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"New Developments in Partial Differential Equations and Applications to Mathematical Physics"
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ON MATHEMATICAL TOOLS FOR STUDYING PARTIAL DIFFERENTIAL EQUATIONS OF CONTINUUM PHYSICS: H-MEASURES AND YOUNG MEASURES

Luc TARTAR

Abstract Some basic results concerning the mathematical tools of YOUNG measures and H-measures are described on a few examples. This includes questions of homogenization, in cases of small amplitude homogenization and in cases where nonlocal effects occur, and questions of propagation of oscillations and concentration effects for a scalar hyperbolic equation and for the wave equation. An application to the relaxation of a functional arising in micromagnetics is given, based on partial knowledge of the relations between YOUNG measures and H-measures.

LEARNING FROM THE PAST

While preparing my lecture for a conference celebrating the 600^{th} anniversary of the University of Ferrara, I thought about its title "New Developments in Partial Differential Equations and Applications to Mathematical Physics" and I wondered what could have meant Mathematical Physics six hundred years ago. In my understanding, Physics is mostly now concerned about Light and Matter in their different forms and, for a mathematician like me, it means a lot of questions in Partial Differential Equations, some much more difficult than others. I knew a question about Light which had been the subject of discussions five hundred years ago, as I had learned about a solution proposed by Leonardo da VINCI[1](1), and this was directly related to the subject of my talk. For what concerned Matter, I chose the question of motion of celestial bodies as adequate for that period, and thought the pioneer to be COPERNICUS.(2) Only after having prepared my lecture was I told that he had studied in Ferrara: he had mostly studied in Bologna and Padua, but he also obtained the degree of doctor of canon law in Ferrara in 1503.

Although not much of our Mathematics and Physics was known at that time, it was certainly not easy in those days to introduce with success a new idea. Not so long ago, PLANCK(3) did not perceive the situation to be much better when he wrote(4) *A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents finally die, and a new generation grows up that is familiar with it.*

To us the system of COPERNICUS seems but a small improvement on the old Ptolemaic system with its circles rolling on circles for explaining the apparent motion of the planets: the Sun indeed had been put at the center, instead of the Earth. This seems unnecessarily complicated, compared to the laws that KEPLER(5) derived using the precise observations of BRAHE.(6) One was still far from imagining

(1) Leonardo da VINCI, 1452 – 1519.
(2) Nicolaus COPERNICUS (Mikołaj KOPERNIK), 1473 – 1543.
(3) Max Karl Ernst Ludwig PLANCK, 1858 – 1947.
(4) Quoted by Clifford TRUESDELL, *Rational Thermodynamics*, second edition, Springer, 1984.
(5) Johannes KEPLER, 1571 – 1630.
(6) Tycho BRAHE, 1546 – 1601.

In those remote days one had to interpret natural phenomena without the help of fancy mathematical models; I learned of such an example in 1982 while I was spending a few days at Scuola Normale Superiore in Pisa. I was advised to go to Florence to see the exhibit of a manuscript of Leonardo da VINCI[1] and one of Leonardo's idea which struck me was related to a question, which I was not even aware of, concerning the light of the Sun reflected by the Moon: if we apply the law of reflection there is only one point of the Moon that can reflect the light from the Sun directly into our eye, and therefore we should only see a bright spot on the Moon and not the totality of the illuminated part. Leonardo's explanation was that there were seas on the Moon and that because of the waves, there was always the possibility to receive light from every illuminated point on the Moon. Of course there were skeptics arguing that there were no waves because there was no wind or, as we would say now, that there were no seas there, but the point is not that we know his hypothesis to be inaccurate, but that it contains the seed for an important improvement: if Leonardo had noticed that the size of the waves did not matter and that only the angles made by the waves were important, he might have discovered that the same result was expected for infinitesimal waves and deduced that rough surfaces reflect the light in every direction. Of course, we know all these facts because we have been told about them, not because we are more intelligent than Leonardo was.

