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Remarks about Gradient Young Measures Generated by Sequences in Soblev Spaces

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CONTENTS

1. INTRODUCTION The oscillatory properties of a sequence of weak* convergent functions may be summarized by the parametrized measure, or Young measure, which it generates, Young [81][1,2]. The parametrized measure generated by a sequence $f^k \in L^\infty(\Omega;\mathbb{R}^N)$ with

$$f^k \to f \text{ in } L^\infty(\Omega;\mathbb{R}^N) \text{ weak*}$$

is a family of probability measures $\nu = (\nu_x)_{x \in \Omega}$ with $\operatorname{supp} \nu_x \subset \mathbb{R}^N$ such that, for every function ψ continuous in λ and measurable in x,

$$\psi(f^k,x) \to \overline{\psi}(x) = \int_{\mathbb{R}^N} \psi(\lambda,x)\, d\nu_x(\lambda) \quad \text{in } L^\infty(\Omega) \text{ weak*}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded. Restricting, if necessary, to a subsequence, every weak* convergent sequence generates a parametrized measure. Thus the parametrized measure describes the weak limit of any continuous function composed with the sequence, which leads to its use in problems where sequences converge weakly but not strongly. In this framework, every family $\nu = (\nu_x)_{x \in \Omega}$ of probability measures with uniformly bounded support is a parametrized measure.

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[2] To appear in Proc. Nonlinear Diff. Eqns. and their Appl., College de France Sem, Brezis, H. and Lions, J.-L., eds.

Variational principles used to study equilibrium configurations of crystalline solids or other materials with order are not lower semicontinuous. Our interest in the foundations of this theory begin with the work of Ericksen, cf. e.g. [27-38]. In these circumstances, the infimum of energy is attained only in some generalized sense while a minimizing sequence may develop finer and finer oscillations, reminiscent of a finely twinned microstructure. The weak limit of the sequence may be inadequate to describe many properties of the configuration, but the parametrized measure has proven useful both to determine macroscopic properties such as energy and stress and microstructural properties such as variant arrangement and location, cf. Ball and James [5,6], Chipot and Kinderlehrer [18], Chipot, Kinderlehrer, and Vergara Caffarelli [19], Fonseca [42-45], James [49], James and Kinderlehrer [50], Kinderlehrer [54], Pedregal [72,73], Matos [65,66], and Battacharya [11,12]. It has also led to computational developments, Collins and Luskin [21-23], Chipot [16], Chipot and Collins [17], Collins, Kinderlehrer and Luskin [24], Luskin and Ma [64], and Nicolaides and Walkington [71]. A recent accounting of some of these and related developments may be found in [39]. In addition, the analysis we discuss here has close connections with the work of Ball and Murat [8,9], Ball and Zhang [10], Brandon and Rogers [14], Firooze and Kohn [41], Kohn [62], James and Kinderlehrer [51-53], Sverak [75-77], and Zhang [82-84].

The use of the Young measure to study possible oscillations of solutions of partial differential equations was initiated by Tartar [78-80].

The two immediate difficulties which arise in applying the ideas leading to (1.1) are

- the variational constraint that $f^k = \nabla u^k$, and
- the (f^k) are not generally bounded in $L^\infty(\Omega,R^N)$, but instead in $L^p(\Omega,R^N)$, for some $p \in [1,\infty)$.

In this note, we would like to discuss our recent thoughts about these questions. Details appear in [57]. Our objective is to characterize the gradient Young measures which arise from sequences bounded in $H^{1,p}(\Omega,R^m)$, $1 \le p \le \infty$. We found subtle differences between the cases $p < \infty$ and $p = \infty$, and since we have reported on the latter case in [55],[56],[58], we concentrate here on the case where p is finite. In the remainder of this section we shall introduce the notions we intend to study and state our prinicpal result, THEOREM 1.1.

As our starting point we recall a more general framework for the study of oscillations described by Ball [3] and also studied by Matos [65]. We retain the convention that $\Omega \subset R^n$ is bounded (and measurable.) Suppose that $f^k \in L^p(\Omega;R^N)$, for some $p \in [1,\infty)$, and

$$\int_\Omega |f^k|^p \, dx \le M. \tag{1.2}$$

Then there is a family $v = (v_x)_{x \in \Omega}$ of probability measures on R^N and a subsequence of the (f^k), not relabelled, such that whenever

$$\psi(f^k) \rightharpoonup \bar{V} \quad \text{in } iMfk), \quad \text{for } y \in C(R^N),$$

then

$$\bar{V}(x) \; m \; \int_{R^N} y(X) dv_x(X) \quad \text{in } Q \text{ a.c.} \tag{1.3}$$

For example, it is obvious from Holder's inequality and the Dunford-Pettis criterion that given a sequence satisfying (1.2), the conclusion (13) is satisfied whenever

$$|y(X)| \leq C(1 + |X|^{W}), \quad J1 \in RN, \tag{1.4}$$

whenever $q < p$. However, it is also obvious from the viewpoint of applications that we wish to have some interpretation of (1.3) when $p = q$. The subsequence (fr) determines a unique parametrized measure. The problem is to decide when it identifies weak limits.

Since (1.3) does not hold for arbitrary sequences bounded in $D\backslash$ we must either impose an additional condition on the sequence or restrict the notion of Young measure as a characterization of oscillatory behavior. What docs this entail? To begin, we shall neglect the gradient constraint and then reinstate it later.

For convenience we set

$$\mathcal{E}P \; « \; \{y \in CXMV.\lim_{t \to \infty} \int_{A} |y^t|^p \text{ exists}\}. \tag{1.5}$$

$\mathcal{E}P$ is isomorphic to the continuous functions on the one point compactification of M and is separable. For technical reasons, this has an advantage over the inseparable space of functions suggested by (1.4) of the same growth rate

$$X = \{y \in C(R^N): |y(X)| \leq CXI + |\lambda|^p), \quad \lambda \in R^N\}$$

and will incur no loss in generality in our considerations.

