SOME RESULTS ON THE BASIS PROBLEM FOR PROPER COMBINATORS WITH PURE EFFECT

by

Doug Ensley
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Research Report No. 88-8 2

April 1988

Some Results on the Basis Problem for Proper Combinators with Pure Effect

by

Doug Ensley

One of the most significant open problems in the theory of combinatory logic is the so-called basis problem. That is, given a finite set $\mathbb{S}$ of proper combinators, is $\mathbb{S}$ combinatorially complete? It is known that this question is undecidable when we only require that $\mathbb{S}$ be a finite set of normal combinators. In this study, we will look at the basis problem modified with respect to certain restrictions on the sorts of combinators that we allow in $\mathbb{S}$. It should be noted that although one may think of combinators as λ terms, we will attempt to consistently treat them as combinators with reduction rules.

In Curry and Feys, the "effect" of a proper combinator is described according to the action on the arguments in the reduction rule of the combinator. The definitions are fairly intuitive: a (non-identity) proper combinator must have at least one of permutative, selective, compositive, or duplicative effect. We will restrict our attention to a special class of proper combinators which we shall say have *pure effect*, meaning that each combinator has only one effect. We should note that for this paper we will consider an identity combinator as having no effect, and hence in particular as not having any pure effect.

We begin by giving a generalization of the following result of Statman: every proper combinator is definable from **B**, **I**, $\mathbf{C}_\star$, any proper combinator with selective effect, and any proper combinator with duplicative effect. We will show that $\mathbf{C}_\star$ may be replaced by any proper combinator with pure permutative effect and the result will be the same.

Remark: (The use of η-conversion) For a proper combinator $\mathbf{P}$ with pure permutative effect, let $p = \mathbf{max}\{i \in 1..\text{order}(\mathbf{P}) : \pi(i) \neq i\}$, where π is the permutation associated with $\mathbf{P}$. Let $\mathbf{X}$ be the proper combinator with pure permutative effect associated with the restriction of π to 1..p. Then clearly $\mathbf{X} =_{\eta} \mathbf{P}$. Thus, we may assume that $\mathbf{P}$ has the property that its last argument is not "fixed" under β-reduction. We also assume that $\mathbf{I}$ is present since every identity combinator is η-equivalent to $\mathbf{I}$ in this way. //

Let $\mathbb{S}$ be the set $\{\mathbf{B}, \mathbf{I}, \mathbf{P}\}$, where $\mathbf{P}$ is any proper combinator with pure permutative effect of order p such that the corresponding permutation $\pi \in S_p$ does not fix p. Then we have the following :

Lemma 1 : If there exist $\mathbf{Q}_o, .., \mathbf{Q}_k \in \mathbb{S}^+$ such that $\mathbf{Q}_o x \mathbf{Q}_1 \cdots \mathbf{Q}_k y = yx$, then $\mathbf{C}_* \in \mathbb{S}^+$.

Proof (By induction on k)

(Basis of the induction)

If $k = 0$, then $\mathbf{C}_* = \mathbf{Q}_o \in \mathbb{S}^+$.

(Inductive step)

Let $k \in \mathbb{N}$ be given and suppose the result holds for this value of k. Let $\mathbf{Q}_o, .., \mathbf{Q}_k, \mathbf{Q}_{k+1} \in \mathbb{S}^+$ be given such that $\mathbf{Q}_o x \mathbf{Q}_1 \cdots \mathbf{Q}_k \mathbf{Q}_{k+1} y = yx$. Then note that

$$\mathbf{Q}_o x \mathbf{Q}_1 \ {}_\beta\!\leftarrow \mathbf{Q}_o x (\underbrace{\mathbf{I} \cdots \mathbf{I}}_{p-\pi(p)-1} \mathbf{Q}_1)$$

$${}_\beta\!\leftarrow (\mathfrak{B} \mathbf{Q}_o x) \mathbf{I} \cdots \mathbf{I} \mathbf{Q}_1 \text{ , where } \mathfrak{B} \text{ is an appropriate applicative combination of } \mathbf{B}\text{'s ,}$$

$${}_\beta\!\leftarrow \underbrace{\mathbf{I} \cdots \mathbf{I}}_{\pi(p)-1} (\mathfrak{B} \mathbf{Q}_o x) \mathbf{I} \cdots \mathbf{I} \mathbf{Q}_1$$

$$\beta\!\leftarrow \mathbf{P}\mathfrak{U}_1\cdots\mathfrak{U}_{p-1}(\mathfrak{B}Q_o x) \quad \text{, for some } \mathfrak{U}_1,..,\mathfrak{U}_{p-1} \in \mathbb{S}^+,$$

$$\beta\!\leftarrow \mathbf{B}(\mathbf{P}\mathfrak{U}_1\cdots\mathfrak{U}_{p-1})(\mathfrak{B}Q_o)x \ .$$

Hence, $\mathbf{B}(\mathbf{P}\mathfrak{U}_1\cdots\mathfrak{U}_{p-1})(\mathfrak{B}Q_o)xQ_2\cdots Q_{k+1}y = yx$, so by the inductive hypothesis, $\mathbf{C}_* \in \mathbb{S}^+$. ■

<u>Theorem 1</u> : **B** and **I** along with any proper combinator with pure permutative effect generate all proper combinators without selective or duplicative effect.

