On the Slow Dynamics for the Cahn-Hilliard Equation in One Space Dimension

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Abstract: We study the limiting behavior of the solution of the Cahn-Hilliard equation using "energy-type methods". We assume that the initial data has a "transition layer structure", i.e., u^e w ± 1 except near finitely many transition points. We show that, in the limit as e —> 0, the solution maintains its transition layer structure, and the transition layers move slower than any power of e.

§1. Introduction

The Cahn-Hilliard equation

$$u_t = MA\,(-2KAu + W'(u)), \qquad (1.1)$$

where W is a double-well potential, was proposed in [CH] and [C] as a simple model for the processes of phase separation and coarsening in a binary alloy at fixed temperature. The function u represents the concentration of one of the metallic component of the alloy and hence the composition of the mixture if we assume that the total density is constant. The coefficient M in (1.1) is a positive kinetic coefficient, and K is the gradient energy coefficient; it is proportional to the square of the interaction distance, which is assumed to be small compared to characteristic dimensions on the laboratory scale. Since the total free energy of the mixture must decrease in time (for thermodynamical reasons) and since the mixture cannot pass through the wall of the container, the natural boundary conditions are

$$d_n u = d_n(-2KAu + f(u)) = 0 \quad \text{on dft, (or equivalently } d_n u = d_n Au = 0 \quad \text{on } \partial\Omega).$$

where d_n denotes the normal derivative on the boundary of the vessel Q. Here we are assuming that the alloy and the vessel walls do not react. These conditions also ensure that the mass of the mixture is preserved.

We describe now the dynamics modeled by the Cahn-Hilliard equation, beginning with the "spinodal decomposition". A uniform liquid mixture at high temperature and of concentration $\bar{u}$ is suddenly quenched to a lower temperature. If $\bar{u}$ lies in the spinodal interval (where $W''\{u) < 0$) corresponding to this new temperature, the uniform mixture $u = \bar{u}$ becomes very unstable and phase separation takes place. After spinodal decomposition, the alloy has a fine grained separated structure with u « u\ in the solid regions and

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$u \approx u_2$ in the liquid ones, where u_1 and u_2 are the positions of the two local minima of W. For mathematical results on the spinodal decomposition of the Cahn-Hilliard equation, we refer to [G]. (See [E] for the case of one-dimensional Cahn-Morral systems.)

The solution then evolves on a much slower time scale. In the one dimensional case, there exist stationary solutions with this fine grain property, but they are unstable (see e.g. [BF]). Therefore even though the solution will be typically drawn toward the associated unstable manifold ([G]), it will move very slowly away from this stationary solution until it reaches the neighborhood of a coarser-grained starionary solution. One expects this "cascading" process to continue, with coarser grains corresponding to even longer time scale ([BF]), and is known to the metallurgists as coarsening.

At the end of this coarsening process, the solution will generically approach a stable state, which minimizes the free energy. In one space dimension, convergence to a stationary solution has been proven by [Z]. Furthermore, [CGS] have shown that all local minimizers of the free energy are monotone functions. Consequently, grains must eventually coalesce to produce a (single transition) stationary state, but one must wait an extremely long time to observe this. It is the phenomenon of "slow motion" of interfaces which we study in this paper. (See below for an explanation of this slow process based on "energy".)

In order to study the dynamics of this problem in one-space dimension, we introduce the dimensionless Cahn-Hilliard equation

$$\begin{aligned} u_t &= \left(-\varepsilon^2 u_{xx} + 2(u^3 - u)\right)_{xx} \qquad \text{in } (a,b) \times \mathbf{R}^+ \\ u(x,0) &= u_0^\varepsilon(x) \qquad x \in (a,b) \end{aligned} \tag{1.2}$$

where ε^2 is the ratio of K to characteristic dimensions, and is therefore small. We shall study the limiting behavior as $\varepsilon \to 0$ of this problem with either a Neumann type boundary condition

$$u_x(a,t) = u_x(b,t) = 0 \quad \text{and } u_{xxx}(a,t) = u_{xxx}(b,t) = 0 \qquad t > 0, \tag{1.3}$$

or a Dirichlet type boundary condition of the form

$$u(a,t) = \pm 1, u(b,t) = \pm 1 \quad \text{and } u_{xx}(a,t) = u_{xx}(b,t) = 0 \qquad t > 0. \tag{1.4}$$

We assume that spinodal decomposition has occured, and that $u_0^\varepsilon(x) \approx \pm 1$ except near finitely many transition points.

