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Remarks about Gradient Young Measures Generated by Sequences in Soblev Spaces

David Kinderlehrer
Carnegie Mellon University

Pablo Pedregal
Carnegie Mellon University
and
Universidad Complutense de Madrid
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David Kinderlehrer and Pablo Pedregal

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1. INTRODUCTION  The oscillatory properties of a sequence of weak* convergent
functions may be summarized by the parametrized measure, or Young measure, which it generates,
Young [81]. The parametrized measure generated by a sequence $f_k \in L^\infty(\Omega;\mathbb{R}^N)$ with

$$f_k \rightharpoonup f \text{ in } L^\infty(\Omega;\mathbb{R}^N) \text{ weak}^*$$

is a family of probability measures $v = (v_x)_{x \in \Omega}$ with supp $v_x \subset \mathbb{R}^N$ such that, for every
function $\psi$ continuous in $\lambda$ and measurable in $x$,

$$\psi(f_k,x) \rightharpoonup \overline{\psi(x)} = \int_{\mathbb{R}^N} \psi(\lambda,x) \, dv_x(\lambda) \text{ in } L^\infty(\Omega) \text{ weak}^*, \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ is bounded. Restricting, if necessary, to a subsequence, every weak* convergent
sequence generates a parametrized measure. Thus the parametrized measure describes the weak
limit of any continuous function composed with the sequence, which leads to its use in problems
where sequences converge weakly but not strongly. In this framework, every family
$v = (v_x)_{x \in \Omega}$ of probability measures with uniformly bounded support is a parametrized
measure.

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Variational principles used to study equilibrium configurations of crystalline solids or other materials with order are not lower semicontinuous. Our interest in the foundations of this theory begin with the work of Ericksen, cf. e.g. [27-38]. In these circumstances, the infimum of energy is attained only in some generalized sense while a minimizing sequence may develop finer and finer oscillations, reminiscent of a finely twinned microstructure. The weak limit of the sequence may be inadequate to describe many properties of the configuration, but the parametrized measure has proven useful both to determine macroscopic properties such as energy and stress and microstructural properties such as variant arrangement and location, cf. Ball and James [5,6], Chipot and Kinderlehrer [18], Chipot, Kinderlehrer, and Vergara Caffarelli [19], Fonseca [42-45], James [49], James and Kinderlehrer [50], Kinderlehrer [54], Pedregal [72,73], Matos [65,66], and Babadjarya [11,12]. It has also led to computational developments, Collins and Luskin [21-23], Chipot [16], Chipot and Collins [17], Collins, Kinderlehrer and Luskin [24], Luskin and Ma [64], and Nicolaides and Walkington [71]. A recent accounting of some of these and related developments may be found in [39]. In addition, the analysis we discuss here has close connections with the work of Ball and Murat [8,9], Ball and Zhang [10], Brandon and Rogers [14], Firooze and Kohn [41], Kohn [62], James and Kinderlehrer [51-53], Sverak [75-77], and Zhang [82-84].

The use of the Young measure to study possible oscillations of solutions of partial differential equations was initiated by Tartar [78-80].

The two immediate difficulties which arise in applying the ideas leading to (1.1) are

1. the variational constraint that \( f^k = \nabla u^k \), and
2. the \( f^k \) are not generally bounded in \( L^\infty(\Omega, \mathbb{R}^N) \), but instead in \( L^p(\Omega, \mathbb{R}^N) \), for some \( p \in [1, \infty) \).

In this note, we would like to discuss our recent thoughts about these questions. Details appear in [57]. Our objective is to characterize the gradient Young measures which arise from sequences bounded in \( H^{1,p}(\Omega, \mathbb{R}^m) \), \( 1 \leq p \leq \infty \). We found subtle differences between the cases \( p < \infty \) and \( p = \infty \), and since we have reported on the latter case in [55],[56],[58], we concentrate here on the case where \( p \) is finite. In the remainder of this section we shall introduce the notions we intend to study and state our principal result, THEOREM 1.1.

As our starting point we recall a more general framework for the study of oscillations described by Ball [3] and also studied by Matos [65]. We retain the convention that \( \Omega \subset \mathbb{R}^n \) is bounded (and measurable.) Suppose that \( f^k \in L^p(\Omega, \mathbb{R}^N) \), for some \( p \in [1, \infty) \), and

\[
\int_\Omega |f^k|^p \, dx \leq M.
\]
Then there is a family $v = (v_x)_{x \in \mathbb{N}}$ of probability measures on $\mathbb{R}^N$ and a subsequence of the $(\mathfrak{B})$, not relabelled, such that whenever

$$\mathfrak{B}(\mathfrak{B}) \nRightarrow \mathfrak{V} \text{ in } \mathcal{M}(\mathfrak{B}), \text{ for } y \in C(\mathbb{R}^N),$$

then

$$\mathfrak{V}(x) \quad \text{m} \quad f(y) \text{d}v_x(X) \text{ in } Q \text{ a.c.} \quad (1.3)$$

For example, it is obvious from Holder's inequality and the Dunford-Pettis criterion that given a sequence satisfying (1.2), the conclusion (13) is satisfied whenever

$$|y(X)| \leq C(1 + |X|^q), \quad \text{J} l \in \mathbb{R}^N, \quad (1.4)$$

whenever $q < p$. However, it is also obvious from the viewpoint of applications that we wish to have some interpretation of (1.3) when $p = q$. The subsequence $(f_r)$ determines a unique parametrized measure. The problem is to decide when it identifies weak limits.