Proof :

Let **P** be a proper combinator with pure permutative effect as before and let $j := \pi^{-1}(p)$. Then clearly,

$$yx \ \beta\!\leftarrow y(\underbrace{\mathbf{I}\cdots\mathbf{I}}_{p-\pi(p)-1}x)$$

$$\beta\!\leftarrow \mathfrak{B}y\mathbf{I}\cdots\mathbf{I}x \text{ , where } \mathfrak{B} \text{ is an appropriate applicative combination of } \mathbf{B}\text{'s ,}$$

$$\beta\!\leftarrow \underbrace{\mathbf{I}\cdots\mathbf{I}}_{\pi(p)-1}(\mathfrak{B}y)\mathbf{I}\cdots\mathbf{I}x$$

$$\beta\!\leftarrow \mathbf{P}\mathfrak{U}_1\cdots\mathfrak{U}_{j-1}x\mathfrak{U}_{j+1}\cdots\mathfrak{U}_p(\mathfrak{B}y) \text{ , for some } \mathfrak{U}_1,..,\mathfrak{U}_p \in \mathbb{S}^+$$

$$\beta\!\leftarrow \mathbf{B}(\mathbf{P}\mathfrak{U}_1\cdots\mathfrak{U}_{j-1}x\mathfrak{U}_{j+1}\cdots\mathfrak{U}_p)\mathfrak{B}y$$

$$\beta\!\leftarrow \mathfrak{B}'\mathbf{B}\mathbf{P}\mathfrak{U}_1\cdots\mathfrak{U}_{j-1}x\mathfrak{U}_{j+1}\cdots\mathfrak{U}_p\mathfrak{B}y \text{ , where } \mathfrak{B}' \text{ is an appropriate applicative combination of } \mathbf{B}\text{'s.}$$

Hence, $\mathfrak{B}'\mathbf{B}\mathbf{P}\mathfrak{U}_1\cdots\mathfrak{U}_{j-1}x\mathfrak{U}_{j+1}\cdots\mathfrak{U}_p\mathfrak{B}y = yx$, so by Lemma 1, $\mathbf{C}_* \in \{\mathbf{B},\mathbf{I},\mathbf{P}\}^+$, and since $\{\mathbf{B},\mathbf{I},\mathbf{C}_*\}$ generates all proper combinators without selective or duplicative effect, then so does $\{\mathbf{B},\mathbf{I},\mathbf{P}\}$. ■

We may now combine this result with the previously mentioned lemma of Statman to get the following characterization of a class of combinatorially complete sets :

Corollary : A set containing **B**, any identity combinator, any purely permutative proper combinator, any selective proper combinator, and any duplicative proper combinator is (with η) combinatorially complete. ■

An obvious question to follow this result is to ask if the condition that **B** be present is really necessary. The answer to this question is as yet unknown. But if we relax the problem to one of finding sets of proper combinators with pure non-duplicative effect that generate all non-duplicative proper combinators, then we will show that one can answer this question efficiently. Clearly, from the above results, we can establish that any set of proper combinators containing **B**, **I**, a combinator with pure permutative effect, and a combinator with pure selective effect generates all proper combinators with non-duplicative effect. But if we replace **B** with an arbitrary combinator with pure compositive effect, then we do not necessarily have a set of combinators that generates all non-duplicative proper combinators. For example, it will be a consequence of this paper that $\mathbf{B} \notin \{\mathbf{I},\mathbf{P},\mathbf{S},\boldsymbol{\beta}\}^+$, where the reduction rules of **P**, **S**, and $\boldsymbol{\beta}$ are $\mathbf{P}xyz \rightarrow zyx$, $\mathbf{S}xyz \rightarrow x$, and $\boldsymbol{\beta}wxyz \rightarrow w(xyz)$, respectively.

Remark (Addendum to remark on the use of η) Since η is present, we may assume (by reasoning similar to the previous remark) that each proper combinator with pure compositive effect does not leave the last argument unbound by parentheses in its reduction rule. //

In order to give a precise charaterization of when such a set generates all non-duplicative combinators, we first must describe each proper combinator with pure effect in terms of certain parameters.

Parameterization of proper combinators with pure effect:

The parameters that follow each correspond to the change in the number of terms following a particular "marked" argument of the combinator in its reduction rule.

(i) (Permutative effect) Let **P** be a proper combinator with pure permutative effect, and let p be the order of **P**. Then there is a permutaion $\pi \in S_p$ such that the reduction rule for **P** is $\mathbf{P}x_1 \cdots x_p \rightarrow x_{\pi(1)} \cdots x_{\pi(p)}$. Let $d(j) = \pi(j)-j$ for each $j \in 1..p$. Note that since $\pi(p) \neq p$, then $d(p) < 0$.

Example: Let **P** be the combinator with the reduction rule $\mathbf{P}x_1x_2x_3x_4 \rightarrow x_4x_1x_3x_2$. Then d(1)=1, d(2)=2, d(3)=0, and d(4)=-3.

(ii) (Selective effect) Let **S** be a proper combinator with pure selective effect, and let k be the order of **S**. Then there is a positive integer $t < k$ and a monotone function $s:1..t \rightarrow 1..m$ such that the reduction rule for **P** is $\mathbf{P}x_1 \cdots x_k \rightarrow x_{s(1)} \cdots x_{s(t)}$. Let $\sigma(j) = (k-s(j))-(t-j)$ for each $j \in 1..t$. Note that **S** is parameterized in terms of the result rather than the argument of its reduction rule.

Example: Let **S** be the combinator with the reduction rule $\mathbf{S}x_1x_2x_3 \rightarrow x_1x_3$. Then $\sigma(1)=1$ and $\sigma(2)=0$.

