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# Memory Effects in Homogenisation Linear Second Order Equation

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## 1. Introduction

We would like to study memory effects in homogenisation of the following sequence of equations:

$$(1) \qquad -\partial_x^2 u^\varepsilon(x,t) + b^\varepsilon(t)\partial_x u^\varepsilon(x,t) + c^\varepsilon(t)u^\varepsilon(x,t) = v(x,t) \,,$$

where the sequences $(b^\varepsilon)$ and $(c^\varepsilon)$ are bounded in $\mathrm{L}^\infty$:

$$(2) \qquad \begin{array}{c} -\gamma \le b^\varepsilon \le \gamma \\ 0 < \alpha \le c^\varepsilon \le \beta \,. \end{array}$$

The above equation is essentially an ordinary differential equation with constant coefficients (in $x$), while coefficients depend on the parameter $t$ (and they *oscillate* in $t$). Before turning to the above problem, let us briefly describe the method on a simpler example ([Tpri] and [Mlhp]).

**First order equation: a simple example.** We consider the following sequence of initial value problems:

$$\begin{cases} \partial_x u^\varepsilon(x,t) + a^\varepsilon(t)u^\varepsilon(x,t) & = 0 \\ u^\varepsilon(0,.) & = 1 \,. \end{cases}$$

This *Cauchy problem* for an ordinary differential equation of first order is equivalent to the integral equation:

$$u^\varepsilon(x,t) + a^\varepsilon(t)\int_0^x u^\varepsilon(\xi,t)d\xi = 1 \,.$$

The solution can be explicitly written as: $u^\varepsilon(x,t) = e^{-a^\varepsilon(t)x}$. Assuming that the sequence $(a^\varepsilon)$ is bounded in $\mathrm{L}^\infty$, so that it has a weakly $*$ convergent subsequence (same notation), we find that the limit of the (sub)sequence (of) $u^\varepsilon$ is given by: $u^0(x,t) = \int e^{-x\lambda}d\nu_t(\lambda)$, where $\nu_t$ denotes a *Young* measure corresponding to a subsequence $(a^\varepsilon)$ (see [Tcca], [Ewcm]). The question is: *which equation does $u^0$ satisfy?*

Let us search for the equation for $u^0$ in the form: $u^0 + K * u^0 = 1$, for $x \ge 0$; where the convolution is taken in $x$. We should find the *kernel* $K$. First, the function $u^0$ should be extended to the whole of real line, more precisely define:

$$S(x,t) := \begin{cases} 0, & x < 0 \\ u^0(x,t), & x \ge 0 \,. \end{cases}$$

Now the equation becomes: $S + K * S = \chi_{[0,\infty\rangle}$. Differentiating, we shall obtain an analogous equation to the differential equation we started from. We denote: $S' = \delta_0 + g$, where

$$g(x,t) := \begin{cases} 0, & x \le 0 \\ \partial_x u^0(x,t), & x > 0 \,. \end{cases}$$

Finally, we obtain:

$$K * (\delta_0 + g) = -g \,,$$

and after inverting $\delta_0 + g$ (in the convolution algebra), under assumption that $\| g \|_{\mathrm{L}^1(0,R)} < 1$, for some $R > 0$, we obtain: $K := -g + g*g - g*g*g \pm \ldots$.

## 2. Fundamental solutions of the $\varepsilon$–problem

Now we return to the study of the equation (1). Let us first find the fundamental solution of the above differential equation. We are looking for the solutions in the form:

$u(x,t) = e^{r(t)x}$. The characteristic equation is ($t$ is a parameter): $-r^2 + b^\varepsilon r + c^\varepsilon = 0$, so the two solutions are given by:

$$r_+^\varepsilon(t) = \frac{b^\varepsilon(t)}{2} + \sqrt{\frac{(b^\varepsilon(t))^2}{4} + c^\varepsilon(t)}$$
$$r_-^\varepsilon(t) = \frac{b^\varepsilon(t)}{2} - \sqrt{\frac{(b^\varepsilon(t))^2}{4} + c^\varepsilon(t)} \, .$$

In order to have the fundamental solution bounded, we shall take it to decay at both $-\infty$ and $+\infty$; we can use the fact that $r_-^\varepsilon < 0$ and $r_+^\varepsilon > 0$. This leads to the following *ansatz:*

$$E^\varepsilon(x,t) = C^\varepsilon(t) \begin{cases} e^{r_+^\varepsilon(t)x}, & x \leq 0 \\ e^{r_-^\varepsilon(t)x}, & x > 0 \end{cases} .$$

We should find a $C^\varepsilon$ such that the following equation is satisfied:

$$-\partial_x^2 E^\varepsilon + b^\varepsilon \partial_x E^\varepsilon + c^\varepsilon E^\varepsilon = \delta_0 \, ,$$

where $\delta_0$ is the Dirac mass at $x = 0$. Away from 0 the equation is clearly satisfied. At 0 we have $\partial_x E^\varepsilon(0^-,t) = C^\varepsilon(t) r_+(t)$ and $\partial_x E^\varepsilon(0^+,t) = C^\varepsilon(t) r_-(t)$; so for the second derivative the singular term is $C^\varepsilon(t)(r_-(t) - r_+(t))\delta_0$, and the right choice for $C^\varepsilon$ is $[(b^\varepsilon(t))^2 + 4c^\varepsilon(t)]^{-1/2}$. Thus, the fundamental solution is of the form:

$$E^\varepsilon(x,t) = \frac{1}{r_+(t) - r_-(t)} \begin{cases} e^{r_+^\varepsilon(t)x}, & x \leq 0 \\ e^{r_-^\varepsilon(t)x}, & x > 0 \end{cases} = \frac{1}{\sqrt{(b^\varepsilon(t))^2 + 4c^\varepsilon(t)}} \begin{cases} e^{r_+^\varepsilon(t)x}, & x \leq 0 \\ e^{r_-^\varepsilon(t)x}, & x > 0 \end{cases} .$$

The fundamental solution enables us to write down the formula for the solution of non-homogeneous equation $u^\varepsilon(x,t) = [E^\varepsilon(.,t) * v(.,t)](x)$ (convolution is taken in $x$).

**Remark.** Let us consider the sequence of solutions of a bit more complicated equations:

$$-a^\varepsilon(t)\partial_x^2 u^\varepsilon(x,t) + b^\varepsilon(t)\partial_x u^\varepsilon(x,t) + c^\varepsilon(t) u^\varepsilon(x,t) = v(x,t) \, , \tag{3}$$

where the sequences $a^\varepsilon$, $b^\varepsilon$ and $c^\varepsilon$ are bounded in $L^\infty$:

$$\begin{aligned} -\gamma \leq\, & b^\varepsilon \leq \gamma \\ 0 < \alpha_1 \leq\, & a^\varepsilon \leq \beta_1 \\ 0 < \alpha_2 \leq\, & c^\varepsilon \leq \beta_2 \, . \end{aligned} \tag{4}$$

Dividing the equation by $a^\varepsilon$, we would have the coefficients satisfying inequalities as required in (2):

$$\begin{aligned} -\frac{\gamma}{\alpha_1} \leq\, & \frac{b^\varepsilon}{a^\varepsilon} \leq \frac{\gamma}{\alpha_1} \\ 0 < \frac{\alpha_2}{\beta_1} \leq\, & \frac{c^\varepsilon}{a^\varepsilon} \leq \frac{\beta_2}{\alpha_1} \, . \end{aligned} \tag{5}$$

Unfortunately, we cannot just reduce this problem to the problem (1), as division by $a^\varepsilon$ would introduce oscillations in the right hand side term $v$; but the same abstract procedure as for the problem (1) can be applied for the problem (3). The fundamental solution would be:

$$E^\varepsilon(x,t) = \frac{1}{a^\varepsilon(t)(r_+(t) - r_-(t))} \begin{cases} e^{r_+^\varepsilon(t)x}, & x \leq 0 \\ e^{r_-^\varepsilon(t)x}, & x > 0 \end{cases} = \frac{1}{\sqrt{(b^\varepsilon(t))^2 + 4a^\varepsilon(t)c^\varepsilon(t)}} \begin{cases} e^{r_+^\varepsilon(t)x}, & x \leq 0 \\ e^{r_-^\varepsilon(t)x}, & x > 0 \end{cases} .$$

Of course, the roots $r_\pm^\varepsilon$ of the equation $-a^\varepsilon r^2 + b^\varepsilon r + c^\varepsilon = 0$ ($t$ si a parameter) are given by: $r_\pm^\varepsilon = \frac{-b^\varepsilon \pm \sqrt{(b^\varepsilon)^2 + 4a^\varepsilon c^\varepsilon}}{2a^\varepsilon}$. This fundamental solution satisfies the differential equation:

$$-a^\varepsilon \partial_x^2 E^\varepsilon + b^\varepsilon \partial_x E^\varepsilon + c^\varepsilon E^\varepsilon = \delta_0 \, ,$$

and the solution of (3) can be written as: $u^\varepsilon(x,t) = [E^\varepsilon(.,t) * v(.,t)](x)$. ∎

## 3. The limit problem

The sequence *(u^e)* of solutions of (1) is bounded, so there is a subsequence (same notation) that converges weakly to a function $u°$. This function can be written as $u° = F * v$ (convolution in x, while *t* is a parameter), where *F(x,t) := J_N E\|_{li}\|_2(x)di/t(\i^ X2)** The one parameter family of probability measures *v%* is the Young measure* associated to the sequence ($6^£$,$c^e$), while the function *E\|^\|_2* is given (compare the expression for $E^£$) by:

$$\underset{XIMX)}{F} , > = \frac{1}{r_+ - r–t} \begin{cases} e^{r+x} & » \le 0 \\ e - & * > 0 \end{cases} = \frac{1}{\sqrt{\lambda_1^2+4\lambda_2}} \begin{cases} e^{\frac{1}{2}\left(\lambda_1+\sqrt{\lambda_1^2+4\lambda_2}\right)x} & x < 0 \\ e^{\frac{1}{2}\left(\lambda_1-\sqrt{\lambda_1^2+4\lambda_2}\right)x}, & x > 0 \end{cases},$$

where, of course, $r\pm := |(Ai \pm \sqrt{A/A2J + 4A2})$. Clearly, the above function satisfies the equation: —E*iM + XIE XiM + A2^Ai,A2 = *o-