Since (1.3) does not hold for arbitrary sequences bounded in $D\setminus$, we must either impose an additional condition on the sequence or restrict the notion of Young measure as a characterization of oscillatory behavior. What does this entail? To begin, we shall neglect the gradient constraint and then reinstate it later.

For convenience we set

$$\mathcal{E} \mathfrak{P} \subset \{y \in \mathcal{C}(\mathbb{R}^N) : \lim_{x \to 0} \mathfrak{J} \text{ exists} \} \quad (1.5)$$

$\mathcal{E} \mathfrak{P}$ is isomorphic to the continuous functions on the one point compactification of $\mathcal{M}$ and is separable. For technical reasons, this has an advantage over the inseparable space of functions suggested by (1.4) of the same growth rate

$$X = \{y \in C(\mathbb{R}^N) : |y(X)| \leq C(1 + |X|^q), \quad \lambda \in \mathbb{R}^N\}$$

and will incur no loss in generality in our considerations.

A particular circumstance leading to the validity of (1.3) for all $y$ satisfying (1.4) in the case $p = q$ occurs when $f^* \mathfrak{P}$ themselves converge weakly in $L^p(\mathfrak{F})$. This follows by application of the Dunford-Pettis criterion and leads us to the notion of $p$-Young measure:

A family $v = (v_x)_{x \in \mathbb{N}}$ is a $p$-Young measure or $p$-parametrized measure provided there is a sequence $f \in L^p(Q;\mathbb{R}^N)$ for some $p \in (1, \infty)$ and $g \in L^1(\mathfrak{F})$, such that
\[
\lim_{k \to \infty} \int_E \psi(f_k) \, dx = \int_E \int_{\mathbb{R}^N} \psi(\lambda) \, dv_x(\lambda) \, dx \quad \text{for } \psi \in \mathcal{E}'P. \tag{1.6}
\]

Equivalently, we may say that for any \( E \subset \Omega \),

\[
\lim_{k \to \infty} \int_E \psi(f_k) \, dx = \int_E \int_{\mathbb{R}^N} \psi(\lambda) \, dv_x(\lambda) \, dx \quad \text{for } \psi \in \mathcal{E}'P. \tag{1.6}
\]

Alternatively, we may define a biting Young measure for an arbitrary sequence. Recall that if \( f^k \in L^1(\Omega) \) and

\[
\int_{\Omega} |f^k| \, dx \leq M < \infty,
\]

then there is a sequence \( E_j \subset \Omega \) with \( E_{j+1} \subset E_j \), \( |E_j| \to 0 \), and a \( g \in L^1(\Omega) \) such that for a subsequence of the \( (g^*_j) \), not relabelled,

\[
g^k \rightharpoonup g \quad \text{in } L^1(\Omega \setminus E_j) \quad \text{for each } j.
\]

This is the conclusion of the Chacon biting lemma [15], cf. also Ball and Murat [8]. We write that

\[
g^k \succ g \quad \text{in } L^1(\Omega) \tag{1.7}
\]

and say that \( g^k \) converge to \( g \) in the biting sense. Clearly if \( g^k \) converge in the biting sense and \( |\psi(\lambda)| \leq C |\lambda| \), then \( \psi(g^k) \) also converge in the biting sense, again by the Dunford-Pettis criterion. Thus we are led to the notion of biting Young measure:

A family \( \nu = (v_x)_{x \in \Omega} \) is a biting Young measure provided there is a sequence \( f^k \in L^p(\Omega; \mathbb{R}^N) \) and \( g \in L^1(\Omega) \) such that

\[
|f^k|_P \rightharpoonup g \quad \text{in } L^1(\Omega) \quad \text{and}
\]

\[
\psi(f^k) \rightharpoonup -\psi \quad \text{in } L^1(\Omega) \quad \text{where}
\]

\[
\bar{\psi}(x) = \int_{\mathbb{R}^N} \psi(\lambda) \, dv_x(\lambda) \quad \text{in } \Omega \text{ a.c., for } \psi \in \mathcal{E}'P. \tag{1.8}
\]

The bitten sets \( (E_j) \) depend only on the sequence \( (f^k) \) and not on the particular \( \psi \). A p-Young measure is a biting Young measure. Furthermore, evident from the property of weak convergence in \( L^1 \), the Young measure determines the biting limit of a sequence and not its distributional limit.
Suppose that \( \mu \) is a homogeneous biting Young measure, i.e., \( \mu_x \) is independent of \( x \in \Omega \). We may regard \( \mu \) as an element of \( E^p \)', the dual of \( E^p \), with

\[
\| \mu \| = \int_\Omega \left( 1 + |\lambda| \right) d\mu(\lambda) = \langle \mu, (1 + 1 \cdot \|\|) \rangle.
\]  

Even though \( \mu \) is a probability measure, it is not necessarily in the unit ball of \( E^p \'). Indeed, consider the simple example with \( \Omega = (0,1) \)

\[
f_k = k^{1/2} \chi_{(0,1/k)}.
\]

These \( f_k \) have the properties that

\[
f_k \to 0 \text{ in } L^2((0,1)) \text{ weakly and in the biting sense and}
\]

\[
|f_k|^2 \text{ do not converge weakly in } L^1((0,1)),
\]

\[
|f_k|^2 \nrightarrow 0.
\]