(iii) (Compositive effect) We will give two equivalent formalizations of the parameters for a proper combinator with pure compositive effect in order to simplify later arguments. Again the strategy is to count the change in terms trailing a marked variable in the reduction rule of the combinator. Both (a) and (b) below count this change and hence are equivalent.

(a) Let $\boldsymbol{\beta}$ be a proper combinator with pure compositive effect, and let n be the order of $\boldsymbol{\beta}$. Then for each $j \in 1..n$, there exist terms $\mathfrak{Y}_1,..,\mathfrak{Y}_{m(j)}$ such that $\boldsymbol{\beta}\underbrace{\mathbf{I}\cdots\mathbf{I}}_{j-1}\underline{z}x_{j+1}\cdots x_n \rightarrow_\beta \underline{z}\mathfrak{Y}_1\cdots\mathfrak{Y}_{m(j)}$. Let $b(j) = (n-j)-m(j)$ for each $j \in 1..n$.

(b) Let $\boldsymbol{\beta}$ be a proper combinator with pure compositive effect, and let n be the order of $\boldsymbol{\beta}$. Then for each $j \in 1..n$, there exist terms $\Theta_1,..,\Theta_{m+1}$ such that $\underline{z}$ is a subterm of each Θ_h, and $\boldsymbol{\beta}x_1\cdots x_{j-1}\underline{z}x_{j+1}\cdots x_n \rightarrow_\beta \Theta_1$, where $\Theta_{m+1} \equiv \underline{z}$, and for each $h \in 1..m$, $\Theta_h \equiv \mathfrak{X}^h{}_1\cdots\mathfrak{X}^h{}_{j(h)-1}\Theta_{h+1}\mathfrak{X}^h{}_{j(h)+1}\cdots\mathfrak{X}^h{}_{l(h)}$ for suitable terms $\mathfrak{X}^h{}_1,..,\mathfrak{X}^h{}_{j(h)-1},\mathfrak{X}^h{}_{j(h)+1},..,\mathfrak{X}^h{}_{l(h)}$, where $\mathfrak{X}^h{}_1$ is always one of the variables x_i if $j(h)>1$. Certainly, $\sum_{h=1}^{m}(l(h)-j(h))$ counts the number of terms trailing variable x_j on the right hand side of the reduction rule for $\boldsymbol{\beta}$ and hence is equal to $m(j)$ from (a) above.

Thus $b(j) = (n-j)-\sum_{h=1}^{m}(l(h)-j(h))$ for each $j \in 1..n$.

Also, note that since the final argument x_n of $\boldsymbol{\beta}$ is assumed to be bound by some parentheses, then $\exists\, \mathfrak{m} \in 2..n-1$ such that $\boldsymbol{\beta}\underbrace{\mathbf{I}\cdots\mathbf{I}}_{n-\mathfrak{m}-1}x_{n-\mathfrak{m}}\cdots x_n \equiv x_{n-\mathfrak{m}}(x_{n-\mathfrak{m}+1}\cdots x_n)$. Denote $\boldsymbol{\beta}\underbrace{\mathbf{I}\cdots\mathbf{I}}_{n-\mathfrak{m}-1}$ by $\boldsymbol{\beta}^*$ and note that $b(n-\mathfrak{m}) = \mathfrak{m}-1$.

<u>Example</u>: Let $\mathbf{V}$ be the combinator with the reduction rule

$\mathbf{V}tuvwxyz \to t(u(vw)x)(yz)$. Then the third argument v is followed by four terms before the reduction and three terms (w, x, and (yz)) after the reduction for a change of one trailing term. Hence, b(3)=1. Similarly, b(1)=4, b(2)=2, b(4)=1, b(5)=1, and b(6)=b(7)=0. //

So let $\mathbf{S}$ be a set of proper combinators with pure non-duplicative effect along with $\mathbf{I}$ and let $\mathbf{N} \subseteq \mathbb{Z}$ be the set of parameters associated with the combinators in $\mathbf{S}$. If we denote the greatest common (positive) divisor of the elements in $\mathbf{N}$ by g, then we have the following property.

<u>Lemma 2</u> : Given any $\underline{X}, \mathfrak{X}_o, .., \mathfrak{X}_\mathfrak{r}, \mathfrak{U}_o, .., \mathfrak{U}_\mathfrak{s} \in \mathbf{S}^+$ such that

$$\mathfrak{X}_o \underline{X} \mathfrak{X}_1 \cdots \mathfrak{X}_\mathfrak{r} \twoheadrightarrow_h \mathfrak{U}_o \underline{X} \mathfrak{U}_1 \cdots \mathfrak{U}_\mathfrak{s} \text{ , then } \mathfrak{r} \equiv \mathfrak{s} \ \mathbf{mod}\ g.$$

Proof (By induction on the length L of the head reduction sequence from $\mathfrak{X}_o \underline{X} \mathfrak{X}_1 \cdots \mathfrak{X}_\mathfrak{r}$ to $\mathfrak{U}_o \underline{X} \mathfrak{U}_1 \cdots \mathfrak{U}_\mathfrak{s}$)

(Basis of the induction)

If L=0, then $\mathfrak{r} = \mathfrak{s}$, so $\mathfrak{r} \equiv \mathfrak{s}\ \mathbf{mod}\ g$ trivially holds.