Elliot and French [EF] and McKinney [Mc] have done numerical computations on this problem which suggest that the evolution of the transition layers is in fact exponentially slow ($e^{-\frac{C}{\varepsilon}}$). Using a dynamical systems approach, Alikakos, Bates and Fusco [ABF] have studied the above dynamical picture using (1.2) and (1.3), with the extra hypothesis that the initial data has only two transitions layers. Their approach is similar in spirit with that of Carr and Pego ([CP]) and Fusco and Hale ([FH], [F]) for the Allen-Cahn equation and they obtain similar metastable patterns. Their results on the evolution of the transition points show that the velocity of displacement of the transition points is of the order of $e^{-\frac{Cl}{\varepsilon}}$, where C is a constant and l is the minimum distance between the transitions in the initial data. The analysis of Alikakos, Bates and Fusco is based on an ansatz for the form

of u^ε, and on estimates for the linearization of (1.2) about this ansatz. We use "energy-type" estimates to show that as $\varepsilon \to 0$, the transition points move slower than any power of ε. Our result is weaker than that of [ABF] but our analysis has the advantages that it is far more elementary, it can handles any number of transition layers with essentially no changes and it places less stringent requirement on the form of the initial data. We can also handle both boundary conditions (1.3) and (1.4) with equal ease; however the Neumann type boundary condition (1.3) is more physical since in that case the mass is preserved.

The energy-type method we use was developed in [BK] in the context of slow evolution for the Allen-Cahn equation. This method relies essentially on the fact that the Allen-Cahn equation is a gradient flow in $L^2(a,b)$ for an appropriate Lyapunov functional F_ε. Fife [Fi] derived equation (1.2) with (1.3) as the gradient flow in $(H^1(a,b))'$ for the same Lyapunov functional F_ε

$$F_\varepsilon[v] = \int_a^b \frac{\varepsilon^2}{2}|v_x|^2 + \frac{1}{2}(v^2-1)^2\,dx. \tag{1.5}$$

(See [Fi] and Lemma 2.1.) Therefore the results that we obtain are very close to the results in [BK] for the Allen-Cahn equation and the methods of proof are similar. However extra difficulties arise from the fact that we have to work in either $(H^1(a,b))'$ or in $H^{-1}(a,b)$. Furthermore we cannot characterize the interfaces by following the zeroes of the function u^ε in time, as was done by [BK] for the Allen-Cahn equation. Instead we use some methods developed in [B].

The method developed in [BK] is based on Γ–convergence properties of the Lyapunov functional (1.5) (see [M1,2], [MM], [S], [KS], [LM]) which has been normalized so as to keep it positive and finite as $\varepsilon \to 0$. The results of [MM], [M1], or [S] asserts that in one space dimension the minimum energy of a transition between 1 and -1 is asymptotically

$$c_0 = \int_{-1}^{1} (1-s^2)\,ds = \frac{4}{3}. \tag{1.6}$$

In other words, if we consider a sequence $\{v^\varepsilon\}$ which converges in $L^1(a,b)$ to a limit v^0, and if v^0 has N transitions between 1 and -1, then letting

$$E_\varepsilon[v] = \int_a^b \frac{\varepsilon}{2}|v_x|^2 + \frac{1}{2\varepsilon}(v^2-1)^2\,dx \tag{1.7}$$

be the normalized energy, we have

$$\liminf_{\varepsilon\to 0} E_\varepsilon[v^\varepsilon] \geq N c_0, \tag{1.8}$$

with equality if the sequence $\{v^\varepsilon\}$ is properly chosen. The important step in this approach is an improvement of this result. The following error estimate for (1.8) can be found in [BK]:

$$E_\varepsilon[v^\varepsilon] \geq N c_0 - C\varepsilon^k, \tag{1.9}$$

for any $k > 0$ and for ε sufficiently small. (See Proposition 2.3.)

The idea of the method is that the velocity of the transition points is related to the dissipation of energy through the equality:

$$E_\varepsilon[u^\varepsilon](0) - E_\varepsilon[u^\varepsilon](T) = \varepsilon^{-1} \int_0^T \|u_t^\varepsilon\|_{-1}^2 \, dx dt \tag{1.10}$$

where $\|\cdot\|_{-1}$ denote either the $(H^1)'$ or the H^{-1} norm. (See Lemma 2.1.) As essentially all of the energy is accounted for by the *existence* of the transition via the lower bound (1.9), there is very little excess energy to be dissipated in the motion of the transition points and hence the speed will be very slow.

In higher space dimensions, one expects a very different behavior. We refer to Pego [P] for formal results in this case.

Finally the results in this paper carry easily to the more general equation

$$u_t = (-\varepsilon^2 u_{xx} + W'(u))_{xx}$$

where W is a bistable potential with wells of equal depth. The case of wells of non equal depth can be reduced to this case by a linear transformation. We focus on (1.2) only for the sake of simplicity.

§2. A bound on the time derivative

We consider only initial data with a "transition layer structure"; more precisely we fix an integer $N \geq 1$ and we let

$$v(x) = \pm 1 \text{ a.e.}$$

be a piecewise constant function with exactly N discontinuities. We suppose that

$$\lim_{\varepsilon \to 0} u^\varepsilon(x,0) = v(x) \qquad \text{in } L^1(a,b) \tag{2.1}$$

and moreover that for all sufficiently small ε

$$E_\varepsilon[u^\varepsilon](0) \leq N c_0 + C\varepsilon^k, \qquad k \in \mathbf{N} \tag{2.2}$$

with c_0 as in (1.6), and E_ε as in (1.7). The first condition gives the number of transitions of the initial data and their relative positions as $\varepsilon \to 0$. The second condition requires "efficient" transitions, i.e. that the excess energy in making the transitions is at most $C\varepsilon^k$ over the minimum possible (see Proposition 2.3). Given any such function v and any $k > 0$, one can easily construct such initial data provided $\varepsilon \leq \varepsilon_0(k)$ by arguing as in [S] or [M1].