For any \( \psi \in E^2 \),

\[
\int_0^1 \psi(f_k) \, dx = \frac{1}{k} \psi(k^{1/2}) + (1 - \frac{1}{k}) \psi(0).
\]

The limit of these functionals defines an operation which is not a probability measure, but the functional

\[
\langle T, \psi \rangle = \psi(0) + \lim_{\lambda \to -\infty} \psi(\lambda)/\lambda^2.
\]  

(1.10)

However the biting limit determines the probability measure \( \delta_0 \) which is generated by some sequence of oscillations in \( L^2((0,1)) \). Each function \( f_k \) determines the parametrized measure \( \delta_{f_k} \) given by

\[
\langle \delta_{f_k}, \psi \rangle = \int_0^1 \psi(f_k) \, dx
\]

In view of (1.10), this shows that the Young measures are not closed in \( E^2 \). It is easy to see that they also are not bounded in the norm (1.9). Conditions which ensure the convergence of a sequence of measures in \( E^p \' \) to another probability measure are analogous to tightness conditions, cf. Billingsley [13].

Let us now impose the constraint that the functions \( f_k \) which generate the measures are gradients. We agree to call the associated measures \( H^1.P \) - Young measures and \( H^1.P \) - biting Young measures, respectively.
Recall that a continuous function \( \varphi(A) \), where \( A \in M \), the \( m \times n \) matrices, is quasiconvex provided

\[
\varphi(F) \leq \frac{1}{|\Omega|} \int_{\Omega} \varphi(F + \nabla \zeta) \, dx \quad \text{for all} \quad \zeta \in C^0_c(\Omega; \mathbb{R}^m) \quad \text{and} \quad F \in M. \tag{1.11}
\]

A result of Acerbi and Fusco [1], which generalizes the theorem of Morrey [67], is that if \( \varphi \in \mathcal{E} \) is quasiconvex and bounded from below, then

\[
\int_{\Omega} \varphi(\nabla u) \, dx \leq \liminf_{k \to \infty} \int_{\Omega} \varphi(\nabla u_k) \, dx \tag{1.12}
\]

whenever

\[
u^k \rightharpoonup u \quad \text{in} \quad H^{1,p}(\Omega; \mathbb{R}^m).
\]

The proof of (1.12) in this case is a direct generalization of Morrey's and does not require the machinery developed by Acerbi and Fusco, cf. Evans [40], Dacorogna [25]. In fact, in the case of linear growth of \( \varphi(A) \), the lower semicontinuity (1.12) remains true even when we assume only that

\[
u^k, u \in H^{1,1}(\Omega; \mathbb{R}^m) \quad \text{and} \quad u^k \rightharpoonup u \quad \text{in} \quad \mathcal{D}'(\Omega).
\]

Deep generalizations of Morrey's Theorem relevant to this situation have recently been proved by Fonseca and Müller [46,47], to whom we refer for additional references.

Assuming \( \Omega \) given, (1.12) also holds for smooth subdomains \( \Omega' \subset \Omega \), in particular for \( \Omega' = B_r(a) \subset \Omega \). Suppose that \( (\nabla u^k) \) generate an \( H^{1,p} \)-Young measure \( \nu \). Then (1.12) implies that

\[
\int_{B_r(a)} \varphi(\nabla u) \, dx \leq \int_{B_r(a)} \varphi(x) \, dx \quad \text{for} \quad \varphi(x) = \int_M \varphi(A) \, dv_x(A),
\]

and thus by Lebesgue's Theorem, \( \varphi(\nabla u(a)) \leq \varphi(a) \) in \( \Omega \). Expressed differently,

\[
\varphi(\nabla u(x)) \leq \int_M \varphi(A) \, dv_x(A), \quad \text{where} \quad \nabla u(x) = \int_M A \, dv_x(A), \tag{1.13}
\]

whenever \( \varphi \in \mathcal{E} \) is quasiconvex and bounded below. So if \( \nu \) is an \( H^{1,p} \)-Young measure, Jensen's Inequality holds for quasiconvex \( \varphi \in \mathcal{E} \) and bounded below.

Another consequence of (1.12) concerns the Jensen Inequality for biting Young measures and is not elementary. We know that if \( \varphi \in \mathcal{E} \) is quasiconvex and \( \varphi \geq 0 \), then

\[
\int_E \varphi(\nabla u) \, dx \leq \liminf_{k \to \infty} \int_E \varphi(\nabla u_k) \, dx, \quad \text{for} \quad E \subset \Omega, \tag{1.14}
\]
whenever \( u^* \to u \in H^\alpha \text{PtfcR}^{111} \). This result does require a substantial part of the Acertri-Fusco apparatus and unfortunately we have been unable to find a simpler proof than, say, [59]. Let \( v = (v_x)_{x \in Q} \) denote the biting Young measure generated by \( (Vu^k) \) and \( \bar{y} \) the biting limit given by (1.8). Whenever \( E \subset Q \setminus E_j \), we than have from (1.12) that

\[
\int \phi(Vu) dx \leq \int \bar{\phi} dx.
\]

As \( \lim \|E_j\| = 0 \), we find again that

\[
\phi(Vu) \leq \bar{\phi} \text{ in } Q \text{ a.e.}
\]

By adding a constant to \( \phi \), this is seen to hold for quasiconvex functions bounded below and by truncation it is seen to hold for arbitrary quasiconvex functions with growth of order \( p \). This has been discussed by Ball and Zhang [10] as well. Otherwise stated, for an \( H^{1*} \)-biting Young measure \( v = (v_x)_{x \in Q} \), whenever \( \phi \in \mathcal{P} \) is quasiconvex,

\[
\langle KVu(x) \rangle \leq \mathcal{F}_{Q}^{\alpha}(A) d\nu(A), \text{ where } Vu(x) = \int_M \nu(x) d\nu(A). \tag{1.15}
\]

Thus Jensen's Inequality for quasiconvex functions of suitable growth holds for biting Young measures as well as \( H^{1*} \)-Young measures.