A particular circumstance leading to the validity of (1.3) for all y satisfying (1.4) in the case $p = q$ occurs when $I f^*$ IP themselves converge weakly in LKQ). This follows by application of the Dunford-Pettis criterion and leads us to the notion of p-Young measure:

> A family $v = (v_x)_{x \in \Omega}$ is* p-Young measure or p-parametrized measure provided there is a sequence $\mathcal{E} \in LP(Q;R^N)>$ for some $p \in [1,\infty)_f$ anda $g \in LHfi)$ such that

$|f^k|^p \rightharpoonup g$ in $L^1(\Omega)$ and

$\psi(f^k) \rightharpoonup \bar{\psi}$ in $L^1(\Omega)$, where

$$\bar{\psi}(x) = \int_{R^N} \psi(\lambda)\, d\nu_x(\lambda) \quad \text{in } \Omega \text{ a.e., for } \psi \in E^p.$$

Equivalently, we may say that for any $E \subset \Omega$,

$$\lim_{k\to\infty} \int_E \psi(f^k)\, dx = \int_E \int_{R^N} \psi(\lambda)\, d\nu_x(\lambda) dx \quad \text{for } \psi \in E^p. \tag{1.6}$$

Alternatively, we may define a biting Young measure for an arbitrary sequence. Recall that if $g^k \in L^1(\Omega)$ and

$$\int_\Omega |g^k|\, dx \le M < \infty,$$

then there is a sequence $E_j \subset \Omega$ with $E_{j+1} \subset E_j$, $|E_j| \to 0$, and a $g \in L^1(\Omega)$ such that for a subsequence of the (g^k), not relabelled,

$$g^k \rightharpoonup g \quad \text{in } L^1(\Omega \setminus E_j) \text{ for each } j.$$

This is the conclusion of the Chacon biting lemma [15], cf. also Ball and Murat [8]. We write that

$$g^k \overset{b}{\rightharpoonup} g \quad \text{in } L^1(\Omega) \tag{1.7}$$

and say that g^k converge to g in the biting sense. Clearly if g^k converge in the biting sense and $|\psi(\lambda)| \le C|\lambda|$, then $\psi(g^k)$ also converge in the biting sense, again by the Dunford-Pettis criterion. Thus we are led to the notion of biting Young measure:

A family $\nu = (\nu_x)_{x\in\Omega}$ is a biting Young measure provided there is a sequence $f^k \in LP(\Omega;R^N)$ and $g \in L^1(\Omega)$ such that

$|f^k|^p \overset{b}{\rightharpoonup} g$ in $L^1(\Omega)$ and

$\psi(f^k) \overset{b}{\rightharpoonup} \bar{\psi}$ in $L^1(\Omega)$ where

$$\bar{\psi}(x) = \int_{R^N} \psi(\lambda)\, d\nu_x(\lambda) \quad \text{in } \Omega \text{ a.e., for } \psi \in E^p. \tag{1.8}$$

The bitten sets (E_j) depend only on the sequence (f^k) and not on the particular ψ. A p-Young measure is a biting Young measure. Furthermore, evident from the property of weak convergence in L^1, the Young measure determines the biting limit of a sequence and not its distributional limit.

Suppose that μ is a homogeneous biting Young measure, i.e., μ_x is independent of $x \in \Omega$. We may regard μ as an element of $E^{p\prime}$, the dual of E^p, with

$$\|\mu\| = \int_\Omega (1 + |\lambda|^p)\, d\mu(\lambda) = \langle \mu, (1 + |\cdot|^p) \rangle . \tag{1.9}$$

Even though μ is a probability measure, it is not necessarily in the unit ball of $E^{p\prime}$. Indeed, consider the simple example with $\Omega = (0,1)$

$$f^k = k^{1/2} \chi_{(0,1/k)} .$$

These f^k have the properties that

$f^k \to 0$ in $L^2((0,1))$ weakly and in the biting sense and

$|f^k|^2$ do not converge weakly in $L^1((0,1))$,

$|f^k|^2 \overset{b}{\to} 0$.

For any $\psi \in E^2$,

$$\int_0^1 \psi(f^k)\, dx = \tfrac{1}{k} \psi(k^{1/2}) + (1 - \tfrac{1}{k})\, \psi(0).$$

The limit of these functionals defines an operation which is not a probability measure, but the functional

$$\langle T, \psi \rangle = \psi(0) + \lim_{\lambda \to \infty} \psi(\lambda)/\lambda^2 . \tag{1.10}$$

However the biting limit determines the probability measure δ_0 which *is* generated by some sequence of oscillations in $L^2((0,1))$. Each function f^k determines the parametrized measure δ_{f^k} given by

$$\langle \delta_{f^k}, \psi \rangle = \int_0^1 \psi(f^k)\, dx$$

In view of (1.10), this shows that the Young measures are not closed in $E^{2\prime}$. It is easy to see that they also are not bounded in the norm (1.9). Conditions which ensure the convergence of a sequence of measures in $E^{p\prime}$ to another probability measure are analogous to tightness conditions, cf. Billingsley [13].

Let us now impose the constraint that the functions f^k which generate the measures are gradients. We agree to call the associated measures $H^{1,p}$ - Young measures and $H^{1,p}$ - biting Young measures, respectively.

Recall that a continuous function $\varphi(A)$, where $A \in \mathbb{M}$, the $m \times n$ matrices, is quasiconvex provided

$$\varphi(F) \leq \frac{1}{|\Omega|} \int_\Omega \varphi(F + \nabla\zeta)\, dx \quad \text{for all} \quad \zeta \in C_0^\infty(\Omega;\mathbb{R}^m) \text{ and } F \in \mathbb{M}. \tag{1.11}$$

A result of Acerbi and Fusco [1], which generalizes the theorem of Morrey [67], is that if $\varphi \in E^p$ is quasiconvex and bounded from below, then

$$\int_\Omega \varphi(\nabla u)\, dx \leq \liminf_{k\to\infty} \int_\Omega \varphi(\nabla u^k)\, dx \tag{1.12}$$

whenever

$$u^k \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^m).$$

The proof of (1.12) in this case is a direct generalization of Morrey's and does not require the machinery developed by Acerbi and Fusco, cf. Evans [40], Dacorogna [25]. In fact, in the case of linear growth of $\varphi(A)$, the lower semicontinuity (1.12) remains true even when we assume only that

$$u^k, u \in W^{1,1}(\Omega;\mathbb{R}^m) \text{ and } u^k \to u \text{ in } \mathcal{D}'(\Omega).$$

Deep generalizations of Morrey's Theorem relevant to this situation have recently been proved by Fonseca and Müller [46,47], to whom we refer for additional references.

Assuming Ω given, (1.12) also holds for smooth subdomains $\Omega' \subset \Omega$, in particular for $\Omega' = B_r(a) \subset \Omega$. Suppose that (∇u^k) generate an $W^{1,p}$ - Young measure ν. Then (1.12) implies that

$$\int_{B_r(a)} \varphi(\nabla u)\, dx \leq \int_{B_r(a)} \bar{\varphi}\, dx \quad \text{for} \quad \bar{\varphi}(x) = \int_{\mathbb{M}} \varphi(A)\, d\nu_x(A),$$

and thus by Lebesgue's Theorem, $\varphi(\nabla u(a)) \leq \bar{\varphi}(a)$ in Ω. Expressed differently,

$$\varphi(\nabla u(x)) \leq \int_{\mathbb{M}} \varphi(A)\, d\nu_x(A), \quad \text{where} \quad \nabla u(x) = \int_{\mathbb{M}} A\, d\nu_x(A), \tag{1.13}$$

whenever $\varphi \in E^p$ is quasiconvex and bounded below. So if ν is an $W^{1,p}$ - Young measure, Jensen's Inequality holds for quasiconvex $\varphi \in E^p$ and bounded below.