Next we recall definitions of norms in $(H^1)'$ and in H^{-1}.

(i) Norm in $(H^1)'$. We use this norm when the boundary condition is given by (1.3). Let $w \in \{v \in L^2(a,b), \int_a^b v(x)\,dx = 0\}$. We let ψ be the solution of

$$\begin{aligned} -\psi_{xx} &= w \qquad \text{in } (a,b) \\ \psi_x(a) &= \psi_x(b) = 0, \end{aligned} \tag{2.3}$$

which satisfies

$$\int_a^b \psi(x)\,dx = 0.$$

Then

$$\|w\|^2_{-1,N} := \int_a^b \psi_x^2. \tag{2.4}$$

For later purposes, we define $Z_N := -\psi_x$, that is

$$Z_N(x) = \int_a^x w(y)\,dy. \tag{2.5}$$

(ii) Norm in $\mathbf{H^{-1}}$. We use this norm when the boundary condition is given by (1.4). Let $w \in L^2(a,b)$ and define ψ by

$$\begin{aligned} -\psi_{xx} &= w \qquad \text{in } (a,b) \\ \psi(a) &= \psi(b) = 0. \end{aligned} \tag{2.6}$$

Then

$$\|w\|^2_{-1,D} := \int_a^b \psi_x^2 \tag{2.7}$$

Here also we introduce the function $Z_D(x) := -\psi_x(x)$, which is given by

$$Z_D(x) = \int_a^x w(y)\,dy - \frac{1}{b-a}\int_a^b dy \int_a^y w(s)\,ds. \tag{2.8}$$

For the remainder of this paper we indiscriminently use $\|\cdot\|_{-1}$ to mean either the $(H^1)'$ or the H^{-1} norm. Also for simplicity of notation we often write $u(t)$ instead of $u^\varepsilon(x,t)$.

In addition to (2.1) and (2.2), we suppose for the sake of simplicity that:

$$u^\varepsilon(\cdot,0) \in H^4(a,b), \text{ and satisfy (1.3) or (1.4).} \tag{2.9}$$

This hypothesis ensures that $u^\varepsilon \in C(\mathbf{R}^+; H^2) \cap L^2(0,T;H^4)$, $\forall T > 0$ (see [T]) and that u_t and u_{xt} are square integrable.

Finally, keeping in mind (2.5) and (2.8), we define in the case of the Neumann boundary condition (1.3)

$$Z(x,t) = \int_a^x (u(y,t) - \overline{u}(t))\,dy, \tag{2.10}$$

where

$$\overline{u}(t) = \frac{1}{b-a}\int_a^b u(x,t)\,dx,$$

while for the Dirichlet condition (1.4), we define

$$Z(x,t) = \int_a^x u(y,t)\,dy - \frac{1}{b-a}\int_a^b dy \int_a^y u(s,t)\,ds \tag{2.11}$$

Then it follows from (2.4), (2.7) and the fact that $j\,u\% = 0$ for the Neumann boundary condition (1.3), that $f(Z_t)^2 dx = \mathrm{I\,M}\wedge$.

Now we can prove a key lemma which says that the functional

$$E_\epsilon[u^\epsilon](t) = \int_a^b \frac{\epsilon}{2}|u_x^\epsilon|^2 + \frac{1}{2\epsilon}\left((u^\epsilon)^2 - 1\right)^2 dx \tag{2.12}$$

is a Lyapunov functional.

LEMMA 2.1. *Let u^ϵ be a solution of (1.2) with either boundary condition (1.3) or (1.4), and assume that* $\mathbf{tz^*(x,0)}$ *satisfy (2.1), (2.2) and (2.9). Then*

$$\frac{d}{dt}E_\epsilon[u^\epsilon](t) = -\frac{1}{\epsilon}\|u_t^\epsilon\|_{-1}^2.$$

PROOF:

$$\begin{aligned}
\frac{d}{dt}E_\epsilon[u^\epsilon](t) &= \int_a^b \epsilon u_x u_{xt} + \frac{1}{\epsilon}(u^2-1)uu_t\,dx \\
&= \int_a^b \left(-\epsilon u_{xx} + \frac{2}{\epsilon}(u^3 - u)\right)u_t\,dx \\
&= \int_a^b \left(-\epsilon u_{xx} + \frac{2}{\epsilon}(u^3 - u)\right)\left(-\epsilon u_{xx} + \frac{2}{\epsilon}(u^3 - u)\right)_{xx} dx \\
&= \frac{1}{\epsilon}\int_a^b (2\%)^2 dx \\
&= -\frac{1}{\epsilon}\|u_t\|_{-1}^2.
\end{aligned}$$

The main Proposition of this section gives a bound on the time derivative of u^ϵ and hence a control on the speed of the transitions. It is the building block for our results and it is the only technically intricate part of this approach.