Our principle result is that (1.15) characterizes the \( H^{1*} \)-Young measures. Thus the \( H^{1,P} \)-Young measures are the same as the \( H^{\alpha} \)-biting Young measures and are the same as the measures which satisfy (1.15). Of course, the sequence that generates the measure as an \( H^{\alpha} \)-Young measure cannot usually be the same as an arbitrary one that generates it as a biting measure.

**THEOREM 1.1** Let \( v = (v_x)_{x \in Q} \) be a family of probability measures in \( C(M) \). Then 
\( v \ll (v_x)_{x \in Q} \) is an \( H^{p} \)-gradient Young measure if and only if

i) there is a \( u \in H^{\alpha}(\mathbb{R}^{m_1}) \) such that

\[
Vu(x) = \int_M ADv_x(A) \quad \text{in Q a.e.,}
\]

ii) Jensen's Inequality (1.13) holds for all \( \phi \in \mathcal{P} \) quasiconvex and bounded below, and

iii) the function

\[
\Psi(x) = \int_{\mathbb{R}^m} \mathcal{F}_{Q}^{\alpha}(A) \, d\nu(A) \in L^{1}(Q).
\]

The last condition is necessary. For example, in the case \( n < m = 1 \), let \( g \in C(0,1) \) be nonnegative with
\[
\frac{1}{g} \, dx = \infty \quad \text{and} \quad v_x = \frac{1}{2} (\delta_{g(x)} + \delta_{-g(x)}).
\]

Then \( v \) satisfies i) and ii) but is not a 1 - Young measure. Analogously in the \( p = \infty \) case, we were obliged to assume that \( \cup \operatorname{supp} v_x \) was bounded. It suffices in ii) to require the Jensen inequality for the more restricted set of quasiconvex in \( E_P \) which are bounded below.

For completeness and comparison, we give the \( H^{1,\infty} \) result as well.

**Theorem 1.2** Let \( v = (v_a)_{a \in \Omega} \) be a family of probability measures in \( C(M) \). Then \( v = (v_a)_{a \in \Omega} \) is an \( H^{1,\infty} \) gradient Young measure, or simply, a gradient Young measure if and only if

i) there is a \( u \in H^{1,\infty}(\Omega; \mathbb{R}^m) \) such that

\[
\nabla u(x) = \int_M A \, dv_x(A) \quad \text{in} \ \Omega \ \text{a.e.},
\]

ii) Jensen's Inequality (1.13) holds for all quasiconvex \( \varphi \in C(M) \) and

iii) \( \operatorname{supp} v_a \subset K, \ a \in \Omega \ \text{a.e.}, \) where \( K \) is a fixed compact.

\[\text{2. WEAK CONVERGENCE AND BITING CONVERGENCE} \]

There is a straightforward way to understand the relationship between weak convergence and biting convergence, which, in fact, has been used implicitly by us [59] and by Ball and Zhang [10].

**Proposition 2.1** Let \( g^k \in L^1(\Omega), \ g^k \geq 0, \) with the property that

\[g^k \rightharpoonup g \ \text{in} \ L^1(\Omega).
\]

A subsequence \( (g^{k'}) \) of the \( (g^k) \) satisfies

\[g^{k'} \rightharpoonup g \ \text{in} \ L^1(\Omega),
\]

if and only if

\[
\liminf_{k \to \infty} \int_{\Omega} g^k \, dx \leq \int_{\Omega} g \, dx.
\]

Moreover, \( (g^k) \) is weakly relatively compact in \( L^1(\Omega) \) if and only if

\[
\limsup_{k \to \infty} \int_{\Omega} g^k \, dx \leq \int_{\Omega} g \, dx.
\]
To prove the proposition, assume biting convergence and the failure of the Dunford-Pettis criterion. This leads to the failure of (2.1).

Consider a biting Young measure \( v = (v_x)_{x \in \Omega} \) generated by \( f^k \in L^p(\Omega; \mathbb{R}^N) \). Suppose that \( \varphi \in \mathcal{E}^p \) majorizes \( 1 + 1 \lambda \mathbb{P} \), i.e., \( 1 + 1 \lambda \mathbb{P} \leq \varphi(\lambda) \), and

\[
\lim_{k \to \infty} \int_{\Omega} \varphi(f^k) \, dx = \int_{\Omega} \int_{\mathbb{R}^N} \varphi(\lambda) \, dv_x(\lambda) \, dx \quad (2.3)
\]

The proposition then implies that

\[
\varphi(f^k) \to \int_{\mathbb{R}^N} \varphi(\lambda) \, dv_x(\lambda) \quad \text{in} \; L^1(\Omega), \quad (2.4)
\]

whence, by application of the Dunford-Pettis criterion again, \( v = (v_x)_{x \in \Omega} \) is a \( p \)-Young measure.