(Inductive step)

Let $L \in \mathbb{N}$ be given and assume the result holds whenever the head reduction sequence has length $\leq$ L. Let $\underline{X}, \mathfrak{X}_o, .., \mathfrak{X}_\mathfrak{r}, \mathfrak{U}_o, .., \mathfrak{U}_\mathfrak{s} \in \mathbf{S}^+$ be given such that the head reduction sequence $\mathbf{X}$ from $\mathfrak{X}_o \underline{X} \mathfrak{X}_1 \cdots \mathfrak{X}_\mathfrak{r}$ to $\mathfrak{U}_o \underline{X} \mathfrak{U}_1 \cdots \mathfrak{U}_\mathfrak{s}$ has length L+1. Note that we may assume that $\mathfrak{X}_o$ is head normal since otherwise contracting a redex in $\mathfrak{X}_o$ (to get $\mathfrak{X}_o{}^*$, say) leaves a reduction sequence of length L from $\mathfrak{X}_o{}^*\underline{X}\mathfrak{X}_1 \cdots \mathfrak{X}_\mathfrak{r} \twoheadrightarrow_h \mathfrak{U}_o\underline{X}\mathfrak{U}_1 \cdots \mathfrak{U}_\mathfrak{s}$, which by the inductive hypothesis implies that $\mathfrak{r} \equiv \mathfrak{s}$ $\mathfrak{mod}$ g. Also, we may assume that $\mathfrak{X}_o\underline{X}\mathfrak{X}_1 \cdots \mathfrak{X}_\mathfrak{r}$ is not a head normal form since otherwise all reductions are internal implying that $\mathfrak{r} = \mathfrak{s}$.

We consider the four possible cases for the form of $\mathfrak{X}_o$.

Case 1 : Suppose $\mathfrak{X}_o \equiv \mathbf{I}Q_1 \cdots Q_{i-1}$ for some i $\in \mathbb{N}$ and $Q_1, .., Q_{i-1} \in \mathbf{S}^+$. Since $\mathfrak{X}_o$ is head normal, then i = 1. Hence $\mathbf{X}$ must be of the form

$$\mathbf{I}\underline{X}\mathfrak{X}_1 \cdots \mathfrak{X}_\mathfrak{r} \rightarrow_h \underline{X}\mathfrak{X}_1 \cdots \mathfrak{X}_\mathfrak{r} \twoheadrightarrow_h \mathfrak{U}_o\underline{X}\mathfrak{U}_1 \cdots \mathfrak{U}_\mathfrak{s}$$

. But the head reduction sequence from $\underline{X}\mathfrak{X}_1 \cdots \mathfrak{X}_\mathfrak{r}$ to $\mathfrak{U}_o\underline{X}\mathfrak{U}_1 \cdots \mathfrak{U}_\mathfrak{s}$ has length L so by the inductive hypothesis $\mathfrak{r} \equiv \mathfrak{s}$ $\mathfrak{mod}$ g.

Case 2 : Suppose $\mathfrak{X}_o \equiv \mathbf{P}Q_1 \cdots Q_{i-1}$ for some i $\in \mathbb{N}$, $Q_1, .., Q_{i-1} \in \mathbf{S}^+$, and pure permutor $\mathbf{P} \in \mathbf{S}$. Recall that we associate the parameter d(j) with the j^{th} argument of $\mathbf{P}$ for each j $\in$ 1..p, where p is the order of $\mathbf{P}$. Since we assume that $\mathfrak{X}_o$ is head normal and $\mathfrak{X}_o\underline{X}\mathfrak{X}_1 \cdots \mathfrak{X}_\mathfrak{r}$ is not, then we may assume that $\mathfrak{r} \geq$ p-i $\geq$ 0. Hence $\mathbf{X}$ must be of the form

$$\mathbf{P}Q_1 \cdots Q_{i-1}\underline{X}\mathfrak{X}_1 \cdots \mathfrak{X}_\mathfrak{r} \rightarrow_h \mathfrak{Y}_1 \cdots \mathfrak{Y}_{j-1}\underline{X}\mathfrak{Y}_{j+1} \cdots \mathfrak{Y}_p\mathfrak{X}_{p-i+1} \cdots \mathfrak{X}_\mathfrak{r}$$
$$\twoheadrightarrow_h \mathfrak{U}_o\underline{X}\mathfrak{U}_1 \cdots \mathfrak{U}_\mathfrak{s} \text{ , where } j = i+d(i).$$

But the head reduction sequence from $\mathfrak{Y}_1 \cdots \mathfrak{Y}_{j-1}\underline{X}\mathfrak{Y}_{j+1}\mathfrak{Y}_p\mathfrak{X}_{p-i+1} \cdots \mathfrak{X}_\mathfrak{r}$ to $\mathfrak{U}_o\underline{X}\mathfrak{U}_1 \cdots \mathfrak{U}_\mathfrak{s}$ has length L, so by the induction hypothesis (p-j)+$\mathfrak{r}$-(p-i) $\equiv \mathfrak{s}$ $\mathfrak{mod}$ g. That is, $\mathfrak{r}$-d(i) $\equiv \mathfrak{s}$ $\mathfrak{mod}$ g, which, since g divides d(i) by definition of g, implies that $\mathfrak{r} \equiv \mathfrak{s}$ $\mathfrak{mod}$ g.