Geometrical optics gave the impression that Light was a question for geometers, and certainly reflection of light or SNELL's[(18)] law of refraction of light is a matter of sines. DESCARTES,[(19)] to whom this law is attributed in France because he published it first, went further than finding it experimentally because he tried an explanation of that law from more basic principles, but his analogy with the propagation of sound in solids was wrong and justly criticized by FERMAT,[(20)] whose own derivation needed a finite propagation speed for light, a fact only accepted after RØMER's[(21)] explanation in 1676 of the anomalies in the eclipses of the moons of Jupiter. HUYGENS[(22)] later showed the wave nature of light, but MALUS's[(23)] discovery of polarized light cannot be explained in the same framework of a scalar wave equation. MAXWELL's[(24)] system of equations, introduced for unifying electromagnetism, does explain polarization of light, but does not explain some more recent discoveries.

Should one believe the theory of quanta of light imagined by PLANCK, the ondulatory nature of electrons shown by de BROGLIE,[(25)] the spin of the electron and the existence of the positron explained through DIRAC's[(26)] system of equations, or even accept the rules of quantum mechanics and SCHRÖDINGER's[(27)] equation, as it is not even an hyperbolic system. Indeed, is it still reasonable to prefer NEWTON's idea of action at distance and a world described by ordinary differential equations to EINSTEIN's idea where a particle only feels a local field but tells of its presence through a system of partial differential equations, hyperbolic and probably semilinear for having only the speed of light as characteristic velocity?

Should one ask again the obvious question: what is a particle, anyway? Or in a simpler way, what is the meaning of the physicists' saying that an electron cannot be a point because a point would radiate energy? What was meant in mathematical terms was that there is no solution of Maxwell's equation corresponding to a point mass, stationary or moving around. Was it then assumed that Maxwell's equation could also describe electrons, although it had been only designed for describing electromagnetic effects, i.e. Light? One could argue now that DIRAC's equation is considered more appropriate for discussing about electrons.

There is not so much reason to be surprised when one remembers that a ray of light is not a solution of the wave equation or of MAXWELL's equation either, and that it is only an approximation valid for high frequencies. With the new mathematical tool of H-measures which I have developed for questions

(18) Willebrord van SNEL van ROYEN, 1580 – 1626.
(19) René DESCARTES, 1596 – 1650.
(20) Pierre de FERMAT, 1601 – 1665.
(21) Ole Christensen RØMER, 1644 – 1710.
(22) Christiaan HUYGENS, 1629 – 1695.
(23) Etienne Louis MALUS, 1775 – 1812.
(24) James Clerk MAXWELL, 1831 – 1879.
(25) Louis Victor Pierre Raymond, duc de BROGLIE, 1892 – 1987.
(26) Paul Adrien Maurice DIRAC, 1902 – 1984.
(27) Erwin SCHRÖDINGER, 1887 – 1961.

mathematical equations for describing physical phenomena or designing experiments for discovering physical laws in the spirit of GALILEO,[7] but in order to give a rational derivation of the motion of the planets and explain the efficiency of KEPLER's laws, one had to wait for NEWTON's[8] law of gravitation and a crucial addition to Mathematics by LEIBNIZ[9] and NEWTON: Infinitesimal Calculus. Actually, the system of COPERNICUS was efficient enough, and we explain now its accuracy in relation with FOURIER's[10] result that any periodic motion is the sum of circular motions, but the accuracy of Celestial Mechanics based on the law of gravitation finally led to a further advance: Uranus was found in 1781 by a systematic survey of the sky by HERSCHEL,[11] and its irregular motion led ADAMS[12] and Le VERRIER[13] to apply the theoretical work of LAGRANGE[14] and to discover the position of Neptune, observed in 1846. If Pluton was discovered in 1930 in the same way, it went otherwise for the anomalies in the motion of Mercury that Le VERRIER had tried in 1855 to explain in a similar way: the 1919 expedition to the island of Príncipe led by EDDINGTON[15] for observing a total eclipse of the sun and measuring of how much the light coming from Mercury would be bent near the Sun confirmed the computations of EINSTEIN[16] based on his general theory of relativity.

