DYNAMIC EQUILIBRIUM IN A MULTI-AGENT ECONOMY: CONSTRUCTION AND UNIQUENESS

by

Ioannis Karatzas & Peter Lakner
Department of Statistics
Columbia University
New York, NY 10027

John P. Lehoczky, Department of Statistics
Steven E. Shreve, Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Research Report No. 88-9 2

April 1988

1. INTRODUCTION

In the recent article [10] we developed a complete theory, based on stochastic calculus, for the consumption/investment problem of a small investor with a general utility function, in a financial market where stock prices are modelled by semimartingales. Such generality notwithstanding, we were able to provide explicit expressions for the optimal consumption policy and terminal wealth of the agent. The present paper draws on the methodology of [10] to construct equilibrium in a multi-agent economy, and to establish uniqueness.

Our model of equilibrium is inspired by the work of Duffie and Huang [2], [3], [4], [9]. We suppose there is a finite number n of agents (small investors) who receive endowment streams denominated in units of a single, infinitely divisible commodity; the latter is traded at a "spot price" ψ, and each agent attempts to maximize his expected total utility from consumption of this commodity, over a finite horizon $[0,T]$. The agents can borrow and invest in the financial assets in order to hedge the risk associated with their endowments. An <u>equilibrium spot price</u> process ψ is one which, when accepted by the individual agents in the determination of their optimal policies, calls for the commodity to be entirely consumed as it enters the economy and for all the financial assets to be in zero net supply.

In the stochastic, dynamic model under consideration, we provide a very precise characterization of equilibrium in terms of the vector $\Lambda = (\lambda_1, \ldots, \lambda_n)$ of weights used by a fictitious "representative" of the n agents. Roughly speaking, this representative agent acts as a proxy for the individual agents (with utility functions $U_k(t,\cdot)$, $1 \leq k \leq n$) by receiving their aggregate endowment, solving his own optimization problem with utility

DYNAMIC EQUILIBRIUM IN A MULTI-AGENT ECONOMY: CONSTRUCTION AND UNIQUENESS

by

IOANNIS KARATZAS*
Department of Statistics
Columbia University
New York, NY 10027

PETER LAKNER*
Department of Statistics
Columbia University
New York, NY 10027

JOHN P. LEHOCZKY**
Department of Statistics
Carnegie Mellon University
Pittsburgh, PA 15213

STEVEN E. SHREVE**
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Abstract

We study an economy with several agents, who receive endowment streams denominated in units of a certain commodity over a finite horizon. The agents can consume the commodity, and they can also trade it at a certain "spot price" ψ; the proceeds of these transactions can be invested in financial assets, whose prices are modelled by continuous semimartingales. The objective of each agent is to choose a consumption/investment strategy that will maximize his expected utility from consumption and allow him to post a nonnegative wealth at the terminal time. We provide explicit information about the optimal strategies of the individual agents when the price ψ is given. We also show how to determine ψ according to the law of "supply and demand", which mandates that the commodity be consumed entirely as it enters the economy and that the net demand for each financial asset be zero.

*Research supported by the National Science Foundation under grant DMS-87-23078.
**Research supported by the National Science Foundation under grant DMS-84-03166.

function

$$(1.1) \qquad U(t,c;\ \Lambda) \triangleq \max_{\substack{c_1 \geq 0, \ldots, c_n \geq 0 \\ c_1 + \ldots + c_n = c}} \sum_{k=1}^{n} \lambda_k U_k(t, c_k),$$

and then apportioning his optimal commodity consumption process to the agents, instead of actually consuming it. The search for equilibrium is reduced to a search for an appropriate vector $\Lambda \in (0,\infty)^n$ in (1.1); cf. sections 7, 8. This allows for equilibrium to be constructed in $\mathbb{R}^n$, rather than in some infinite-dimensional functional space (as, for instance, in [4], [5]).

Under the condition that $c \mapsto cU_k'(t,c)$ is nondecreasing for every $k \in \{1,\ldots,n\}$, the lattice fixed point theorem can be used to establish the existence of equilibrium, and a separate simple argument settles the question of uniqueness (section 9). It turns out that the equilibrium spot price is determined up to a multiplicative constant, in a way that is affected by the coefficients of the financial market; by constrast, the equilibrium allocation of the commodity among agents is uniquely determined, and only by the individual endowments and utilities (the so-called <u>primitives</u> of the model). Some explicit computations are carried out in section 10.

For simplicity and economy of exposition, we consider utility functions with $U_k'(t,0+) = \infty$ and exclude capital assets from the model. A full account of the theory without these restrictions appears in [11].

