)\zeta(t)\psi(t)) \\ &= I_j(t;\ \eta_j(\tilde{\Lambda})\tilde{\zeta}(t)\tilde{\psi}(t)) = \tilde{c}_j^*(t),\ 0 \le t \le T. \quad \square \end{aligned}$$

12. Proof of Existence

In this section we show that there exists $\Lambda = (\lambda_1, \ldots, \lambda_J) \in (0,\infty)^J$ such that the numbers $\eta_1(\Lambda), \ldots, \eta_J(\Lambda)$ determined by (10.19) satisfy (10.17). This is the existence part of Theorem 11.1. For $j = 1, \ldots, J$, and $\Lambda \in (0,\infty)^J$, define the function

$$(12.1)\quad S_j(\mu;\Lambda) \triangleq E\int_0^T \frac{1}{\mu} U'(t, \hat{c}(t); \Lambda)[I_j(t, \frac{1}{\mu} U'(t, \hat{c}(t); \Lambda)) - \hat{c}_j(t)]dt;\ \mu \in (0,\infty).$$

Lemma 12.1. For each $\Lambda \in (0,\infty)^J$ and each $j \in \{1, \ldots, J\}$, the function $S_j(\cdot;\Lambda)$ is strictly increasing and satisfies

$$\lim_{\mu \downarrow 0} S_j(\mu;\Lambda) = -\infty, \quad \lim_{\mu \to \infty} \mu S_j(\mu;\Lambda) = \infty.$$

Proof: Because $\lim_{y \downarrow 0} I_j(t,y) = \infty;\ \forall\ t \in [0,T]$, we have $\lim_{\mu \to \infty} \mu S_j(\mu;\Lambda) = \infty$. Suppose $0 < y_1 < y_2 < U_j'(t,0)$ and define $c_i \triangleq I_j(t,y_i)$, $i = 1,2$. Then $0 < c_2 < c_1 < \infty$, and according to (3.7),

$$y_1 I_j(t,y_1) = c_1 U_j'(t,c_1) \geq c_2 U_j'(t,c_2) = y_2 I_j(t,y_2).$$

If $y_2 \geq U_j'(t,0)$, then for $y_1 \in (0,y_2)$,

$$y_1 I_j(t,y_1) \geq 0 = y_2 I_j(t,y_2).$$

We conclude that for each $t \in [0,T]$, the mapping

$$\phi_j(t,y) \triangleq y\, I_j(t,y); \quad y \in (0,\infty) \tag{12.2}$$

is nonincreasing in y. Since

$$S_j(\mu;\Lambda) = E\int_0^T \phi_j(t, \frac{1}{\mu} U'(t, \hat{c}(t); \Lambda))dt \tag{12.3}$$

$$- \frac{1}{\mu} E \int_0^T U'(t, \hat{c}(t); \Lambda)\hat{c}_j(t)dt$$

and $\hat{c}_j(\cdot)$ is not identically zero, $S_j(\cdot;\Lambda)$ is strictly increasing and $\lim_{\mu \downarrow 0} S_j(\mu;\Lambda) = -\infty$. □

Lemma 12.1 implies that for each $\Lambda \in (0,\infty)^J$ and each $j \in \{1,\dots,J\}$, there is a unique positive number $L_j(\Lambda)$ such that

$$S_j(L_j(\Lambda);\, \Lambda) = 0. \tag{12.4}$$

Comparison with (10.19) shows, in fact, that

$$L_j(\Lambda) = \frac{1}{\eta_j(\Lambda)} . \tag{12.5}$$

We define $L = (L_1,\dots,L_J)\colon (0,\infty)^J \to (0,\infty)^J$ and note that $\eta_j(\Lambda)$ satisfies (10.17) if and only if Λ is a fixed point of L.

Since $U'(t,c;\, \Lambda)$ is positively homogeneous in Λ, we have $S_j(\gamma\mu;\, \gamma\Lambda) = S_j(\mu,\Lambda)$ for every $\gamma \in (0,\infty)$, $\mu \in (0,\infty)$ and $\Lambda \in (0,\infty)^J$. Therefore

$S_j(\gamma L_j(\Lambda);\ \gamma\Lambda) = S_j(L_j(\Lambda);\ \Lambda) = 0$, and thus

$$L_j(\gamma\Lambda) = \gamma\, L_j(\Lambda).$$

Because of this positive homogeneity, any fixed point Λ^* for L will lead to a one-parameter family of fixed points $\{\gamma\Lambda^* | \gamma \in (0,\infty)\}$. If $J \geq 2$, we can therefore reduce by one the dimension of the fixed point problem and be confident that we have not significantly changed it. In this spirit, let us define the mapping $R = (R_2,\ldots,R_J)\colon (0,\infty)^{J-1} \rightarrow (0,\infty)^{J-1}$ by

$$(12.6)\qquad R_j(\lambda_2,\ldots,\lambda_J) = L_j(1,\ \lambda_2,\ldots,\lambda_J);\quad j = 2,\ldots,J.$$

<u>12.2 Lemma</u>: Assume $J \geq 2$. If $(\lambda_2,\ldots,\lambda_J)$ is a fixed point for R, then $(1,\ \lambda_2,\ldots,\lambda_J)$ is a fixed point for L. If $(\lambda_1,\lambda_2,\ldots,\lambda_J)$ is a fixed point for L, then $(\frac{\lambda_2}{\lambda_1},\ldots,\frac{\lambda_J}{\lambda_1})$ is a fixed point for R.

<u>Proof</u>: The second assertion follows immediately from the positive homogeneity of L. As for the first, let $(\lambda_2\ ,\ldots,\ \lambda_J) \in (0,\infty)^{J-1}$ be a fixed point for R, and set $\lambda_1 = 1$, $\Lambda = (\lambda_1,\lambda_2,\ldots,\lambda_J)$. Then

$$(12.7)\qquad \frac{1}{\eta_j(\Lambda)} = L_j(\Lambda) = R_j(\lambda_2\ ,\ldots,\lambda_J) = \lambda_j;\ j = 2,\ldots,J.$$

It remains to show that $L_1(\Lambda) = \frac{1}{\eta_1(\Lambda)} = 1$, i.e., $S_1(1;\Lambda) = 0$. But (12.4), (12.7), (10.3) and (10.8) imply

$$(12.8)\quad S_1(1;\Lambda) = \sum_{j=1}^{J} \lambda_j S_j(\lambda_j;\Lambda)$$

$$= E\int_0^T U'(t,\ \hat{c}(t);\ \Lambda) \sum_{j=1}^{J} [I_j(t,\ \frac{1}{\lambda_j} U'(t,\hat{c}(t);\ \Lambda)) - \hat{c}_j(t)]dt$$

$$= E\int_0^T U'(t,\ \hat{c}(t);\ \Lambda)\ [I(t,U'(t,\hat{c}(t);\ \Lambda)) - \hat{c}(t)]dt$$

$$= 0. \qquad \square$$

<u>12.3 Remark</u>. If $J = 1$, equation (12.8) is still valid, with $\Lambda = 1$. In particular, 1 is then a fixed point of $L: (0,\infty) \to (0,\infty)$, and because of positive homogeneity, L is the identity mapping on $(0,\infty)$. This validates Theorem 11.1 in this case. □

If $J \geq 2$, Lemma 12.2 shows that Theorem 11.1 is equivalent to the assertion that R has a unique fixed point. We shall obtain the existence of this fixed point as a consequence of the following version of the Knaster-Tarski Fixed Point Theorem (see, e.g., Dugundji & Granas [7], p. 14 or Birkhoff [1], p. 54).

<u>12.4 Theorem</u>. Let $\leq$ denote the partial ordering on $(0,\infty)^{J-1}$ given by

$$(\lambda_2,\ldots,\lambda_J) \leq (\mu_2,\ldots,\mu_J) \iff \lambda_j \leq \mu_j,\quad \forall\ j \in \{2,\ldots,J\}.$$

Let $\mathfrak{R}: (0,\infty)^{J-1} \to (0,\infty)^{J-1}$ be an isotone mapping and assume that there exist

$\Lambda_\ell, \Lambda_u \in (0,\infty)^{J-1}$ such that

$$(12.9) \qquad \Lambda_\ell \le \mathfrak{R}(\Lambda_\ell), \quad \Lambda_\ell \le \Lambda_u, \quad \mathfrak{R}(\Lambda_u) \le \Lambda_u.$$

Then $\mathfrak{R}$ has a fixed point Λ^* satisfying $\Lambda_\ell \le \Lambda^* \le \Lambda_u$.