PROPOSITION 2.2. **Assume** *that the initial data* $\mathbf{u^c(x,0)}$ *satisfy (2.1), (2.2) for some choice of* $k > 0$, *and (2.9). Let* $0 < 6 < 1$ *be an arbitrary constant, then there exist positive constants C\ and C2 such that*

$$\int_0^{\frac{C_1}{\epsilon^{k+\delta}}} \|u_t^\epsilon\|_{-1}^2\,dt \le C_2\epsilon^{k+1} \tag{2.13}$$

for all **sufficiently** *small e. The values of C\ and* $\mathbf{C_2}$ *depend only on v and k but not on e.*

Another Proposition is needed before we can prove Proposition 2.2. This Proposition contains Proposition 2.1 and Lemma 4.2 in Bronsard-Kohn [BK], and gives, in particular, the lower bound (1.9) on the energy which is due to the presence of N transitions. This result is purely variational in character; it applies equally well for the Cahn-Hilliard equation and for the Allen-Cahn equation since both have the same Lyapunov functional (1.12). Let v be as in (2.1):

PROPOSITION 2.3 [BK]. *Let k be a positive integer. There exist positive constants δ_k and c_k with the following property: if w is an H^1 function on (a,b) satisfying*

$$\int_a^b |w - v|\, dx \le \delta_k$$

and

$$E_\epsilon[w] \le Nc_0 + \epsilon^k$$

then

$$E_\epsilon[w] \ge Nc_0 - c_k\epsilon^k:$$

Moreover, there exist positive constants ϵ_0, β sufficiently small and $x_i, y_i \in (a,b)$, $1 \le i \le N$, close to the jump discontinuities of v, such that

$$x_i < y_i < x_{i+1}, \qquad y_i - x_i < 4k\delta_k,$$
$$\left(w^2(x_i) - 1\right)^2 \le C\epsilon^k, \qquad \left(w^2(y_i) - 1\right)^2 \le C\epsilon^k,$$

and

$$\left(w^2(x) - 1\right)^2 \le \beta \text{ for } x \notin \cup_{i=1}^N (x_i, y_i),$$

when $\epsilon \le \epsilon_0$.

This Proposition says essentially that w has a transition layer structure and the interfaces are in the intervals (x_i, y_i).

We are now ready for the proof of Proposition 2.2.

PROOF OF PROPOSITION 2.2: We prove this Proposition in two steps: we first show that if $T = T_\epsilon$ satisfies

$$\int_0^{T_\epsilon} \int_a^b |Z_t^\epsilon|\, dxdt \le \epsilon^\alpha \qquad \alpha = \frac{1-\delta}{2}, \tag{2.14}$$

where Z is given by either (2.10) or (2.11), then u^ϵ satisfies the property

$$\int_0^{T_\epsilon} \|u_t\|_{-1}^2\, dt \le C\epsilon^{k+1}. \tag{2.15}$$

Then we prove that (2.14) holds for $T_\epsilon = \dfrac{C}{\epsilon^{k+\delta}}$. (Here and throughout, C represents a constant that is independent of ϵ, whose value may change from line to line.)

In order to show (2.15) we use (2.14) to obtain the appropriate lower bound on the energy of $u^\epsilon(x, T_\epsilon)$. First using (2.14) we have

$$\int_a^b |Z^\epsilon(x,0) - Z^\epsilon(x,T_\epsilon)|\, dx \le \int_0^{T_\epsilon} \int_a^b |Z_t^\epsilon|\, dxdt,$$

which tends to zero as $\epsilon \to 0$. Moreover using (2.1) we find

$$\lim_{\epsilon \to 0} \int_a^b |Z_0(x) - Z^\epsilon(x,0)|\, dx = 0,$$

where Z_0 is defined as in (2.10) or (2.11), respectively for the boundary condition (1.3) or (1.4). We deduce that

$$\lim_{\varepsilon \to 0} \int_a^b |Z_0(x) - Z^\varepsilon(x, T_\varepsilon)|\, dx = 0. \tag{2.16}$$

Next it follows from Lemma 2.1 and (2.2) that for all $t \geq 0$

$$\int_a^b \frac{\varepsilon}{2} u_x^2 + \frac{1}{2\varepsilon}(u^2 - 1)^2\, dx \leq C \tag{2.17}$$

so that if we define

$$g(s) = \int_0^s |\tau^2 - 1|\, d\tau \tag{2.18}$$

and use the inequality $2|a||b| \leq a^2 + b^2$, we obtain

$$\begin{aligned} \int_a^b |\frac{d}{dx} g(u(x, T_\varepsilon))|\, dx &= \int_a^b |\left(u(x, T_\varepsilon)^2 - 1\right)|\, |u_x(x, T_\varepsilon)|\, dx \\ &\leq C. \end{aligned}$$