Another consequence of (1.12) concerns the Jensen Inequality for biting Young measures and is not elementary. We know that if $\varphi \in E^p$ is quasiconvex and $\varphi \geq 0$, then

$$\int_E \varphi(\nabla u)\, dx \leq \liminf_{k\to\infty} \int_E \varphi(\nabla u^k)\, dx, \quad \text{for} \quad E \subset \Omega, \tag{1.14}$$

whenever u* -* u in H^PtffcR[111]). This result does require a substantial part of the Acertri - Fusco apparatus and unfortunately we have been unable to find a simpler proof than, say, [59]. Let v = $(V_X)X_{\epsilon Q}$ denote the biting Young measure generated by (Vu^k) and $\bar{9}$ the biting limit given by (1.8). Whenever E c Q\Ej, we than have from (1.12) that

$$\int_E <p(Vu)dx \; £ \; \int_E <\bar{p}\, dx.$$

As limlEjl = 0, we find again that

$$<p(Vu) \; £ \; \bar{q}> \text{ in } Q \text{ a.e.}$$

By adding a constant to <p, this is seen to hold for quasiconvex functions bounded below and by truncation it is seen to hold for arbitrary quasiconvex functions with growth of order p. This has been discussed by Ball and Zhang [10] as well. Otherwise stated, for an H^1* - biting Young measure v = $(v_x)x_{\epsilon}$ Q, whenever <pe £P is quasiconvex,

$$<KVu(x)) \; \leq \; \int_M fq>(A)dv_x(A), \quad \text{where} \quad Vu(x) = \int_M U\, dv_x(A). \qquad (1.15)$$

Thus Jensen's Inequality for quasiconvex functions of suitable growth holds for biting Young measures as well as H^1•P - Young measures.

Our principle result is that (1.15) characterizes the Hi* - Young measures. Thus the $H^{1,p}$ - Young measures are the same as the H^P - biting Young measures and are the same as the measures which satisfy (1.15). Of course, the sequence that generates the measure as an H^P - Young measure cannot usually be the same as an arbitrary one that generates it as a biting measure.

THEOREM 1.1 *Let* v = $(v_a)a_{\epsilon}a$ *be afamily cfprobability measures in* C(M)\ *Then* v « $(v_x)x$ € Q *is an* H^P - *gradient Young measure ifand only if*

i) *there is a* ue H^P^R"[1]) *such that*

$$Vu(x) = \int_M jA dv_x(A) \quad \text{inQa.e.},$$

ii) *Jensen's Inequality* (1.13) *holds for all* <pe £P *quasiconvex and bounded below, and*

iii) *thefunction*

$$\Psi(x) = \int_{\Omega} |A|^p dv_x(A) \; € \; LUO).$$

The last cemdition is necessary. For example, in the case n « m = l,let ge C(0,l) be nonnegativc with

$$\int_0^1 g\,dx = \infty \quad \text{and} \quad \nu_x = \tfrac{1}{2}(\delta_{g(x)} + \delta_{-g(x)}).$$

Then ν satisfies i) and ii) but is not a 1 - Young measure. Analogously in the $p = \infty$ case, we were obliged to assume that $\cup \operatorname{supp} \nu_x$ was bounded. It suffices in ii) to require the Jensen inequality for the more restricted set of quasiconvex in E^p which are bounded below.

For completeness and comparison, we give the $H^{1,\infty}$ result as well.

THEOREM 1.2 *Let* $\nu = (\nu_a)_{a \in \Omega}$ *be a family of probability measures in* $C(M)'$. *Then* $\nu = (\nu_a)_{a \in \Omega}$ *is an* $H^{1,\infty}$ *- gradient Young measure, or simply, a gradient Young measure if and only if*

i) *there is a* $u \in H^{1,\infty}(\Omega;R^m)$ *such that*

$$\nabla u(x) = \int_M A\, d\nu_x(A) \quad \text{in } \Omega \text{ a.e.},$$

ii) *Jensen's Inequality* (1.13) *holds for all quasiconvex* $\varphi \in C(M)$ *and*

iii) $\operatorname{supp} \nu_a \subset K$, $a \in \Omega$ *a.e., where* K *is a fixed compact.*

2. WEAK CONVERGENCE AND BITING CONVERGENCE There is a straight forward way to understand the relationshiup between weak convergence and biting convergence, which, in fact, has been used implicitly by us [59] and by Ball and Zhang [10].

PROPOSITION 2.1 *Let* $g^k \in L^1(\Omega)$, $g^k \geq 0$, *with the property that*

$$g^k \overset{b}{\rightharpoonup} g \quad \textit{in } L^1(\Omega).$$

A subsequence $(g^{k'})$ *of the* (g^k) *satisfies*

$$g^{k'} \rightharpoonup g \quad \textit{in } L^1(\Omega)$$

if and only if

$$\liminf_{k \to \infty} \int_\Omega g^k\,dx \leq \int_\Omega g\,dx. \tag{2.1}$$

Moreover, (g^k) *is weakly relatively compact in* $L^1(\Omega)$ *if and only if*

$$\limsup_{k \to \infty} \int_\Omega g^k\,dx \leq \int_\Omega g\,dx. \tag{2.2}$$

To prove the proposition, assume biting convergence and the failure of the Dunford-Pettis criterion. This leads to the failure of (2.1).

Consider a biting Young measure $\nu = (\nu_x)_{x \in \Omega}$ generated by $f^k \in L^p(\Omega;\mathbb{R}^N)$. Suppose that $\varphi \in E^p$ majorizes $1 + |\lambda|^p$, i.e., $1 + |\lambda|^p \leq \varphi(\lambda)$, and

$$\lim_{k \to \infty} \int_\Omega \varphi(f^k)\, dx = \int_\Omega \int_{\mathbb{R}^N} \varphi(\lambda)\, d\nu_x(\lambda) dx \tag{2.3}$$

The proposition then implies that

$$\varphi(f^k) \rightharpoonup \int_{\mathbb{R}^N} \varphi(\lambda)\, d\nu_x(\lambda) \quad \text{in } L^1(\Omega), \tag{2.4}$$

whence, by application of the Dunford-Pettis criterion again, $\nu = (\nu_x)_{x \in \Omega}$ is a p-Young measure.