2. THE FINANCIAL MARKET

Let us consider a market in which $d+1$ assets are traded continuously. One of them is a pure discount <u>bond</u>, with price $P_0(t) = \exp\{\int_0^t r(s)ds\}$ at time

t. The remaining d assets are risky <u>stocks</u>, and the price-per-share $P_i(t)$ of the i^{th} stock is modelled by the linear stochastic equation

$$(2.1)\qquad dP_i(t) = P_i(t)[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)]; \quad i = 1,\dots,d.$$

Here $W = (W_1,\dots,W_d)^*$ is an $\mathbb{R}^d$-valued Brownian motion, the components of which can be thought of as modelling the sources of uncertainty in the market; W is defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ and on a finite horizon $[0,T]$, and we shall denote by $\{\mathcal{F}_t\}$ the augmentation of its natural filtration $\mathcal{F}_t^W = \sigma(W(s);\ 0 \le s \le t)$. The <u>interest rate</u> $r(\cdot)$ of the bond, the <u>appreciation rate</u> vector $b(\cdot) = (b_1(\cdot),\dots,(b_d(\cdot))^*$ of the stocks, and the <u>volatility</u> coefficient matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1\le i,j\le d}$, will all be bounded processes, progressively measurable with respect to $\{\mathcal{F}_t\}$. In addition, we shall impose the strong nondegeneracy condition

$$(2.2)\qquad \xi^*\sigma(t,\omega)\sigma^*(t,w)\xi \ge \delta\ \|\xi\|^2; \quad \forall\ \xi \in \mathbb{R}^d,\ (t,\omega) \in [0,T] \times \Omega$$

for some $\delta > 0$. Under (2.2), the inverses of both $\sigma(\cdot)$ and $\sigma^*(\cdot)$ exist and are bounded; in particular, the relative risk process $\theta(t) \triangleq (\sigma(t))^{-1}[b(t)-r(t)\underset{\sim}{1}]$, $0 \le t \le T$, is bounded and progressively measurable.

It follows then from the Girsanov theorem (e.g. [12], section 3.5) that the exponential supermartingale

$$(2.3)\qquad Z(t) \triangleq \exp\{-\int_0^t \theta^*(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\},\ \mathcal{F}_t; \quad 0 \le t \le T,$$

is actually a martingale, and that

$$(2.4) \qquad \tilde{W}(t) \triangleq W(t) + \int_0^t \theta(s)ds; \quad 0 \le t \le T,$$

is Brownian motion under the probability measure $\tilde{\mathbb{P}}(A) \triangleq E(Z_T 1_A)$; $A \in \mathcal{F}_T$. Under this measure, the discounted stock price processes

$$(2.5) \quad Q_i(t) = \beta(t)P_i(t) \quad , \quad \text{with} \ \beta(t) \triangleq (P_0(t))^{-1} = \exp\{-\int_0^t r(s)ds\}$$

are martingales, a fact of great importance in the modern theory of continuous trading (cf. [7], [8], [13] for its connections with the notions of "absence of arbitrage opportunities" and "completeness" in the market model). We shall see in Remark 6.1 that the process

$$(2.6) \qquad \zeta(t) \triangleq \beta(t)Z(t) \quad ; \quad 0 \le t \le T$$

acts as a "deflator", in the sense that multiplication by $\zeta(t)$ converts wealth held at time t to the equivalent amount of wealth at time zero.

3. THE ECONOMY

The economy we envision consists of

(i) the financial market of section 2,

(ii) a single consumption good or "commodity", traded at the spot price $\psi = \{\psi(t);\ 0 \le t \le T\}$, and

(iii) a finite number n of agents (small investors). Each one of these receives an exogenous endowment at the rate $\epsilon_k = \{\epsilon_k(t);\ 0 \leq t \leq T\}$, denominated in units of the commodity; he can either consume this endowment, or turn it into cash and invest the proceeds in the financial market. The goal of each agent is to maximize his expected total utility from consumption, subject to having nonnegative terminal wealth.

The equilibrium problem for such an economy is to determine a spot price ψ so that the markets "clear" when each agent behaves optimally and the commodity is traded at the price ψ. We shall provide a very precise solution to this problem (sections 7-9), after having presented the explicit solution of the individual agent's optimization problem in the manner of [10] (section 6).

Let us list our basic assumptions: the commodity endowments $(\epsilon_1,\ldots,\epsilon_n)$ are positive and $\{\mathcal{F}_t\}$-progressively measurable processes, and the aggregate endowment $\epsilon(t) \triangleq \sum_{k=1}^{n} \epsilon_k(t)$ satisfies

$$0 < k \leq \epsilon(t) \leq K\ ; \quad \forall\ 0 \leq t \leq T, \tag{3.1}$$

for two finite constants $K > k$. On the other hand, the spot price process ψ is supposed to be positive, $\{\mathcal{F}_t\}$-progressively measurable, and such that the "deflated" spot price $\zeta\psi$ is bounded away from zero and from above (as in (3.1)).

4. THE k^{th} AGENT'S SITUATION

Each agent (say the k^{th}) acts as a price-taker. He views the price ψ

as given, and has at his disposal the choice of an $\mathbb{R}^d$-valued portfolio process $\pi_k(t) = (\pi_{k1}(t),\ldots,\pi_{kd}(t))^*$ and of a nonnegative consumption rate process $c_k(t)$, $0 \le t \le T$. These processes are both progressively measurable with respect to $\{\mathcal{F}_t\}$ and satisfy $\int_0^T \{c_k(t)\psi(t) + \|\pi_k(t)\|^2\}dt < \infty$, almost surely. The interpretation here is that $\pi_{ki}(t)$ represents the amount invested, at time t, by the k^{th} investor in the i^{th} stock.