Moreover since $g(s) \approx Cs^3$ for large s,

$$|g(u)| \leq c_1 + c_2 \left(u^2 - 1\right)^2,$$

for a suitable choice of the constants c_1 and c_2; it follows using (2.17) that

$$\int_a^b |g(u(x,t))|\, dx \leq C.$$

Thus $g(u^\varepsilon(x, T_\varepsilon))$ is bounded in $BV(a,b)$ and (since $BV \hookrightarrow L^1$ compactly) there exist ε_n and $\chi \in L^1(a,b)$ such that

$$g(u^{\varepsilon_n}(x, T_{\varepsilon_n})) \to \chi(x) \text{ in } L^1(a,b) \tag{2.19}$$

and

$$g(u^{\varepsilon_n}(x, T_{\varepsilon_n})) \to \chi(x) \text{ a.e.}$$

as $\varepsilon_n \to 0$. Since g is strictly monotone, it follows

$$g^{-1}\left[g(u^{\varepsilon_n}(x, T_{\varepsilon_n}))\right] = u^{\varepsilon_n}(x, T_{\varepsilon_n}) \to g^{-1}(\chi(x)) \text{ a.e. in } (a,b). \tag{2.20}$$

Applying (2.16) and (2.17) we now claim that

$$u^{\varepsilon_n}(x, T_{\varepsilon_n}) \to v(x) \text{ in } L^4(a,b) \tag{2.21}$$

as $\varepsilon_n \to 0$. In the case of the Dirichlet boundary condition (1.4), this is immediate since u^ε are uniformly bounded in $L^4(a,b)$. In the case of the Neumann boundary condition (1.3), we have $u^\varepsilon(x,T_\varepsilon) - \overline{u^\varepsilon}(T_\varepsilon) \rightharpoonup v(x) - \overline{v}$ in $L^4(a,b)$ as $\varepsilon \to 0$, hence

$$u^\varepsilon(x,T_\varepsilon) \rightharpoonup v(x) + C_1 \text{ in } L^4(a,b). \tag{2.22}$$

Using (2.20) and (2.22), it follows that $u^{\varepsilon_n}(x,T_{\varepsilon_n}) \to v(x) + C_1$ a.e., whereas by (2.17) $(u^\varepsilon(x,T_\varepsilon))^2 \to 1$ a.e.. We conclude that C_1 must be zero.
It follows from (2.20) and (2.21) that $g^{-1}(\chi) = v$, and so using (2.19) we have

$$g(u^\varepsilon(x,T_\varepsilon)) \to g(v(x)) \text{ in } L^1(a,b) \text{ as } \varepsilon \to 0.$$

Again using that $g(s) \approx Cs^3$ for large s, we finally deduce that

$$u^\varepsilon(x,T_\varepsilon) \to v(x) \text{ in } L^1(a,b) \text{ as } \varepsilon \to 0. \tag{2.23}$$

Next we apply Proposition 2.3. By (2.23) there exists a positive constant ε_1 such that for all $\varepsilon \le \varepsilon_1$

$$\int_a^b |u^\varepsilon(x,T_\varepsilon) - v(x)|\,dx \le \delta_k$$

so that

$$E_\varepsilon[u^\varepsilon](T_\varepsilon)) \ge Nc_0 - c_k\varepsilon^k.$$

Using Lemma 2.1 and (2.2), we deduce that

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^{T_\varepsilon} \|u_t\|_{-1}^2\,dt &= E_\varepsilon[u^\varepsilon](0) - E_\varepsilon[u^\varepsilon](T_\varepsilon)\\ &\le Nc_0 + C\varepsilon^k - Nc_0 + c_k\varepsilon^k\\ &\le C\varepsilon^k,\end{aligned}$$

so that

$$\int_0^{T_\varepsilon} \|u_t\|_{-1}^2\,dt \le C\varepsilon^{k+1}.$$

Thus to prove (2.13), we must simply show that (2.14) holds with $T_\varepsilon = \dfrac{C}{\varepsilon^{k+\delta}}$. Either

$$\int_0^\infty \int_a^b |Z_t|\,dxdt \le \varepsilon^\alpha$$

and there is nothing to prove, or there exists $T_{1\varepsilon}$ such that

$$\int_0^{T_{1\varepsilon}} \int_a^b |Z_t|\,dxdt = \varepsilon^\alpha.$$

Then

$$\begin{aligned}\varepsilon^{\alpha} &= \int_0^{T_{1\varepsilon}} \int_a^b |Z_t|\, dx dt \\ &\leq \sqrt{T_{1\varepsilon}}\sqrt{b-a}\left(\int_0^{T_{1\varepsilon}} \int_a^b (Z_t)^2\, dx dt\right)^{\frac{1}{2}} \\ &= \sqrt{T_{1\varepsilon}}\sqrt{b-a}\left(\int_0^{T_{1\varepsilon}} \|u_t\|_{-1}^2\, dt\right)^{\frac{1}{2}}.\end{aligned}$$