Here is the situation in which we shall apply (2.3), (2.4). Suppose that $\mu \in E^{p'}$ is a probability measure and that $f^k \in L^p(\Omega)$, $|\Omega| = 1$, is a sequence of functions such that

$$\lim_{k \to \infty} \int_\Omega \psi(f^k)\, dx = \int_{\mathbb{R}^N} \psi(\lambda)\, d\mu(\lambda), \qquad \text{whenever } \psi \in E^p. \tag{2.5}$$

Since we may evaluate (2.5) on $\varphi_o(\lambda) = 1 + |\lambda|^p$, the sequence (f^k) is bounded in L^p and generates a biting Young measure ν, which we may assume after rescaling to be homogeneous, cf. THEOREM 3.1 in the sequel. We claim that $\nu = \mu$. This will tell us that μ is a p-Young measure by (2.3), (2.4). We interpret the left hand side of (2.5) by observing that

$$\langle \mu^k, \psi \rangle = \int_\Omega \psi(f^k)\, dx$$

is just the average of the parametrized measure which is the Dirac mass at f^k,

$$\langle \delta_{f^k(x)}, \psi \rangle = \psi(f^k(x)).$$

Thus a probability measure in the closure of the averages of Dirac masses of L^p functions is a p-Young measure which is generated by (f^k).

To show that $\mu = \nu$, suppose first that $\psi \geq 0$. By the Lebesgue Theorem and the Monotone Convergence Theorem,

$$\int_{\mathbb{R}^N} \psi(\lambda)\, d\mu(\lambda) = \lim_{\alpha \to 1-} \Big\{ \int_{\{\psi \leq 1\}} \psi^\alpha(\lambda)\, d\mu(\lambda) + \int_{\{\psi > 1\}} \psi^\alpha(\lambda)\, d\mu(\lambda) \Big\}$$

$$= \lim_{\alpha \to 1-} \int_{\mathbb{R}^N} \psi^\alpha(\lambda)\, d\mu(\lambda)$$

$$= \lim_{\alpha \to 1-} \lim_{k \to \infty} \int_\Omega \psi^\alpha(f^k)\, dx \ .$$

For $\alpha < 1$, $\psi^\alpha \in E^q \subset E^p$, $q = p\alpha$. The sequence $(\psi^\alpha(f^k))$ satisfies the Dunford-Pettis criterion and hence is weakly (pre)compact in $L^1(\Omega)$. Thus by (1.3),

$$\lim_{k \to \infty} \int_\Omega \psi^\alpha(f^k)\, dx \ = \ \int_{\mathbb{R}^N} \psi^\alpha(\lambda)\, d\nu(\lambda) \ .$$

Again by the Lebesgue Theorem and the Monotone Convergence Theorem,

$$\lim_{\alpha \to 1-} \int_{\mathbb{R}^N} \psi^\alpha(\lambda)\, d\nu(\lambda) \ = \ \int_{\mathbb{R}^N} \psi(\lambda)\, d\nu(\lambda) \ .$$

Thus $\mu = \nu$ on nonnegative ψ. By decomposing an arbitrary ψ into its positive and negative parts, we deduce that $\mu = \nu$.

An important observation is that if the sequence $f^k = \nabla u^k$ above is a sequence of gradients, then μ is an $H^{1,p}$- Young measure. We have shown

PROPOSITION 2.2 *Suppose that $\mu \in E^{p'}$ is a probability measure and that $f^k \in L^p(\Omega;\mathbb{R}^N)$, $|\Omega| = 1$, is a sequence of functions such that*

$$\lim_{k \to \infty} \int_\Omega \psi(f^k)\, dx \ = \ \int_{\mathbb{R}^N} \psi(\lambda)\, d\mu(\lambda), \qquad \textit{whenever } \psi \in E^p. \quad (2.5)$$

Then μ is a p-Young measure. If the $f^k = \nabla u^k$ are gradients, then μ is an $H^{1,p}$ - Young measure.

For completeness we give the corresponding L^∞ result.

PROPOSITION 2.3 *Suppose that $\mu \in C(\mathbb{R}^N)'$ is a probability measure and that $f^k \in L^\infty(\Omega;\mathbb{R}^N)$, $|\Omega| = 1$, is a sequence of functions such that*

$$\lim_{k \to \infty} \int_\Omega \psi(f^k)\, dx \ = \ \int_{\mathbb{R}^N} \psi(\lambda)\, d\mu(\lambda), \qquad \textit{whenever } \psi \in C(\mathbb{R}^N). \quad (2.6)$$

Then μ is a Young measure. If the $f^k = \nabla u^k$ are gradients, then μ is an $H^{1,\infty}$ - Young measure, or simply, a gradient Young measure.

A consequence of (2.6) is that for each $\psi \in C(\mathbb{R}^N)$,

$$\sup_k \ \left| \int_\Omega \psi(f^k)\, dx \right| \ < \ +\infty. \quad (2.7)$$

Suppose that $\lim_{k \to \infty} \sup |f^k| = +\infty$. We may suppose without loss in generality that

$$E_k = \{ |f^k| \geq k \} \quad \text{has measure} \quad |E_k| = \alpha_k > 0.$$

Let $\varphi \in C(R^+)$ be any monotone function satisfying

$$\lim_{k \to \infty} \alpha_k \varphi(k) = \infty.$$

Then

$$\alpha_k \varphi(k) \leq \int_\Omega \varphi(|f^k|)\, dx \to \int_{R^N} \varphi(|\lambda|)\, d\mu(\lambda),$$

which violates (2.7). This argument also shows that $\operatorname{supp} \mu = K$ is compact.

Thus the sequence (f^k) is bounded in L^∞ and generates a Young measure, which, it is easy to check, is μ.

3. STRUCTURE OF $H^{1,p}$ - YOUNG MEASURES

Recall the convention that

$$E^p = \{ \psi \in C(M): \lim_{|A| \to \infty} \frac{\psi(A)}{1 + |A|^p} \text{ exists} \}. \tag{3.1}$$

The homogeneous $H^{1,p}$ - Young measures are contained in E^p.