The main result is given by the following:

**Theorem 1.** *The limit u° of the sequence of solutions ($u^£$) of the problems (1), under the assumptions (2), satisfy the equation:*

$$-6\%u°(x,t) + b_{ej}t(t)d_xu°(x,t) + \mathbf{Ceff(<)u°(x,t)} + [K(.,t) * u\%,t)](x) = v(x,t)$$

*where the kernel K (containing the memory term) is given by its Fourier transform:* $\hat{K}$ = A/(l+A), *whereF^x.t)* := / G A I, A 2 (M)<MAI, A$_2$) *and* $F_2$ := - ( - ^ + $6_{eff}d_x$+Ceff£o)*Fi, wiiiie tie *function* $G\|_{l9}\|_2$, *as well as the effective coefficients* 6eff and ceff, is *defined below. This kernel K is in any* 1/ *space, for p* 6 [2,00). ∎

Let us determine the coefficients 6eff and ceff first. Using the expression for ti°, we can *cancel* the function t; and obtain a functional equation:

$$(6) \qquad \left( - \partial_x^2 + b_{\text{eff}}(t)\partial_x + c_{\text{eff}}(t)\delta_0 + K \right) * F = \delta_0 .$$

We recall that $£_0 * E_{Xl,\|3} = E_{Xu\|2}(0)$ and $d_x * E_{XIM} = E'_{XIM} = r\pm E_{XIM}$. Thus, the convolution of the first three terms with *F* gives us:

$$\begin{aligned}\left( - \partial_x^2 + b_{\text{eff}}(t)\partial_x + c_{\text{eff}}(t)\delta_0 \right) * F &= \int \left( - \partial_x^2 + b_{\text{eff}}(t)\partial_x + c_{\text{eff}}(t)\delta_0 \right) * E_{\lambda_1,\lambda_2} d\nu_t(\text{Al}, A_2) \\ &= \delta_0 + \int [(b_{\text{eff}}(t) - \lambda_1)\partial_x + (c_{\text{eff}}(t) - A_2)] * E_{XIM} du_t\{\|_u\|_2),\end{aligned}$$

where we have used the equation: —$E_{x'|X2}$ + $XiE'_{XiX2}$ + A2^Ai,A$_2$ = *o in order to express the term involving $E''_{x_1,Aj}$.

We define $G_{XuXi}\{.,t)$ := [($6_{eff}$(<) - $X\|)d_x$ + (crf(0 - A$_2$)] * $E_{XuX2}$, and consider it as a function of a; (take <, Ai and A$_2$ as parameters).

* **More details on such applications of Young measures can be found in [Ewcm].**

Using the explicit formula for $E_{\lambda_1,\lambda_2}$ we can write the above defined function $G_{\lambda_1,\lambda_2}$ explicitly:

$$G_{\lambda_1,\lambda_2}(x,t) = \begin{cases} \frac{-r_+^2+b_{\text{eff}}(t)r_+ + c_{\text{eff}}(t)}{r_+-r_-}e^{r_+x}, & x<0 \\ \frac{-r_-^2+b_{\text{eff}}(t)r_- + c_{\text{eff}}(t)}{r_+-r_-}e^{r_-x}, & x\ge 0 \end{cases} = \begin{cases} \frac{(b_{\text{eff}}(t)-\lambda_1)r_+ + c_{\text{eff}}(t)-\lambda_2}{r_+-r_-}e^{r_+x}, & x<0 \\ \frac{(b_{\text{eff}}(t)-\lambda_1)r_- + c_{\text{eff}}(t)-\lambda_2}{r_+-r_-}e^{r_-x}, & x\ge 0 \ . \end{cases}$$

In the second equality we have used the fact that $r_+$ and $r_-$ are the solutions of the quadratic equation: $-r^2+\lambda_1 r+\lambda_2 = 0$. This allowed us to express the quadratic term: $r_\pm^2 = \lambda_1 r_\pm + \lambda_2$.

The first and second derivatives of $G_{\lambda_1,\lambda_2}$ can easily be computed as well:

$$\partial_x G_{\lambda_1,\lambda_2}(x,t) = \begin{cases} \frac{r_+\left(-r_+^2+b_{\text{eff}}(t)r_+ + c_{\text{eff}}(t)\right)}{r_+-r_-}e^{r_+x}, & x<0 \\ \frac{r_-\left(-r_-^2+b_{\text{eff}}(t)r_- + c_{\text{eff}}(t)\right)}{r_+-r_-}e^{r_-x}, & x\ge 0 \end{cases} + (\lambda_1 - b_{\text{eff}})\delta_0$$

$$\partial_x^2 G_{\lambda_1,\lambda_2}(x,t) = \begin{cases} \frac{r_+^2\left(-r_+^2+b_{\text{eff}}(t)r_+ + c_{\text{eff}}(t)\right)}{r_+-r_-}e^{r_+x}, & x<0 \\ \frac{r_-^2\left(-r_-^2+b_{\text{eff}}(t)r_- + c_{\text{eff}}(t)\right)}{r_+-r_-}e^{r_-x}, & x\ge 0 \end{cases}$$
$$+ (\lambda_1^2 + \lambda_2 - b_{\text{eff}}\lambda_1 - c_{\text{eff}})\delta_0 + (\lambda_1 - b_{\text{eff}})\delta_0' \ .$$

**Lemma 1.** *In order that the integral of $G_{\lambda_1,\lambda_2}$ be continuously differentiable in $x$, it is necessary and sufficient that $b_{\text{eff}}(t) = \lim b^\varepsilon(t) = \int \lambda_1 d\nu_t(\lambda_1,\lambda_2)$ and that $b_{\text{eff}}^2(t) + c_{\text{eff}}(t) = \int(\lambda_1^2+\lambda_2)d\nu_t(\lambda_1,\lambda_2)$.*

Dem. We only need to check continuity at $x = 0$. Using the formula for $E_{\lambda_1,\lambda_2}$ obtained above, we have:

$$G_{\lambda_1,\lambda_2}(0^\pm,t) = \frac{(b_{\text{eff}}(t)-\lambda_1)r_\mp + (c_{\text{eff}}(t)-\lambda_2)}{r_+ - r_-} \ ,$$

so that the jump of $G_{\lambda_1,\lambda_2}$ at 0 is $\lambda_1 - b_{\text{eff}}$; and the continuity reduces to the condition: $\int \lambda_1 d\nu_t(\lambda_1,\lambda_2) = b_{\text{eff}}(t)$. Next, for the continuity of the derivative, we first compute limits from left and right for the derivative:

$$\partial_x G_{\lambda_1,\lambda_2}(0^\pm,t) = \frac{(b_{\text{eff}}(t)-\lambda_1)r_\mp^2 + (c_{\text{eff}}(t)-\lambda_2)r_\mp}{r_+ - r_-} \ .$$

Using the fact that $r_+ + r_- = \lambda_1$, the condition for the continuity can be written as: $\int (\lambda_1(b_{\text{eff}}(t)-\lambda_1) + c_{\text{eff}}(t) - \lambda_2)d\nu_t(\lambda_1,\lambda_2) = 0$. The required result follows, because $\nu$ is a probability measure.

**Q.E.D.**

**Remark.** Let us see what would the formulae be for the problem given in (3). As for (1), we have the convergence, and $u^0 = F * v$, where $F(x,t) := \int_N E_{\lambda_1,\lambda_2,\lambda_3}(x)d\nu_t(\lambda_1,\lambda_2,\lambda_3)$.

$$E_{\lambda_1,\lambda_2,\lambda_3}(x) = \frac{1}{\lambda_1}\frac{1}{r_+ - r_-}e^{r_\pm x} = \frac{1}{\sqrt{\lambda_2^2+4\lambda_1\lambda_3}}e^{r_\pm x} \ ,$$

where, again, $r_\pm := \frac{\lambda_2 \pm\sqrt{\lambda_2^2+4\lambda_1\lambda_3}}{2\lambda_1}$. For the *effective coefficients* we get:

$$\frac{1}{a_{\text{eff}}(t)} = \int \frac{d\nu_t(\lambda_1,\lambda_2,\lambda_3)}{\lambda_1}$$
$$\frac{b_{\text{eff}}(t)}{a_{\text{eff}}(t)^2} = \int \frac{\lambda_2}{\lambda_1^2} d\nu_t(\lambda_1,\lambda_2,\lambda_3)$$
$$\frac{b_{\text{eff}}(t)^2}{a_{\text{eff}}(t)^3} + \frac{c_{\text{eff}}(t)}{a_{\text{eff}}(t)^2} = \int \left(\frac{\lambda_2^2}{\lambda_1^3} + \frac{\lambda_3}{\lambda_1^2}\right) d\nu_t(\lambda_1,\lambda_2,\lambda_3) \ .$$

The limit problem is, of course:

$$-a_{\text{eff}}(t)\partial_x^2 u^0(x,t) + b_{\text{eff}}(t)\partial_x u^0(x,t) + c_{\text{eff}}(t)u^0(x,t) + [K(.,t) * u^0(.,t)](x) = v(x,t) \ .$$