For a mathematician, if the measurement in a physical experiment compares with accuracy to the prediction of a mathematical model, it does not prove that Nature follows that precise model. A mathematician knows that every continuous function on a compact interval can be approximated uniformly by polynomials, but he does not deduce that polynomials are important, as they could be replaced by many other classes of functions: he embeds the question into a more general framework, the theory of approximation. Even if physicists transform into dogma a set of rules which has given good results on a list of interesting physical questions, mathematicians should remain skeptical, and this has been well expressed by PENROSE when he wrote[17] *Quantum theory, it may be said, has two things in its favour and only one against it. First, it agrees with all the experiments. Second, it is a theory of astonishing and profound mathematical beauty. The only thing to be said against the theory is that it makes absolutely no sense. Indeed, such a bizarre collection of ideas would hardly have been put forward had it not been the case that an equally bizarre and seemingly contradictory collection of experimental facts had forced themselves on the attention of the physics community.*

It is important to notice that some complicated rule can be transformed into some quite simple result once a new mathematical theory has been developed. COPERNICUS having studied canon law at Ferrara, was aware of the complicated laws of a necessarily human Church, in opposition with God's laws for the motion of celestial bodies which he thought probably simple, but it was probably difficult for him to imagine that they were even simpler when expressed using 18^{th} century or even 20^{th} century Mathematics. Having this example in mind, it is then surprising to find so many physicists and even mathematicians who think that the world is described by ordinary differential equations, preferably in hamiltonian form, as if God did not know better. Why is it that they would not learn about partial differential equations and look for developing the 21^{th} century Mathematics which will certainly simplify most of what we think we have understood? Probably because their new religion forbids them to do so.

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(7) Galileo GALILEI, 1564 – 1642.
(8) Sir Isaac NEWTON, 1643 – 1727.
(9) Gottfried Wilhelm LEIBNIZ, 1646 – 1716.
(10) Baron Jean Baptiste Joseph FOURIER, 1768 – 1830.
(11) Sir William Frederick HERSCHEL (Friedrich Wilhelm), 1738 – 1822.
(12) John Couch ADAMS, 1819 – 1892.
(13) Urbain Jean Joseph Le VERRIER, 1811 – 1877.
(14) Comte Joseph Louis LAGRANGE (Giuseppe Luigi LAGRANGIA), 1736 – 1813.
(15) Sir Arthur Stanley EDDINGTON, 1882 – 1944.
(16) Albert EINSTEIN, 1879 – 1955.
(17) From Roger PENROSE's review of "The Quantum World" by J. C. POLKINGHORNE, The Times Higher Education Supplement, March 23, 1984. I am grateful to John M. BALL for having sent me a copy of that review.

AN EXAMPLE OF HOMOGENIZATION OF A WAVE EQUATION

In order to describe in a simple way the mathematical tools of YOUNG[2] measures and H-measures[3] we will consider a simple homogenization problem corresponding to the propagation of waves in a material with periodic microstructure, but we will consider a situation where the wave length is long compared to the period. The space variable will belong to R^N, and mathematicians like to use an arbitrary value for N, at least as long as the amount of work is not too excessive compared to the interest of the question. The period cell will be denoted by $Y = \{y = \sum_{i=1}^N \theta_i y^i,\ 0 \le \theta_i \le 1,\ i = 1, \ldots, N\}$ where $y^1, \ldots, y^N$ are linearly independent vectors of R^N, and its volume will be denoted $|Y|$; we will say that a measurable function f is Y-periodic if for almost all $z \in R^N$ and all $i = 1, \ldots, N$, one has $f(z + y^i) = f(z)$. The density ρ is assumed to be a measurable Y-periodic function satisfying

$$0 < \rho_- \le \rho(x) \le \rho_+ < \infty \text{ almost everywhere} \tag{1}$$

and the acoustic tensor a is assumed to be a measurable Y-periodic symmetric tensor satisfying

$$\alpha|\xi|^2 \le \sum_{i,j=1}^N a_{ij}(y)\xi_i\xi_j \le \beta|\xi|^2 \text{ for all } \xi \in R^N \text{ and almost all } y \in Y, \tag{2}$$