Case 3 : Suppose $\mathfrak{X}_o \equiv \mathbf{s}Q_1 \cdots Q_{i-1}$ for some $i \in \mathbb{N}$, $Q_1, \ldots, Q_{i-1} \in \mathbb{S}^+$, and pure selector $\mathbf{s} \in \mathbb{S}$. Recall that we associate the parameter $\sigma(j)$ with the j^{th} variable in the result of the reduction rule of $\mathbf{s}$ for each $j \in 1..t$. Since we assume that $\mathfrak{X}_o$ is head normal and $\mathfrak{X}_o \underline{X} \mathfrak{X}_1 \cdots \mathfrak{X}_r$ is not, then we may assume that $r \geq k-i \geq 0$. Note that if $i \neq s(m)$ $\forall$ $m \in 1..t$, then $\mathfrak{X}_o \underline{X} \mathfrak{X}_1 \cdots \mathfrak{X}_r \to_h \mathfrak{M}$ and $\mathfrak{M} \twoheadrightarrow_h \mathfrak{U}_o \underline{X} \mathfrak{U}_1 \cdots \mathfrak{U}_\varsigma$, which is a contradiction since $\underline{X}$ does not occur in $\mathfrak{M}$. Thus $\exists$ $m \in 1..t$ such that $i = s(m)$. Hence $\mathbf{X}$ must be of the form

$$PQ_1 \cdots Q_{i-1} \, \underline{X} \mathfrak{X}_1 . \mathfrak{X}_r \to_h \mathfrak{Y}_1 \cdots \mathfrak{Y}_{m-1} \underline{X} \mathfrak{Y}_{m+1} \cdots \mathfrak{Y}_t \mathfrak{X}_{k-i+1} \cdots \mathfrak{X}_r$$
$$\twoheadrightarrow_h \mathfrak{U}_o \underline{X} \mathfrak{U}_1 \cdots \mathfrak{U}_\varsigma \, .$$

But the head reduction sequence from $\mathfrak{Y}_1 \cdots \mathfrak{Y}_{j-1} \underline{X} \mathfrak{Y}_{j+1} \cdots \mathfrak{Y}_t \mathfrak{X}_{k-i+1} \cdots \mathfrak{X}_r$ to $\mathfrak{U}_o \underline{X} \mathfrak{U}_1 \cdots \mathfrak{U}_\varsigma$ has length L, so by the induction hypothesis $(t-m)+r-(k-s(m)) \equiv \varsigma \ \mathbf{mod}\ g$. That is, $r-\sigma(m) \equiv \varsigma \ \mathbf{mod}\ g$, which, since g divides $\sigma(m)$ by definition of g, implies that $r \equiv \varsigma \ \mathbf{mod}\ g$.

Case 4 : Suppose $\mathfrak{X}_o \equiv \boldsymbol{\beta} Q_1 \cdots Q_{i-1}$ for some $i \in \mathbb{N}$, $Q_1, \ldots, Q_{i-1} \in \mathbb{S}^+$, and pure compositor $\boldsymbol{\beta} \in \mathbb{S}$. Recall that we associate the parameter $b(j)$ with the j^{th} argument of $\boldsymbol{\beta}$ for each $j \in 1..n$, where n is the order of $\boldsymbol{\beta}$. Since we assume that $\mathfrak{X}_o$ is head normal and $\mathfrak{X}_o \underline{X} \mathfrak{X}_1 \cdots \mathfrak{X}_r$ is not, then we must have that $r \geq n-i \geq 0$. Note that in order for the head reduction sequence to terminate in the form $\mathfrak{U}_o \underline{X} \mathfrak{U}_1 \cdots \mathfrak{U}_\varsigma$, the Θ_i's associated with $\boldsymbol{\beta}$ and $\underline{X}$ (as described in the preceding parameterization discussion (b)) must come to the head in order in $\mathbf{X}$. Hence $\mathbf{X}$

must have the following form :

$$\beta Q_1 \cdots Q_{i-1} X \mathfrak{X}_1 \cdots \mathfrak{X}_r \rightarrow_h \Theta_1 \mathfrak{X}_{n-i+1} \cdots \mathfrak{X}_r$$

$$\equiv \mathfrak{Y}^1{}_1 \cdots \mathfrak{Y}^1{}_{j(1)-1} \Theta_2 \mathfrak{Y}^1{}_{j(1)+1} \cdots \mathfrak{Y}^1{}_{l(1)} \mathfrak{X}_{n-i+1} \cdots \mathfrak{X}_r$$

$$\twoheadrightarrow_h \Theta_2 \mathfrak{L}^2{}_1 \cdots \mathfrak{L}^2{}_{t(2)} \quad \text{(Call this reduction sequence } \mathbf{X}_2\text{)}$$

$$\equiv \mathfrak{Y}^2{}_1 \cdots \mathfrak{Y}^2{}_{j(2)-1} \Theta_3 \mathfrak{Y}^2{}_{j(2)+1} \cdots \mathfrak{Y}^2{}_{l(2)} \mathfrak{L}^2{}_1 \cdots . \mathfrak{L}^2{}_{t(2)}$$

$$\vdots$$

$$\twoheadrightarrow_h \Theta_h \mathfrak{L}^h{}_1 \cdots \mathfrak{L}^h{}_{t(h)} \quad \text{(Call this reduction sequence } \mathbf{X}_h\text{)}$$

$$\equiv \mathfrak{Y}^h{}_1 \cdots \mathfrak{Y}^h{}_{j(h)-1} \Theta_{h+1} \mathfrak{Y}^1{}_{j(h)+1} \cdots \mathfrak{Y}^h{}_{l(h)} \mathfrak{L}^h{}_1 \cdots \mathfrak{L}^h{}_{t(h)}$$

$$\vdots$$

$$\twoheadrightarrow_h \Theta_m \mathfrak{L}^m{}_1 \cdots \mathfrak{L}^m{}_{t(m)} \quad \text{(Call this reduction sequence } \mathbf{X}_m\text{)}$$

$$\equiv \mathfrak{Y}^m{}_1 \cdots \mathfrak{Y}^m{}_{j(m)-1} X \mathfrak{Y}^1{}_{j(m)+1} \cdots \mathfrak{Y}^m{}_{l(m)} \mathfrak{L}^m{}_1 \cdots \mathfrak{L}^m{}_{t(m)}$$

$$\twoheadrightarrow_h \mathfrak{U}_o X \mathfrak{U}_1 \cdots \mathfrak{U}_{\varsigma}\text{, where } X \text{ is a subterm of } \Theta_h \text{ for each } h \in 1..m.$$