Let $\nu = (\nu_x)_{x \in \Omega}$ be an $H^{1,p}$ - Young measure. Note that

$$\langle \nu, \psi \otimes \zeta \rangle = \int_\Omega \int_{R^N} \psi(A)\, \zeta(x)\, d\nu_x(A) dx \ , \ \zeta \in L^1(\Omega), \ \psi \in E^p. \tag{3.2}$$

The average $\bar{\nu}$ of ν is also a measure. It is given by

$$\langle \bar{\nu}, \psi \otimes \zeta \rangle = \frac{1}{|\Omega|} \int_\Omega \int_{R^N} \psi(A)\, d\nu_x(A) dx \int_\Omega \zeta(x) dx \ , \ \zeta \in L^1(\Omega), \ \psi \in E^p. \tag{3.3}$$

THEOREM 3.1 *Assume that* $|\partial\Omega| = 0$. *Let* ν *be an* $H^{1,p}$ - *Young measure with underlying deformation* $y(x)$ *satisfying* $y|_{\partial\Omega} = y_o$, *where* y_o *is affine. Then* $\bar{\nu}$ *defined by* (3.3) *is also an* $H^{1,p}$ - *Young measure.*

We sketch the proof of this basic fact. Let (∇u^k) generate ν. It is elementary to check that we may assume that $u^k|_{\partial\Omega} = y_o$ and that $y_o(x) = F_o x$, $F_o \in M$. For each j, the collection of sets $\{ a + \varepsilon\bar{\Omega}: a \in \Omega, \varepsilon < 1/j \}$ forms a Vitali cover of $\bar{\Omega}$ from which we may extract a countable (or finite) subcover $\{ a_i + \varepsilon_i\bar{\Omega}: a_i \in \Omega, \varepsilon_i < 1/j \}$ of pairwise disjoint sets such that

$$\bar{\Omega} = \cup (a_i + \varepsilon_i\bar{\Omega}) \cup N, \ |N| = 0. \tag{3.4}$$

Assume that IQI = 1. Let j = k in (3.4) and set

$$J^{*}\text{»} - \underset{I}{j} \text{«}\bullet^{**}?\text{»} \bullet \text{«}^{*} \quad \underset{F_0x}{\text{"«}^{**}\text{>}} \quad \underset{\text{otherwise}}{} \tag{3.5}$$

For y e £P and ∈ (C(fr),

$$\int_{\Omega} \psi(\nabla y^k)\,\zeta\,dx = \sum \int_{a_i + \varepsilon_i\Omega} \psi(\nabla y^k(x))\,\zeta(x)\,dx$$

$$= \sum \int_{a_i + \varepsilon_i\Omega} \psi\left(\nabla u^k\left(\frac{x - a_i}{\varepsilon_i}\right)\right)\zeta(x)\,dx$$

$$= \sum \varepsilon_i^n \int_{\Omega} V(Vu^*(x))\,C(a_i + e_i x)\,dx$$

$$= \int_{\Omega} V(Vu^k(x))\,dx \times e?\,C(i_i + \varepsilon_i x),$$

for a choice of fii e ai + EiQ. Since (Vu^k) generate the H^1^ - Young measure v and the second term is a Riemann sum for the integral of £.

$$\lim_{k\to\infty} \int_{\Omega} \psi(\nabla y^k)\,\zeta\,dx = \lim_{k\to} \int_{\Omega} fv(Vu^k(x))\,dx \sum \varepsilon_i^n\,\zeta(\bar{a}_i + \varepsilon_i x)$$

$$= \int_{\Omega} \bar{\psi}\,dx \int_{\Omega} \zeta\,dx$$

$$= \underline{J}v(A)d\tilde{v} \int_{\Omega} U dx.$$

Thus (2.5) is satisfied for the measure $\bar{v}$ and the sequence (Vy^k) so the conclusion follows from PROPOSITION 2.2.

As a parenthetical remark, we note that the construction above produces any homogeneous Young measure as a "self-similar" structure, but that this is quite different and not equivalent to being a "laminar structure", Pedregal [73], Sverak [77].

For fixed p, $1 \leq p < \infty$, let Afp denote the set of homogeneous $H^{*'P}$ - Young measures with first moment F, i.e.,

$$A\ dv\ .$$

PROPOSITION 3.2 *A#F Is convex.*

This is a consequence of the averaging theorem. As before, suppose that $|\Omega| = 1$. Given $\nu, \nu' \in M_F$ and τ, $0 < \tau < 1$, choose a subset $\Omega' \subset \Omega$ with smooth boundary and $|\Omega'| = \tau$. If (u^k) generate ν and (u'^k) generate ν', one may choose a sequence of cut-off functions η^j and a subsequence k_j of the k so that

$$w^j = (1-\eta^j)\, u^{k_j} + \eta^j\, u'^{k_j}$$

generate the $H^{1,p}$ - Young measure $\mu = (\mu_x)_{x \in \Omega}$ given by

$$\mu_x = \begin{cases} \nu' & x \in \Omega' \\ \nu & x \in \Omega \setminus \Omega' \end{cases} .$$

Since the underlying deformation of μ is $y(x) = Fx$, which is affine, $\overline{\mu}$ is again an $H^{1,p}$ - Young measure. Inspection shows that

$$\overline{\mu} = (1-\tau)\nu + \tau\nu' .$$

Note that for $\nu \in M_F$ generated by (∇u^k), the formula

$$\int_M \psi(A)\, d\nu = \lim_{k\to\infty} \int_\Omega \psi(\nabla u^k)\, dx , \quad \psi \in F^p,$$

is a special case of the definition, eg., (1.6). Hence the special $H^{1,p}$ – Young measures $\mu = \overline{\delta}_{\nabla u}$, $u \in H^{1,p}(\Omega;\mathbb{R}^m)$, $u|_{\partial\Omega} = Fx$, are dense in M_F. That is, the averages of Dirac masses (with underlying deformation Fx) are a special class of $H^{1,p}$ - Young measures dense in M_F.

4. THE HOMOGENEOUS CASE We use the Hahn-Banach Theorem. Suppose that $\mu \in E^{p'}$ is a probability measure for which

$$\varphi(F) \le \int_M \varphi(A)\, d\mu(A), \quad \text{where} \quad F = \int_M A\, d\mu(A), \tag{4.1}$$

whenever $\varphi \in E^p$ is quasiconvex.

Let T be a linear functional on $E^{p'}$ in the weak* topology such that $T \ge 0$ on M_F, a convex set. Then there is a $\psi \in E^p$ such that

$$0 \le \langle T, \nu \rangle = \langle \nu, \psi \rangle = \int_M \psi(A)\, d\nu(A), \quad \nu \in M_F. \tag{4.2}$$

In particular, (4.2) holds for $\nu = \overline{\delta}_{\nabla u}$, $u \in H^{1,p}(\Omega;R^m)$, $u|_{\partial\Omega} = Fx$, that is

$$0 \leq \int_\Omega \psi(\nabla u)\,dx, \quad u \in H^{1,p}(\Omega;R^m),\ u|_{\partial\Omega} = Fx. \tag{4.3}$$

Let $\psi^\#$ denote the quasiconvexification or relaxation of ψ. Thus, assuming that $|\Omega| = 1$,

$$\psi^\#(F) = \inf_A \int_\Omega \psi(\nabla u)\,dx,$$

$$A = \{ u \in H^{1,p}(\Omega;R^m):\ u|_{\partial\Omega} = Fx \}\ .$$

Additional details about $\psi^\#$ and its relationship to ψ may be found in Dacorogna [25] or [59]. By (4.3), $\psi^\#(F) \geq 0$. Note that $\psi^\# \leq \psi$. Thus by (4.1),

$$0 \leq \psi^\#(F) \leq \int_M \psi^\#(A)\,d\mu(A) \leq \int_M \psi(A)\,d\mu(A) = \langle T, \mu \rangle.$$

Thus μ cannot be separated from M_F. It follows from the separability of F^p and the density of the averaged Dirac masses that there is a sequence $u^k \in A$ such that

$$\int_M \psi(A)\,d\mu(A) = \lim_{k\to\infty} \int_\Omega \psi(\nabla u^k)\,dx \quad \text{for any } \psi \in F^p.$$

By PROPOSITION 2.2, μ is an $H^{1,p}$ - Young measure.