Here is the situation in which we shall apply (2.3), (2.4). Suppose that \( \mu \in \mathcal{E}^p' \) is a probability measure and that \( f^k \in L^p(\Omega), \| \Omega \| = 1 \), is a sequence of functions such that

\[
\lim_{k \to \infty} \int_{\Omega} \psi(f^k) \, dx = \int_{\mathbb{R}^N} \psi(\lambda) \, d\mu(\lambda), \quad \text{whenever} \; \psi \in \mathcal{E}^p. \quad (2.5)
\]

Since we may evaluate (2.5) on \( \varphi_{0}(\lambda) = 1 + 1 \lambda \mathbb{P} \), the sequence \( (f^k) \) is bounded in \( L^p \) and generates a biting Young measure \( v \), which we may assume after rescaling to be homogeneous, cf. THEOREM 3.1 in the sequel. We claim that \( v = \mu \). This will tell us that \( \mu \) is a \( p \)-Young measure by (2.3), (2.4). We interpret the left hand side of (2.5) by observing that

\[
\langle \mu^k, \psi \rangle = \int_{\Omega} \psi(f^k) \, dx
\]

is just the average of the parametrized measure which is the Dirac mass at \( f^k \),

\[
\langle \delta_{f^k(x)}, \psi \rangle = \psi(f^k(x)).
\]

Thus a probability measure in the closure of the averages of Dirac masses of \( L^p \) functions is a \( p \)-Young measure which is generated by \( (f^k) \).

To show that \( \mu = v \), suppose first that \( \psi \geq 0 \). By the Lebesgue Theorem and the Monotone Convergence Theorem,

\[
\int_{\mathbb{R}^N} \psi(\lambda) \, d\mu(\lambda) = \lim_{\alpha \to 1^-} \left\{ \int_{\psi \leq 1} \psi^{\alpha}(\lambda) \, d\mu(\lambda) + \int_{\psi > 1} \psi^{\alpha}(\lambda) \, d\mu(\lambda) \right\}
\]

\[
= \lim_{\alpha \to 1^-} \int_{\mathbb{R}^N} \psi^{\alpha}(\lambda) \, d\mu(\lambda)
\]
For $\alpha < 1$, $\psi^\alpha \in E^q \subset E^p$, $q = p\alpha$. The sequence $(\psi^\alpha(f^k))$ satisfies the Dunford-Pettis criterion and hence is weakly (pre)compact in $L^1(\Omega)$. Thus by (1.3),

$$\lim_{k \to \infty} \int_{\Omega} \psi^{\alpha}(f^k) \, dx = \int_{\mathbb{R}^N} \psi^{\alpha}(\lambda) \, d\nu(\lambda).$$

Again by the Lebesgue Theorem and the Monotone Convergence Theorem,

$$\lim_{\alpha \to 1^-} \int_{\mathbb{R}^N} \psi^{\alpha}(\lambda) \, d\nu(\lambda) = \int_{\mathbb{R}^N} \psi(\lambda) \, d\nu(\lambda).$$

Thus $\mu = \nu$ on nonnegative $\psi$. By decomposing an arbitrary $\psi$ into its positive and negative parts, we deduce that $\mu = \nu$.

An important observation is that if the sequence $f^k = \nabla u^k$ above is a sequence of gradients, then $\mu$ is an $H^{1,p}$-Young measure. We have shown

**PROPOSITION 2.2** Suppose that $\mu \in \mathcal{E}^p$ is a probability measure and that $f^k \in L^p(\Omega;\mathbb{R}^N)$, $|\Omega| = 1$, is a sequence of functions such that

$$\lim_{k \to \infty} \int_{\Omega} \psi(f^k) \, dx = \int_{\mathbb{R}^N} \psi(\lambda) \, d\mu(\lambda), \quad \text{whenever } \psi \in \mathcal{E}^p. \quad (2.5)$$

Then $\mu$ is a $p$-Young measure. If the $f^k = \nabla u^k$ are gradients, then $\mu$ is an $H^{1,p}$-Young measure.

For completeness we give the corresponding $L^\infty$ result.

**PROPOSITION 2.3** Suppose that $\mu \in \mathcal{C}(\mathbb{R}^N)'$ is a probability measure and that $f^k \in L^\infty(\Omega;\mathbb{R}^N)$, $|\Omega| = 1$, is a sequence of functions such that

$$\lim_{k \to \infty} \int_{\Omega} \psi(f^k) \, dx = \int_{\mathbb{R}^N} \psi(\lambda) \, d\mu(\lambda), \quad \text{whenever } \psi \in \mathcal{C}(\mathbb{R}^N). \quad (2.6)$$

Then $\mu$ is a Young measure. If the $f^k = \nabla u^k$ are gradients, then $\mu$ is an $H^{1,\infty}$-Young measure, or simply, a gradient Young measure.

A consequence of (2.6) is that for each $\psi \in \mathcal{C}(\mathbb{R}^N)$,

$$\sup_k \left| \int_{\Omega} \psi(f^k) \, dx \right| < +\infty. \quad (2.7)$$

Suppose that $\lim_{k \to \infty} \sup |f^k| = +\infty$. We may suppose without loss in generality that
\[ E_k = \{ |f_k| \geq k \} \text{ has measure } |E_k| = \alpha_k > 0. \]

Let \( \varphi \in C(\mathbb{R}^+) \) be any monotone function satisfying
\[
\lim_{k \to \infty} \alpha_k \varphi(k) = \infty.
\]

Then
\[
\alpha_k \varphi(k) \leq \int_{\Omega} \varphi(|f_k|) \, dx \to \int_{\mathbb{R}^N} \varphi(|\lambda|) \, d\mu(\lambda),
\]
which violates (2.7). This argument also shows that \( \text{supp } \mu = K \) is compact.