Proof: Let $Q = \{\Lambda \in (0,\infty)^{J-1} \mid \Lambda_\ell \le \Lambda \le \Lambda_u, \Lambda \le \mathfrak{R}(\Lambda)\}$. Then $\Lambda_\ell \in Q$ and Q is bounded, so we may define $\Lambda^* \triangleq \sup Q$, where the supremum is taken componentwise. For every $\Lambda \in Q$ we have $\Lambda \le \Lambda^*$, which implies $\Lambda \le \mathfrak{R}(\Lambda) \le \mathfrak{R}(\Lambda^*)$ and thus $\mathfrak{R}(\Lambda^*) \ge \sup Q = \Lambda^*$. Furthermore, $\Lambda_\ell \le \Lambda^* \le \Lambda_u$, so $\Lambda^* \in Q$. But we also have $\Lambda_\ell \le \mathfrak{R}(\Lambda_\ell) \le \mathfrak{R}(\Lambda^*) \le \mathfrak{R}(\Lambda_u) \le \Lambda_u$ and $\mathfrak{R}(\mathfrak{R}(\Lambda^*)) \ge \mathfrak{R}(\Lambda^*)$, whence $\mathfrak{R}(\Lambda^*) \in Q$. Thus, $\mathfrak{R}(\Lambda^*) \le \sup Q = \Lambda^*$. It follows that Λ^* is a fixed point for $\mathfrak{R}$. □

12.5 Lemma. The mapping $R = (R_2,\ldots,R_J)$ defined by (12.6) is isotone.

Proof: It suffices to prove the isotonicity of L. Let $\Lambda, M \in (0.\infty)^J$ be given with $\Lambda \le M$. Then (10.3) shows that $I(t,\cdot;\Lambda) \le I(t,\cdot;M)$, so $U'(\cdot;\Lambda) \le U'(\cdot;M)$. The representation (12.3) of S_j, where $\phi_j(t,\cdot)$ is nonincreasing, yields $S_j(L_j(\Lambda);\, M) \le S_j(L_j(\Lambda);\, \Lambda) = 0$, $j = 1,\ldots,J$. Recause $S_j(\cdot;M)$ is increasing and $S_j(L_j(M);\, M) = 0$, we must have $L_j(M) \ge L_j(\Lambda)$, $j = 1,\ldots,J$.

□

12.6 Theorem. The mapping $L = (L_1,\ldots,L_J)$ defined by (12.1), (12.4) has a fixed point.

Proof: We proceed by induction on the number of agents J. The case $J = 1$

was dealt with in Remark 12.3. Assume the existence of a fixed point for the counterpart of L constructed for agents $2,\ldots,J$. In other words, assume the existence of $(\lambda_2,\ldots,\lambda_J) \in (0,\infty)^{J-1}$ such that

$$(12.10)\quad E\int_0^T \frac{1}{\lambda_j} U'(t, \sum_{i=2}^{J} \hat{c}_i(t);(0,\lambda_2,\ldots,\lambda_J))[I_j(t,\frac{1}{\lambda_j} U'(t, \sum_{i=2}^{J} \hat{c}_i(t);(0,\lambda_2,\ldots,\lambda_J))) - \hat{c}_j(t)]\, dt = 0;\ j = 2,\ldots,J.$$

Here,

$$(12.11)\qquad U(t,c;(0,\lambda_2,\ldots,\lambda_J)) \triangleq \max_{\substack{c_2\geq 0,\ldots,c_J\geq 0 \\ c_2+\ldots+c_J=c}} \sum_{j=2}^{J} \lambda_j U_j(t,c_j)$$

$$= \lim_{\lambda_1 \downarrow 0} U(t,c;(\lambda_1,\lambda_2,\ldots,\lambda_J)).$$

(Note: The induction hypothesis implies the existence of such a vector $(\lambda_2,\ldots,\lambda_J)$ if the counterpart of (3.4) is satisfied, i.e., $\sum_{j=2}^{J} \hat{c}_j(t)$ is bounded away from zero. Since $\hat{c}(t) \triangleq \sum_{j=1}^{J} \hat{c}_j(t)$ is bounded away from zero, it is also the case that for some choice of i, $\hat{c}(t) - \hat{c}_i(t)$ is bounded away from zero. The construction of this section is predicated on the assumption that we may choose $i = 1$. This assumption can be made without loss of generality.)

We may rewrite (12.10) in terms of the nonincreasing function $\phi_j(t,\cdot)$ defined in (12.2) as

$$(12.12)\qquad E\int_0^T \phi_j\Big(t, \frac{1}{\lambda_j} U'\big(t, \sum_{i=2}^J \hat{c}_i(t);\ (0,\lambda_2,\dots,\lambda_J)\big)dt$$

$$- \frac{1}{\lambda_j} E\int_0^T U'\big(t, \sum_{i=2}^J \hat{c}_i(t);(0,\lambda_2,\dots,\lambda_J)\big)\hat{c}_j(t)dt = 0,$$

for every $j = 2,\dots,n$. Because for every $t \in [0,T]$,

$$U'(t, \hat{c}(t);\ (0,\lambda_2,\dots,\lambda_J)) \le U'(t, \sum_{i=2}^J \hat{c}_i(t);(0,\lambda_2,\dots,\lambda_J)),$$

a.s., and this inequality is strict on a subset of $[0,T] \times \Omega$ having positive Lebesgue $\times$ P measure, (12.12) implies that for every $j = 2,\dots,J$;

$$E\int_0^T \phi_j\Big(t, \frac{1}{\lambda_j} U'(t, \hat{c}(t);\ (0,\lambda_2,\dots,\lambda_J)\Big)dt$$

$$- \frac{1}{\lambda_j} E\int_0^T U'(t,\hat{c}(t);(0,\lambda_2,\dots,\lambda_J))\hat{c}_j(t)dt > 0.$$

We may choose a sufficiently small positive λ_1 such that

$$S_j(\lambda_j;\ (\lambda_1,\lambda_2,\dots,\lambda_J)) > 0;\quad j = 2,\dots,J,$$

and (12.4) and Lemma 12.1 now show that

$$L_j(\lambda_1,\lambda_2,\dots,\lambda_J) \le \lambda_j;\quad j = 2,\dots,J.$$

Set $\Lambda_u = (\frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_J}{\lambda_1}) \in (0,\infty)^{J-1}$. We have

$$R_j(\Lambda_u) = L_j(1, \frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_J}{\lambda_1}) = \frac{1}{\lambda_1} L_j(\lambda_1, \lambda_2, \ldots, \lambda_J) \le \frac{\lambda_j}{\lambda_1}, \quad j = 2, \ldots, J,$$

so $R(\Lambda_u) \le \Lambda_u$.

With Λ_u as above and for $\alpha \in (0,1)$, define $\Lambda(\alpha) \triangleq (1, \frac{\alpha\lambda_2}{\lambda_1}, \ldots, \frac{\alpha\lambda_J}{\lambda_1})$. For $t \in [0,T]$, we have from (10.3) that $\lim_{\alpha \downarrow 0} I(t,\cdot;\, \Lambda(\alpha)) = I_1(t,\cdot)$, so

$$\lim_{\alpha \downarrow 0} U'(t,\cdot;\, \Lambda(\alpha)) = U_1'(t,\cdot)$$

and

$$\lim_{\alpha \downarrow 0} \frac{\alpha\lambda_j}{\lambda_1} S_j(\frac{\alpha\lambda_j}{\lambda_1};\, \Lambda(\alpha)) = - E \int_0^T U_1'(t, \hat{c}(t)) \hat{c}_j(t) dt < 0, \; j = 2, \ldots, J.$$

Therefore, for sufficiently small positive α, we have $L_j(\Lambda(\alpha)) > \frac{\alpha\lambda_j}{\lambda_1}$; $j = 2, \ldots, J$, so we may choose $\bar{\alpha} \in (0,1)$ such that $R(\Lambda(\bar{\alpha})) \ge \Lambda(\bar{\alpha})$. Furthermore, $\Lambda(\bar{\alpha}) \le \Lambda_u$.

We set $\Lambda_\ell = \Lambda_{\bar{\alpha}}$ and cite Theorem 12.4 for the existence of a fixed point for R. The existence of a fixed point for L then follows from Lemma 12.2.