4.1 Remark: If we denote by $X_k(t)$ the wealth of the k^{th} investor at time t, then $X_k(t) - \sum_{i=1}^d \pi_{ki}(t)$ is the amount that he invests in the bond. Neither this amount nor the individual $\pi_{ki}(t)$'s are constrained to be nonnegative; this means that unlimited borrowing at the interest rate $r(\cdot)$, and short-selling of stocks, are permitted.

Obviously, the wealth X_k corresponding to a given portfolio/consumption process pair (π_k, c_k) satisfies the equation

$$dX_k(t) = \psi(t)[\epsilon_k(t)-c_k(t)]dt + \sum_{i=1}^d \pi_{ki}(t)[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_j(t)]$$

$$(4.1) \qquad\qquad +(X_k(t) - \sum_{i=1}^d \pi_{ki}(t))r(t)dt$$

$$= r(t)X_k(t)dt + \psi(t)[\epsilon_k(t)-c_k(t)]dt + \pi_k^*(t)\sigma(t)d\tilde{W}(t)$$

(cf. (2.4)), whose solution is given as

$$(4.2)\quad \beta(t)X_k(t) = \int_0^t \beta(s)\psi(s)[\epsilon_k(s) - c_k(s)]ds + \int_0^t \beta(s)\pi_k^*(s)\sigma(s)d\tilde{W}(s).$$

4.2 Definition: A portfolio/consumption process pair (π_k, c_k) is called <u>admissible</u>, if for the corresponding wealth process X_k we have that ζX_k is bounded from below and that $X_k(T) \geq 0$ holds, almost surely. □

Now suppose that the k^{th} agent is also endowed with a $C^{0,2}$ <u>utility function</u> $U_k: [0,T] \times (0,\infty) \to \mathbb{R}$, which enjoys the following properties for every $t \in [0,T]$:

(i) $U_k(t,\cdot)$ is strictly increasing and strictly concave,

(ii) the derivative $U_k'(t,c) \triangleq \frac{\partial}{\partial c} U_k(t,c)$ satisfies

$\lim_{c\to\infty} U_k'(t,c) = 0$ and $U_k'(t,0+) \triangleq \lim_{c\downarrow 0} U_k'(t,c) = \infty$.

The k^{th} agent's <u>optimization problem</u> is to maximize the expected total utility from sonsumption $E\int_0^T U_k(t,c_k(t))dt$, over all admissible pairs (π_k, c_k) that satisfy

$$(4.3)\qquad E\int_0^T U_k^-(t,c_k(t))dt < \infty.$$

We shall let $(\hat{\pi}_k, \hat{c}_k)$ denote an optimal pair for this problem, and $\hat{X}_k$ the associated wealth process.

5. EQUILIBRIUM CONSIDERATIONS

We are now in a position to define the notion of equilibrium.

5.1 Definition: A spot price process ψ is called an equilibrium spot price process, if in the notation of section 4 we have

(5.1) $$\sum_{k=1}^{n} \hat{c}_k(t) = \epsilon(t) \ ; \quad \forall\ 0 \leq t \leq T,$$

(5.2) $$\sum_{k=1}^{n} \hat{\pi}_{ki}(t) = 0 \quad ; \quad \forall\ 0 \leq t \leq T \ \text{ and } \ i = 1,\dots,d,$$

(5.3) $$\sum_{k=1}^{n} \hat{X}_k(t) = 0 \quad ; \quad \forall\ 0 \leq t \leq T,$$

almost surely. □

These conditions amount to the "clearing" of the spot market, the stock markets and the bond market, respectively.

6. SOLUTION OF THE k^{th} AGENT'S PROBLEM

Let us consider an admissible pair (π_k, c_k), evaluate the corresponding wealth process X_k at the stopping time

$$\tau_m = \inf\{t \in [0,T];\ \int_0^t \beta^2(s)\|\pi^*(s)\sigma(s)\|^2 ds \geq m\} \wedge T$$

for an arbitrary positive integer m, and take expectations in the resulting expression (4.2) with respect to $\tilde{\mathbb{P}}$. We obtain

$$(6.1)\qquad E\int_0^{\tau_m}\psi(s)\zeta(s)c_k(s)ds = E\int_0^{\tau_m}\psi(s)\zeta(s)\epsilon_k(s)ds + E[\zeta(\tau_m)X_k(\tau_m)].$$

Now we let $m \to \infty$; admissibility and Fatou's lemma give $\underline{\lim}_{m\to\infty}\ E[\zeta(\tau_m)X_k(\tau \geq E[\zeta(T)X_k(T)] \geq 0$, which coupled with monotone convergence yields in (6.1):

$$(6.2)\qquad E\int_0^T \zeta(s)\psi(s)c_k(s)ds \leq E\int_0^T \zeta(s)\psi(s)\epsilon_k(s)ds.$$

6.1 Remark: This inequality has the form (and the significance, as we show in Proposition 6.2) of a budget constraint, which justifies the terminology "deflator" for the process ζ of (2.6). It mandates that the expected total value of consumption, deflated down to the original time, does not exceed the coresponding quantity for the value of endowment.