Thus the sequence \((f^k)\) is bounded in \( L^\infty \) and generates a Young measure, which, it is easy to check, is \( \mu \).

3. STRUCTURE OF \( H^{1,p} \)-YOUNG MEASURES

Recall the convention that
\[
E^P = \{ \psi \in C(M) : \lim_{|A| \to 0} \frac{\psi(A)}{1 + |A|^p} \text{ exists} \}. \tag{3.1}
\]
The homogeneous \( H^{1,p} \)-Young measures are contained in \( E^P \).

Let \( \nu = (\nu_x)_{x \in \Omega} \) be an \( H^{1,p} \)-Young measure. Note that
\[
\langle \nu, \psi \otimes \zeta \rangle = \int_{\Omega} \int_{\mathbb{R}^N} \psi(A) \zeta(x) \, dv_x(A) \, dx, \quad \zeta \in L^1(\Omega), \quad \psi \in E^P. \tag{3.2}
\]
The average \( \bar{\nu} \) of \( \nu \) is also a measure. It is given by
\[
\langle \bar{\nu}, \psi \otimes \zeta \rangle = \frac{1}{|\Omega|} \int_{\Omega} \int_{\mathbb{R}^N} \psi(A) \, dv_x(A) \, dx \int_{\Omega} \zeta(x) \, dx, \quad \zeta \in L^1(\Omega), \quad \psi \in E^P. \tag{3.3}
\]

**THEOREM 3.1**

Assume that \( |\partial \Omega| = 0 \). Let \( \nu \) be an \( H^{1,p} \)-Young measure with underlying deformation \( y(x) \) satisfying \( y|_{\partial \Omega} = y_0 \), where \( y_0 \) is affine. Then \( \bar{\nu} \) defined by (3.3) is also an \( H^{1,p} \)-Young measure.

We sketch the proof of this basic fact. Let \((\nu_{u^k})\) generate \( \nu \). It is elementary to check that we may assume that \( u^k|_{\partial \Omega} = y_0 \) and that \( y_0(x) = F_0x, \quad F_0 \in M \). For each \( j \), the collection of sets
\[
\{ a + \varepsilon \tilde{A} : a \in \Omega, \varepsilon < 1/j \}
\]
forms a Vitali cover of \( \tilde{A} \) from which we may extract a countable (or finite) subcover \( \{ a_i + \varepsilon_i \tilde{A} : a_i \in \Omega, \varepsilon_i < 1/j \} \) of pairwise disjoint sets such that
\[
\tilde{A} = \bigcup (a_i + \varepsilon_i \tilde{A}) \cup N, \quad N \setminus A = 0. \tag{3.4}
\]
Assume that $IQI = 1$. Let $j = k$ in (3.4) and set

$$J^* - j \begin{cases} \int_{F_0}^x & \text{otherwise} \\ 1 & \end{cases} \quad \tag{3.5}$$

For $y \in \mathcal{P}$ and $e \in C(\mathcal{F}),$

$$\int_\Omega \psi(\nabla y) \zeta \, dx = \sum_{a_i + \varepsilon_i \Omega} \int \psi(\nabla y(x)) \zeta(x) \, dx$$

$$= \sum_{a_i + \varepsilon_i \Omega} \int \psi(\nabla u(x) (\frac{\zeta}{\varepsilon_i})) \zeta(x) \, dx$$

$$= \sum_{a_i + \varepsilon_i \Omega} \int \psi(\nabla u(x)) \zeta(x) \, dx$$

$$= \int_{\Omega} \psi(x) \, dx \int_{\mathcal{F}} \zeta(x) \, dx$$

Thus (2.5) is satisfied for the measure $\overline{\nu}$ and the sequence $(\nabla y)$ so the conclusion follows from PROPOSITION 2.2.

As a parenthetical remark, we note that the construction above produces any homogeneous Young measure as a "self-similar" structure, but that this is quite different and not equivalent to being a "laminar structure", Pedregal [73], Sverak [77].

For fixed $p$, $1 \leq p \leq \infty$, let $A_{f_{\nu}}$ denote the set of homogeneous $H^{\nu}P$ - Young measures with first moment $F$, i.e.,