□

13. Proof of Uniqueness when $U_j'(t,0) = \infty$ for all t and j

The study of uniqueness of equilibrium requires an analysis of the sensitivity of the representative agent utility function (10.1) with respect to the parameter Λ. Throughout this section, we assume that

$$(13.1) \qquad U_j'(t,0) = \infty; \quad \forall \quad t \in [0,T],\ j \in \{1,\dots,J\}.$$

The proof of uniqueness in the absence of (13.1) is relegated to the appendix; it is conceptually similar but technically more difficult than the proof of this section. We also assume throughout this section that $J \geq 2$; if $J = 1$, Remark 12.3 applies and Theorem 11.1 and its corollaries hold.

In the presence of (13.1), the function $I(t,y;\ \Lambda)$ defined by (10.3) is differentiable in y and Λ for all $t \in [0,T]$, so the Implicit Function Theorem applied to the identity

$$I(t,\ U'(t,c;\ \Lambda);\ \Lambda) = c; \quad \forall \quad t \in [0,T],\ c > 0,$$

guarantees the differentiability in c and Λ of $U'(t,c;\ \Lambda)$ for all $t \in [0,T]$.

For $t \in [0,T]$, $c \in (0,\infty)$ and $\Lambda \in (0,\infty)^J$, let the vector $(c_1(t,c;\ \Lambda),\dots,c_J(t,c;\ \Lambda))$ denote the maximizing argument in (10.1). From (10.5), (10.7), we obtain the formula

$$(13.2) \qquad c_j(t,c;\ \Lambda) = I_j(t,\ \frac{1}{\lambda_j} U'(t,c;\ \Lambda));\ j = 1,\dots,J,$$

or equivalently

(13.3) $$U'(t,c;\ \Lambda) = \lambda_j U_j'(t, c_j(c,t;\ \Lambda)),\ j = 1,\ldots,J.$$

We also have

(13.4) $$c_1(t,c;\ \Lambda) + \ldots + c_J(t,c;\ \Lambda) = c.$$

Differentiation of (13.3), followed by division by (13.3), results in

$$\frac{\frac{\partial}{\partial\lambda_1} U'(t,c;\ \Lambda)}{U'(t,c;\ \Lambda)} = \frac{1}{\lambda_1} + \frac{U_1''(t,c_1)}{U_1'(t,c_1)} \cdot \frac{\partial c_1}{\partial\lambda_1} = \frac{U_2''(t,c_2)}{U_2'(t,c_2)} \cdot \frac{\partial c_2}{\partial\lambda_1}$$

$$= \ldots = \frac{U_J''(t,c_J)}{U_J'(t,c_J)} \cdot \frac{\partial c_J}{\partial\lambda_1}.$$

$$\frac{\frac{\partial}{\partial\lambda_2} U'(t,c;\ \Lambda)}{U'(t,c;\ \Lambda)} = \frac{U_1''(t,c_1)}{U_1'(t,c_1)} \cdot \frac{\partial c_1}{\partial\lambda_2} = \frac{1}{\lambda_2} + \frac{U_2''(t,c_2)}{U_2'(t,c_2)} \cdot \frac{\partial c_2}{\partial\lambda_2}$$

$$= \ldots = \frac{U_J''(t,c_J)}{U_J'(t,c_J)} \cdot \frac{\partial c_J}{\partial\lambda_2}.$$

$$\vdots$$

$$\frac{\frac{\partial}{\partial\lambda_J} U'(t,c;\ \Lambda)}{U'(t,c;\ \Lambda)} = \frac{U_1''(t,c_1)}{U_1'(t,c_1)} \cdot \frac{\partial c_1}{\partial\lambda_J} = \frac{U_2''(t,c_2)}{U_2'(t,c_2)} \cdot \frac{\partial c_2}{\partial\lambda_J}$$

$$= \ldots = \frac{1}{\lambda_J} + \frac{U_J''(t,c_J)}{U_J'(t,c_J)} \cdot \frac{\partial c_J}{\partial\lambda_J}.$$

Denote $\nabla = (\frac{\partial}{\partial\lambda_1},\ldots,\frac{\partial}{\partial\lambda_J})$ and let $v = (v_1,\ldots,v_J) \in \mathbb{R}^J$ be given; if we multiply equation j above by v_j and sum the resulting expressions componentwise, we obtain

$$(13.5) \qquad \frac{1}{U'(t,c;\Lambda)}(v \cdot \nabla U'(t,c;\Lambda)) = \frac{v_1}{\lambda_1} + \left[\frac{U_1''(t,c_1)}{U_1'(t,c_1)}\right](v \cdot \nabla c_1)$$

$$= \dots = \frac{v_J}{\lambda_J} + \left[\frac{U_J''(t,c_J)}{U_J'(t,c_J)}\right](v \cdot \nabla c_J).$$

On the other hand, differentiation of (13.4) yields

$$(13.6) \qquad v \cdot \nabla c_1 + \dots + v \cdot \nabla c_J = 0.$$

Equations (13.5), (13.6) provide a system of J equations for the J quantities $v \cdot \nabla c_1, \dots, v \cdot \nabla c_J$. With $x_j := v \cdot \nabla c_j$, $k_j := \frac{U_j''(t,c_j)}{U_j'(t,c_j)}$, $r_j = \frac{v_j}{\lambda_j}$; $j = 1,\dots,J$, we may write this system in matrix form as

$$(13.7) \qquad \begin{bmatrix} k_1 & -k_2 & & & \\ & k_2 & -k_3 & & \\ & & k_3 & -k_4 & \\ & & & \ddots & \ddots \\ & & & k_{J-1} & -k_J \\ 1 & 1 & 1 & \dots\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{J-1} \\ x_J \end{bmatrix} = \begin{bmatrix} r_2 - r_1 \\ r_3 - r_2 \\ r_4 - r_3 \\ \vdots \\ r_J - r_{J-1} \\ 0 \end{bmatrix}.$$

It is easily verified that

$$(13.8) \qquad x_j := \left[\sum_{i=1}^{J} \frac{1}{k_i}\right]^{-1} \sum_{i=1}^{J} \frac{r_i - r_j}{k_i k_j}, \qquad j = 1,\dots,J,$$

is a solution; in particular,

$$\sum_{j=1}^{J} x_j = [\sum_{i=1}^{J} \frac{1}{k_i}]^{-1} \cdot \sum_{j=1}^{J} \sum_{i<j} [\frac{r_i - r_j}{k_i k_j} + \frac{r_j - r_i}{k_j k_i}] = 0.$$

It remains to show that (13.8) is the only solution to (13.7), and for that we show by induction on J that the determinant of the coefficient matrix in (13.7) is $\sum_{j=1}^{J} \prod_{i \neq j} k_i$, which is nonzero because $k_i < 0;\ 1 \leq i \leq J$. For $J = 2$, the coefficient matrix is $\begin{bmatrix} k_1 & -k_2 \\ 1 & 1 \end{bmatrix}$, whose determinant is $k_1 + k_2$. For $J \geq 3$, we assume the result for $J-1$, and expand the determinant of the $J \times J$ coefficient matrix down the first column to obtain

$$k_1 \det \begin{bmatrix} k_2 & -k_3 & & & \\ & k_3 & -k_4 & & \\ & & \ddots & \ddots & \\ & & & k_{J-1} & -k_J \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix} + (-1)^{J-1} \det \begin{bmatrix} -k_2 & & & \\ k_2 & -k_3 & & \\ & \ddots & \ddots & \\ & & k_{J-1} & -k_J \end{bmatrix}$$

$$= k_1 \sum_{j=2}^{J} \prod_{\substack{i=2 \\ i \neq j}}^{J} k_i + \prod_{j=2}^{J} k_i = \sum_{j=1}^{J} \prod_{i \neq j} k_i .$$

We summarize with a lemma.

13.1 Lemma. Under the condition (13.1), we have for all $j \in \{1, \ldots, J\}$, $t \in [0,T]$, $c \in (0,\infty)$, $\Lambda \in (0,\infty)^J$ and $v = (v_1, \ldots, v_J) \in \mathbb{R}^J$:

$$(13.9)\quad \sum_{i=1}^{J} v_i \frac{\partial c_j}{\partial \lambda_i}(t,c;\ \Lambda)$$

$$= \left[\sum_{i=1}^{J} \frac{U_i'(t,c_i(t,c;\ \Lambda))}{U_i''(t,c_i(t,c;\ \Lambda))}\right]^{-1} \sum_{i=1}^{J} \left(\frac{v_i}{\lambda_i} - \frac{v_j}{\lambda_j}\right) \frac{U_i'(t,c_i(t,c;\ \Lambda))U_j'(t,c_j(t,c;\ \Lambda))}{U_i''(t,c_i(t,c;\ \Lambda))U_j''(t,c_j(t,c;\ \Lambda))}.$$

<u>13.2 Theorem</u>. Assume condition (13.1) and also that $J \geq 2$. If $\Lambda, \tilde{\Lambda}$ are both fixed points of the operator L defined by (12.1), (12.4), then we have $\Lambda = \gamma\tilde{\Lambda}$ for some $\gamma > 0$.