Using the implication $\{(2.14) \Rightarrow (2.15)\}$ we deduce that

$$\varepsilon^{\alpha} \leq \sqrt{T_{1\varepsilon}} C \varepsilon^{\frac{k+1}{2}}$$

so that using $\alpha = \dfrac{1-\delta}{2}$

$$\begin{aligned}T_{1\varepsilon} &\geq C\varepsilon^{2\alpha-k-1} \\ &\geq C\varepsilon^{-(k+\delta)}.\end{aligned}$$

§3. Slow motion

In this section, we deduce the slow motion of the profile of u from Proposition 2.2 ; we show that "nothing happens on a time scale of order ε^{-k+1} ":

THEOREM 3.1. *Assume that the initial data $u^{\varepsilon}(x,0)$ satisfies (2.1), (2.2) for some $k > 0$ and (2.9). Then for any $m > 0$ and $0 < \eta < 1$*

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq \varepsilon^{-k+\eta} m} \|u^{\varepsilon}(t) - v\|_{L^1} = 0 \tag{3.1}$$

We first prove the following inequality which will be needed for the proof of the Theorem.

LEMMA 3.2. *Let w be in $H^1(a,b)$. If $w \in H_0^1(a,b)$ define Z by (2.8), or else define Z in a similar way as (2.10). Then*

$$\int_a^b |w(x)|^2\, dx \leq \left(\int_a^b |Z(x)|^2\, dx\right)^{\frac{1}{2}} \left(\int_a^b |w_x(x)|^2\, dx\right)^{\frac{1}{2}}.$$

PROOF: Using that $Z_x(x) = w$, and the properties of Z and w, we find

$$\begin{aligned}\int_a^b |w(x)|^2\, dx &= \int_a^b Z_x(x)\, w(x)\, dx \\ &= [Z(x)\, w(x)]_{x=a}^{x=b} - \int_a^b Z(x)\, w_x(x)\, dx \\ &\leq \left(\int_a^b |Z(x)|^2\, dx\right)^{\frac{1}{2}} \left(\int_a^b |w_x(x)|^2\, dx\right)^{\frac{1}{2}}.\end{aligned}$$

We are now ready for the proof of Theorem 3.1.

PROOF OF THEOREM 3.1: First we remark that Proposition 2.2 is equivalent to

$$\int_0^{\frac{C_1}{\varepsilon^{k+\delta}}} \int_a^b (Z_t^\varepsilon)^2 \, dx dt \leq C_2 \varepsilon^{k+1}.$$

Therefore under the change in time scale

$$\tilde{Z}^\varepsilon(x,\tau) = Z^\varepsilon(x, \varepsilon^{-k+\eta}\tau)$$

we have

$$\int_0^{\frac{C_1}{\varepsilon^{\delta+\eta}}} \int_a^b (\tilde{Z}_\tau^\varepsilon)^2 \, dx d\tau \leq C_2 \varepsilon^{1+\eta}.$$

Thus for $\tau \leq m$

$$\begin{aligned}
\int_a^b \left(\tilde{Z}^\varepsilon(x,\tau) - \tilde{Z}^\varepsilon(x,0)\right)^2 dx &\leq \int_a^b \left(\int_0^\tau \tilde{Z}_\tau^\varepsilon \, d\tau\right)^2 dx \\
&\leq \tau \int_a^b \int_0^\tau \left(\tilde{Z}_\tau^\varepsilon\right)^2 d\tau dx \\
&\leq m \int_0^m \int_a^b \left(\tilde{Z}_\tau^\varepsilon\right)^2 dx d\tau \\
&\leq C_2 \varepsilon^{1+\eta} m, \qquad (3.2)
\end{aligned}$$

provided that ε is sufficiently small and that $m \leq \frac{C_1}{\varepsilon^{\delta+\eta}}$. Moreover, letting $\tilde{u}(x,\tau) = u^\varepsilon(x, \varepsilon^{-k+\eta}\tau)$, we have

$$E_\varepsilon[\tilde{u}](\tau) \leq C \text{ for all } \tau > 0$$

so that

$$\int_a^b (\tilde{u}_x(x,\tau))^2 \, dx \leq \frac{C}{\varepsilon} \text{ and } \quad \int_a^b (\tilde{u}_x(x,0))^2 \, dx \leq \frac{C}{\varepsilon}. \qquad (3.3)$$

Therefore Lemma 3.2, inequalities (3.2) and (3.3) yield

$$\begin{aligned}
\int_a^b (\tilde{u}(x,\tau) - \tilde{u}(x,0))^2 \, dx &\leq \left(\int_a^b \left(\tilde{Z}^\varepsilon(x,\tau) - \tilde{Z}^\varepsilon(x,0)\right)^2 dx\right)^{\frac{1}{2}} \\
&\qquad \left(\int_a^b (\tilde{u}(x,\tau) - \tilde{u}(x,0))_x^2 \, dx\right)^{\frac{1}{2}} \\
&\leq C\varepsilon^{\frac{\eta}{2}}.
\end{aligned}$$