Since each head reduction sequence $\mathbf{X}_h$ has length $\leq L$, then by the induction hypothesis (applied to each sequence), we have that $t(h+1) \equiv t(h)+(l(h)-j(h))$ **mod** $g \quad \forall\ h \in 2..m-1$, $t(2) \equiv \mathfrak{r}-(n-i)+(l(1)-j(1))$ **mod** g, and $t(m)+(l(m)-j(m)) \equiv \varsigma$ **mod** g. Hence by iterated substitution it is easy to see that

$t(2)+\sum_{h=2}^{m}(l(h)-j(h)) \equiv \varsigma$ **mod** g. That is, $\mathfrak{r}-(n-i)+\sum_{h=1}^{m}(l(h)-j(h))$ $\equiv \varsigma$ **mod** g, which is precisely $\mathfrak{r}-b(i) \equiv \varsigma$ **mod** g. And since g divides $b(i)$ by definition of g, we have that $\mathfrak{r} \equiv \varsigma$ **mod** g. ■

We next give a constructive algorithm (similar to Curry's abstraction algorithms) for building combinatory terms in a predetermined way. Each step in the algorithm takes as arguments a term of the form $\mathfrak{Q}_o\underline{Y}\mathfrak{Q}_1\cdots\mathfrak{Q}_q$ with "marked" subterm $\underline{Y}$, a proper combinator with pure non-duplicative effect, and a natural number that refers to one of the arguments (or variables in the result, in the case of selective effect) in the reduction rule of the combinator. The algorithm then returns at each step a term of the form $\mathfrak{W}_o\underline{Y}\mathfrak{W}_1\cdots\mathfrak{W}_{q'}$. We refer to the subterms following the marked subterm as the trailing terms. We should note that we are generally only interested in the number of trailing terms, not in their actual respective forms.

<u>Description of the steps in the β-expansion algorithm</u> :

Let a term of the form $\mathfrak{Q}_o\underline{Y}\mathfrak{Q}_1\cdots\mathfrak{Q}_q$ be given. The following labels serve as references to each step in the algorithm.

[**P**:j] If $q \geq p-j$, then we note that $\mathfrak{Q}_o\underline{Y}\mathfrak{Q}_1\cdots\mathfrak{Q}_q$

$$\beta\leftarrow \underbrace{\mathbf{I}\cdots\mathbf{I}}_{j-1}(\underbrace{\mathbf{I}\cdots\mathbf{I}}_{m-2}\mathfrak{Q}_o\underline{Y})\mathfrak{Q}_1\cdots\mathfrak{Q}_q$$

$$\beta\leftarrow \mathbf{P}\mathfrak{U}_1\cdots\mathfrak{U}_{j-d(j)-1}(\mathbf{I}\cdots\mathbf{I}\mathfrak{Q}_o\underline{Y})\mathfrak{U}_{j-d(j)+1}\cdots\mathfrak{U}_p\mathfrak{Q}_{p-j+1}\cdots\mathfrak{Q}_q$$

$$\beta\leftarrow \beta^*(\mathbf{P}\mathfrak{U}_1\cdots\mathfrak{U}_{j-d(j)-1})\mathbf{I}\cdots\mathbf{I}\mathfrak{Q}_o\underline{Y}\mathfrak{U}_{j-d(j)+1}\cdots\mathfrak{U}_p\mathfrak{Q}_{p-j+1}\cdots\mathfrak{Q}_q .$$

Hence, let $\mathfrak{W}_o$ be $\beta^*(\mathbf{P}\mathfrak{U}_1\cdots\mathfrak{U}_{j-d(j)-1})\underbrace{\mathbf{I}\cdots\mathbf{I}}_{m-2}\mathfrak{Q}_o$, and the number of trailing terms becomes $q + d(j)$.

[**S**:j] If $q \geq k-j$, then we note that $\mathfrak{Q}_o\underline{Y}\mathfrak{Q}_1\cdots\mathfrak{Q}_q$

$$\beta\leftarrow \underbrace{\mathbf{I}\cdots\mathbf{I}}_{j-1}(\underbrace{\mathbf{I}\cdots\mathbf{I}}_{m-2}\mathfrak{Q}_o\underline{Y})\mathfrak{Q}_1\cdots\mathfrak{Q}_q$$

$$\beta\leftarrow \mathbf{S}\mathfrak{U}_1\cdots\mathfrak{U}_{j-\sigma(j)-1}(\mathbf{I}\cdots\mathbf{I}\mathfrak{Q}_o\underline{Y})\mathfrak{U}_{j-\sigma(j)+1}\cdots\mathfrak{U}_k\mathfrak{Q}_{k-j+1}\cdots\mathfrak{Q}_q$$

$\beta \leftarrow \boldsymbol{\beta}^* (\mathbf{S}\mathfrak{U}_1 \cdots \mathfrak{U}_{j-\sigma(j)-1}) \mathbf{I} \cdots \mathbf{I}\mathfrak{Q}_o \underline{\mathbf{Y}} \mathfrak{U}_{j-\sigma(j)+1} \cdots \mathfrak{U}_k \mathfrak{Q}_{k-j+1} \cdots \mathfrak{Q}_q$.