<u>Proof</u>: Let $\Lambda = (\lambda_1,\ldots,\lambda_J)$ and $\tilde{\Lambda} = (\tilde{\lambda}_1,\ldots,\tilde{\lambda}_J)$ be fixed points of L, and define for $\alpha \in [0,1]$; $j = 1,\ldots,J$:

$$(13.10)\qquad \Lambda(\alpha) = (\lambda_1(\alpha),\ldots,\lambda_J(\alpha)) \triangleq (1-\alpha)\Lambda + \alpha\,\tilde{\Lambda},$$

$$(13.11)\quad F_j(\alpha) \triangleq E\int_0^T U'(t,\hat{c}(t);\ \Lambda(\alpha))[I_j(\frac{1}{\lambda_j(\alpha)}U'(t,\hat{c}(t);\Lambda(\alpha))) - \hat{c}_j(t)]dt.$$

Because Λ and $\tilde{\Lambda}$ are fixed points of L, we have

$$(13.12)\qquad F_j(0) = F_j(1) = 0,\quad j = 1,\ldots,J.$$

From (13.2), (13.3), we may write

$$(13.13)\quad F_j(\alpha) = E\int_0^T \lambda_j(\alpha)U_j'(t,c_j(t,\hat{c}(t);\ \Lambda(\alpha)))[c_j(t,\hat{c}(t);\ \Lambda(\alpha)) - \hat{c}_j(t)]dt.$$

Choose $j_0 \in \{1,\ldots,J\}$ such that

$$(13.14) \qquad \frac{\lambda_{j_0}}{\tilde{\lambda}_{j_0}} = \min\{\frac{\lambda_i}{\tilde{\lambda}_i} \mid i = 1,\ldots,J\}.$$

According to Lemma 13.1 applied with $v = \tilde{\Lambda} - \Lambda$, for every $t \in [0,T]$, $c \in (0,\infty)$ we have

$$(13.15) \qquad \frac{d}{d\alpha} c_{j_0}(t,c;\ \Lambda(\alpha)) = \sum_{i=1}^{J} (\tilde{\lambda}_i - \lambda_i) \frac{\partial c_{j_0}}{\partial \lambda_i}(t,c;\ \Lambda(\alpha)) \geq 0,$$

because $U_i' > 0$, $U_i'' < 0$ and

$$(13.16) \qquad \frac{\tilde{\lambda}_i - \lambda_i}{\lambda_i(\alpha)} - \frac{\tilde{\lambda}_{j_0} - \lambda_{j_0}}{\lambda_{j_0}(\alpha)} = \frac{\tilde{\lambda}_i \tilde{\lambda}_{j_0}}{\lambda_i(\alpha)\lambda_{j_0}(\alpha)} \left(\frac{\lambda_{j_0}}{\tilde{\lambda}_{j_0}} - \frac{\lambda_i}{\tilde{\lambda}_i}\right) \leq 0$$

for all $i \in \{1,\ldots,J\}$. Indeed, the inequality in (13.15) is strict, unless

$$(13.17) \qquad \frac{\lambda_i}{\tilde{\lambda}_i} = \frac{\lambda_{j_0}}{\tilde{\lambda}_{j_0}}; \quad \forall \ i \in \{1,\ldots,J\}.$$

If (13.17) fails, then the strict version of (13.14) gives $c_{j_0}(t,c;\ \Lambda) < c_{j_0}(t,c;\ \tilde{\Lambda})$ for all $t \in [0,T]$, $c \in (0,\infty)$, and therefore

$$E \int_0^T U_{j_0}'(t, c_{j_0}(t,\hat{c}(t);\ \Lambda);\ \Lambda))\hat{c}_{j_0}(t)dt > E \int_0^T U_{j_0}'(t, c_{j_0}(t,\hat{c}(t);\ \tilde{\Lambda}))\hat{c}_{j_0}(t)dt.$$

Condition (3.7) guarantees that

$$E\int_0^T U'_{j_0}(t,c_{j_0}(t,\hat{c}(t);\ \Lambda))\ c_{j_0}(t,\hat{c}(t);\ \Lambda)dt$$

$$\leq E\int_0^T U'_{j_0}(t,c_{j_0}(t,c_{j_0}(t,\hat{c}(t);\ \Lambda))c_{j_0}(t,\hat{c}(t);\ \tilde{\Lambda})\ dt.$$

Together with (13.13), these inequalites imply that

$$\frac{1}{\lambda_{j_0}} F_{j_0}(0) < \frac{1}{\tilde{\lambda}_{j_0}} F_{j_0}(1),$$

which contradicts (13.12). We deduce that (13.17) holds, and setting $\gamma \triangleq \frac{\lambda_{j_0}}{\tilde{\lambda}_{j_0}}$ we obtain $\Lambda = \gamma\tilde{\Lambda}$. □

14. Examples.

We conclude with three examples in which equilibrium can be computed explicitly. The first is the extension of the example developed in Section 2.

14.1 Example (Logarithmic utility functions). Let the number of agents J be arbitrary, and assume that each agent has the same time-independent utility function $U_j(t,c) = \log c$. Then $I_j(t,y) = \frac{1}{y}$, and the optimal consumption process given by (9.12) is

$$c_j^*(t) = \frac{1}{\eta_j \zeta(t)\psi(t)} \; ; \quad 0 \leq t \leq T,$$

where η_j is chosen so that (see (9.7))

$$\mathcal{X}_j(\eta_j) = \frac{T}{\eta_j} = \xi_j := E \int_0^T \zeta(t)\psi(t)\hat{c}_j(t)dt.$$

In other words,

$$c_j^*(t) = \frac{\xi_j}{T\,\zeta(t)\psi(t)} \; ; \quad 0 \leq t \leq T, \tag{14.1}$$

and this expression should be compared with (2.4). Equilibrium requires

$$\frac{1}{T\,\zeta(t)\psi(t)} \sum_{j=1}^{J} \xi_j = \sum_{j=1}^{J} c_j^*(t) = \hat{c}(t); \quad 0 \leq t \leq T,$$

from which we conclude that there is a constant $\gamma > 0$ such that

(14.2) $$\zeta(t)\psi(t)\hat{c}(t) = \gamma \ ; \quad 0 \le t \le T,$$

(compare with (2.5)). Substitution of (14.2) into (14.1) yields

(14.3) $$c_j^*(t) = \lambda_j \hat{c}(t), \quad 0 \le t \le T,$$

where

(14.4) $$\lambda_j = \frac{1}{T} E \int_0^T \frac{\hat{c}_j(t)}{\hat{c}(t)}\, dt$$

(compare with (2.6), (2.7)). The vector $\Lambda = (\lambda_1, \ldots, \lambda_J)$ is a fixed point of L defined by (12.1), (12.4). Indeed $U'(c;\Lambda) = \frac{1}{c}$; $c > 0$, and thus

$$S_j(\lambda_j;\ \Lambda) = E \int_0^T \frac{1}{\lambda_j \hat{c}(t)} [\lambda_j \hat{c}(t) - \hat{c}_j(t)]dt$$

$$= T - \frac{1}{\lambda_j} E \int_0^T \frac{\hat{c}_j(t)}{\hat{c}(t)}\, dt = 0; \quad j = 1, \ldots, J. \quad \square$$

<u>14.2 Example</u> (<u>Power utility functions</u>.) Let $\delta \in (0,1)$ be given, and let each agent have the utility function $U_j(t,c) = c^\delta$. Then $I_j(t,y) = \left(\frac{y}{\delta}\right)^{\frac{1}{\delta-1}}$ and the optimal consumption process given by (9.12) is

$$c_j^*(t) = \left[\frac{\eta_j}{\delta} \zeta(t)\psi(t)\right]^{\frac{1}{\delta-1}} ; \quad 0 \le t \le T,$$

where η_j is chosen so that (see (9.7))

$$\mathfrak{X}_j(\eta_j) = \left[\frac{\eta_j}{\delta}\right]^{\frac{1}{\delta-1}} E \int_0^T [\zeta(t)\psi(t)]^{\frac{\delta}{\delta-1}} dt = \xi_j := E \int_0^T \zeta(t)\psi(t)\hat{c}_j(t)dt.$$