$$A_{f_{\nu}}$$

PROPOSITION 3.2 $A_{f_{\nu}}$ is convex.
This is a consequence of the averaging theorem. As before, suppose that \( |\Omega| = 1 \).
Given \( \nu, \nu' \in M_F \) and \( \tau, 0 < \tau < 1 \), choose a subset \( \Omega' \subset \Omega \) with smooth boundary and \( |\Omega'| = \tau \). If \( (u^k) \) generate \( \nu \) and \( (u'^k) \) generate \( \nu' \), one may choose a sequence of cut-off functions \( \eta^j \) and a subsequence \( k_j \) of the \( k \) so that
\[
\nu^j = (1 - \eta^j) \nu + \eta^j \nu^j
\]
generate the \( H^{1,p} \)-Young measure \( \mu = (\mu_x)_{x \in \Omega} \) given by
\[
\mu_x = \begin{cases} 
\nu' & x \in \Omega' \\
\nu & x \in \Omega \setminus \Omega'.
\end{cases}
\]
Since the underlying deformation of \( \mu \) is \( y(x) = Fx \), which is affine, \( \overline{\mu} \) is again an \( H^{1,p} \)-Young measure. Inspection shows that
\[
\overline{\mu} = (1 - \tau) \nu + \tau \nu'.
\]
Note that for \( \nu \in M_F \) generated by \( (\nabla u^k) \), the formula
\[
\int_M \psi(A) \, d\nu = \lim_{k \to \infty} \int_\Omega \psi(\nabla u^k) \, dx, \quad \psi \in F_P,
\]
is a special case of the definition, e.g., (1.6). Hence the special \( H^{1,p} \)-Young measures
\[
\mu = \overline{\delta}_\nu, \quad \nu \in H^{1,p}(\Omega; \mathbb{R}^m), \quad u|_{\partial \Omega} = Fx, \quad \text{are dense in } M_F.
\]
That is, the averages of Dirac masses (with underlying deformation \( Fx \)) are a special class of \( H^{1,p} \)-Young measures dense in \( M_F \).

4. **THE HOMOGENEOUS CASE** We use the Hahn-Banach Theorem. Suppose that \( \mu \in E_P' \) is a probability measure for which
\[
\varphi(F) \leq \int_M \varphi(A) \, d\mu(A), \quad \text{where} \quad F = \int_M A \, d\mu(A),
\]
whenever \( \varphi \in E_P \) is quasiconvex.

Let \( T \) be a linear functional on \( E_P' \) in the weak* topology such that \( T \geq 0 \) on \( M_F \), a convex set. Then there is a \( \psi \in E_P \) such that
\[
0 \leq \langle T, \nu \rangle = \langle \nu, \psi \rangle = \int_M \psi(A) \, dv(A), \quad \nu \in M_F.
\]
In particular, (4.2) holds for \( v = \delta_{V^*} \), \( u \in H^1(P(\Omega; \mathbb{R}^m)) \), \( u|_{\partial \Omega} = Fx \), that is
\[
0 \leq \int_{\Omega} \psi(\nabla u) \, dx, \quad u \in H^1(P(\Omega; \mathbb{R}^m)), \quad u|_{\partial \Omega} = Fx. \quad (4.3)
\]

Let \( \psi^\# \) denote the quasiconvexification or relaxation of \( \psi \). Thus, assuming that \( |\Omega| = 1 \),
\[
\psi^\#(F) = \inf_A \int_{\Omega} \psi(\nabla u) \, dx,
\]
\[\quad A = \{ u \in H^1(P(\Omega; \mathbb{R}^m)) : u|_{\partial \Omega} = Fx \}.\]

Additional details about \( \psi^\# \) and its relationship to \( \psi \) may be found in Dacorogna [25] or [59].

By (4.3), \( \psi^\#(F) \geq 0 \). Note that \( \psi^\# \leq \psi \). Thus by (4.1),
\[
0 \leq \psi^\#(F) \leq \int_M \psi^\#(A) \, d\mu(A) \leq \int_M \psi(A) \, d\mu(A) = \langle T, \mu \rangle.
\]

Thus \( \mu \) cannot be separated from \( M_P \). It follows from the separability of \( F_P \) and the density of the averaged Dirac masses that there is a sequence \( u^k \in A \) such that
\[
\int_M \psi(A) \, d\mu(A) = \lim_{k \to \infty} \int_{\Omega} \psi(\nabla u^k) \, dx \quad \text{for any} \quad \psi \in F_P.
\]

By PROPOSITION 2.2, \( \mu \) is an \( H^{1,P} \) - Young measure.

As we remarked earlier, by truncation it suffices to assume (4.1) for quasiconvex functions \( \varphi \in E_P \) which are bounded below.

The general case of THEOREM 1.1 is proved from the homogeneous one by covering lemmas and approximation. This requires that
\[
\int \int_{\Omega_M} 1_A \, dv_A(A) \, dx < +\infty.
\]
For details, we refer to [57].

In the proof of THEOREM 1.2, it is necessary to retain the framework of the (inseparable) locally convex space \( C(M) \) and \( C(M) \), cf. [55]. An interesting part of the argument involves truncating a sequence \( u^k \in H^{1,\infty}(\Omega; \mathbb{R}^m) \) to a uniformly bounded one which generates the same Young measure. This is accomplished with the aid of a generalization of a lemma of Zhang [83], itself derived from [2],[63], cf. [55] Proposition 5.3:

**LEMMA 4.1** Let \( u \in C_0^0(\Omega; \mathbb{R}^m) \) and \( L > 0 \). Then there is a \( w \in H^{1,\infty}_0(\Omega; \mathbb{R}^m) \) such that
where \( C_1 \) and \( C_2 \) depend only on \( n \) and \( m \).

Note that \( \nabla w = \nabla u \) in \( \{ w = u \} \).

5. SOME APPLICATIONS

We give a few simple applications. Let \( \psi \in C(M) \) satisfy

\[
\max (|A|^p, 0) \leq \psi(A) \leq C(1 + |A|^p), \quad A \in M,
\]

where \( 1 < p < \infty \) and consider the functional

\[
\Psi(v) = \int_{\Omega} \psi(\nabla v) \, dx
\]

for \( v \in A = u_0 + H^{1,p}_0(\Omega; \mathbb{R}^m) \), with \( u_0 \) given.