As we remarked earlier, by truncation it suffices to assume (4.1) for quasiconvex functions $\varphi \in E^p$ which are bounded below.

The general case of THEOREM 1.1 is proved from the homogeneous one by covering lemmas and approximation. This requires that

$$\int_\Omega \int_M |A|^p\,d\nu_x(A)dx < +\infty.$$

For details, we refer to [57].

In the proof of THEOREM 1.2, it is necessary to retain the framework of the (inseparable) locally convex space $C(M)$ and $C(M)'$, cf. [55]. An interesting part of the argument involves truncating a sequence $u^k \in H^{1,\infty}(\Omega;R^m)$ to a unformly bounded one which generates the same Young measure. This is accomplished with the aid of a generalization of a lemma of Zhang [83], itself derived from [2],[63], cf. [55] Proposition 5.3:

LEMMA 4.1 *Let* $u \in C_o^\infty(\Omega;R^m)$ *and* $L > 0$. *Then there is a* $w \in H_o^{1,\infty}(\Omega;R^m)$ *such that*

$$\| w \|_{W_0^{1,\infty}(\Omega;\mathbb{R}^m)} \leq C_1 L \quad \textit{and} \tag{4.4}$$

$$|\{ w \neq u \}| \leq \frac{C_2}{L} \Big\{ \int_{\{|\nabla u| \geq L\} \cap \Omega} |\nabla u| \, dx + \int_{\Omega} |u| \, dx \Big\}, \tag{4.5}$$

where C_1 *and* C_2 *depend only on* n *and* m.

Note that $\nabla w = \nabla u$ in $\{ w = u \}$.

5. SOME APPLICATIONS We give a few simple applications. Let $\psi \in C(\mathbb{M})$ satisfy

$$c \max(|A|^p, 0) \leq \psi(A) \leq C(1 + |A|^p), \quad A \in \mathbb{M}, \tag{5.1}$$

where $1 < p < \infty$ and consider the functional

$$\Psi(v) = \int_{\Omega} \psi(\nabla v) \, dx \tag{5.2}$$

for $v \in A = u_0 + W_0^{1,p}(\Omega;\mathbb{R}^m)$, with u_0 given.

THEOREM 5.1 *If* ψ *satisfies* (5.1), *then the problem*

$$\min_A \Psi(v)$$

admits a minimizing sequence $u^k \in A$ *such that* $(|\nabla u^k|^p)$ *is weakly convergent in* $L^1(\Omega)$.

The proof is a direct application of THEOREM 1.1. An arbitrary minimizing sequence is bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$ and thus determines a biting Young measure on E^p. By the theorem, the biting Young measure is an $W^{1,p}$ - Young measure which is generated by a sequence obeying the conclusion of THEOREM 5.1. The conclusion of the theorem fails when $p = 1$, as is well known. Recent results about the relaxation of functionals with linear growth are due to Dal Maso [26] and Fonseca and Müller [47].

THEOREM 5.2 *Let* φ *be quasiconvex satisfy*

$$0 \leq \varphi(A) \leq C(1 + |A|^p).$$

If $u^k \rightharpoonup u$ *in* $W^{1,p}(\Omega;\mathbb{R}^m)$ *and*

$$\int_{\Omega} \varphi(\nabla u^k) \, dx \to \int_{\Omega} \varphi(\nabla u) \, dx \quad \textit{as } k \to \infty,$$

then

$$9(Vuk) \text{ -*> } <p(Vu) \quad \text{in } L^*(G).$$

Let $v = (v_x)_{x \in n}$ denote the biting Young measure generated by (Vu^k). Then there is another sequence (Vw^k) bounded in LP(Q) which generates $v = (v_x)_{x \in n}$ asan H^{1*}-Young measure. Let

$$\overline{9}(x) - \int_M <p(A)dv_x(A).$$

By Jensen's Inequality, $<KVu) \leq \overline{<p}$. Let (E_j) denote the bitten sets from the Biting Lemma. Then

$$f<p(Vu^k)dx \quad £ \quad fq>(Vu^k)dx \quad -» \quad f\ \overline{9}\,dx$$

Hence, letting k -> 00 and then '>-» », we see that

$$\int_\Omega fcp(Vu)dx \quad ^ \quad \int_\Omega f\ \overline{^}\,dx.$$

From Jensen's Inequality quoted above, we deduce that $<p(Vu) = \overline{9}$. Now we have that

$$<p(Vu^k) \quad ^ \quad <p(Vu) \quad \text{in } L^*(Q) \text{ and}$$

$$\text{limsup}_{k-*«} \int_\Omega Jq>(Vu^k)dx = \lim_{k-\bullet -} \int_\Omega f9(Vu^k)dx = \int_\Omega f(p(Vu)dx.$$

The conclusion now follows from PROPOSITION 2.1. A direct proof of this was given in [59].

As a special case of this theorem, we give another proof of Mailer's observation about weak continuity of det Vu in the limit case where $u \in H^{!>n}(Q;R^n)$, $Q \subset R^n$, cf. Coifman, lions, Meyer, and Semmes [20], Iwaniec and Sbordooc [48], and Miiller himself [69,70].

COROLLARY 5.3 *Let* $u, u^k \in H^{\wedge}Q^{*11})$, $Q \subset R^n_f$ *satitfy*

$1^{\wedge}$ -$^{\wedge}$ u *in* $H^{\wedge}ftR^{11})$ *and*

$detVu^k \ ^{\wedge} \ 0$ *in CL*

Then

det Vi* -$^{\wedge}$ detVu *in* $L^{\wedge}(\Omega)$.

To verify this, first note that since $u^k \rightharpoonup u$ in $W^{1,n}(\Omega;R^n)$, it is elementary to verify by integration by parts that

$$\det \nabla u^k \rightharpoonup \det \nabla u \text{ in } \mathcal{D}'_{loc}(\Omega).$$

In particular

$$\int_\Omega \det \nabla u\, \zeta\, dx \geq 0 \text{ for } 0 \leq \zeta \in C_0^\infty(\Omega),$$

so $\det \nabla u \geq 0$ in Ω.