In other words,

$$(14.5) \quad c_j^*(t) = \left[E \int_0^T [\zeta(t)\psi(t)]^{\frac{\delta}{\delta-1}} dt\right]^{-1} \xi_j[\zeta(t)\psi(t)]^{\frac{1}{\delta-1}}; \quad 0 \le t \le T.$$

Equilibrium requires

$$\left[E \int_0^T [\zeta(t)\psi(t)]^{\frac{\delta}{\delta-1}} dt\right]^{-1} \left(\sum_{j=1}^{J} \xi_j\right) [\zeta(t)\psi(t)]^{\frac{1}{\delta-1}} = \hat{c}(t); \quad 0 \le t \le T,$$

from which we conclude that there is a constant $\gamma > 0$ such that

$$(14.6) \qquad [\zeta(t)\psi(t)]^{\frac{1}{1-\delta}} \hat{c}(t) = \gamma; \quad 0 \le t \le T.$$

Substitution of (14.6) into (14.5) yields

$$(14.7) \qquad c_j^*(t) = \lambda_j^{\frac{1}{1-\delta}} \hat{c}(t); \quad 0 \le t \le T,$$

where

$$(14.8) \qquad \lambda_j = \left[\frac{E \int_0^T \hat{c}^{\delta-1}(t)\, \hat{c}_j(t)dt}{E \int_0^T \hat{c}_j^{\delta}(t)dt} \right]^{1-\delta} .$$

Note that formulas (14.2) - (14.3) are obtained if we set $\delta = 0$ in (14.6) - (14.8). The vector $\Lambda = (\lambda_1, \ldots, \lambda_J)$ is a fixed point of L defined by (12.1), (12.4). Indeed $U'(c;\Lambda) = \delta\, c^{\delta-1} \sum_{j=1}^{J} \lambda_j^{\frac{1}{1-\delta}} = \delta\, c^{\delta-1}$; $c > 0$, and thus

$$S_j(\lambda_j;\Lambda) = E \int_0^T \frac{\delta}{\lambda_j} \hat{c}^{\delta-1}(t) \left[\lambda_j^{\frac{1}{1-\delta}} \hat{c}(t) - \hat{c}_j(t)\right]dt = 0; \quad j = 1,\ldots,J.$$

□

If agents have different utility functions, one cannot in general compute closed form solutions to the equilibrium problem. One special case in which this computation can be done is the model with $J = 2$, $U_1(c) = \log c$, $U_2(c) = \sqrt{c}$. Another special case is the following.

<u>14.3 Example</u> (<u>Constant aggregate income</u>.) Let the number of agents J be arbitrary, and let each agent j have his individual, time-independent utility function $U_j(c)$. Assume that there is a positive number $\hat{c}$ such that $P[\sum_{j=1}^{J} \hat{c}_j(t) = \hat{c}] = 1$, $0 \le t \le T$. We show that the <u>equilibrium deflated spot price $\zeta(t)\psi(t)$ is constant</u>, and <u>each agent's optimal equilibrium consumption is constant</u> and equal to

$$(14.9) \qquad c_j^* \triangleq \frac{1}{T} E \int_0^T \hat{c}_j(t)dt, \qquad j = 1,\ldots,J.$$

To do this, we define $\Lambda = (\lambda_1,\ldots,\lambda_J)$, where

$$(14.10) \qquad \lambda_j \triangleq \frac{1}{U'(c_j^*)}.$$

According to (10.3),

$$I(1;\Lambda) = \sum_{j=1}^{J} I_j(\frac{1}{\lambda_j}) = \sum_{j=1}^{J} c_j^* = \hat{c},$$

so $U'(\hat{c};\Lambda) = 1 = \lambda_j U'(c_j^*)$. From (12.1) we have

$$S_j(\frac{1}{\lambda_j};\Lambda) = T\lambda_j I_j(\frac{1}{\lambda_j}) - \lambda_j \int_0^T \hat{c}_j(t)dt = 0,$$

so Λ is a fixed point of the operator L defined by (12.4). In other words, with $\eta_j(\Lambda)$; $j = 1,\ldots,J$, as described in Theorem 10.2, relation (10.17) holds. It follows from that theorem that $\psi(t) \triangleq \frac{1}{\zeta(t)}$ is the (unique up to a multiplicative constant) equilibrium spot price and $c_j^* = I_j(\frac{1}{\lambda_j})$ is the (unique) optimal equilibrium consumption for agent j. Note in this example that agents' income processes can be random and time-varying, so although their optimal equilibrium consumption processes are constant, they will in general need nonconstant portfolio processes to finance this consumption. □

In the absence of condition (3.9), there can be equilibrium spot price processes which differ from one another by more than a multiplicative constant. When this occurs, we are unable in any generality to prove uniqueness of the optimal equilibrium consumption processes for the individual agents. Such uniqueness is present, however, in the following example. It is an open question whether this uniqueness is always present when all the conditions of our model except (3.9) hold.

<u>14.4 Example</u>. Let $J = 2$ and define

$$U_1(t,c) \triangleq \begin{cases} \log c; & 0 \le t \le \frac{T}{2}, \\ \log (c+1); & \frac{T}{2} < t \le T, \end{cases}$$

$$U_2(t,c) \triangleq \begin{cases} \log (c+1); & 0 \le t \le \frac{T}{2}, \\ \log c\ ; & \frac{T}{2} < t \le T. \end{cases}$$

Direct computation reveals that

$$I_1(t,y) = \begin{cases} \frac{1}{y}\ ; & 0 \le t \le \frac{T}{2}\ , \quad y > 0, \\ (\frac{1-y}{y})^+; & \frac{T}{2} < t \le T, \ y > 0, \end{cases}$$

$$I_2(t,y) = \begin{cases} (\frac{1-y}{y})^+; & 0 \le t \le \frac{T}{2}, \quad y > 0, \\ \frac{1}{y}; & \frac{T}{2} < t \le T, \quad y > 0, \end{cases}$$

$$U'(t,c;\ \lambda_1,\lambda_2) = \begin{cases} \dfrac{\lambda_1}{c}\ ; & 0 \le t \le \frac{T}{2},\quad 0 < c \le \dfrac{\lambda_1}{\lambda_2}, \\ \dfrac{\lambda_1+\lambda_2}{c+1}; & 0 \le t \le \frac{T}{2},\quad c > \dfrac{\lambda_1}{\lambda_2}, \\ \dfrac{\lambda_2}{c}; & \frac{T}{2} < t \le T,\quad 0 < c \le \dfrac{\lambda_2}{\lambda_1}, \\ \dfrac{\lambda_1+\lambda_2}{c+1}; & \frac{T}{2} < t \le T,\quad c > \dfrac{\lambda_2}{\lambda_1}, \end{cases}$$

and so if $0 < c \le \min\{\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}\}$, we have

$$c_1(t,c;\ \lambda_1,\lambda_2) = I_1(\frac{1}{\lambda_1} U'(t,c;\ \lambda_1,\lambda_2)) = \begin{cases} c\ ; & 0 \le t \le \frac{T}{2}, \\ 0\ ; & \frac{T}{2} < t \le T, \end{cases}$$

$$c_2(t,c;\ \lambda_1,\lambda_2) = I_2(\frac{1}{\lambda_2} U'(t,c;\ \lambda_1,\lambda_2)) = \begin{cases} 0\ ; & 0 \le t \le \frac{T}{2}, \\ c\ ; & \frac{T}{2} < t \le T. \end{cases}$$

Now take the income processes to be

$$\hat{c}_1(t) = \begin{cases} \frac{1}{2}\ ; & 0 \le t \le \frac{T}{2}, \\ 0; & \frac{1}{2} < t \le \frac{T}{2}, \end{cases}$$

$$\hat{c}_2(t) = \frac{1}{2} - \hat{c}_1(t);\quad 0 \le t \le T.$$

If $\lambda_1 > 0$, $\lambda_2 > 0$ are chosen to satisfy

$$\min\{\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}\} \ge \frac{1}{2}, \tag{14.11}$$

then the equilibrium conditions

$$(14.12)\quad \int_0^T U'(t, \tfrac{1}{2}; \lambda_1,\lambda_2)c_j(t, \tfrac{1}{2}; \lambda_1,\lambda_2) = \int_0^T U'(t, \tfrac{1}{2}; \lambda_1,\lambda_2)\hat{c}_j(t)dt; \; j = 1,2,$$

are satisfied. In particular, the corresponding equilibrium spot price is

$$(14.13)\qquad \psi(t) \triangleq \begin{cases} \dfrac{2\lambda_1}{\zeta(t)}; & 0 \leq t \leq \dfrac{T}{2}, \\ \dfrac{2\lambda_2}{\zeta(t)}; & \dfrac{T}{2} < t \leq T, \end{cases}$$

which is not determined up to a multiplicative constant. In fact, (14.12) can be used to show that all the equilibrium spot price processes are given by (14.13), where (14.11) is satisfied. Consequently, the unique optimal equilibrium consumption processes are

$$c_1^*(t) = \hat{c}(t), \quad c_2^*(t) = \hat{c}(t); \qquad 0 \leq t \leq T.$$

15.1 Appendix. Proof of Uniqueness when $U_j'(t,0) < \infty$ for some t and j

The proof of uniqueness of equilibrium in Section 13 was given under the assumption (13.1). In this section, we provide the modification of that proof required when (13.1) is no longer assumed. As in Section 13, we assume here without loss of generality that $J \geq 2$.