THEOREM 5.1

If \( \psi \) satisfies (5.1), then the problem

\[
\min_A \Psi(v)
\]

admits a minimizing sequence \( u^k \in A \) such that \( (\nabla u^k)^p \) is weakly convergent in \( L^1(\Omega) \).

The proof is a direct application of THEOREM 1.1. An arbitrary minimizing sequence is bounded in \( H^{1,p}(\Omega; \mathbb{R}^m) \) and thus determines a biting Young measure on \( \mathcal{P} \). By the theorem, the biting Young measure is an \( H^{1,p} \) Young measure which is generated by a sequence obeying the conclusion of THEOREM 5.1. The conclusion of the theorem fails when \( p = 1 \), as is well known. Recent results about the relaxation of functionals with linear growth are due to Dal Maso [26] and Fonseca and Müller [47].

THEOREM 5.2

Let \( \varphi \) be quasiconvex satisfy

\[
0 \leq \varphi(A) \leq C(1 + |A|^p).
\]

If \( u^k \rightharpoonup u \) in \( H^{1,p}(\Omega; \mathbb{R}^m) \) and

\[
\int_{\Omega} \varphi(\nabla u^k) \, dx \rightharpoonup \int_{\Omega} \varphi(\nabla u) \, dx \quad \text{as} \quad k \to \infty
\]
then

\[ 9(Vu_k) \Rightarrow <p(Vu) \text{ in } L^*(G). \]

Let \( v = (v_x)_x \in \mathbb{N} \) denote the biting Young measure generated by \( (Vu_k) \). Then there is another sequence \( (Vw_k) \) bounded in \( L^P(Q) \) which generates \( v = (v_x)_x \in \mathbb{N} \) as an \( H^1 \)-Young measure. Let

\[ \overline{J}(x) = \int<p(A)d\nu_x(A). \]

By Jensen's Inequality, \( <KVu) \leq \overline{J} \). Let \( (E_j) \) denote the bitten sets from the Biting Lemma. Then

\[ \int<p(Vu_k)dx \leq \int\overline{J}(Vu_k)dx \rightarrow \int \overline{J} dx. \]

Hence, letting \( k \rightarrow 0^+ \) and then \( \Rightarrow \rightarrow \), we see that

\[ \int cp(Vu)dx \leq \int \overline{J} dx. \]

From Jensen's Inequality quoted above, we deduce that \( <p(Vu) = \overline{J} \). Now we have that

\[ <p(Vu_k) \leq <p(Vu) \text{ in } L^*(Q) \text{ and } \]

\[ \limsup_{k \rightarrow \infty} \int q(Vu_k)dx = \lim_{k \rightarrow \infty} \int 9(Vu_k)dx = \int <p(Vu)dx. \]

The conclusion now follows from Proposition 2.1. A direct proof of this was given in [59].

As a special case of this theorem, we give another proof of Mailer's observation about weak continuity of \( \det Vu \) in the limit case where \( u \in H_{t_0}^n(Q;R^n) \), \( Q \subset R^n \), cf. Coifman, Lions, Meyer, and Semmes [20], Iwaniec and Sbordone [48], and Müller himself [69,70].

**Corollary 5.3** Let \( u, u_k \in H^1(Q;R^n) \) satisfy

\[ 1^* \hookrightarrow u \text{ in } H^1(Q;R^n) \text{ and } \]

\[ \det Vu_k \rightarrow 0 \text{ in } C^L \]

Then

\[ \det V \rightarrow \det Vu \text{ in } L^*(Q). \]
To verify this, first note that since $u^k \to u$ in $H^{1,n}(\Omega;\mathbb{R}^n)$, it is elementary to verify by integration by parts that

$$\det V u^k \to \det V u \text{ in } L^1_{bc}(\Omega).$$

In particular

$$\int_\Omega \det V u \zeta \, dx \geq 0 \text{ for } 0 \leq \zeta \in C_0^\infty(\Omega),$$

so $\det V u \geq 0$ in $\Omega$.

Let $\varphi(A) = \max(\det A, 0)$, a quasiconvex function satisfying

$$0 \leq \varphi(A) \leq C(1 + |A|^p), \quad A \in M.$$

From the above, $\varphi(V u^k) = \det V u^k$ and $\varphi(V u) = \det V u$. For any smooth subdomain $\Omega' \subset \Omega$, choose a Lipschitz cut-off function $\eta$ with $\eta = 1$ in $\Omega'$. Then

$$\limsup_{k \to \infty} \int_{\Omega} \det V u^k \, dx \leq \lim_{k \to \infty} \int_{\Omega} \eta \det V u^k \, dx$$

$$= \int_{\Omega} \eta \det V u \, dx$$

$$= \int_{\Omega'} \det V u \, dx + \int_{\Omega \setminus \Omega'} \eta \det V u \, dx$$

(5.3)

Choose a sequence $(\eta^j)$ with $\eta^j \to 0$ pointwise in $\Omega \setminus \Omega'$. By the dominated convergence theorem the second integral in (5.3) tends to zero, so we obtain that

$$\limsup_{k \to \infty} \int_{\Omega} \det V u^k \, dx \leq \int_{\Omega} \det V u \, dx$$

The conclusion follows from THEOREM 5.2.

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Addresses of the authors:

D.K. Department of Mathematics and Center for Nonlinear Analysis Carnegie Mellon University Pittsburgh, PA 15213 - 3890, USA

P.P. Departamento de Matemática Aplicada Universidad Complutense de Madrid 28040 Madrid, Spain