Let $\varphi(A) = \max(\det A, 0)$, a quasiconvex function satisfying

$$0 \leq \varphi(A) \leq C(1 + |A|^n), \quad A \in M.$$

From the above, $\varphi(\nabla u^k) = \det \nabla u^k$ and $\varphi(\nabla u) = \det \nabla u$. For any smooth subdomain $\Omega' \subset \Omega$, choose a Lipschitz cut-off function η with $\eta = 1$ in Ω'. Then

$$\begin{aligned} \limsup_{k\to\infty} \int_{\Omega'} \det \nabla u^k\, dx &\leq \lim_{k\to\infty} \int_\Omega \eta \det \nabla u^k\, dx \\ &= \int_\Omega \eta \det \nabla u\, dx \\ &= \int_{\Omega'} \det \nabla u\, dx + \int_{\Omega\setminus\Omega'} \eta \det \nabla u\, dx \qquad (5.3) \end{aligned}$$

Choose a sequence (η^j) with $\eta^j \to 0$ pointwise in $\Omega \setminus \Omega'$. By the dominated convergence theorem the second integral in (5.3) tends to zero, so we obtain that

$$\limsup_{k\to\infty} \int_{\Omega'} \det \nabla u^k\, dx \leq \int_{\Omega'} \det \nabla u\, dx$$

The conclusion follows from THEOREM 5.2.

REFERENCES

[1] Acerbi, E. and Fusco, N. 1984 Semicontinuity problems in the calculus of variations, Arch. Rat. Mech. Anal., 86, 125 - 145

[2] Acerbi, E. and Fusco, N. 1988 An approximation lemma for $W^{1,p}$ functions, *Proc. Symp. Material instabilities in continuum mechanics*, Heriot-Watt (Ball, J. M. ed.) Oxford, 1-5

[3] Ball, J. M. 1989 A version of the fundamental theorem for Young measures, *PDE's and continuum models of phase transitions*, Lecture Notes in Physics, 344,(Rascle, M., Serre, D., and Slemrod, M., eds.) Springer, 207-215

[4] Ball, J. M. 1990 Sets of gradients with no rank-one connections, J. math pures et appl., 69, 241-259

[5] Ball, J. M. and James, R. 1987 Fine phase mixtures as minimizers of energy, Arch. Rat. Mech. Anal.,100, 15-52

[6] Ball, J. M. and James, R. 1991 Proposed experimental tests of a theory of fine microstructure and the two well problem, Phil. Trans. R. Soc. London A (to appear)

[7] Ball, J. M. and Murat, F. 1984 $W^{1,p}$ - quasiconvexity and variational problems for multiple integrals, J. Fnal Anal, 58,225-253

[8] Ball, J. M. and Murat, F. 1990 Remarks on Chacon's biting lemma, (to appear)

[9] Ball, J. M. and Murat, F. Remarks on rank-one convexity and quasiconvexity, to appear

[10] Ball, J. M. and Zhang, K. 1990 Lower semicontinuity of multiple integrals and the biting lemma, Proc. Royal Soc. Edinburgh, 114A, 367-379

[11] Battacharya, K. 1991 Wedge-like microstructure in martensite, Acta Metal., 39, 2431-2444

[12] Battacharya, K. Self accomodation in martensite, Arch. Rat. Mech. Anal. (to appear)

[13] Billingsley, P. 1986 *Probability and measure*, Wiley

[14] Brandon, D. and Rogers, R. Nonlocal regularization of L. C. Young's tacking problem (to appear)

[15] Brooks, J. K. and Chacon, R. V. 1980 Continuity and compactness of measures, Adv. in Math., 37, 16-26

[16] Chipot, M. 1991 Numerical analysis of oscillations in nonconvex problems,Num. Math.

[17] Chipot, M. and Collins, C. Numerical approximation in variational problems with potential wells, (to appear)

[18] Chipot, M. and Kinderlehrer, D. 1988 Equilibrium configurations of crystals, Arch. Rat. Mech. Anal., 103, 237-277

[19] Chipot, M., Kinderlehrer, D., and Vergara-Caffarelli, G. 1986 Smoothness of linear laminates, Arch. Rat. Mech. and Anal., 96, 81-96

[20] Coifman, P.-L. Lions, Meyer, and Semmes,S. 1989 Compacité par compensation et espaces de Hardy, CRAS Paris, 309, 945-949

[21] Collins, C. and Luskin, M. 1989 The computation of the austenitic-martensitic phase transition,*PDE's and continuum models of phase transitions*, Lecture Notes in Physics, 344, (Rascle, M., Serre, D., and Slemrod, M., eds.) Springer, 34-50

[22] Collins, C. and Luskin, M. Numerical modeling of the microstructure of crystals with symmetry-related variants, *Proc. ARO US-Japan Workshop on Smart/Intelligent Materials and Systems*, Technomic

[23] Collins, C. and Luskin, M. Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem, to appear

[24] Collins, C., Kinderlehrer, D., and Luskin, M. 1991 Numerical approximation of the solution of a variational problem with a double well potential, SIAM J. Numer. Anal., 28, 321-333

[25] Dacorogna, B. 1989 *Direct methods in the Calculus of Variations*, Springer

[26] Dal Maso, G. 1980 Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals, Manus. Math. 30, 387-416

[27] Ericksen, J. L. 1979 On the symmetry of deformable crystals, Arch. Rat. Mech. Anal. 72, 1-13

[28] Ericksen, J. L. 1980 Some phase transitions in crystals, Arch. Rat. Mech. Anal. 73, 99-124

[29] Ericksen, J. L. 1981 Changes in symmetry in elastic crystals, IUTAM Symp. Finite Elasticity (Carlson, D.E. and Shield R.T., eds.) M. Nijhoff, 167-177

[30] Ericksen, J. L. 1981 Some simpler cases of the Gibbs phenomenon for thermoelastic solids, J.of thermal stresses, 4 , 13-30

[31] Ericksen, J. L. 1982 Crystal lattices and sublattices, Rend. Sem. Mat. Padova, 68, 1-9

[32] Ericksen, J. L. 1983 Ill posed problems in thermoelasticity theory, Systems of Nonlinear Partial Differential Equations, (Ball, J., ed) D. Reidel, 71-95

[33] Ericksen, J. L. 1984 The Cauchy and Born hypotheses for crystals, *Phase Transformations and Material Instabilities in Solids*, (Gurtin, M., ed) Academic Press, 61-78

[34] Ericksen, J. L. 1986 Constitutive theory for some constrained elastic crystals, Int. J. Solids Structures, 22, 951 - 964

[35] Ericksen, J. L. 1986 Stable equilibrium configurations of elastic crystals, Arch. Rat. Mech. Anal. 94, 1-14

[36] Ericksen, J. L. 1987 Twinning of crystals I, *Metastability and Incompletely Posed Problems*, IMA Vol. Math. Appl. 3,(Antman, S., Ericksen, J.L., Kinderlehrer, D., Müller, I.,eds) Springer, 77-96

[37] Ericksen, J. L. 1988 Some constrained elastic crystals, *Material Instabilities in Continuum Mechanics*, (Ball, J. ed.) Oxford, 119 - 136

[38] Ericksen, J. L. 1989 Weak martensitic transformations in Bravais lattices, Arch. Rat. Mech. Anal, 107, 23 - 36

[39] Ericksen, J. L., James, R. D., Kinderlehrer, D., and Luskin, M. (eds) *Microstructure and phase transitions*, IMA Volume, Springer, (to appear)

[40] Evans, L. C. 1990 *Weak convergence methods for nonlinear partial differential equations*, C B M S 74, Amer. Math. Soc.