For $t \in [0,T]$, $c \in [0,\infty)$ and $\Lambda \in (0,\infty)^J$, let $(c_1(t,c;\ \Lambda),\ldots,c_J(t,c;\ \Lambda))$ denote the maximizing argument in (10.1), which is given by (10.4). (Recall (10.7) in this connection.) Then (13.2) and (13.4) are valid, but instead of (13.3), we have the conditions for $j = 1,\ldots,J$:

$$c_j(t,c;\ \Lambda) \geq 0, \tag{15.1}$$

$$U'(t,c;\ \Lambda) - \lambda_j U_j'(t,c_j(t,c;\ \Lambda)) \geq 0, \tag{15.2}$$

$$c_j(t,c;\ \Lambda)[U'(t,c;\ \Lambda) - \lambda_j U_j'(t,c_j(t,c;\ \Lambda))] = 0. \tag{15.3}$$

<u>15.1 Lemma</u>. For each $t \in [0,T]$, the functions $I(t,\cdot;\ \cdot)$, $U'(t,\cdot;\ \cdot)$ and $c_j(t,\cdot;\ \cdot)$ are Lipschitz continuous on compact subsets of $(0,\infty) \times (0,\infty)^J$.

<u>Proof</u>: Let $t \in [0,T]$ be fixed. Each function $I_j(t,\cdot)$ is piecewise continuously differentiable, and on compact subsets of $(0,\ U_j'(t,0))$, $I_j'(t,\cdot)$ is bounded. The Lipschitz continuity of $I(t,\cdot;\ \cdot)$ on compact subsets of $(0,\infty) \times (0,\infty)^J$ follows immediately from (10.3).

Define $M(\Lambda) \triangleq \max_{1 \leq j \leq J} \lambda_j U_j'(t,0)$ for all $\Lambda \in (0,\infty)^J$, and set $\mathcal{M} \triangleq \{(y,\Lambda) \in (0,\infty) \times (0,\infty)^J \mid y < M(\Lambda)\}$. For fixed Λ, $U'(t,\cdot;\ \Lambda)$ maps $(0,\infty)$ onto $(0,M(\Lambda))$, and the inverse $I(t,\cdot;\ \Lambda)$ is piecewise continuously differentiable. If Γ is a compact subset of M, then there exists a positive constant $\alpha(\Gamma)$ such that $I'(t,y;\Lambda) \leq -\alpha(\Gamma)$ for all $(y,\Lambda) \in \Gamma$ wherever this derivative (with respect to the y-variable) is defined. Therefore, for every $\epsilon(\Gamma) > 0$ chosen so that $(y,\Lambda) \in \Gamma$ implies $(y \pm \epsilon, \Lambda) \in M$, for every $\epsilon \in (0, \epsilon(\Gamma))$, there is a positive number $\alpha(\epsilon(\Gamma), \Gamma)$ such that

$$(15.4) \quad |I(t,\tilde{y};\Lambda) - I(t,y;\Lambda)| \geq \alpha(\epsilon(\Gamma),\Gamma)\,|\tilde{y}-y|;\ \forall\ (y,\Lambda) \in \Gamma,\ \tilde{y} \in [y-\epsilon, y+\epsilon].$$

We first prove (nonuniform) Lipschitz continuity of $U'(t,\cdot;\cdot)$. Define a norm on $(0,\infty)^J$ by $\|\lambda_1,\ldots,\lambda_J\| \triangleq \max_{1\leq j\leq J} |\lambda_j|$. Let $(c,\Lambda) \in (0,\infty) \times (0,\infty)^J$ be given, set $y \triangleq U'(t,c;\Lambda)$, and let $\epsilon > 0$ be such that $(y \pm \epsilon, \Lambda) \in M$. We may then choose $\alpha(\epsilon) > 0$ such that

$$(15.5) \qquad |I(t,\tilde{y};\Lambda) - I(t,y;\Lambda)| \geq \alpha(\epsilon)\,|\tilde{y}-y|;\quad \forall\ \tilde{y} \in [y-\epsilon, y+\epsilon].$$

The local Lipschitz continuity of $I(t,\cdot;\cdot)$ allows us to choose $\rho > 0$, $K > 0$ such that for all $\tilde{y} \in [y-\epsilon, y+\epsilon]$ and all $\tilde{\Lambda}$ satisfying $\|\tilde{\Lambda}-\Lambda\| \leq \rho$, we have

$$(15.6) \qquad |I(t,\tilde{y};\tilde{\Lambda}) - I(t,\tilde{y};\Lambda)| \leq K\,\|\tilde{\Lambda}-\Lambda\|.$$

Decreasing ρ if necessary, we assume without loss of generality that $K\rho \leq \frac{1}{2}\alpha(\epsilon)\epsilon$. Now suppose that $(\tilde{c},\tilde{K}) \in (0,\infty) \times (0,\infty)^J$ satisfies $|\tilde{c}-c| \leq \frac{1}{2}\alpha(\epsilon)\epsilon$, $\|\tilde{\Lambda}-\Lambda\| \leq \rho$, and set $\tilde{y} \triangleq U'(t,\tilde{c};\tilde{\Lambda})$. We show that

$$(15.7) \quad |U'(t,\tilde{c};\tilde{\Lambda}) - U'(t,c;\Lambda)| = |\tilde{y}-y| \leq \frac{2}{\alpha(\epsilon)} \max\{|\tilde{c}-c|,\ K\|\tilde{\Lambda}-\Lambda\|\}.$$

Let $\gamma \triangleq \frac{2}{\alpha(\epsilon)} \max \{|\tilde{c}-c|, K\|\tilde{\Lambda}-\Lambda\|\}$, and note that $\gamma \leq \epsilon$. If $\tilde{y} < y-\gamma$, then from (15.6), (15.5), we would have

$$|\tilde{c}-c| = I(t,\tilde{y};\ \tilde{\Lambda}) - I(t,y;\ \Lambda)$$

$$> [I(t,y-\gamma;\ \tilde{\Lambda}) - I(t,y-\gamma;\ \Lambda)] + [I(t,y-\gamma;\ \Lambda) - I(t,y;\ \Lambda)]$$

$$\geq -K\ \|\tilde{\Lambda}-\Lambda\| + \alpha(\epsilon)\gamma \geq |\tilde{c}-c|,$$

a contradiction. On the other hand, if $\tilde{y} > y+\gamma$, then

$$|\tilde{c}-c| = I(t,y;\ \Lambda) - I(t,\tilde{y};\ \tilde{\Lambda})$$

$$> [I(t,y;\ \Lambda) - I(t,\ y+\gamma;\ \Lambda)] + [I(t,y+\gamma;\ \Lambda) - I(t,y+\gamma;\ \tilde{\Lambda})]$$

$$\geq \alpha(\epsilon)\gamma - K\ \|\tilde{\Lambda}-\Lambda\| \geq |\tilde{c}-c|.$$

It follows that $\tilde{y} \in [y-\epsilon,\ y+\epsilon]$, which proves (15.7) and thereby the Lipschitz continuity of $U'(t,\cdot;\ \cdot)$ at (c,Λ).

Now let D be a compact subset of $(0,\infty) \times (0,\infty)^J$ and define $\Gamma \triangleq \{(U'(t,c;\ \Lambda),\Lambda)\ |\ (c,\Lambda) \in D\}$. Because $U'(t,\cdot;\ \cdot)$ is continuous, Γ is a compact subset of $\mathbf{M}$, and we can repeat the above proof with $\epsilon(\Gamma)$, $\alpha(\epsilon(\Gamma),\Gamma)$ and (15.4) substituted for ϵ, $\alpha(\epsilon)$ and (15.5), and with ρ,K chosen so that (15.6) holds for all $(\tilde{y},\tilde{\Lambda})$, $(\tilde{y},\Lambda)$; here $\tilde{y} \in [y - \epsilon(\Gamma),\ y + \epsilon(\Gamma)]$.