6.2 Proposition: Let a spot price process ψ be given. If (π_k, c_k) is an admissible pair for the k^{th} agent, then (6.2) is satisfied; conversely, for any consumption process c_k satisfying (6.2), there exists a portfolio process π_k such that the pair (π_k, c_k) is admissible.

Proof: It remains to justify the second claim; for any consumption process c_k satisfying (6.2), introduce the random variable

$$(6.3)\qquad D_k \triangleq \int_0^T \beta(s)\psi(s)[\epsilon_k(s)-c_k(s)]ds$$

and observe that (6.2) amounts to $\tilde{E}D_k \geq 0$. Now the $\tilde{\mathbb{P}}$-martingale

(6.4) $$M_k(t) \triangleq \tilde{E}D_k - \tilde{E}(D_k|\mathcal{F}_t); \quad 0 \leq t \leq T,$$

can be written as a stochastic integral

(6.5) $$M_k(t) = \int_0^t \beta(s)\pi_k^*(s)\sigma(s)d\tilde{W}(s)$$

for a suitable portfolio process π_k, by virtue of the martingale representation theorem (cf. [12], Problem 3.4.16 and proof of Proposition 5.8.6). Finally, the process

(6.6) $$X_k(t) = \frac{1}{\beta(t)}\{\int_0^t \beta(s)\psi(s)[\epsilon_k(s)-c_k(s)]ds + M_k(t)\}$$

is obviously, from (6.5) and (4.2), the wealth associated with the pair (π_k, c_k) and satisfies

$$\zeta(t)X_k(t) = Z(t)\tilde{E}D_k - E\{\int_t^T \zeta(s)\psi(s)[\epsilon_k(s)-c_k(s)]ds|\mathcal{F}_t\}; \quad 0 \leq t \leq T,$$

a.s. Both requirements of Definition 4.2 for admissibility follow easily from this representation.

□

We conclude from Proposition 6.2 that the k^{th} agent's optimization problem can be cast thus: <u>to maximize the expected utility from consumption</u> $E\int_0^T U_k(t,c_k(t))dt$ <u>over consumption processes</u> c_k <u>which satisfy (6.2) and (4.3)</u>.

The solution to this problem is straightforward; denoting by $I_k(t,\cdot)$ the inverse of the strictly decreasing mapping $U_k'(t,\cdot)$ from $(0,\infty)$ onto itself, and using the consequence of the concavity of $U_k(t,\cdot)$:

$$U_k(t,I_k(t,y)) - yI_k(t,y) = \max_{c\geq 0}[U_k(t,c)-yc]; \quad \forall\ (t,y) \in [0,T] \times (0,\infty),$$

one can show as in [10] that a consumption process of the form

$$(6.7) \qquad \hat{c}_k(t) = I_k(t,y_k\zeta(t)\psi(t)); \quad 0 \leq t \leq T$$

satisfies (4.3) and is in fact optimal, provided that the constant $y_k > 0$ is chosen so that the budget constraint (6.2) is satisfied as an equality by the process $\hat{c}_k$, i.e.,

$$(6.8)\ E\int_0^T \zeta(t)\psi(t)I_k(t,y_k\zeta(t)\psi(t))dt = E\int_0^T \zeta(t)\psi(t)\epsilon_k(t)dt\ ; \quad k = 1,\dots,n.$$

In fact, it is not hard to show that for a given spot price process ψ, $y_k > 0$ <u>is uniquely determined by (6.8)</u>: the function

$$\mathcal{X}(y) \triangleq E\int_0^T \zeta(t)\psi(t)I_k(t,y\zeta(t)\psi(t))dt\ ; \quad y \in (0,\infty)$$

is continuous and strictly decreasing, and maps $(0,\infty)$ onto itself. There is exactly one value $y_k \in (0,\infty)$ for which $\mathcal{X}(y_k)$ equals the right-hand side of (6.8).

7. CONSTRUCTION OF EQUILIBRIUM

The question now is whether one can find a spot price process ψ for which (5.1)-(5.3) are satisfied.

7.1 Proposition: Suppose that ψ is an equilibrium spot price process and that the positive numbers $y_1,\ldots,y_n$ are defined in terms of it by (6.8). Then ψ and $(y_1,\ldots,y_n)$ must satisfy

$$\sum_{k=1}^{n} I_k(t, y_k \zeta(t)\psi(t)) = \epsilon(t); \quad 0 \le t \le T \tag{7.1}$$

as well. Conversely, suppose that there exist a spot price process ψ and a vector $(y_1,\ldots,y_n) \in (0,\infty)^n$ for which (6.8) and (7.1) are satisfied; then ψ is an equilibrium spot price process.