Written in a more abstract form (no $v$) the problem reads: $(-a_{\text{eff}}(t)\partial_x^2 + b_{\text{eff}}(t)\partial_x + c_{\text{eff}}(t)\delta_0 + K) * F = \delta_0$, and:

$$(-a_{\text{eff}}(t)\partial_x^2 + b_{\text{eff}}(t)\partial_x + c_{\text{eff}}(t)\delta_0) * F = \delta_0 + \int G_{\lambda_1,\lambda_2,\lambda_3}(.,t)d\nu_t(\lambda_1,\lambda_2,\lambda_3) \ ,$$

where:

$$G_{\lambda_1,\lambda_2,\lambda_3}(x,t) = \frac{-a_{\text{eff}}(t)r_\pm^2 + b_{\text{eff}}(t)r_\pm + c_{\text{eff}}(t)}{\lambda_1(r_+ - r_-)}e^{r_\pm x} = \frac{(b_{\text{eff}}(t) - \lambda_2\frac{a_{\text{eff}}(t)}{\lambda_1})r_\pm + c_{\text{eff}}(t) - \lambda_3\frac{a_{\text{eff}}(t)}{\lambda_1}}{\lambda_1(r_+ - r_-)}e^{r_\pm x} \ .$$

∎

## 4. $L^1$ setting

We defined: $F_1(x,t) := \int G_{\lambda_1,\lambda_2}(x,t)d\nu_t(\lambda_1,\lambda_2)$. The convolution equation can now be written as: $K * F = -F_1$. Applying the operator $(-\partial_x^2 + b_{\text{eff}}(t)\partial_x + c_{\text{eff}}(t)\delta_0)$ on both sides, after using the relation just established, we obtain: $K*(\delta_0+F_1) = -(-\partial_x^2+b_{\text{eff}}\partial_x+c_{\text{eff}}\delta_0)*F_1$. Denote the right hand side by $F_2$, and finally we obtain the expression: $K = F_2*(\delta_0+F_1)^{-1}$.

If the $L^1$ norm of $F_1$ is less than 1, then the inverse can be written in the form of a series: $(\delta_0 + F_1)^{-1} = \delta_0 - F_1 + F_1 * F_1 - F_1 * F_1 * F_1 \pm \ldots$.

Let us estimate the $L^1$ norm of $F_1$. We have:

$$\begin{aligned}\int_{-\infty}^{\infty} |G_{\lambda_1,\lambda_2}(x,t)|dx &= \frac{|(b_{\text{eff}}(t)-\lambda_1)r_+ + c_{\text{eff}} - \lambda_2|}{r_+ - r_-}\int_{-\infty}^{0} e^{r_+ x}dx \\ &\quad + \frac{|(b_{\text{eff}}(t)-\lambda_1)r_- + c_{\text{eff}} - \lambda_2|}{r_+ - r_-}\int_{0}^{\infty} e^{r_- x}dx \\ &= \frac{|(b_{\text{eff}}(t)-\lambda_1)r_+ + c_{\text{eff}} - \lambda_2|}{r_+(r_+ - r_-)} - \frac{|(b_{\text{eff}}(t)-\lambda_1)r_- + c_{\text{eff}} - \lambda_2|}{r_-(r_+ - r_-)} \\ &\leq 2\frac{|b_{\text{eff}}(t)-\lambda_1|}{r_+ - r_-} + |c_{\text{eff}}(t) - \lambda_2|\frac{r_- - r_+}{r_+ r_-(r_+ - r_-)} \ ,\end{aligned}$$

After taking into account $r_+ - r_- = \sqrt{\lambda_1^2 + 4\lambda_2}$ and $-r_+r_- = \lambda_2$ we obtain:

$$\| F_1(.,t) \|_{L^1(\mathbf{R})} \leq 2\int \frac{|b_{\text{eff}}(t)-\lambda_1|}{\sqrt{\lambda_1^2+4\lambda_2}}d\nu_t(\lambda_1,\lambda_2) + c_{\text{eff}}(t)\int \left|\frac{1}{\lambda_2} - \frac{1}{c_{\text{eff}}(t)}\right| d\nu_t(\lambda_1,\lambda_2) \ .$$

This estimate can be used to prove the invertibility of $\delta_0 + F_1$ in a special case of *small* first derivative term.

Let us compute $F_2 = -(-\partial_x^2 + b_{\text{eff}}\partial_x + c_{\text{eff}}\delta_0) * F_1$ explicitly. Changing the order of integration, we can use the expressions for the derivatives of $G_{\lambda_1,\lambda_2}$ computed in the previous section.

After integration in $\lambda_1, \lambda_2$ with respect to $\nu$, because of the previous lemma, the terms involving Dirac masses drop out. Thus, we have:

$$F_2(x,t) = \int [\partial_x^2 G_{\lambda_1,\lambda_2}(x,t) - b_{\rm eff}(t)\partial_x G_{\lambda_1,\lambda_2}(x,t) - c_{\rm eff}(t) G_{\lambda_1,\lambda_2}(x,t)] d\nu_t(\lambda_1,\lambda_2)$$
$$= \int \frac{-(-r_\pm^2 + b_{\rm eff}(t) r_\pm + c_{\rm eff}(t))^2}{r_+ - r_-} e^{r_\pm x} d\nu_t(\lambda_1,\lambda_2)\,.$$

This gives that $F_2$ is a *nice* function in $x$ — it decays exponentially both at $\infty$ and $-\infty$, and has only one point of discontinuity ($x = 0$), where it has finite limits on both sides. The difference of the limits at zero (the *jump*) is equal to: $\int (\lambda_1 - b_{\rm eff})[\lambda_1(\lambda_1 - b_{\rm eff}) + 2(\lambda_2 - c_{\rm eff})] d\nu_t(\lambda_1,\lambda_2)$.

As a function of $x$, $F_2$ is in the intersection $\mathrm{L}^1 \cap \mathrm{L}^\infty$, so its Fourier transform is in $\mathrm{L}^2 \cap \mathrm{L}^\infty$.

*If we assume that* $c_{\rm eff} \geq \beta$ *(which may indeed happen), then from* $\frac{2\gamma}{\sqrt{\alpha}} < \frac{\sqrt{5}-1}{2}$ *we can conclude that* $\| F_1(.,t) \|_{\mathrm{L}^1(\mathbf{R})} < 1$.

## 5. $\mathrm{L}^2$ setting

**Lemma 2.** *The function* $1 + \hat{F}_1$ *is nowhere zero.*

Dem. In order to simplify the notation, we shall suppress explicit writing of the parameter $t$, that appears in $b_{\rm eff}, c_{\rm eff}$ and $\nu$.

We shall first compute the Fourier transform of the function $G_{\lambda_1,\lambda_2}$. Then we can use the formula: $\hat{F}_1(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi}(\int G_{\lambda_1,\lambda_2}(x) d\nu(\lambda_1,\lambda_2)) dx = \int \hat{G}_{\lambda_1,\lambda_2}(\xi) d\nu(\lambda_1,\lambda_2)$ in order to compute $\hat{F}_1$.

As a function of $x$ ($\lambda_1$ and $\lambda_2$ are parameters only), $G_{\lambda_1,\lambda_2}$ consists of two exponential functions, joined at $x = 0$.

The Fourier transform of $G_{\lambda_1,\lambda_2}$ can be computed now:

$$\hat{G}_{\lambda_1,\lambda_2}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi} G_{\lambda_1,\lambda_2}(x) dx$$
$$= \frac{-r_+^2 + b_{\rm eff} r_+ + c_{\rm eff}}{r_+ - r_-} \int_{-\infty}^{0} e^{-2\pi i x\xi} e^{r_+ x} dx + \frac{-r_-^2 + b_{\rm eff} r_- + c_{\rm eff}}{r_+ - r_-} \int_0^{\infty} e^{-2\pi i x\xi} e^{r_- x} dx$$
$$= \frac{-r_+^2 + b_{\rm eff} r_+ + c_{\rm eff}}{r_+ - r_-} \frac{1}{r_+ - 2\pi i\xi} e^{(r_+ - 2\pi i\xi)x}\Big|_{-\infty}^{0}$$
$$+ \frac{-r_-^2 + b_{\rm eff} r_- + c_{\rm eff}}{r_+ - r_-} \frac{1}{r_- - 2\pi i\xi} e^{(r_- - 2\pi i\xi)x}\Big|_{0}^{\infty}$$
$$= \frac{1}{r_+ - r_-}\left[\frac{-r_+^2 + b_{\rm eff} r_+ + c_{\rm eff}}{r_+ - 2\pi i\xi} - \frac{-r_-^2 + b_{\rm eff} r_- + c_{\rm eff}}{r_- - 2\pi i\xi}\right],$$

where the fact that $r_+ > 0$ and $r_- < 0$ has been used.