with

$$0 < \alpha \le \beta < \infty. \tag{3}$$

For a characteristic length $\varepsilon > 0$, we look for a solution u^ε of the wave equation

$$\frac{\partial}{\partial t}(\rho(\frac{x}{\varepsilon})\frac{\partial u^\varepsilon}{\partial t}) - \sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a_{ij}(\frac{x}{\varepsilon})\frac{\partial u^\varepsilon}{\partial x_j}) = 0, \tag{4}$$

where the equation is taken in the sense of distributions as our coefficients may be discontinuous, and we ask u^ε to satisfy the initial conditions

$$u^\varepsilon(x,0) = v(x), \frac{\partial u^\varepsilon}{\partial t}(x,0) = w(x) \tag{5}$$

where the data v and w are independent of ε and correspond to a finite energy $E(0)$,

$$\frac{1}{2}\int_{R^N}(|w|^2 + \sum_{i,j=1}^N a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j})dx = E(0) < \infty, \tag{6}$$

so that the solution will have the same finite energy

$$\frac{1}{2}\int_{R^N}(|\frac{\partial u^\varepsilon}{\partial t}|^2 + \sum_{i,j=1}^N a_{ij}\frac{\partial u^\varepsilon}{\partial x_i}\frac{\partial u^\varepsilon}{\partial x_j})dx = Constant = E(0). \tag{7}$$

The case of data depending upon ε and such that part of the energy is sent into wavelengths of order ε, for example $v(x)$ being replaced by $\varepsilon^{1-N/2}v(x/\varepsilon)$ and $w(x)$ being replaced by $\varepsilon^{-N/2}w(x/\varepsilon)$, is not entirely understood.

The question is to understand what the solution u^ε looks like and to identify its limit u^0 as ε tends to 0, usually in a weak topology. This particular problem offers no surprise and u^0 satisfies an effective equation of the same type

$$\frac{\partial}{\partial t}(\rho^{eff}\frac{\partial u^0}{\partial t}) - \sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a_{ij}^{eff}\frac{\partial u^0}{\partial x_j}) = 0, \tag{8}$$

of homogenization, one can give a precise meaning to what a beam of light is for the wave equation, or a beam of polarized light for MAXWELL'S equation. I have not checked what is the analogous result for DIRAC's equation, but the result will still be incomplete anyway because my approach cannot yet explain completely what happens for semilinear systems. In that spirit, as ray of lights are ideal objects which are useful for describing the solutions of the wave equation, electrons could be ideal objects useful for describing the solutions of DIRAC's equation, without being solutions themselves. If the obstacles due to the semilinear character were overcome, the equation of propagation for these objects would probably involve the mass and the spin of such an electron, the mass being probably entirely made of pure electromagnetic energy/[28]) with EINSTEIN's relation $c = me^2$, of course.

Quantum mechanics was invented for explaining the surprising effects of absorption and spontaneous emission at specific frequencies in experiments of spectroscopy. Because physicists thought that they had to find a list of numbers, which were thought to be proportional to $1/n^2 — 1/m^2$ in the case of hydrogen, they were quite happy when the spectrum of an operator related to their problem appeared to give all the $1/n^2$, and they invented an argument about eigenvalues being levels of energy together with a recipe for creating the desired operator in other situations. We know now that with a more accurate experimental setting one finds a density of absorption for large bands of frequency: there are indeed peaks, but they are quite far from being localized at well defined frequencies, and therefore the numbers that the physicists were trying to recover do not even exist.

A more reasonable approach for a mathematician would be to say that an experiment of spectroscopy consists in sending a wave into a gas which contains objects having a size comparable to the wavelength used, so that one expects some resonance effects to occur, an extra difficulty being that these objects move and that one does not even know what shapes they have. The mathematical problem is then to study the solutions of an hyperbolic system in an heterogeneous material when the characteristic size of the inhomogeneities is comparable to the wavelength. Needless to say, mathematicians do not yet know how to solve such a general question of homogenization, but the partial results already obtained show some analogy with what physicists say; for instance, the mathematical meaning for the absorption and spontaneous emission rules is that effective equations often have extra nonlocal terms in space and time. In some instances, the corrections to be added to macroscopic equations can be computed by integrating *H*-measures, and are therefore quadratic corrections, very similar to some which are computed by using the rules of quantum mechanics; in other cases, //-measures only enter into the first correction of an expansion, with some similarities with the summation of diagrams in quantum field theory.