Proof: For the first claim, recall that the optimal consumption processes are given by (6.7), for $k \in \{1,\ldots,n\}$; the spot market clearing condition (5.1) leads then to (7.1).

For the second claim, notice that for the spot price process ψ in question the optimal consumption processes $\hat{c}_k$, $1 \le k \le n$ are again given by (6.7). Denote by $\hat{D}_k$, $\hat{M}_k$, $\hat{\pi}_k$ and $\hat{X}_k$ the corresponding processes in (6.3)-(6.6), which now satisfy $\tilde{E}\hat{D}_k = 0$, $\hat{X}_k(T) = 0$, and observe the a.s. identities: $\Sigma_{k=1}^{n}\hat{D}_k = 0$ from (6.3), (7.1); $\sum_{k=1}^{n} \hat{M}_k(t) = 0$ from (6.4); and

$\sum_{k=1}^{n} \hat{X}_k(T) = 0$ from (6.6). Thus (5.1), (5.3) are satisfied, and it can easily be seen from (6.5) that the portfolios $\hat{\pi}_k$, $1 \le k \le n$ can be chosen so that (5.2) is satisfied as well.

□

In order to achieve a further reduction in the characterization of equilibrium, let us define for every $\Lambda \in (0,\infty)^n$ the function

$$(7.2) \qquad I(t,h;\ \Lambda) \triangleq \sum_{k=1}^{n} I_k(t,h\lambda_k^{-1});\ (t,h) \in [0,T] \times (0,\infty).$$

For every $t \in [0,T]$, $I(t,\cdot;\ \Lambda)$ is a continuous, strictly decreasing mapping of $(0,\infty)$ onto itself with $\lim_{h\to\infty} I(t,h;\ \Lambda) = 0$, $\lim_{h\downarrow 0} I(t,h;\ \Lambda) = \infty$; thus, this mapping has an inverse $H(t,\cdot;\ \Lambda)$, in terms of which (7.1), (6.8) are re-written equivalently as

$$(7.3) \qquad \psi(t) \equiv \psi(t;\Lambda) = \frac{1}{\zeta(t)} H(t,\epsilon(t);\Lambda) \ ;\ 0 \le t \le T,$$

$$(7.4) \quad E\int_0^T H(t,\epsilon(t);\Lambda) I_k(t, \frac{1}{\lambda_k} H(t,\epsilon(t);\Lambda))dt = E\int_0^T H(t,\epsilon(t);\Lambda)\epsilon_k(t)dt; 1 \le k \le n,$$

respectively, with the identification $\Lambda \triangleq (\lambda_1,\ldots,\lambda_n) = (\frac{1}{y_1},\ldots,\frac{1}{y_n}) \in (0,\infty)$.

Consequently, <u>the search for equilibrium has been reduced to the search for a vector $\Lambda \in (0,\infty)^n$ which satisfies (7.4)</u>; once such a vector has been found, the corresponding equilibrium spot price is given by (7.3), and the optimal consumption policies of the individual agents as

$$(7.5)\qquad \hat{c}_k(t;\Lambda) \triangleq I_k(t, \frac{1}{\lambda_k} H(t,\epsilon(t);\ \Lambda));\quad 0 \le t \le T,\ 1 \le k \le n$$

by virtue of (6.7).

From the properties of the function H in conjunction with (3.1), it follows that $\zeta(\cdot)\psi(\cdot;\Lambda)$ is bounded, both from above and away from the origin.

8. INTERPRETATION OF $H(t,c;\ \Lambda)$

For every $\Lambda \in (0,\infty)^n$, let us introduce the function

$$(8.1)\qquad U(t,c;\ \Lambda) = \max_{\substack{c_1\ge 0,\dots,c_n\ge 0\\ c_1+\dots+c_n=0}} \sum_{k=1}^{n} \lambda_k U_k(t,c_k);\ (t,c) \in [0,T] \times (0,\infty),$$

which inherits the basic properties of the individual utility functions $U_k(t,c)$: for every $t \in [0,T]$, the function $U(t,\cdot;\ \Lambda)$ is strictly increasing and concave, and the derivative $U'(t,c;\ \Lambda) \triangleq \frac{\partial}{\partial c} U(t,c,;\ \Lambda)$ satisfies $U'(t,\infty;\ \Lambda) = 0$, $U'(t,0^+;\ \Lambda) = \infty$. For this reason, $U(\cdot,\cdot;\ \Lambda)$ is called the <u>utility function of a representative agent</u>, who assigns weights $\lambda_1,\dots,\lambda_n$ to the individual agents in the economy. Furthermore, U enjoys the <u>positive homogeneity</u> property

$$(8.2)\qquad U(t,c;\ \rho\Lambda) = \rho\, U(t,c,;\ \Lambda);\quad \forall\ \rho > 0.$$

It is easily checked that the maximization in (8.1) is achieved by $\bar{c}_k = I_k(t, \frac{1}{\lambda_k} H(t,c,;\ \Lambda))$, so that $U(t,c;\ \Lambda) = \sum_{k=1}^{n} \lambda_k U_k(t,\ I_k(t, \frac{1}{\lambda_k} H(t,c;\ \Lambda)))$. A differentiation with respect to c

yields the interpretation

(8.3) $$H(t,c;\ \Lambda) = U'(t,c;\ \Lambda)$$

of $H(t,\cdot;\ \Lambda)$ as the derivative of the utility function of the representative agent with weights $\lambda_1,\ldots,\lambda_n$.