After taking the difference of the two fractions in the brackets, the numerator takes the form:

$$-r_+ r_-(r_+ - r_-) + c_{\rm eff}(r_- - r_+) - 2\pi i\xi(r_-^2 - r_+^2 + b_{\rm eff}(r_+ - r_-)) =$$
$$= (r_+ - r_-)(-r_+ r_- - c_{\rm eff} + 2\pi i\xi(r_+ + r_- - b_{\rm eff}))\,.$$

Thus, the Fourier transform of $G_{\lambda_1,\lambda_2}$ takes a simpler form:

$$\hat{G}_{\lambda_1,\lambda_2}(\xi) = \frac{-r_+r_- - c_{\text{eff}} + 2\pi i\xi(r_+ + r_- - b_{\text{eff}})}{r_+r_- - 4\pi^2\xi^2 - 2\pi i\xi(r_+ + r_-)} .$$

Using the definition of $r_+$ and $r_-$ (in the terms of $\lambda_1$ and $\lambda_2$) we obtain:

$$\hat{G}_{\lambda_1,\lambda_2}(\xi) = \frac{c_{\text{eff}} - \lambda_2 + 2\pi i\xi(b_{\text{eff}} - \lambda_1)}{4\pi^2\xi^2 + \lambda_2 + 2\pi i\lambda_1\xi} .$$

Next, $\nu$ is a probability measure and we can put the constant under the integral sign, obtaining: $1 + \hat{F}_1 = \int(1 + \hat{G}_{\lambda_1,\lambda_2})d\nu(\lambda_1, \lambda_2)$. Let us first try to simplify the integrand.

$$\begin{aligned} 1 + \hat{G}_{\lambda_1,\lambda_2}(\xi) &= \frac{c_{\text{eff}} + 4\pi^2\xi^2 + 2\pi i\xi b_{\text{eff}}}{4\pi^2\xi^2 + \lambda_2 + 2\pi i\lambda_1\xi} \\ &= \frac{(c_{\text{eff}} + 4\pi^2\xi^2)(4\pi^2\xi^2 + \lambda_2) + 4\pi^2\xi^2\lambda_1 b_{\text{eff}}}{(4\pi^2\xi^2 + \lambda_2)^2 + 4\pi^2\lambda_1^2\xi^2} \\ &\quad + 2\pi i\xi\frac{-\lambda_1(c_{\text{eff}} + 4\pi^2\xi^2) + b_{\text{eff}}(4\pi^2\xi^2 + \lambda_2)}{(4\pi^2\xi^2 + \lambda_2)^2 + 4\pi^2\lambda_1^2\xi^2} . \end{aligned}$$

We have to prove that $1 + \hat{F}_1$ is nowhere zero. Assume that its imaginary part *is* zero: $\operatorname{Im}(1 + \hat{F}_1) = 0$. Under the assumption $\xi \neq 0$ (otherwise the expression reduces to $c_{\text{eff}} \int \lambda_2^{-1} d\nu \geq c_{\text{eff}}/\beta$), this gives us:

$$(7) \quad \int \frac{\lambda_1(c_{\text{eff}} + 4\pi^2\xi^2)}{(4\pi^2\xi^2 + \lambda_2)^2 + 4\pi^2\lambda_1^2\xi^2} d\nu(\lambda_1, \lambda_2) = b_{\text{eff}} \int \frac{4\pi^2\xi^2 + \lambda_2}{(4\pi^2\xi^2 + \lambda_2)^2 + 4\pi^2\lambda_1^2\xi^2} d\nu(\lambda_1, \lambda_2) .$$

We can now use the right hand side and insert it in the expression for the real part (the case $b_{\text{eff}} = 0$ immediately gives real part positive):

$$\operatorname{Re}(1 + \hat{F}_1(\xi)) = \left[\frac{(c_{\text{eff}} + 4\pi^2\xi^2)^2}{b_{\text{eff}}} + 4\pi^2\xi^2 b_{\text{eff}}\right] \int \frac{\lambda_1 d\nu(\lambda_1, \lambda_2)}{(4\pi^2\xi^2 + \lambda_2)^2 + 4\pi^2\lambda_1^2\xi^2} .$$

The expression in brackets is not zero, while the integral can be expressed from (7), where the integral on the right hand side of (7) is certainly positive. And this proves the theorem.

(It might be interesting to note that if $1 + \hat{F}_1(\xi)$ is real, it must be positive.)

**Q.E.D.**

As in the lemma, we can prove the following:

**Corollary 1.** *The Fourier transform of the function $F_2$ is given by:*

$$\int \frac{1}{r_+ - r_-} \left[\frac{(-r_-^2 + b_{\text{eff}}(t)r_- + c_{\text{eff}}(t))^2}{r_- - 2\pi i\xi} - \frac{(-r_+^2 + b_{\text{eff}}(t)r_+ + c_{\text{eff}}(t))^2}{r_+ - 2\pi i\xi}\right] d\nu_t(\lambda_1, \lambda_2) .$$

∎

**Remark.** *For (almost) every $t$, there is a constant $K$ (depending on $t$) such that $|1+\hat F_1|(\xi,t) \ge K$ (uniformly in $\xi$).*

Indeed, the function $1+\hat F_1$ has limit 1 for $\xi \to \pm\infty$. By the continuity in $\xi$, there is a neighbourhood of $\pm\infty$ such that the absolute value of the function is more than 1/2 there. On the complement, due to its boundedness and continuity of the function, and the result of the previous lemma, the absolute value attains its minimum. If we take $K$ to be the smaller of that minimum and 1/2, we have proved the claim. ∎

**Remark.** The statement of the lemma 2 is true for the problem given in (3). Similar computations as above give us:

$$\hat G_{\lambda_1,\lambda_2,\lambda_3}(\xi) = \frac{c_{\rm eff} - \lambda_3\frac{a_{\rm eff}}{\lambda_1} + 2\pi i\xi\left(b_{\rm eff} - \lambda_2\frac{a_{\rm eff}}{\lambda_1}\right)}{\lambda_3 + 4\pi^2\xi^2\lambda_1 + 2\pi i\xi\lambda_2} .$$

In order to simplify the expression for $1+\hat F_1 = \int (1+\hat G_{\lambda_1,\lambda_2,\lambda_3}) d\nu_t(\lambda_1,\lambda_2,\lambda_3)$, we express $\int 1 d\nu_t(\lambda_1,\lambda_2,\lambda_3) = a_{\rm eff}\int\frac{1}{\lambda_1} d\nu_t(\lambda_1,\lambda_2,\lambda_3)$. Now, we can use the same idea to continue:

$$\begin{aligned}
1+\hat F_1(\xi,t) &= \int\left[\hat G_{\lambda_1,\lambda_2,\lambda_3}(\xi) + \frac{a_{\rm eff}(t)}{\lambda_1}\right] d\nu_t(\lambda_1,\lambda_2,\lambda_3)\\
&= \int \frac{c_{\rm eff}(t) + 4\pi^2\xi^2 a_{\rm eff}(t) + 2\pi i\xi b_{\rm eff}(t)}{\lambda_3 + 4\pi^2\xi^2\lambda_1 + 2\pi i\xi\lambda_2} d\nu_t(\lambda_1,\lambda_2,\lambda_3)\\
&= \int \frac{\left(c_{\rm eff}(t) + 4\pi^2\xi^2 a_{\rm eff}(t)\right)\left(\lambda_3 + 4\pi^2\xi^2\lambda_1\right)}{\left(\lambda_3 + 4\pi^2\xi^2\lambda_1\right)^2 + 4\pi^2\xi^2\lambda_2^2} d\nu_t(\lambda_1,\lambda_2,\lambda_3)\\
&\quad + 2\pi i\xi\int\frac{-\lambda_2\left(c_{\rm eff}(t) + 4\pi^2\xi^2 a_{\rm eff}(t)\right) + b_{\rm eff}(t)\left(\lambda_3 + 4\pi^2\xi^2\lambda_1\right)}{\left(\lambda_3 + 4\pi^2\xi^2\lambda_1\right)^2 + 4\pi^2\xi^2\lambda_2^2} d\nu_t(\lambda_1,\lambda_2,\lambda_3) .
\end{aligned}$$

The proof concludes in the same way as the proof of lemma 2.

For $F_2$ we obtain:

$$F_2(x,t) = \int \frac{[illegible]}{\lambda_1(r_+ - r_-)} [illegible] \, d\nu_t(\lambda_1,\lambda_2,\lambda_3) ,$$

and for its Fourier transform:

$$\hat F_2(\xi,t) = \int\left[\frac{\left(-a_{\rm eff}(t) r_-^2 + b_{\rm eff}(t) r_- + c_{\rm eff}(t)\right)}{\lambda_1(r_+ - r_-)(r_- - 2\pi i\xi)} - \frac{\left(-a_{\rm eff}(t) r_+^2 + b_{\rm eff}(t) r_+ + c_{\rm eff}(t)\right)}{\lambda_1(r_+ - r_-)(r_+ - 2\pi i\xi)}\right] d\nu_t(\lambda_1,\lambda_2,\lambda_3) .$$

∎

We can do even better—obtain an uniform bound in $t$.

**Lemma 3.** *With the bounds for $b^\varepsilon$ and $c^\varepsilon$ given in (2) above, the following inequalities hold:*

$$|2\pi\lambda_1\xi| \le \frac{\gamma}{2\sqrt\alpha}(4\pi^2\xi^2 + \lambda_2) ,$$

$$|1+\hat F_1| \ge C := \frac{4\alpha}{4\alpha+\gamma^2}\min\left\{1, \frac{c_{\rm eff}}{\beta}\right\} .$$

Dem. The assumptions we have are: $|\lambda_1| \le \gamma$ and $\lambda_2 \ge \alpha$. Thus, it is enough to prove that:

$$2\pi\gamma|\xi| \le \frac{\gamma}{2\sqrt\alpha}(4\pi^2\xi^2 + \alpha) ,$$

and the difference is $\frac{\gamma}{2\sqrt\alpha}(2\pi|\xi| - \sqrt\alpha)^2 \ge 0$.

This proves the first inequality. For the second, let us first note that:

$$|1+\hat{F}_1(\xi,t)| = |c_{\text{eff}} + 4\pi^2\xi^2 + 2\pi i\xi b_{\text{eff}}| \left| \int \frac{d\nu_t(\lambda_1,\lambda_2)}{4\pi^2\xi^2+\lambda_2+2\pi i\lambda_1\xi} \right| .$$

The second factor is certainly bigger than its real part. For the integrand we have, using the first inequality:

$$\frac{(4\pi^2\xi^2+\lambda_2)}{(4\pi^2\xi^2+\lambda_2)^2+4\pi^2\lambda_1^2\xi^2} \geq \frac{1}{1+\frac{\gamma^2}{4\alpha}}\frac{1}{4\pi^2\xi^2+\lambda_2} .$$

Thus, we have that:

$$|1+\hat{F}_1(\xi,t)| \geq \frac{1}{1+\frac{\gamma^2}{4\alpha}}\frac{4\pi^2\xi^2+c_{\text{eff}}}{4\pi^2\xi^2+\beta} .$$

But, the rational function is 1 at $\pm\infty$, and the only critical point is at zero, so it is greater than the minimum of 1 and $c_{\text{eff}}/\beta$.