$\|\tilde{\Lambda} - \Lambda\| \leq \rho$, and (y,Λ) is an arbitrary point in Γ. Then for $(c,\Lambda) \in \Gamma$ and $(\tilde{c},\tilde{\Lambda}) \in (0,\infty) \times (0,\infty)^J$ such that $|\tilde{c} - c| \leq \frac{1}{2}\alpha(\epsilon(\Gamma),\Gamma)\ \epsilon(\Gamma)$ and $\|\tilde{\Lambda} - \Lambda\| \leq \rho$, relation (15.7) holds.

Being the composition of locally Lipschitz functions (see (13.2)), $c_j(t,\cdot;\cdot)$ is itself locally Lipschitz for each j. □

For $t \in [0,T]$, $c \in (0,\infty)$ and $j \in \{1,\ldots,J\}$, we partition $(0,\infty)^J$ into three sets:

(15.8) $$P_j(t,c) \triangleq \{\Lambda \in (0,\infty)^J \mid U'(t,c;\ \Lambda) < \lambda_j U_j'(t,0)\},$$

(15.9) $$B_j(t,c) \triangleq \{\Lambda \in (0,\infty)^J \mid U'(t,c;\ \Lambda) = \lambda_j U_j'(t,0)\},$$

(15.10) $$Z_j(t,c) \triangleq \{\Lambda \in (0,\infty)^J \mid U'(t,c;\ \Lambda) > \lambda_j U_j'(t,0)\}.$$

Conditions (15.1) - (15.3) show that

(15.11) $$\Lambda \in P_j(t,c) \iff c_j(t,c;\ \Lambda) > 0.$$

Both $P_j(t,c)$ and $Z_j(t,c)$ are open, while $B_j(t,c)$ is relatively closed in $(0,\infty)^J$.

Given a nonempty set $K \subset \{1,\ldots,J\}$, we define the open set

$$D_K(t,c) \triangleq [\bigcap_{j\in K} P_j(t,c)] \cap [\bigcap_{j\notin K} Z_j(t,c)].$$

For $\Lambda \in D_K(t,c)$, the definition (10.1) for the representative agent utility function reduces to

$$(15.12)\quad U(t,c;\ \Lambda) = \max\{\sum_{j\in K} \lambda_j U_j(t,c_j) \mid c_j > 0\ \ \forall\, j \in K,\ \sum_{j\in K} c_j = c\}.$$

We can use this representation in the proof of Lemma 13.1 to obtain the following extension of that result.

15.2 Lemma. Let K be a nonempty subset of $\{1,\dots,J\}$. For all $t \in [0,T]$, $c \in (0,\infty)$, $\Lambda \in D_K(t,c)$ and $v = (v_1,\dots,v_J) \in \mathbb{R}^J$, we have for all $j \in K$:

$$(15.13)\quad \sum_{i\in K} v_i \frac{\partial c_j}{\partial \lambda_i}(t,c;\ \Lambda)$$

$$= \left[\sum_{i\in K} \frac{U_i'(t,c_i(t,c;\ \Lambda))}{U_i''(t,c_i(t,c;\ \Lambda))}\right]^{-1} \sum_{i\in K} \left(\frac{v_i}{\lambda_i} - \frac{v_j}{\lambda_j}\right) \frac{U_i'(t,c_i(t,c;\Lambda))U_j'(t,c_j(t,c;\Lambda))}{U_i''(t,c_i(t,c;\Lambda))U_j''(t,c_j(t,c;\Lambda))}.$$

15.3 Lemma. Let $\Lambda, \tilde{\Lambda} \in (0,\infty)^J$ be given, choose $j_0 \in \{1,\dots,J\}$ as in (13.14), and define $\Lambda(\alpha);\ 0 \le \alpha \le 1$ by (13.10). Then for all $t \in [0,T]$, $c \in (0,\infty)$, the function

$$(15.14)\qquad y_{j_0}(t,c;\ \Lambda(\alpha)) \triangleq \frac{1}{\lambda_{j_0}(\alpha)} U'(t,c;\ \Lambda(\alpha)),\qquad 0 \le \alpha \le 1,$$

is nonincreasing. Furthermore, if there exist $i_0 \in \{1,\dots,J\}$ and $\bar{\alpha} \in [0,1]$ with the properties

$$(15.15)\qquad \frac{\lambda_{j_0}}{\tilde{\lambda}_{j_0}} < \frac{\lambda_{i_0}}{\tilde{\lambda}_{i_0}},\qquad c_{i_0}(t,c;\ \Lambda(\bar{\alpha})) > 0,$$

then

$$(15.16) \qquad y_{j_0}(t,c;\ \Lambda) > y_{j_0}(t,c;\ \tilde{\Lambda}).$$

Proof: We simplify notation by writing $y(\alpha) = y_{j_0}(t,c;\ \Lambda(\alpha))$. We know from Lemma 15.1 that this function is almost everywhere differentiable on [0,1]. Direct computation yields for $i = 1,\dots,J$:

$$(15.17) \quad \frac{d}{d\alpha}\left[\frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)}\right] = y(\alpha)\frac{\tilde{\lambda}_i\tilde{\lambda}_{j_0}}{\lambda_i^2(\alpha)}\left(\frac{\lambda_i}{\tilde{\lambda}_i} - \frac{\lambda_{j_0}}{\tilde{\lambda}_{j_0}}\right) + y'(\alpha)\frac{\lambda_{j_0}(\alpha)}{\lambda_i(\alpha)},$$

a.e. $\alpha \in [0,1]$.

From (15.14) we have

$$(15.18) \qquad c = I(t,\ \lambda_{j_0}(\alpha)y(\alpha)) = \sum_{i=1}^{J} I_i\left(t,\ \frac{\lambda_{j_0}(\alpha)y(\alpha)}{\lambda_i(\alpha)}\right),$$

an expression we wish to differentiate with respect to α. Now

$$I_i'(t,y) = \begin{cases} \dfrac{1}{U_i''(t,\ I_i(t,y))}, & 0 < y < U_i'(t,0), \\ 0, & y > U_i'(t,0), \end{cases}$$

and $I_i'(t,\cdot)$ has left- and right-hand derivatives at $U_i'(t,0)$ equal to $\frac{1}{U_i''(t,0)}$ and 0, respectively. Define

$$N_i \triangleq \{\alpha \in [0,1] \mid \frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)} = U_i'(t,0)\}.$$

Let $\alpha \in [0,1]$ be such that $y'(\alpha)$ is defined, and let $\{\alpha_k\}_{k=1}^{\infty} \subset [0,1]\backslash\{\alpha\}$ be a sequence converging to α. We may choose k large enough so that for those values of i for which $\alpha \notin N_i$, the interval between α and α_k does not intersect N_i. For such k, choose $\theta_k^{(i)}$ between $\frac{\lambda_{j_0}(\alpha_k)\, y(\alpha_k)}{\lambda_i(\alpha_k)}$ and $\frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)}$ so that

$$I_i(t, \frac{\lambda_{j_0}(\alpha_k)\, y(\alpha_k)}{\lambda_i(\alpha_k)}) - I_i(t, \frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)})$$

$$= I_i'(t, \theta_k^{(i)}) \left[\frac{\lambda_{j_0}(\alpha_k)\, y(\alpha_k)}{\lambda_i(\alpha_k)} - \frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{y_i(\alpha)}\right].$$

If $\alpha \notin N_i$, then $\theta_k^{(i)} \notin U_i'(t,0)$. From (15.18), (15.17), we have

$$(15.19) \quad 0 = \lim_{k\to\infty} \sum_{i=1}^{J} I_i'(t,\theta_k^{(i)})(\frac{1}{\alpha_k-\alpha}) \left[\frac{\lambda_{j_0}(\alpha_k)\, y(\alpha_k)}{\lambda_i(\alpha_k)} - \frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)}\right]$$

$$\leq \sum_{i=1}^{J} \overline{\lim_{k\to\infty}}\, I_i'(t,\theta_k^{(i)})(\frac{1}{\alpha_k-\alpha}) \left[\frac{\lambda_{j_0}(\alpha_k)\, y(\alpha_k)}{\lambda_i(\alpha_k)} - \frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)}\right]$$

$$\leq \sum_{\{i \mid \alpha\in N_i\}} \frac{1}{|U_i''(t,0)|} \left[y(\alpha) \frac{\tilde{\lambda}_i\tilde{\lambda}_{j_0}}{\lambda_i^2(\alpha)} (\frac{\lambda_i}{\tilde{\lambda}_i} - \frac{\lambda_{j_0}}{\tilde{\lambda}_{j_0}}) + y'(\alpha) \frac{\lambda_{j_0}(\alpha)}{\lambda_i(\alpha)}\right]^-$$

$$+ \sum_{\{i \mid \alpha\notin N_i\}} I_i'(t, \frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)}) \left[y(\alpha) \frac{\tilde{\lambda}_i\tilde{\lambda}_{j_0}}{\lambda_i^2(\alpha)} (\frac{\lambda_i}{\tilde{\lambda}_i} - \frac{\lambda_{j_0}}{\tilde{\lambda}_{j_0}}) + y'(\alpha) \frac{\lambda_{j_0}(\alpha)}{\lambda_i(\alpha)}\right].$$