[41] Firooze, N. and Kohn, R. 1991 Geometric parameters and the relaxation of multiwell energies, IMA preprint Series 765

[42] Fonseca, I. 1985 Variational methods for elastic crystals, Arch. Rat. Mech. Anal., 97, 189-220

[43] Fonseca, I. 1988 The lower quasiconvex envelope of the stored energy function for an elastic crystal, J. Math. pures et appl, 67, 175-195

[44] Fonseca, I. 1991 The Wulff Theorem revisited, Proc. R. Soc. Lond., 432, 125-145

[45] Fonseca, I. 1992 Lower semicontinuity of surface energies, Proc. R. Acad. Edin., 120A, 99-115

[46] Fonseca, L. and Müller, S. (to appear)

[47] Fonseca, L. and Müller, S. Quasiconvex integrands and lower semicontinuity in L^1, SIAM J. Math Anal. (to appear)

[48] Iwaniec, T. and Sbordone, C. On the integrability of the Jacobian under minimal hypothesis (to appear)

[49] James, R. D. 1988 Microstructure and weak convergence, *Proc. Symp. Material Instabilities in Continuum Mechanics*, Heriot-Watt, (Ball, J. M., ed.), Oxford, 175-196

[50] James, R. D. and Kinderlehrer, D. 1989 Theory of diffusionless phase transitions, *PDE's and continuum models of phase transitions*, Lecture Notes in Physics, 344,(Rascle, M., Serre, D., and Slemrod, M., eds.) Springer, 51-84

[51] James, R. D. and Kinderlehrer, D. 1990 Frustration in ferromagnetic materials, Cont. Mech. Therm. 2, 215-239

[52] James, R. D. and Kinderlehrer, D. A theory of magnetostriction with application to TbDyFe2 (to appear)

[53] James, R. D. and Kinderlehrer, D. Frustration and microstructure: an example in magnetostriction, Proc. First Europ. Conf. Elliptic Parab. Problems, Pont a Mousson

[54] Kinderlehrer, D. 1988 Remarks about the equilibrium configurations of crystals, *Proc. Symp. Material instabilities in continuum mechanics*, Heriot-Watt (Ball, J. M. ed.) Oxford, 217-242

[55] Kinderlehrer, D. and Pedregal, P. 1991 Characterizations of Young measures generated by gradients, Arch. Rat. Mech. Anal., 115, 329-365

[56] Kinderlehrer, D. and Pedregal, P. 1991 Charactérisation des mesures de Young associées à un gradient, C.R.A.S. Paris, 313, 765-770

[57] Kinderlehrer, D. and Pedregal, P. 1992 Gradient Young measures generated by sequences in Sobolev spaces (to appear)

[58] Kinderlehrer, D. and Pedregal, P. 1992 Remarks about the analysis of gradient Young measures (to appear)

[59] Kinderlehrer, D. and Pedregal, P. 1992 Weak convergence of integrands and the Young measure representation, SIAM J. Math Anal.,23,1 - 19

[60] Kinderlehrer, D. and Pedregal, P. Remarks about Young measures supported on two wells, (to appear)

[61] Kinderlehrer, D. and Vergara-Caffarelli, G. 1989 The relaxation of functionals with surface energies, Asymptotic Analysis 2, 279-298

[62] Kohn, R. V. 1991 The relaxation of a double-well energy, Cont. Mech. Therm., 3, 193-236

[63] Liu, F.-C. 1977 A Luzin type property of Sobolev functions, Ind. Math. J., 26, 645-651

[64] Luskin, M. and Ma, L. 1990 Analysis of the finite element approximation of microstructure in micromagnetics, UMSI report 90/164

[65] Matos, J. 1990 Thesis, University of Minnesota

[66] Matos, J. The absence of fine microstructure in α–β quartz, (to appear)

[67] Morrey, C. B., Jr. 1966 *Multiple Integrals in the Calculus of Variations*, Springer

[68] Müller, S. 1988 Weak continuity of determinants an nonlinear elasticity, CRAS Paris, 307, 501-506

[69] Müller, S. 1989 A surprising higher integrability property of mappings with positive determinant, Bull. AMS 21, 245-248

[70] Müller, S. 1990 Higher integrability of determinants and weak convergence in L^1, J. Reine Angew. Math., 412, 20 - 34

[71] Nicolaides, R. A. and Walkington, N. Computation of microstructure utilizing Young measure representations, Proc. Recent Adv. Adaptive Sensory Materials and their Appl.

[72] Pedregal, P. 1989 Weak continuity and weak lower semicontinuity for some compensation operators, Proc. Royal Soc. Edin. 113, 267 - 279

[73] Pedregal, P. Laminates and microstructure (to appear)

[74] Slemrod, M. Dynamics of measure valued solutions to a backward-forward heat equation (to appear)

[75] Sverak, V. On the regularity of the Monge-Ampère equation without convexity, (to appear)

[76] Sverak, V. Quasiconvex functions with subquadratic growth, to appear

[77] Sverak, V. Rank one convexity does not imply quasiconvexity, to appear

[78] Tartar, L. 1979 Compensated compactness and applications to partial differential equations, *Nonlinear analysis and mechanics: Heriot Watt Symposium, Vol I V*(Knops, R., ed.) Pitman Res. Notes in Math. 39, 136-212

[79] Tartar, L. 1983 The compensated compactness method applied to systems of conservation laws, *Systems of nonlinear partial differential equations* (Ball, J. M., ed) Riedel

[80] Tartar, L. 1984 Étude des oscillations dans les équations aux dérivées partielles nonlinéaires, Springer Lect. Notes Physics, 195, 384-412

[81] Young, L. C. 1969 *Lectures on calculus of variations and optimal control theory*, W.B. Saunders

[82] Zhang, K. 1990 Biting theorems for Jacobians and their applications, Anal. Nonlineare, 7, 345-366

[83] Zhang, K. A construction of quasiconvex functions with linear growth at infinity (to appear)

[84] Zhang, K. Rank-one connections and the three "well" problem, (to appear)

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