**Q.E.D.**

Let us first recall that $u^0 = F * v$. So, it is enough to prove that (6) is satisfied by $F$. We have already obtained that $K * (\delta_0 + F_1) = F_2$. Applying the Fourier transform (in $x$) to this equality, we get: $\hat{K}(1+\hat{F}_1) = \hat{F}_2$, or:

$$\hat{K} = \frac{\hat{F}_2}{1+\hat{F}_1} .$$

We should only check that the above makes sense. Clearly, everything except division is justified in the space $\mathcal{S}'$ of tempered distributions.

The previous lemma gives us that $|1+\hat{F}_1| \geq C$ (almost everywhere), so $|1+\hat{F}_1|^{-1} \leq 1/C$ and the reciprocal is in the space $L^\infty(\mathbf{R})$. From the explicit form of $\hat{F}_2$ (corollary 1), we conclude that it is in the space $L^q$, for every $q \in \langle 1,2]$. Thus, $\hat{K} \in L^q$ as well.

The inverse Fourier transform of $\hat{K}$ is in $L^p$, for every $p \in [2,\infty\rangle$; and this completes the proof of the theorem 1.

## 6. Boundary value problems

### One space dimension

Let us consider the following sequence of boundary value problems on the line $[0,\pi]$ (the length is taken to be $\pi$ just for notational convenience), in the time $t \in [0,T]$:

$$\begin{aligned} -a^\varepsilon(t)\partial_x^2 u^\varepsilon(x,t) + c^\varepsilon(t)u^\varepsilon(x,t) &= f(x,t) \\ u^\varepsilon(0,.) &= 0 \\ u^\varepsilon(\pi,.) &= 0 . \end{aligned} \tag{8}$$

In the unbounded domain, this problem was studied by L. Tartar and F. Murat (see [Trho]).

The solution can be sought in the form of a series (eigenfunction expansion for the rectangle, after extension of all the functions to $[-\pi, \pi]$ as odd functions):

$$u^\varepsilon(x,t) = \sum_{k=1}^{\infty} b_k^\varepsilon(t) \sin kx \ .$$

The second derivative in $x$ can be computed easily to have the expansion:

$$\partial_x^2 u^\varepsilon(x,t) = -\sum_{k=1}^{\infty} k^2 b_k^\varepsilon(t) \sin kx \ .$$

For (almost) every fixed $t$, $u^\varepsilon$ and $\partial_x^2 u^\varepsilon$ are odd functions (on $[-\pi, \pi]$), so $f$ has to be odd in $x$ as well. This is a compatibility condition. We can write:

$$f(x,t) = \sum_{k=1}^{\infty} d_k(t) \sin kx \ ,$$

where coefficients $d_k$ can be computed as integrals: $d_k(t) = \frac{2}{\pi}\int_0^\pi f(x,t)\sin kx\, dx$.

Inserting the expansions in the equation (the boundary conditions are satisfied because of the *odd* extension), after equating the coefficients in front of *eigenvectors*, we obtain: $k^2 a^\varepsilon(t) b_k^\varepsilon(t) + c^\varepsilon(t) b_k^\varepsilon(t) = d_k(t)$, so:

$$b_k^\varepsilon(t) = \frac{d_k(t)}{k^2 a^\varepsilon(t) + c^\varepsilon(t)} \ .$$

If the coefficients $a^\varepsilon$ and $c^\varepsilon$ are uniformly bounded (for almost every $t$); more precisely:

$$0 < \alpha \le a^\varepsilon(t) \le \beta$$
$$0 < \alpha' \le \frac{c^\varepsilon(t)}{a^\varepsilon(t)} \le \beta' \ ;$$

then these sequences converge in $L^\infty(0,T)$ weak $*$ and define the functions $A_0$ by $\frac{1}{a^\varepsilon} \longrightarrow \frac{1}{A_0}$; and $C_1, C_2, \ldots$ by:

$$\frac{(c^\varepsilon)^m}{(a^\varepsilon)^{m+1}} \longrightarrow \frac{C_m}{A_0^{m+1}} \ , \quad m \in \mathbf{N} \ .$$

In order to determine the limit problem, we should study the behaviour of the following sequence of functions:

$$\varphi_t^\varepsilon(z) := \frac{1}{z a^\varepsilon(t) + c^\varepsilon(t)} \ .$$

**Lemma 4.** *The sequence of functions $(\varphi_\cdot^\varepsilon(z))_{\varepsilon\in\mathbf{R}^+}$ converges in $L^\infty(0,T)$ weak $*$ (for every $z$ close to infinity) to a function $\varphi_\cdot(z)$ with the following expansion around infinity:*

$$\varphi_t(z) = \sum_{i=0}^{\infty} (-1)^i \frac{C_i(t)}{(A_0(t)z)^{i+1}} \ .$$

*Moreover, for almost every $t$, function $\varphi_t$ can be extended to a holomorphic function on the complex plane without the segment $[-\beta', -\alpha']$ of the real line, and it has a representation:*

$$\varphi_t(z) = \int_N \frac{d\nu_t}{z\lambda_1 + \lambda_2} .$$

**Dem.** For a fixed $t$, the function $\varphi_t$ can be represented as a Taylor series around infinity:

$$\begin{aligned} \frac{1}{za^\varepsilon(t) + c^\varepsilon(t)} &= \frac{1}{za^\varepsilon(t)} \frac{1}{1 + \frac{c^\varepsilon(t)}{a^\varepsilon(t)z}} \\ &= \frac{1}{za^\varepsilon(t)} \sum_{i=1}^{\infty} (-1)^i \Big(\frac{c^\varepsilon(t)}{a^\varepsilon(t)z}\Big)^i = \sum_{i=0}^{\infty} (-1)^i \frac{(c^\varepsilon(t))^i}{(a^\varepsilon(t)z)^{i+1}} . \end{aligned}$$

The above expansion is valid for $|\frac{c^\varepsilon(t)}{a^\varepsilon(t)z}| < 1$, and this is certainly the case whenever $|z| > \beta'$. For such a $z$ we can pass to the limit, and obtain (the bounds are uniform in $t$ and locally uniform in $|z| > \beta'$):

$$\varphi^\varepsilon_.(z) \longrightarrow \varphi_.(z) = \sum_{i=0}^{\infty} (-1)^i \frac{C_i(.)}{(A_0(.)z)^{i+1}} .$$

A priori, this expansion is valid only for $|z| > \beta'$. In order to prove that $\varphi_t$ can be extended to the complex plane without the segment $[-\beta', -\alpha']$, we shall need another tool—Young measures.

There exists a probability (Young) measure $\nu_t$ in the variables $(\lambda_1, \lambda_2)$, associated to a subsequence of $(a^\varepsilon, c^\varepsilon)$. Its support is contained in the set $N := \{(\lambda_1, \lambda_2) \in \mathbf{R}^2 : \lambda_1 \in [\alpha, \beta], \lambda_2 \in [\lambda_1\alpha', \lambda_1\beta']\}$ and $\varphi_t$ can be written as:

$$\varphi_t(z) = \int_N \frac{d\nu_t}{z\lambda_1 + \lambda_2} .$$

Because the integrand has poles at $z = -\lambda_2/\lambda_1$, the function $\varphi_t$ is holomorphic for $z \in \mathbf{C} \setminus [-\beta', -\alpha']$.

**Q.E.D.**

The function $\frac{1}{\varphi_t}$ is a *Nevanlinna* function; i.e. for $\operatorname{Im} z > 0$, we have $\operatorname{Im} \frac{1}{\varphi_t(z)} > 0$. Indeed: the imaginary part of $\frac{1}{z\lambda_1+\lambda_2}$ is $\frac{-\operatorname{Im} z\lambda_1}{|z\lambda_1+\lambda_2|^2}$; so:

$$\operatorname{Im} \frac{1}{\varphi_t(z)} = -\frac{\operatorname{Im} \varphi_t(z)}{|\varphi_t(z)|^2} = \operatorname{Im} z \int \frac{\lambda_1 d\nu_t}{|z\lambda_1 + \lambda_2|^2} |\varphi_t(z)|^{-2} .$$

This function is regular on $\mathbf{C} \setminus [-\beta', -\alpha']$, and it is real for $z \in \mathbf{R} \setminus [-\beta', -\alpha']$.

The classical representation theorem for Nevanlinna functions (see [A&Klpm]) assures that there is a nonnegative measure $\mu_t$, supported on the interval $[-\beta', -\alpha']$, and real numbers $A_t \in \mathbf{R}^+$ and $C_t \in \mathbf{R}$ such that:

$$\frac{1}{\varphi_t(z)} = A_t z + C_t - \int \frac{d\mu_t(\lambda)}{z - \lambda} .$$

On the other hand, the function $\frac{1}{\varphi_t(z)}$ can be expressed as the sum $A_0(t)z + g_t(z)$, where the expansion around infinity is given by:

$$g_t(z) := C_1(t) + \frac{C_1^2(t) - C_2(t)}{A_0(t)z} + \frac{C_1^3(t) - 2C_1(t)C_2(t) + C_3(t)}{A_0^2(t)z^2} + O\Big(\frac{1}{z^3}\Big) .$$

Now, we determine the constants $A_t = A_0(t)$ and $C_t = C_1(t)$ by comparison in the expansion around infinity.