9. EXISTENCE AND UNIQUENESS OF EQUILIBRIUM

In this section we formulate and prove the basic result of this paper.

9.1 Theorem: Suppose that for every $t \in [0,T]$ and $k \in \{1,\ldots,n\}$,

(9.1) the function $c \mapsto cU_k'(t,c)$ is nondecreasing.

Then there exists a vector $\Lambda \in (0,\infty)^n$ satisfying (7.4). If Λ and $\tilde{\Lambda}$ are two such vectors, then we have

(9.2) $$\Lambda = \gamma\tilde{\Lambda} \text{ and } \psi(t;\Lambda) = \gamma\psi(t;\tilde{\Lambda});\quad 0 \le t \le T$$

for some $\gamma > 0$, as well as

(9.3) $$\hat{c}_k(t;\Lambda) = \hat{c}_k(t;\tilde{\Lambda});\quad 0 \le t \le T,\quad k = 1,\ldots,n.$$ □

The relation (9.2) expresses the fact that equilibrium prices can be determined only up to a multiplicative constant, because the currency can always be re-valued; however, as relation (9.2) stresses, this will not affect how <u>real</u> wealth (measured in optimal consumption units of the commodity) is distributed among the agents.

In order to set the stage for the proof, let us rewrite the equation (7.4) in the form $S_k(\lambda_k;\Lambda) = 0$, where

$$(9.4)\quad S_k(\mu;\Lambda) \triangleq E\int_0^T \frac{1}{\mu} H(t,\epsilon(t);\Lambda)[I_k(t, \frac{1}{\mu} H(t,\epsilon(t);\Lambda)) - \epsilon_k(t)]dt;\quad 0 < \mu < \infty.$$

Because of condition (9.1), which amounts to imposing that $y \mapsto yI_k(t,y)$ is nonincreasing, we see that $S_k(\cdot;\Lambda)$ is strictly increasing, with $S_k(0+;\Lambda) = -\infty$ and $\lim_{\mu\to\infty} \mu S_k(\mu;\Lambda) = \infty$, for every $\Lambda \in (0,\infty)^n$. Thus there exists exactly one number $L_k(\Lambda) \in (0,\infty)$ such that $S_k(L_k(\Lambda);\ \Lambda) = 0$, and <u>solving (7.4) amounts to finding a fixed point for the operator</u>

$$(9.5)\qquad L = (L_1,\dots,L_n):\ (0,\infty)^n \to (0,\infty)^n.$$

This operator is <u>positively homogeneous</u>: $L(\rho\Lambda) = \rho L(\Lambda)$, $\forall\ \rho \in (0,\infty)$. Indeed, from (8.2), (8.3) and (9.4) we deduce $S_k(\rho\mu;\ \rho\Lambda) = S_k(\mu;\Lambda)$, whence $S_k(\rho L_k(\Lambda);\ \rho\Lambda) = 0$. In other words, if Λ is a fixed point of L, then the entire ray $\{\rho\Lambda;\ \rho \in (0,\infty)\}$ is a family of fixed points. We shall show that under (9.1) there is exactly one such ray.

Proof of existence: We introduce the usual partial ordering in $(0,\infty)^n$: $\Lambda \le M$ if and only if $\lambda_k \le \mu_k$, $\forall\ k \in \{1,\dots,n\}$, and we write $\Lambda < M$ if $\Lambda \le M$ and $\Lambda \ne M$. In particular, notice in (7.2) the implications

$$(9.6)\qquad \Lambda \underset{(<)}{\le} M \Rightarrow I(t,h;\Lambda) \underset{(<)}{\le} I(t,h;M);\ \forall\ (t,h) \in [0,T] \times (0,\infty).$$

Now for $\Lambda \le M$ we have from (9.6): $H(t,\epsilon(t);\ \Lambda) \le H(t,\epsilon(t);\ M)$, and the condition (9.1) yields $S_k(L_k(\Lambda);\ M) \le S_k(L_k(\Lambda);\ \Lambda) = 0 = S_k(L_k(M);\ M)$, whence $L_k(\Lambda) \le L_k(M)$, $\forall\ k \in \{1,\dots,n\}$. It develops that the operator L of (9.5) is <u>isotone</u>: $\Lambda \le M \Rightarrow L(\Lambda) \le L(M)$.

Exactly as in [11], section 12, one can find Λ_ℓ, Λ_u in $(0,\infty)^n$ such that $\Lambda_\ell \le \Lambda_u$, $\Lambda_\ell \le L(\Lambda_\ell)$ and $L(\Lambda_u) \le \Lambda_u$. From the Knaster-Tarski lattice fixed point theorem ([6], p.14 or [1], p. 54), one obtains then the existence of $\Lambda \in (0,\infty)^n$ with $\Lambda_\ell \le \Lambda \le \Lambda_u$, such that $L(\Lambda) = \Lambda$.