We conclude

$$\sup_{0\leq\tau\leq m} \|\tilde{u}(\tau) - \tilde{u}(0)\|_{L^1(a,b)} \leq C\varepsilon^{\frac{\eta}{4}}, \qquad (3.4)$$

which together with (2.1) yields

$$\lim_{\varepsilon\to 0} \sup_{0\leq\tau\leq m} \|\tilde{u}(\tau) - v\|_{L^1(a,b)} = 0.$$

We remark that an argument analogous to the proof of Theorem 3.1 shows that u^ε is Hölder continuous in t: $\int_a^b |u^\varepsilon(x,t_2) - u^\varepsilon(x,t_1)| \, dx \leq C(t_2 - t_1)^{\frac{1}{4}}$.

§4. **The** transitions move **slowly**

In this Section, we give a more direct description of the motion of the "transition points" by showing that each "interface" remains in an arbitrarily small interval around the transitions of the function r i n a time scale of order £~*$^{+1}$,

We remark that since we have to deal with a fourth order equation, we cannot characterize the interface by using the zeroes of the function *u* as in [BK]. Instead we follow an approach of [B].

First we rescale in time as in Section 3, and let

$$\tilde{u}(x,\tau) = u^{\varepsilon}(x,\varepsilon^{-k+\eta}\tau).$$

By (3.4) and (2.1), $\sup_{o\le r\le m} ||\tilde{u}(r) - v||_{x,i(a|\&)}$ can be chosen arbitrarily small. Let $6 > 0$, and choose $eo(6_y m)$ so that for $e < So(6,m)$,

$$\sup_{0\le r\le m} ||\tilde{u}(r)-t;||_{Li(af6)} \le <\$*,$$

$$\text{with} 2fc<5* < \frac{\delta}{\sim} \tag{4.1}$$

Without loss of generality we assume that $v(a)` = —1$. Moreover, let $7;$ be the location of the jump discontinuities of v. Then for each r € [0,m], one can find points $Xi(r)$ and $yi(r)$ such that

$$*i(r) \in (7: - 2k6_k, 7t)\,, \text{ and } y;(r) \in (_{7i}, _{7l}- + 2k6_k) \tag{4.2}$$

with

$$\tilde{u}(x_{\dot{t}}(r), r) > 1 — ce^{\frac{1}{2}} \text{ and } \tilde{u}(j/_{\dot{t}}(r), r) < —1 + ce^{\frac{1}{2}*}, \text{ if i is even,}$$

$$\tilde{f}x(x,(r), r) < —1 + ce^{\frac{1}{2}} \text{ and } \tilde{u}(yi(r), r) > 1 — ce^{\frac{1}{2}*}, \text{ if i is odd,}$$

as in Proposition 2.3. (See the proof of Proposition 2.1 in [BK].) Since $\tilde{u}$ is continuous, it takes on all the values between $\tilde{u}(x;,(r),r)$ and $\tilde{f}x(j/i(r),r)$ in the interval $(x_t(r), j/;(r))$. The next Proposition shows that $\tilde{u}$ has exactly N interfaces.

PROPOSITION 4.1. *Let*

$$VJ := \{\$ \in [—5?^{c_0}_{c_0} ir]\backslash 9(^{ue}(^{x}i^{T})) = {}^{s} f^{or} \&*> \textit{least } N+l \textit{ distincts values of} x\},$$

where g is* defined by *(2.18), then Meas(V$_z$*) < Ce

Before proving this Proposition, we prove the main Theorem of this Section. The content of this theorem is that the time it takes for the interfaces to move a distance 6 is extremely long.

THEOREM 4.2. ***Given 6 > 0, m > 0 and 0 < 77 < 1, choose eo(S,m) small enough that (4.1) holds for e < eo(6, m). Then for such e and for all t ≤ m£~*$^{+l?}$, we have***

$$|*.(<) - 7.1 < I \text{ and } |y,(<) - \cdot_{7},| < |, \quad \_J<i\le N.$$

PROOF: It follows from (4.1) and (4.2) that

$$x_i(t) \in (\gamma_i - 2k\delta_k, \gamma_i), \text{ and } y_i(t) \in (\gamma_i, \gamma_i + 2k\delta_k),$$

for $0 \le t \le m\varepsilon^{-k+\eta}$, and hence

$$|x_i(t) - \gamma_i| < 2k\delta_k < \frac{\delta}{2}, \text{ and } |y_i(t) - \gamma_i| < 2k\delta_k < \frac{\delta}{2}$$

for such time.

We conclude from Theorem 4.2 that it takes a time of order at least $\varepsilon^{-k+\eta}$ to see an appreciable change in the position of the interfaces.

We now prove Proposition 4.1.