**Theorem 2.** *The limit $u^0$ of the sequence $(u^\varepsilon)$ satisfies the following equation:*

$$-A_0(t)\partial_x^2 u^0(x,t) + C_1(t)u^0(x,t) - \mathcal{H}u^0(x,t) = f(x,t) , \tag{9}$$

*with the same homogeneous Dirichlet boundary conditions, where the operator $\mathcal{H}$ is the integral operator $\mathcal{H}u^0(x,t) := \int_0^\pi k(x,y,t)u^0(y,t)\,dy$, where the kernel can be expressed as a sum $k(x,y,t) := 2\sum_{k=1}^\infty h_k(t)\sin kx \sin ky$, with $h_k(t) := \frac{1}{\pi}\int \frac{d\mu_t(\lambda)}{k^2-\lambda}$ (of course, the measure $\mu_t$ is the one from above, its existence being the consequence of the representation theorem).*

Dem. From the simple formula: $\int_0^\pi \sin ky \sin ly\,dy = \frac{\pi}{2}\delta_{kl}$, we obtain the following expression for $\mathcal{H}u^0(x,t) = \sum_{k=1}^\infty \pi h_k(t)b_k^0(t)\sin kx$. Expanding all the functions in the Fourier series and equating the coefficients in the equation (9) we obtain:

$$A_0(t)k^2 b_k^0(t) + C_1(t)b_k^0(t) - \pi h_k(t)b_k^0(t) = d_k(t) ,$$

or after division by $b_k^0(t)$ (otherwise $d_k(t)$ has to be 0, and $h_k(t)$ is not defined, so we can take for it any value we choose):

$$A_0(t)k^2 + C_1(t) - \pi h_k(t) = \frac{d_k(t)}{b_k^0(t)} = \frac{1}{\varphi_t(k^2)} = A_0(t)k^2 + C_1(t) - \int \frac{d\mu_t(\lambda)}{k^2-\lambda} ,$$

so $h_k(t) = \frac{1}{\pi}\int \frac{d\mu_t(\lambda)}{k^2-\lambda}$.

**Q.E.D.**

**Remark.** The operator $\mathcal{H}$ can be expressed as a convolution:

$$\mathcal{H}u^0(x,t) = \int_{-\pi}^{\pi} H(x-y,t)u^0(y,t)\,dy ,$$

where the kernel $H$ has an expansion in *eigenfunctions* given by $H(x,t) = \sum_{k=1}^\infty h_k(t)\cos kx$. $H$ is assumed to be extended by periodicity, while $u^0$ is extended to $[-\pi,\pi]$ by the odd extension.

Indeed, by using the orthogonality:

$$\begin{aligned}\int_{-\pi}^{\pi} H(x-y,t)u^0(y,t)\,dy &= \int_{-\pi}^{\pi}\Big(\sum_{k=1}^\infty h_k(t)\cos k(x-y)\Big)\sum_{l=1}^\infty b_l^0(t)\sin ly\,dy \\ &= \sum_{k=1}^\infty h_k(t)\sum_{l=1}^\infty b_l^0(t)\int_{-\pi}^{\pi}(\cos kx\cos ky + \sin kx \sin ky)\sin ly\,dy \\ &= \pi\sum_{k=1}^\infty h_k(t)b_k^0(t)\sin kx .\end{aligned}$$

This expression depends on addition formulae for trigonometric functions, and it is not true in other cases.

∎

**Remark.** The sum appearing in the expression for the kernel $k$ can be computed explicitly. First, note that:

$$k(x,y,t) = \frac{2}{\pi}\sum_{k=1}^{\infty}\int \frac{d\mu_t(\lambda)}{k^2-\lambda}\sin kx \sin ky = \frac{2}{\pi} J \,{}^{\prime}\mathrm{Ig}\; f\; \frac{f\wedge \sin kx \sin ky]}{V\text{-}\backslash \quad 1} ,\; *''(*>$$

If we denote (A is negative, so all the sums are finite):

$$F(\lambda, x, V) = 2\; \frac{\wedge \sin kx \sin ky}{£ \quad *_2 A} = \frac{\wedge\wedge}{2£} \frac{\cos fc(x-y) - \cos fc(x-fy)}{*\text{»}-A}$$

$$G(\lambda, z) := \sum_{k=1}^{\infty}\frac{\cos 2\pi k z}{k^2-\lambda} ,$$

then F can be expressed in terms of the even function G: F(A, x, y) = ^ [G(A, ^ ^ ) — G(A, $\frac{x+y}{2\pi}$)].

Next we define a sum of exponentials:

$$H(\alpha, z) := \frac{1}{2}\sum_{k\in \mathbf{Z}} \frac{e^{2\pi i k z}}{4\pi^2 k^2 + \alpha^2} ,$$

and express G in the terms of *H:*

$$G(\lambda, z) Y = \frac{1}{-} v \wedge \frac{e^{2*ikz}}{} - \frac{4?r^2}{} \sum_{k\neq 0} \frac{c^{2«**}}{4\pi^2k^2 - 4\pi^2\lambda}$$

$$= 2\pi^2 \sum_{k\neq 0}\frac{e^{Jlrikz}}{4\pi^2k^2+\alpha^2} = 2\pi^2\left(H(\alpha,z) - \frac{1}{\alpha^2}\right) ,$$

where $a$ := 27TV—A, and A < 0.

The sum defining $H$ can be computed explicitly. It is a periodic function in $z_f$ with period 1. We shall try to determine $H$ as a solution of an ordinary differential equation.

The second derivative of $H$ in $z$ is:

$$'' \quad \rangle \quad ' \quad \sum_{k\in \mathbf{Z}}^{*} \rightarrow 4\pi^2 k^2) \frac{}{4\pi} |E \quad f, \quad \Lambda$$

so # satisfies: —$H''(a,z)$ -f a²i7(a,2r) = 5Zik€Z e^2w''**z. On one period (say (0,1)), the solution is given by $H(\alpha, z) = C\mathrm{ch}\alpha(z - \quad)$.

From Poisson's formula (valid for sums in distributional sense, see: [Smmp], pg. 98):

$$\sum_{k\in \mathbf{Z}} e^{2\pi i k z} = \sum_{k\in \mathbf{Z}} \delta_k ,$$

we can determine the constant $C$ = 2askT> which gives for $H$:

$$H(\alpha, z) = \frac{\mathrm{ch}\alpha(z-\frac{1}{2})}{2\alpha \mathrm{sh}\frac{1}{2}} .$$

Let us use this expression in order to find $F$. First, we should note that G(A, |z|) = G(A, z), because $G$ is even in $z$. Thus we have:

$$G(\lambda, \frac{x-y}{2\pi}) = 2\pi\left[H\left(\alpha, \frac{|x-y|}{2\pi}\right) - \frac{1}{\alpha^2}\right]$$

$$= \frac{\pi}{2\sqrt{-\lambda}}\frac{\mathrm{ch}\sqrt{-\lambda}(|x-y|-\pi)}{\mathrm{sh}\pi\sqrt{-\lambda}} + \frac{1}{2A}$$

$$G(\lambda, \frac{x+y}{\wedge T}) = \frac{\pi}{2V—A}\frac{\mathrm{ch}\sqrt{=1X}(x+y-T)}{\mathrm{sh}.7TV—A} \quad \frac{J}{\wedge A} ,$$

and finally for $F$:

$$F(\lambda,x,y)=\frac{\pi}{4\sqrt{-\lambda}}\frac{\mathrm{ch}\sqrt{-\lambda}(|x-y|-\pi)-\mathrm{ch}\sqrt{-\lambda}(x+y-\pi)}{\mathrm{sh}\pi\sqrt{-\lambda}}\ .$$

From this explicit computation it directly follows that kernel $k$ is a Lipshitz function, and even of class $C^\infty$ outside of the diagonal $x=y$. ∎

## Several space dimensions

A possible generalisation of the problem (8) to several space dimensions can be obtained by replacing the differential operator $-\partial_x^2$ by another positive operator, $-\Delta$. The new problem reads:

$$\begin{cases}-a^\varepsilon(t)\Delta u^\varepsilon(\mathbf{x},t)+c^\varepsilon(t)u^\varepsilon(\mathbf{x},t)=f(\mathbf{x},t)\\ u_{|\Gamma}=0\ ,\end{cases}\tag{10}$$

where $\mathbf{x}\in\Omega$, while $\Gamma$ is the boundary of $\Omega$.

The proposed method will be described for some specific domains.

**Rectangular domain.** The Laplacian $-\Delta$ on the domain $[0,\pi]\times[0,\pi]$ ($\mathbf{x}=(x^1,x^2)$) has $\lambda=m^2+n^2$, $m,n\in\mathbf{N}$ as eigenvalues, and $v_{m,n}(\mathbf{x})=\sin mx^1\sin nx^2$ as corresponding eigenfunctions.

Taking the *ansatz* $u^\varepsilon(\mathbf{x},t)=\sum_{m,n=1}^\infty b^\varepsilon_{m,n}(t)\sin mx^1\sin nx^2$ for the solution, we have for the Laplacian the following expansion: $-\Delta u^\varepsilon(\mathbf{x},t)=\sum_{m,n=1}^\infty(m^2+n^2)b^\varepsilon_{m,n}(t)\sin mx^1\sin nx^2$.