Proof of uniqueness: Let Λ, $\tilde{\Lambda}$ be two fixed points of L, define $\gamma \triangleq \max_{1\le k\le n} (\lambda_k/\tilde{\lambda}_k)$ and $M \triangleq \gamma\tilde{\Lambda}$, and notice $\Lambda \le M$. If $\Lambda = M$, then the uniqueness claims in (9.2), (9.3) follow immediately from the defining relations (7.3), (7.5) and the positive homogeneity of $H(t,c;\ \cdot)$. Therefore, we have to rule out the case $\Lambda < M$.

Suppose that $\Lambda < M$ holds. Then from (9.6) we obtain $H(t,\epsilon(t);\ \Lambda) < H(t,\epsilon(t);M)$; furthermore, for any integer $j \in \{1,\dots,n\}$ satisfying $\lambda_j = \gamma\tilde{\lambda}_j$ (and hence also $\lambda_j = \mu_j$), we have with $\varphi_j(t;\Lambda) \triangleq H(t,\epsilon(t);\ \Lambda)/\lambda_j$:

$$E\int_0^T \varphi_j(t;\Lambda)\ \epsilon_j(t)dt < E\int_0^T \varphi_j(t;M)\epsilon_j(t)dt\ ,$$

$$E\int_0^T \varphi_j(t;\Lambda)I_j(t,\varphi_j(t;\Lambda))dt \ge E\int_0^T \varphi_j(t;M)I_j(t,\varphi_j(t;M))dt$$

by virtue of (9.1). From these two relations and (9.4) we conclude $S_j(\mu_j;M) < S_j(\lambda_j;\Lambda) = 0$; but we have also $S_j(\tilde{\lambda}_j;\tilde{\Lambda}) = 0$ by assumption, and thus $S_j(\mu_j;M) = S_j(\gamma\tilde{\lambda}_j;\ \gamma\tilde{\Lambda}) = 0$, a contradiction.

9.2 Remark: The financial market of section 2 has minimal effect on equilibrium. Indeed, consider two economies (indexed by $m = 1,2$) with the same primitives (endowments and utilities for the agents) but possibly different coefficients r, b_i, σ_{ij} in their financial markets, and denote by ζ_1, ζ_2 the corresponding deflator processes of (2.6). The functions S_k, $1 \le k \le n$ depend only on the primitives, and thus the same is true for the operator L of (9.5). We have from (7.3), (7.5):
$\psi_m(t)\zeta_m(t) = H(t,\epsilon(t);\Lambda_m)$ and $\hat{c}_{mk}(t) = I_k(t,H(t,\epsilon(t);\ \Lambda_m)/\lambda_m)$ for $0 \le t \le T$, $m = 1,2$, where the vectors Λ_1, Λ_2 are both fixed points of L. It follows that $\Lambda_1 = \gamma\Lambda_2$ for some $\gamma > 0$, and thus

$$\psi_1(t)\zeta_1(t) = \gamma\psi_2(t)\zeta_2(t),\ \hat{c}_{1k}(t) = \hat{c}_{2k}(t).$$

In other words, the choice of the financial market can affect the prices by more than a multiplicative factor, but cannot affect the equilibrium allocation of the commodity among agents.

10. EXAMPLES

We cite a few special cases in which the equilibrium can be computed explicitly.

10.1 Example: $U_k(t,c) = c^\delta$, $\forall\ (t,k) \in [0,T] \times \{1,\ldots,n\}$, **for some** $\delta \in (0,1)$. In this case, the vector $\Lambda = (\lambda_1,\ldots,\lambda_n) \in (0,\infty)^n$ with

$$\lambda_k = \left(E \int_0^T \epsilon_k(t)(\epsilon(t))^{\delta-1}dt \ /\ E\int_0^T (\epsilon_k(t))^\delta dt \right)^{1-\delta};\quad 1 \le k \le n$$

gives the unique solution of (7.4) subject to $\sum_{k=1}^{n} \lambda_k^{1/(1-\delta)} = 1$.

The equilibrium spot price and optimal consumption processes are given as $\psi(t) = \frac{\text{const.}}{\zeta(t)(\epsilon(t))^{1-\delta}}$, $\hat{c}_k(t) = \lambda_k^{1/(1-\delta)}\epsilon(t)$; $1 \le k \le n$.

10.2 Example: $U_k(t,c) = \log c$, $\forall\ (t,k) \in [0,T] \times \{1,\dots,n\}$. In this case we recover the same formulae as in Example 10.1 but with $\delta = 0$. In particular, the constant $\lambda_k = \frac{1}{T} E \int_0^T \frac{\epsilon_k(t)}{\epsilon(t)}\,dt$ is then a measure of the k^{th} agent's relative importance in the economy, and determines what fraction of the total supply he will be allowed to consume at any given time: $\hat{c}_k(t) = \lambda_k\, \epsilon(t)$, $1 \le k \le n$. Finally, the equilibrium spot price $\psi(t) = \frac{\text{const.}}{\zeta(t)e(t)}$ is proportional to the total supply.