Hence, let $\mathfrak{W}_o$ be $\boldsymbol{\beta}^* (\mathbf{S}\mathfrak{U}_1 \cdots \mathfrak{U}_{j-\sigma(j)-1}) \underbrace{\mathbf{I} \cdots \mathbf{I}}_{m-2} \mathfrak{Q}_o$, and the number of trailing terms becomes $q + \sigma(j)$.

[$\boldsymbol{\beta}$:j] Let $\boldsymbol{\beta}_j \equiv \boldsymbol{\beta}\underbrace{\mathbf{I} \cdots \mathbf{I}}_{j-1}$. Then $\boldsymbol{\beta}_j x_j \cdots x_n = x_j \mathfrak{Y}_1 \cdots \mathfrak{Y}_{m(j)}$ for some terms $\mathfrak{Y}_1, .., \mathfrak{Y}_{m(j)}$ as in the parameterization (a) of $\boldsymbol{\beta}$. If $q \geq m(j)$, then we note that $\mathfrak{Q}_o \underline{\mathbf{Y}} \mathfrak{Q}_1 \cdots \mathfrak{Q}_q$

$\beta \leftarrow (\underbrace{\mathbf{I} \cdots \mathbf{I}}_{m-2} \mathfrak{Q}_o \underline{\mathbf{Y}}) \mathfrak{Q}_1 \cdots \mathfrak{Q}_q$

$\beta \leftarrow (\mathbf{I} \cdots \mathbf{I}\mathfrak{Q}_o \underline{\mathbf{Y}}) ([\vec{x} := \vec{\mathbf{I}}, \mathfrak{Q}_1] \mathfrak{Y}_1) \cdots ([\vec{x} := \vec{\mathbf{I}}, \mathfrak{Q}_{m(j)}] \mathfrak{Y}_{m(j)}) \mathfrak{Q}_{m(j)+1} \cdots \mathfrak{Q}_q$

$\beta \leftarrow \boldsymbol{\beta}_j (\mathbf{I} \cdots \mathbf{I}\mathfrak{Q}_o \underline{\mathbf{Y}}) \mathfrak{U}_1 \cdots \mathfrak{U}_{n-j} \mathfrak{Q}_{m(j)+1} \cdots \mathfrak{Q}_q$

$\beta \leftarrow \boldsymbol{\beta}^* \boldsymbol{\beta}_j \mathbf{I} \cdots \mathbf{I}\mathfrak{Q}_o \underline{\mathbf{Y}} \mathfrak{U}_1 \cdots \mathfrak{U}_{n-j} \mathfrak{Q}_{m(j)+1} \cdots \mathfrak{Q}_q$, where $[\vec{x} := \vec{\mathbf{I}}, \mathbf{Y}]$ **Z** is the result of substituting **Y** for the last variable in **Z** and **I** for all of the other variables in **Z**. Hence, let $\mathfrak{W}_o$ be $\boldsymbol{\beta}^* \boldsymbol{\beta}_j \mathbf{I} \cdots \mathbf{I}\mathfrak{Q}_o$, and the number of trailing terms becomes $q + n-j-m(j)$ or $q + b(j)$. //

<u>Lemma 2</u> : If g = 1 and if there is some pure compositor $\boldsymbol{\beta}$ in $\mathbb{S}$, then $\forall\, \mathfrak{X} \in \mathbb{S}^+$, $\exists\, \mathfrak{U} \in \mathbb{S}^+$, such that $\mathfrak{U}x \twoheadrightarrow_\beta \mathfrak{X}x\mathbf{I}$.

Proof (By construction)

Let $\mathfrak{X} \in \mathbb{S}^+$ be given. Since g = 1, then $\exists\, \mathfrak{a}, \mathfrak{b} \in \mathbb{N}$ such that $\gcd(\mathfrak{a},\mathfrak{b}) = 1$. That is, $\exists\, I,J \in \mathbb{Z}$ such that $I\mathfrak{a} + J\mathfrak{b} = 1$. Let $n_\mathfrak{a} := \mathbf{min}\{m \in \mathbb{N} : -I-m*d(p) \geq 0\}$ and $n_\mathfrak{b} := \mathbf{min}\{m \in \mathbb{N} : -J-m*d(p) \geq 0\}$, and let $I' := -I-n_\mathfrak{a}*d(p)$ and $J' := -J-n_\mathfrak{b}*d(p)$. Then clearly $I', J' \in \mathbb{N}$ and $n_\mathfrak{a}*\mathfrak{a}+n_\mathfrak{b}*\mathfrak{b} \geq 0$. Hence

$$I'*\mathfrak{a} + J'*\mathfrak{b} + 1 = -\, d(p)*(n_\mathfrak{a}*\mathfrak{a}+n_\mathfrak{b}*\mathfrak{b}) . \qquad \text{(i)}$$

The general strategy will be to start with $\mathfrak{X}x\mathbf{I}$ and apply the algorithm step associated with $\mathfrak{a}$ precisely I' times , the step associated with $\mathfrak{b}$ precisely J' times, and then [**P**:p] precisely $(n_{\mathfrak{a}}*\mathfrak{a}+n_{\mathfrak{b}}*\mathfrak{b})$ times to leave no trailing terms. We must however guarantee that there are enough trailing terms at each step to insure that each may in fact be carried out. (Note the restrictions in the algorithm description on the number of original trailing terms necessary to perform each step in the construction.) For ease of exposition, we will use an extreme brute force approach of accomplishing this.