The function $f$ can be expanded in a series: $f(\mathbf{x},t)=\sum_{m,n=1}^\infty d^\varepsilon_{m,n}(t)\sin mx^1\sin nx^2$. Equating the coefficients in the expansion we arrive at:

$$b^\varepsilon_{m,n}(t)=\frac{d_{m,n}(t)}{(m^2+n^2)a^\varepsilon(t)+c^\varepsilon(t)}\ .$$

Under the same assumptions on $(a^\varepsilon)$ and $(c^\varepsilon)$ as in the one-dimensional case, we can apply the lemma 4 and obtain:

$$\varphi_t(z)=\frac{1}{A_0(t)z+C_1(t)-\int\frac{d\mu_t(\lambda)}{z-\lambda}}$$

$$\lim_\varepsilon b^\varepsilon_{m,n}(t)=\varphi_t(m^2+n^2)d_{m,n}(t)=\frac{d_{m,n}(t)}{A_0(t)(m^2+n^2)+C_1(t)-\int\frac{d\mu_t(\lambda)}{(m^2+n^2)-\lambda}}\ .$$

The limit $u^0$ of $(u^\varepsilon)$ satisfies:

$$-A_0(t)\Delta u^0(\mathbf{x},t)+C_1(t)u^0(\mathbf{x},t)-\mathcal{H}u^0(\mathbf{x},t)=f(\mathbf{x},t)\ ,\tag{11}$$

where: $\mathcal{H}u^0(\mathbf{x},t)=\int_\Omega k(\mathbf{x},\mathbf{y},t)u^0(\mathbf{y},t)d\mathbf{y}$, while the kernel is given as a series:

$$k(\mathbf{x},\mathbf{y},t)=\sum_{m,n=1}^\infty h_{m,n}(t)\sin mx^1\sin nx^2\sin my^1\sin ny^2\ ,$$

with:

$$h_{m,n}(t)=\frac{4}{\pi^2}\int\frac{d\mu_t(\lambda)}{(m^2+n^2)-\lambda}\ .$$

The operator $\mathcal{H}$ can be expressed as a convolution:

$$\mathcal{H}u^0(\mathbf{x},t)=\int_{-\pi}^{\pi}H(\mathbf{x}-\mathbf{y},t)u^0(\mathbf{y},t)\,d\mathbf{y}\ ,$$

where: $H(\mathbf{x},t) = \sum_{m,n=1}^{\infty} h_{m,n}(t) \cos mx^1 \cos nx^2$.

The same procedure leads to analogous expressions in higher dimensions.

**Circular domain.** The Laplacian in the domain $\Omega := K[0,1] := \{\mathbf{x} \in \mathbf{R}^2 : |\mathbf{x}| \leq 1\}$ can be written in polar coordinates $x^1 = r\cos\vartheta$, $x^2 = r\sin\vartheta$ in the form: $\Delta = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\vartheta^2$.

Let $(k_{n,m}, m \in \mathbf{N})$ be the zeroes of the Bessel function $J_n$, so $J_n(k_{n,m}) = 0$. Then, $k_{n,m}^2$ are the eigenvalues, while the corresponding eigenfunctions are:

$$J_n(k_{n,m}r)\cos n\vartheta, J_n(k_{n,m}r)\sin n\vartheta \ .$$

We have the following *ansatz* and expansions:

$$\begin{aligned}
u^\varepsilon(r,\vartheta,t) &= \sum_{m=1}^{\infty} \frac{b_{0,m}^\varepsilon(t)}{2} J_0(k_{0,m}r) \\
&\quad + \sum_{n,m=1}^{\infty} \left[ b_{n,m}^\varepsilon(t) J_n(k_{n,m}r)\cos n\vartheta + \bar{b}_{n,m}^\varepsilon(t) J_n(k_{n,m}r)\sin n\vartheta \right] \\
-\Delta u^\varepsilon(r,\vartheta,t) &= \sum_{m=1}^{\infty} k_{0,m}^2 \frac{b_{0,m}^\varepsilon(t)}{2} J_0(k_{0,m}r) \\
&\quad + \sum_{n,m=1}^{\infty} k_{n,m}^2 \left[ b_{n,m}^\varepsilon(t) J_n(k_{n,m}r)\cos n\vartheta + \bar{b}_{n,m}^\varepsilon(t) J_n(k_{n,m}r)\sin n\vartheta \right] \\
f(r,\vartheta,t) &= \sum_{m=1}^{\infty} \left[ \frac{d_{0,m}(t)}{2} J_0(k_{0,m}r) + \sum_{n=1}^{\infty} \left( d_{n,m}(t) J_n(k_{n,m}r)\cos n\vartheta + \bar{d}_{n,m}(t) J_n(k_{n,m}r)\sin n\vartheta \right) \right] ,
\end{aligned}$$

so for the coefficients we get:

$$\begin{aligned}
b_{n,m}^\varepsilon(t) &= \frac{d_{n,m}(t)}{k_{n,m}^2 a^\varepsilon(t) + c^\varepsilon(t)} \\
\bar{b}_{n,m}^\varepsilon(t) &= \frac{\bar{d}_{n,m}(t)}{k_{n,m}^2 a^\varepsilon(t) + c^\varepsilon(t)} \ .
\end{aligned}$$

Lemma 4 gives us:

$$\begin{aligned}
\lim_\varepsilon b_{n,m}^\varepsilon(t) &= \varphi_t(k_{n,m}^2) d_{m,n}(t) = \frac{d_{m,n}(t)}{A_0(t)k_{n,m}^2 + C_1(t) - \int \frac{d\mu_t(\lambda)}{k_{n,m}^2 - \lambda}} \\
\lim_\varepsilon \bar{b}_{n,m}^\varepsilon(t) &= \bar{\varphi}_t(k_{n,m}^2) \bar{d}_{m,n}(t) = \frac{\bar{d}_{m,n}(t)}{A_0(t)k_{n,m}^2 + C_1(t) - \int \frac{d\mu_t(\lambda)}{k_{n,m}^2 - \lambda}} \ .
\end{aligned}$$

The limit $u^0$ of $(u^\varepsilon)$ satisfies the equation (11), of the same form as above, except for the kernel:

$$\begin{aligned}
k(r,\vartheta;R,\Theta,t) &= \sum_{m=1}^{\infty} \frac{h_{0,m}(t)}{2} J_0(k_{0,m}r) J_0(k_{0,m}R) \\
&\quad + \sum_{n,m=1}^{\infty} h_{n,m}(t) J_n(k_{n,m}r) J_n(k_{n,m}R) \left[ \cos n\vartheta \cos n\Theta + \sin n\vartheta \sin n\Theta \right] \\
h_{n,m}(t) &= \frac{2}{\pi \left(J_n'(k_{n,m})\right)^2} \int \frac{d\mu_t(\lambda)}{k_{n,m}^2 - \lambda} \ .
\end{aligned}$$

**Cylindrical domain.** The Laplacian in the domain $K[0,1] \times [0,\pi]$ in $\mathbf{R}^3$ can be written in cylindrical coordinates $x^1 = r\cos\vartheta$, $x^2 = r\sin\vartheta$, $x^3 = z$ in the form: $\Delta = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\vartheta^2 + \partial_z^2$.

The eigenvalues are $l^2 + k^2_{n,m}$, where $k_{n,m}$ are the zeroes of Bessel functions $J_n$, and $l \in \mathbf{N}$. The corresponding eigenfunctions are:

$$J_n(k_{n,m}r)\cos n\vartheta \sin lz,\, J_n(k_{n,m}r)\sin n\vartheta \sin lz\ .$$

The expansions are:

$$\begin{aligned}
u^\varepsilon(r,\vartheta,z,t) &= \sum_{m,l=1}^{\infty} \frac{b^\varepsilon_{0,m,l}(t)}{2} J_0(k_{0,m}r)\sin lz \\
&+ \sum_{n,m,l=1}^{\infty} \left[b^\varepsilon_{n,m,l}(t)J_n(k_{n,m}r)\cos n\vartheta \sin lz + \bar b^\varepsilon_{n,m,l}(t)J_n(k_{n,m}r)\sin n\vartheta \sin lz\right] \\
-\Delta u^\varepsilon(r,\vartheta,z,t) &= \sum_{m,l=1}^{\infty} (k^2_{n,m}+l^2)\frac{b^\varepsilon_{0,m,l}(t)}{2} J_0(k_{0,m}r)\sin lz \\
&+ \sum_{n,m,l=1}^{\infty} (k^2_{n,m}+l^2)\left[b^\varepsilon_{n,m,l}(t)J_n(k_{n,m}r)\cos n\vartheta \sin lz + \bar b^\varepsilon_{n,m,l}(t)J_n(k_{n,m}r)\sin n\vartheta \sin lz\right] \\
f(r,\vartheta,z,t) &= \sum_{m,l=1}^{\infty} \frac{d_{0,m,l}(t)}{2} J_0(k_{0,m}r)\sin lz \\
&+ \sum_{n,m,l=1}^{\infty} \left[d_{n,m,l}(t)J_n(k_{n,m}r)\cos n\vartheta \sin lz + \bar d_{n,m,l}(t)J_n(k_{n,m}r)\sin n\vartheta \sin lz\right]\ ,
\end{aligned}$$

so the coefficients can be expressed as (similar expressions for $\bar b^\varepsilon_{n,m,l}$):

$$b^\varepsilon_{n,m,l}(t) = \frac{d_{n,m,l}(t)}{(k^2_{n,m}+l^2)a^\varepsilon(t) + c^\varepsilon(t)}\ .$$

Lemma 4 gives us:

$$\lim_\varepsilon b^\varepsilon_{m,n,l}(t) = \varphi_t(k^2_{n,m}+l^2)d_{n,m,l}(t) = \frac{d_{n,m,l}(t)}{A_0(t)(k^2_{n,m}+l^2) + C_1(t) - \int \frac{d\mu_t(\lambda)}{(k^2_{n,m}+l^2)-\lambda}}\ .$$

The limit $u^0$ of $(u^\varepsilon)$ satisfies the equation (11), of the same form as above, except for the kernel:

$$\begin{aligned}
k(r,\vartheta,z;R,\Theta,Z,t) &= \sum_{m,l=1}^{\infty} \frac{h_{0,m,l}(t)}{2} J_0(k_{0,m}r)\sin lz J_0(k_{0,m}R)\sin lZ \\
&+ \sum_{n,m,l=1}^{\infty} h_{n,m,l}(t)\big[J_n(k_{n,m}r)\cos n\vartheta \sin lz J_n(k_{n,m}R)\cos n\Theta \sin lZ \\
&\qquad + J_n(k_{n,m}r)\sin n\vartheta \sin lz J_n(k_{n,m}R)\sin n\Theta \sin lZ\big] \\
h_{n,m,l}(t) &= \frac{2}{\pi^2\big(J_n'(k_{n,m})\big)^2}\int \frac{d\mu_t(\lambda)}{(k^2_{n,m}+l^2)-\lambda}\ .
\end{aligned}$$