If agents have different utility functions, it is not possible in general to compute the solution of the equilibrium problem in closed form. A special case, in which such computations can be carried out, arises when $n = 2$, $U_1(c) = \log c$ and $U_2(c) = \sqrt{c}$. Another special case is the following.

10.3 Example: Constant aggregate endowment $\epsilon(t) \equiv \epsilon > 0$ and time-independent utility function. In this case the optimal consumption rates are constant: $\hat{c}_k(t) \equiv \hat{c}_k \triangleq \frac{1}{T} E \int_0^T \epsilon_k(t)dt$, a solution to (7.4) is given by

$$\lambda_k = \frac{1}{U'_k(\hat{c}_k)}; \quad k = 1,\dots,n,$$

and the deflated equilibrium price is constant: $\psi(t) = \frac{\text{const.}}{\zeta(t)}$.

11. EXTENSIONS AND RAMIFICATIONS

The results of this paper hold also in the case $U_k'(t,0+) < \infty$ for some $(t,k) \in [0,T] \times \{1,\dots,n\}$, but the analysis becomes considerably more complex; it is carried out in complete detail in [11].

In addition to the financial assets of section 2, one can allow the agents to trade in capital assets, and one can associate to each one of these assets a dividend process $\delta_m(\cdot)$, $1 \le m \le M$, denominated in units of the commodity. In contrast to financial assets, which are essentially contracts between the agents, capital assets have to maintain a positive net supply. One can show that the prices $S_m(\cdot)$ of these new assets have to be given as

$$(11.1) \qquad \zeta(t)S_m(t) = E\left[\int_t^T \zeta(s)\psi(s)\delta_m(s)ds \,\middle|\, \mathcal{F}_t\right]; \quad 0 \le t \le T$$

in order to prevent "arbitrage opportunities". Once the spot price ψ has been determined by equilibrium considerations, the relation (11.1) allows the endogenous computation of the capital asset prices $S_m(\cdot)$, $1 \le m \le M$. Again, consult [11] for the details.

Consider now an economy with deterministic endowments and no financial market (except for a zero-interest-rate bond); agents can consume but cannot borrow or invest, are bound simply by the budget constraints

$$\int_0^T \psi(s)c_k(s)ds \le \int_0^T \psi(s)\epsilon_k(s)ds; \qquad 1 \le k \le n$$

(the deterministic analogue of (6.2)), and try to maximize their total utilities $\int_0^T U_k(t,c_k(t))dt$ from consumption. Equilibrium amounts to the requirements (5.1), (5.3) alone. In this simple model the results of sections 6-10 are all valid, provided that one sets $\zeta(t) \equiv 1$ and drops the expectation signs in the formulae.

REFERENCES.

[1] BIRKHOFF, G. (1948) **Lattice Theory**. Amer. Math. Soc. Colloq. Publ., Vol. XXV.

[2] DUFFIE, D. (1986) Stochastic equilibria: existence, spanning number, and the "no expected financial gain from trade" hypothesis, *Econometrica* **54**, 1161-1183.

[3] DUFFIE, D. & HUANG, C.F. (1985) Implementing Arrow-Debreu equilibria by continuous trading of a few long-lived securities, *Econometrica* **53**, 1337-1356.

[4] DUFFIE, D. & HUANG, C.F. (1987) Stochastic Production-Exchange Equilibria. Research paper, Grad. School of Business, Stanford University.

[5] DUFFIE, D. & ZAME, W. (1987) The consumption-based capital asset pricing model. Research paper, Grad. School of Business, Stanford University.

[6] DUGUNDJI, J. & GRANAS, A. (1982) **Fixed Point Theory**. Polish Scientific Publishers, Warsaw.

[7] HARRISON, J. & PLISKA, S.R. (1981) Martingales and stochastic integrals in the theory of continuous trading, *Stoch. Proc. Appl.* **11**, 215-260.

[8] HARRISON, J.M. & PLISKA, S.R. (1983), A stochastic calculus model of continuous trading: complete markets, *Stoch. Proc. Appl.* **15**, 313-316.

[9] HUANG, C.F. (1987) An intertemporal general equilibrium asset pricing model: the case of diffusion information, *Econometrica* **55**, 117-142.

[10] KARATZAS, I., LEHOCZKY, J.P. & SHREVE, S.E. (1987) Optimal portfolio and consumption decisions for a "small investor" on a finite horizon, *SIAM J. Control & Optim.* **25**, 1557-1586.

[11] KARATZAS, I., LEHOCZKY, J.P. & SHREVE, S.E. (1988) Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model. Preprint.

[12] KARATZAS, I. & SHREVE, S.E. (1987) **Brownian Motion and Stochastic Calculus**. Springer-Verlag, New York.

[13] TAQQU, M. & WILLINGER, W. (1987) The analysis of finite security markets using martingales, *Adv. Appl. Probab.* **19**, 1-25.