PROOF OF PROPOSITION 4.1: Using Lemma 2.1 and (2.2), we deduce

$$\begin{aligned} Nc_0 + \varepsilon &\ge E_\epsilon[u^\epsilon](\tau) \\ &\ge \int_a^b |\partial_x g(u(x,\tau))|\, dx. \end{aligned}$$

In the following we write x_i and y_i instead of $x_i(\tau)$ and $y_i(\tau)$ for simplicity of notation. Applying the co-area formula (see [Fe], [Mo], [Si]),

$$\int_a^b |\nabla h|\, dx = \int_{\mathbf{R}} H^{n-1}\{x | h(x) = s\}\, ds$$

which holds for any Lipschitz h and where $H^{n-1} = n-1$ dimensional Hausdorff measure, we find

$$\begin{aligned} Nc_0 + \varepsilon &\ge \int_{-\frac{c_0}{2}}^{\frac{c_0}{2}} (\#\text{ points x for which } g(u(x,\tau)) = \eta)\, d\eta \\ &= \int_{V_\epsilon^\tau} (\#\text{ points x for which } g(u(x,\tau)) = \eta)\, d\eta \\ &\qquad + \int_{(V_\epsilon^\tau)^c \cap [-\frac{c_0}{2}, \frac{c_0}{2}]} (\#\text{ points x for which } g(u(x,\tau)) = \eta)\, d\eta \\ &= \int_{V_\epsilon^\tau} (N+1)\, d\eta \\ &\qquad + \int_{(V_\epsilon^\tau)^c \cap [-\frac{c_0}{2}, \frac{c_0}{2}]} (\#\text{ points x for which } g(u(x,\tau)) = \eta)\, d\eta \\ &= meas(V_\epsilon^\tau) + \int_{V_\epsilon^\tau} N\, d\eta \\ &\qquad + \int_{(V_\epsilon^\tau)^c \cap [-\frac{c_0}{2}, \frac{c_0}{2}]} (\#\text{ points x for which } g(u(x,\tau)) = \eta)\, d\eta. \end{aligned}$$

Let

$$A := \left(g(-1+c\varepsilon^{\frac{1}{2}}), g(1-c\varepsilon^{\frac{1}{2}})\right).$$

Using the definition of the x_i's and y_i's, and that g is a continuous non decreasing function, it follows that for $\eta \in (g(u(x_i,\tau)), g(u(y_i,\tau)))$ for i odd and $\eta \in (g(u(y_i,\tau)), g(u(x_i,\tau)))$ for i even, there exists $x \in (x_i, y_i)$ such that $g(u(x,\tau)) = \eta$. Therefore the number of points such that $g(u(x,\tau)) = \eta$ is exactly N whenever $\eta \in (V_\varepsilon^\tau)^c \cap A$. In consequence, we find

$$\begin{aligned} Nc_0 + \varepsilon &\geq meas(V_\varepsilon^\tau) + \int_{V_\varepsilon^\tau \cap A} N\, d\eta \\ &\qquad + \int_{(V_\varepsilon^\tau)^c \cap A} (\#\text{ points x for which } g(u(x,\tau)) = \eta)\, d\eta. \\ &= meas(V_\varepsilon^\tau) + \int_{V_\varepsilon^\tau \cap A} N\, d\eta + \int_{(V_\varepsilon^\tau)^c \cap A} N\, d\eta \\ &= meas(V_\varepsilon^\tau) + N meas(A) \\ &= meas(V_\varepsilon^\tau) + N(c_0 - c\varepsilon). \end{aligned}$$

Hence

$$meas(V_\varepsilon^\tau) < C\varepsilon,$$

and the proof is complete.

This Proposition shows that $\tilde{u}$ has exactly N interfaces and that they are located respectively in the intervals $(x_i(\tau), y_i(\tau))$. One can show easily that the width of the interface is in fact $O(\varepsilon)$: let β be a positive constant in $(0,1)$ and let $A_\beta := \{x \,|\, |u(x,t)| < 1-\beta\}$. Then using Lemma 2.1, we have that

$$\begin{aligned} C &\geq E_\varepsilon[u^\varepsilon](t) \\ &\geq \int_{A_\beta} \frac{1}{\varepsilon}\left(u(x,t)^2 - 1\right)^2 dx \\ &\geq \frac{1}{\varepsilon}\beta^2 m(A_\beta). \end{aligned}$$

Hence $meas(A_\beta) \leq \frac{c}{\beta^2}\varepsilon$.

We remark also that the interfaces can be characterized in a more direct fashion: this proposition implies that there is a value $\eta^\varepsilon(\tau) \in [-\frac{3}{4}, \frac{3}{4}]$ and exactly N numbers $\delta_i^\varepsilon(\tau) \in (x_i(\tau), y_i(\tau))$ such that $u(\delta_i^\varepsilon(\tau), \tau) = \eta^\varepsilon(\tau)$. These points $\delta_i^\varepsilon(\tau)$ characterize the interfaces.

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