If $y'(\alpha) \geq 0$, relation (15.19) can hold only if:

$$(15.20) \quad \forall\ i \in \{1,\dots,J\},\ \frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)} < U_i'(t,0) \Rightarrow \frac{\lambda_i}{\tilde{\lambda}_i} = \frac{\lambda_{j_0}}{\tilde{\lambda}_{j_0}} \text{ and } y'(\alpha) = 0.$$

Because

$$c = \sum_{i=1}^{J} I_i\left(t, \frac{1}{\lambda_i(\alpha)} U'(t,c;\ \Lambda(\alpha))\right) = \sum_{\{i \mid \alpha \in N_i\}} I_i\left(t, \frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)}\right),$$

we must have $I_i\left(t, \frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)}\right) > 0$ for some i; this is equivalent to $\frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)} < U_i'(t,0)$, and shows that $y'(\alpha) \leq 0$ for a.e. $\alpha \in [0,1]$. Furthermore, if i_0 and $\bar{\alpha}$ with properties (15.15) exist, then there exist $0 \leq \delta \leq \bar{\alpha} \leq \beta \leq 1$ with $\delta < \beta$ and $c_{i_0}(t,c;\ \Lambda(\alpha)) > 0;\ \forall\ \alpha \in [\delta,\beta]$. But $c_{i_0}(t,c;\ \Lambda(\alpha)) > 0$ is equivalent to

$$\frac{\lambda_{j_0}(\alpha)\, y(\alpha)}{\lambda_i(\alpha)} = \frac{1}{\lambda_i(\alpha)} U'(t,c;\ \Lambda(\alpha)) < U_i'(t,0),$$

so that (15.20) is violated by $i = i_0$. It follows in this case that $y'(\alpha) < 0;\ \forall\ \alpha \in [\delta,\beta]$. □

<u>15.4 Lemma</u>. Under the assumptions and notation of Lemma 15.3, the function $\alpha \mapsto c_{j_0}(t,c;\ \Lambda(\alpha))$ is nondecreasing on $[0,1]$ for all $t \in [0,T]$ and $c \in (0,\infty)$.

Proof: The Lebesgue measure of the product set

$$\bigcup_{i=1}^{J} \{(c,\alpha) \in (0,\infty) \times [0,1] \mid c = I(t, \lambda_j(\alpha)U_j'(t,0))\} = \{(c,\alpha) \mid \Lambda(\alpha) \in \bigcup_{j=1}^{J} B_j(t,c)\}$$

is zero. We may choose a set $C \subset (0,\infty)$ such that $(0,\infty)\backslash C$ has Lebesgue measure zero, and for every $c \in C$ the set $\{\alpha \in [0,1] \mid \Lambda(\alpha) \in \bigcup_{j=1}^{J} B_j(t,c)\}$ has measure zero. Denote by $A(c)$ the complement in $[0,1]$ of this set. For $c \in C$ and $\alpha \in A(c)$, there exists a nonempty set $K(c,\alpha) \subset \{1,\ldots,J\}$ such that $\Lambda(\alpha) \in D_{K(c,\alpha)}(t,c)$. If $j_0 \notin K(c,\alpha)$, then $c_{j_0}(t,c;\, \Lambda(\cdot))$ is identically zero in a neighborhood of α. If $j_0 \in K(c,\alpha)$, then (13.14), (13.16) and Lemma 15.2 imply that

$$(15.21)\quad \frac{d}{d\alpha} c_{j_0}(t,c;\, \Lambda(\alpha)) = \sum_{i\in K(c,\alpha)} (\tilde{\lambda}_i - \lambda_i) \frac{\partial c_{j_0}}{\partial \lambda_i}(t,c;\, \Lambda(\alpha)) \geq 0.$$

Because $\alpha \mapsto c_{j_0}(t,c;\, \Lambda(\alpha))$ is Lipschitz continuous, integration of (15.21) shows that this function is nondecreasing for every $c \in C$. Since C is dense in $(0,\infty)$ and $c_{j_0}(t,c;\, \Lambda(\alpha))$ is continuous in c, the function $\alpha \mapsto c_{j_0}(t,c;\, \Lambda(\alpha))$ is nondecreasing for all $c \in (0,\infty)$. □

15.5 Theorem. Assume that $J \geq 2$, and that we have $U_j'(t,0) < \infty$ for some $t \in [0,T]$, $j \in \{1,\ldots,J\}$. . If both Λ and $\tilde{\Lambda}$ are fixed points of the operator L defined by (12.1), (12.4), then $\Lambda = \gamma\tilde{\Lambda}$ for some $\gamma > 0$.

Proof: Let Λ and $\tilde{\Lambda}$ be fixed points of L, define $\Lambda(\alpha)$ and $F_j(\alpha)$ by

(13.10), (13.11), respectively, and note that $F_j(\alpha) = \lambda_j(\alpha)[G_j(\alpha) - H_j(\alpha)]$, where

$$G_j(\alpha) \triangleq E \int_0^T \frac{1}{\lambda_j(\alpha)} U'(t, \hat{c}(t); \Lambda(\alpha)) \; c_j(t, \hat{c}(t); \Lambda(\alpha))dt,$$

$$H_j(\alpha) = E \int_0^T \frac{1}{\lambda_j(\alpha)} U'(t, \hat{c}(t); \Lambda(\alpha)) \; \hat{c}_j(t)dt.$$

If $c_j(t, \hat{c}(t); \Lambda(\alpha)) > 0$, then (15.3) shows that

$$\frac{1}{\lambda_j(\alpha)} U'(t, \hat{c}(t); \Lambda(\alpha)) = U_j'(t, c_j(t,\hat{c}(t); \Lambda(\alpha)),$$

so

$$G_j(\alpha) = E \int_0^T U_j'(t,c_j(t,\hat{c}(t); \Lambda(\alpha))) \; c_j(t,\hat{c}(t);\Lambda(\alpha))dt.$$

With j_0 chosen to satisfy (13.14), we have from (3.7) and Lemma 15.4 that $G_{j_0}(0) \leq G_{j_0}(1)$. From Lemma 15.3 we see that $H_{j_0}(0) \geq H_{j_0}(1)$, but because of (13.12), $G_{j_0}(0) - H_{j_0}(0) = G_{j_0}(1) - H_{j_0}(1) = 0$. Taken together, these facts imply

$$H_{j_0}(0) = H_{j_0}(1). \tag{15.22}$$

Suppose that Λ is not a scalar multiple of $\tilde{\Lambda}$; then we may choose

$i_0 \in \{1,\ldots,J\}$ such that $\frac{\lambda_{j_0}}{\tilde{\lambda}_{j_0}} < \frac{\lambda_{i_0}}{\tilde{\lambda}_{i_0}}$. Since $\hat{c}_{i_0}(t,\omega)$ is positive on a set of positive Lebesgue × P measure, $H_{i_0}(0) > 0$. Relation (13.12) now implies that $G_{i_0}(0) > 0$, from which we conclude that for some set $\Gamma \subset [0,T] \times \Omega$ with positive Lebesgue × P measure,

$$c_{i_0}(t,\hat{c}(t,\omega);\ \Lambda) > 0 \quad \forall\ (t,\omega) \in \Gamma.$$

According to Lemma 15.3,

$$\frac{1}{\lambda_{j_0}} U'(t,\hat{c}(t,\omega);\ \Lambda) > \frac{1}{\tilde{\lambda}_{j_0}} U'(t,\hat{c}(t,\omega);\ \tilde{\Lambda});\quad \forall\ (t,\omega) \in \Gamma.$$

Finally, we appeal to (3.9) to assert that $\hat{c}_{j_0}(t,\omega) > 0;\ \ \forall\ (t,\omega) \in \Gamma$. It follows that $H_{j_0}(0) > H_{j_0}(1)$, contradicting (15.22). □

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