**Spherical domain.** The Laplacian in the ball $K[0,1]$ in $\mathbf{R}^3$ can be written in spherical coordinates $x^1 = r\sin\vartheta\cos\varphi$, $x^2 = r\sin\vartheta\sin\varphi$, $x^3 = r\cos\vartheta$ in the form:

$$\Delta = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\vartheta}\partial_\vartheta(\sin\vartheta\partial_\vartheta) + \frac{1}{r^2\sin^2\vartheta}\partial^2_\varphi\ .$$

The eigenvalues are the squares of the zeroes of Bessel functions: $k^2_{n+1/2,m}$, and the eigenfunctions: $\frac{1}{\sqrt{r}}Y_n(\vartheta,\varphi)J_{n+1/2}(k_{n+1/2,m}r)$. The Laplace spherical harmonic $Y_n$ has the form:

$$Y_n(\vartheta,\varphi)=\frac{a_0}{2}P_n(\cos\vartheta)+\sum_{h=1}^{n}\big(a_h\cos h\varphi+b_h\sin h\varphi\big)P_{n,h}(\cos\vartheta)\ .$$

So, for $n$-th eigenvalue we have $2n+1$ eigenfunctions: $P_n(\cos\vartheta)$, $\cos\varphi P_{n,1}(\cos\vartheta)$, $\ldots$, $\cos n\varphi P_{n,n}(\cos\vartheta)$, $\sin\varphi P_{n,1}(\cos\vartheta)$, $\ldots$, $\sin n\varphi P_{n,n}(\cos\vartheta)$.

The expansions are:

$$\begin{aligned}
u^\varepsilon(r,\vartheta,\varphi,t)&=\sum_{m=1}^{\infty}\frac{b^\varepsilon_{0,m,0}(t)}{2}P_0(\cos\vartheta)\frac{1}{\sqrt{r}}J_{1/2}(k_{1/2,m}r)+\sum_{n,m=1}^{\infty}\Big[\frac{b^\varepsilon_{n,m,0}(t)}{2}P_n(\cos\vartheta)\\
&\quad+\sum_{h=1}^{n}P_{n,h}(\cos\vartheta)\big(b^\varepsilon_{n,m,h}(t)\cos h\varphi+\bar b^\varepsilon_{n,m,h}(t)\sin h\varphi\big)\Big]\frac{1}{\sqrt{r}}J_{n+1/2}(k_{n+1/2,m}r)\\
-\Delta u^\varepsilon(r,\vartheta,\varphi,t)&=\sum_{m=1}^{\infty}k_{1/2,m}\frac{b^\varepsilon_{0,m,0}(t)}{2}P_0(\cos\vartheta)\frac{1}{\sqrt{r}}J_{1/2}(k_{1/2,m}r)+\sum_{n,m=1}^{\infty}k_{n+1/2,m}\Big[\frac{b^\varepsilon_{n,m,0}(t)}{2}P_n(\cos\vartheta)\\
&\quad+\sum_{h=1}^{n}P_{n,h}(\cos\vartheta)\big(b^\varepsilon_{n,m,h}(t)\cos h\varphi+\bar b^\varepsilon_{n,m,h}(t)\sin h\varphi\big)\Big]\frac{1}{\sqrt{r}}J_{n+1/2}(k_{n+1/2,m}r)\\
f(r,\vartheta,\varphi,t)&=\sum_{m=1}^{\infty}\frac{d_{0,m,0}(t)}{2}P_0(\cos\vartheta)\frac{1}{\sqrt{r}}J_{1/2}(k_{1/2,m}r)+\sum_{n,m=1}^{\infty}\Big[\frac{d_{n,m,0}(t)}{2}P_n(\cos\vartheta)\\
&\quad+\sum_{h=1}^{n}P_{n,h}(\cos\vartheta)\big(d_{n,m,h}(t)\cos h\varphi+\bar d_{n,m,h}(t)\sin h\varphi\big)\Big]\frac{1}{\sqrt{r}}J_{n+1/2}(k_{n+1/2,m}r)\ ,
\end{aligned}$$

so the coefficients can be expressed as:

$$b^\varepsilon_{n,m,h}(t)=\frac{d_{n,m,h}(t)}{k^2_{n+1/2,m}a^\varepsilon(t)+c^\varepsilon(t)}\ .$$

Lemma 4 gives us (analogous expression for $\bar b^\varepsilon_{m,n,h}$):

$$\lim_\varepsilon b^\varepsilon_{m,n,h}(t)=\varphi_t(k^2_{n+1/2,m})d_{n,m,h}(t)=\frac{d_{n,m,h}(t)}{A_0(t)k^2_{n+1/2,m}+C_1(t)-\int\frac{d\mu_t(\lambda)}{k^2_{n+1/2,m}-\lambda}}\ .$$

The limit $u^0$ of $(u^\varepsilon)$ satisfies the equation (11), of the same form as above, except for the kernel:

$$\begin{aligned}
k(r,\vartheta,\varphi;R,\Theta,\Phi,t)&=\sum_{m=1}^{\infty}\frac{h_{0,m,0}(t)}{2}P_0(\cos\vartheta)\frac{1}{\sqrt{r}}J_{1/2}(k_{1/2,m}r)P_0(\cos\Theta)\frac{1}{\sqrt{R}}J_{1/2}(k_{1/2,m}R)\\
&\quad+\sum_{n,m=1}^{\infty}\Big[\frac{h_{n,m,0}(t)}{2}P_n(\cos\vartheta)P_n(\cos\Theta)\\
&\quad+\sum_{h=1}^{n}P_{n,h}(\cos\vartheta)P_{n,h}(\cos\Theta)h_{n,m,h}(t)\big(\cos h\varphi\cos h\Phi+\sin h\varphi\sin h\Phi\big)\Big]\\
&\quad\frac{1}{\sqrt{r}}J_{n+1/2}(k_{n+1/2,m}r)\frac{1}{\sqrt{R}}J_{n+1/2}(k_{n+1/2,m}R)\\
h_{n,m}(t)&=\frac{2}{\pi\big(J_n'(k_{n,m})\big)^2}\frac{2n+1}{2}\frac{(n-h)!}{(n+h)!}\int\frac{d\mu_t(\lambda)}{k^2_{n+1/2,m}-\lambda}\ .
\end{aligned}$$

## Genera! result

In general, we consider a domain *ft* with boundary F, and the eigenvalue problem for the (negative) Laplacian:

$$(12) \qquad \begin{cases} Aw + Xu = 0 \\ \cdot \,] \, r \cdot \cdot \end{cases}$$

Let $(A_n, v_n)$ be a complete set of eigenpairs for the problem (12). By positivity $A_n > 0$, and by the compactness of the operator $(—A)^{-1}$, the eigenvalues $A_n \longrightarrow \infty$.

Let us consider the problem (10) now. If we expand /(x,<) = *^2dn(t)v$_n$(x)* and $tx^c$(x,<) = *J2bñ(t)vn(x)*, the coefficient $b^e_n$ is given by:

$$W - \frac{*\bullet\kappa}{\backslash_n a^*(t) + c^\varepsilon(t)} .$$

Under the assumptions on $(a^e)$ and $(c^e)$ stated before, and with the same notation, the following theorem is true:

**Theorem 3.** *The limit $u°$ of the sequence $(u^\epsilon)$ satisfies the following equation:*

(13) **-Ao(t)Au°(x,t)** + *d(t)u°(x,t)* - **«ti°(x,0 = /(x,<)** ,

*where the operator H is the integral operator 7iu°(x,t)* := $f_Q$ A:(x,y,i)u°(y,<)*dy, with the kernel* fc(x,y,<) := *Y^Li hn(t)v$_n$(x)v$_n$(y), where h$_n$(t)* := / " ^ z J$^{(\lambda)}$ *(of course, the measure fit is the one from above, its existence being the consequence of the representation theorem).*

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## References

**[A&Klpm]** **N. Ahiezer, M. Krĕin: The L-problem of moments, in** ***Some Questions in the Theory of Moments,*** **AMS, Providence, 1962**

**[C&Hmmp]** **Richard Courant, David Hilbert:** ***Methods of Mathematical Physics I;*** **John Wiley & Sons, New York, 1989**

**[Ewcm]** **Lawrence Craig Evans:** ***Weak convergence methods for Nonlinear Partial Differential Equations;*** **AMS, Providence, 1990**

**[Mlhp]** **Maria-Luisa Mascarenhas: A linear homogenisation problem with time dependent coefficient;** ***Transactions of AMS,*** **281, 1, 1984 (179-195)**

**[Smmp]** **Laurent Schwartz:** ***Mathematics for the Physical Sciences;*** **Hermann, Paris, 1966**

**[Tcca]** **Luc Tartar: Compensated compactness and applications to partial differential equations, in** ***Heriot-Watt Symposium,*** **vol. IV, Pitman, New York, 1979**

**[Trho]** **Luc Tartar: Remarks on homogenization, in** ***Homogenization and effective moduli of materials and media,*** **(ed. J. L. Eriksen et al.), Springer-Verlag, New York, 1986**

**[Tnei]** **Luc Tartar: Nonlocal effects induced by homogenisation, in** ***Partial Differential Equations and the Calculus of Variations, vol II*** **(ed. F. Colombini et al. ) 1988**

**[Tmeh]** **Luc Tartar: Memory Effects and Homogenization;** ***Arch. Rat. Mech.,*** **111,2, 1990 (121-133)**

**[Tpri]** **Luc Tartar:** ***private communication***