Recall that $\mathfrak{m}$ is the positive natural number corresponding to $\boldsymbol{\beta}^*$ $(\boldsymbol{\beta}^* = \boldsymbol{\beta}\underbrace{\mathbf{I}\cdots\mathbf{I}}_{n-\mathfrak{m}-1})$. If we apply $[\boldsymbol{\beta} : n-\mathfrak{m}-1]$ to $\mathfrak{X}$x**I**, precisely B = $(-d(p))*(I'+J')*\mathbf{max}\{n,k,p\}$ times, then there will be $1+B*(\mathfrak{m}-1)$ trailing terms (in particular, trailing **I**'s). Since $\mathfrak{a} \in \mathbf{N}$, then $\exists$ a combinator $\mathbf{Q} \in \mathbf{S}$ such that $\mathfrak{a}$ is the parameter associated with the, say, j^{th} argument (or variable in the result, in the case of selective effect) of the reduction rule of **Q**. Similarly, since $\mathfrak{b} \in \mathbf{N}$, then $\exists$ a combinator $\mathbf{V} \in \mathbf{S}$ such that $\mathfrak{b}$ is the parameter associated with the, say, i^{th} argument (or variable in the result, in the case of selective effect) of the reduction rule of **V**. Since $1+B*(\mathfrak{m}-1) \geq I'*\mathbf{max}\{n,k,p\}$, then if we apply [**Q**:j] I' times to the term with $1+B*(\mathfrak{m}-1)$ trailing terms, then we will get a term with $1+B*(\mathfrak{m}-1)+I'*\mathfrak{a}$ trailing terms. Since $1+B*(\mathfrak{m}-1)+I'*\mathfrak{a} \geq J'*\mathbf{max}\{n,k,p\}$, then if we apply [**V**:i] J' times to the term with $1+B*(\mathfrak{m}-1)+I'*\mathfrak{a}$ trailing terms, then we will get a term with $1+B*(\mathfrak{m}-1)+I'*\mathfrak{a}+J'*\mathfrak{b}$ trailing terms. But from (i) and the definition of B, this is exactly $B*(\mathfrak{m}-1)-d(p)*(n_{\mathfrak{a}}*\mathfrak{a}+n_{\mathfrak{b}}*\mathfrak{b}) = -d(p)*C$ trailing terms, where $C := (n_{\mathfrak{a}}*\mathfrak{a}+n_{\mathfrak{b}}*\mathfrak{b})+(I'+J')*\mathbf{max}\{n,k,p\}$.

Certainly $C \in \mathbb{N}$, so if we apply [**P**:p] C times to the term with $1+B*(\mathfrak{m}-1)+I'*\mathfrak{a}+J'*\mathfrak{b}$ trailing terms, then we will get a term $\mathfrak{U}x$ with 0 trailing terms such that $\mathfrak{U}x \twoheadrightarrow_\beta \mathfrak{X}x\mathbf{I}$, as desired. ■

<u>Theorem</u>: (a) If g=1, then $\mathbf{B} \in \mathbb{S}^+$.

(b) If $g \neq 1$, then $\mathbf{C}_* \notin \mathbb{S}^+$.

Proof:

(a) Suppose g=1 and that there is a compositor $\boldsymbol{\beta} \in \mathbb{S}$. Note that $x(yz) = \boldsymbol{\beta}^* x\underbrace{\mathbf{I}..\mathbf{I}}_{m-2}yz$. But by $\mathfrak{m}-2$ applications of Lemma 3, there exists $\mathbf{Q}_0 \in \mathbb{S}^+$ such that $\mathbf{Q}_0 x = \boldsymbol{\beta}^* x\underbrace{\mathbf{I}..\mathbf{I}}_{m-2}$. Hence, $\mathbf{Q}_0 xyz = x(yz)$, so $\mathbf{B} = \mathbf{Q}_0 \in \mathbb{S}^+$.

(b) Suppose $g \neq 1$. Assume that $\mathbf{C}_* \in \mathbb{S}^+$. That is, there exists a combinatory head reduction $\mathbf{C}_* x\underline{y} \twoheadrightarrow_\beta \underline{y}x$. But by Lemma 2, this implies that $1 \equiv 0 \mod g$, which is impossible since $g \neq 1$. Therefore, $\mathbf{C}_* \notin \mathbb{S}^+$. ■

Therefore, we have found necessary and sufficient conditions for such a set $\mathbb{S}$ to generate all proper combinators without duplicative effect.

<u>Corollary</u> : A set $\mathbb{S}$ (containing an identity) of combinators with pure non-duplicative effect generates all combinators without duplicative effect if and only if combinators with each effect are present in $\mathbb{S}$ and g = 1.

Proof :

(⇐) Suppose combinators of each effect are present and g = 1. Then by Theorem 2 (a), $\mathbf{B} \in \mathbb{S}^+$. Since **B** and **I** are present along with some proper combinator with pure permutative effect and some proper combinator with selective effect in $\mathbb{S}$, then by the corollary to Theorem 1, all combinators without duplicative effect are in $\mathbb{S}^+$.

(⇒) Suppose $\mathbb{S}$ generates all non-duplicative combinators. Then $\mathbb{S}$ certainly contains at least one combinator with each other effect. And, in particular, $\mathbf{C}_\star \in \mathbb{S}^+$. Hence, by Theorem 2 (b), g = 1. ■

References

[1] H.B. Currey & R. Feys, **Combinatory Logic** Vol. 1. North Holland 1968.

[2] R. Statman, "On translating lambda terms into combinators", in Proceeding Symposium on Logic in Computer Science, *I.E.E.E.* 1986.