I have avoided an important trap that many like falling into, which is the postulate that we cannot understand what happens at a microscopic level and that the laws of Physics are probabilistic by nature. In studying oscillating solutions of partial differential equations, many nonnegative measures do appear in a quite natural way but normalizing them and talking about probabilities will not change the perfectly deterministic framework implied by dealing with hyperbolic systems. Of course, with only a partial information about the oscillations at time zero, there is some uncertainty about what can be said about these oscillations at a later time, but the mathematical understanding of the question should also tell us a way to obtain more information.

There is still a lot to be done but once such a mathematical theory will be more developed, it should become the natural framework for discussing many of the physical phenomena which have puzzled physicists in the last century. Once this goal attained, physicists will probably have found new puzzling experimental facts, and a new mathematical theory may have to be developed, which will render elementary this one which I am trying to create, and the quest for simplicity will continue at a higher level of understanding.

([28]) I have thought for a long time that mass should only be a side effect of electromagnetism, but the first written argument which I read in that direction was an article of W. BOSTICK, based on MAXWELL'S equation coupled with some rules about quantum mechanics; in my opinion one should work with DIRAC's equation instead.

Some effective quantities are obtained by taking weak limits: potentials, electric or magnetic or velocity fields, induction or vorticity fields, densities of charge or mass or energy (they are coefficients of differential forms), while others are not obtained by taking weak limits: electrical or thermal conductivity, electric or magnetic permittivity, elastic properties, sound speed, and this happens for mixtures, composite materials, polycrystals.

Physicists often tend to disagree when mathematicians let some physical parameter ε converge to 0, but this is only a first step which consists in identifying what is the right topology for the various quantities involved in order to find a limiting equation whose solution will be near the physical one. One should also remember that there are infinitely many ways to imbed a given problem into a sequence of such problems, and that various scalings may correspond to different physical questions, each having its own limiting behaviour.

Homogenization is a mathematical theory whose first goal is to derive for each situation of interest what are the effective equations valid at a macroscopic level, assuming that one has a complete information about the microscopic level. Its second goal is to deduce what can be said under partial information about the microscopic level. Uncovering new mathematical objects which carry in a concise way some important information about the microstructure is a reward of that approach. The term microstructure is used when dealing with sequences converging only weakly in order to express that something is happening at a microscopic level; one also talks about an oscillating sequence of functions.

The term microgeometry is used when dealing with many oscillating sequences of functions constructed on the same geometrical pattern: an open set Ω of R^N is decomposed as a countable union of disjoint measurable subsets of Ω

$$\Omega = \cup_i \omega_i^\varepsilon, \tag{16}$$

and one only considers sequences U^ε with values in R^p of the form

$$U^\varepsilon(x) = \sum_i \chi_i^\varepsilon(x) V^i \tag{17}$$

where the functions χ_i^ε are the characteristic functions of ω_i^ε and V^i are elements of R^p, often belonging to a closed bounded set K.

The main question is to study how weak limits of U^ε, or effective quantities generated from it, depend upon the values V^i and on the information on the decomposition of Ω in ω_i^ε.

YOUNG measures describe the weak limits of U^ε for all possible choices of V^i. If

$$\chi_i^\varepsilon \text{ converges weakly * to } \theta_i \text{ in } L^\infty(\Omega) \tag{18}$$

so that one has

$$0 \le \theta_i(x) \le 1 \text{ and } \sum_i \theta_i(x) = 1 \text{ almost everywhere in } \Omega, \tag{19}$$

then for every continuous function F on R^p, one has

$$F(U^\varepsilon) \text{ converges weakly * to } \sum_i \theta_i F(V^i) \text{ in } L^\infty(\Omega). \tag{20}$$

If one defines the probability measure ν_x on R^p by

$$\langle \nu_x, G \rangle = \sum_i \theta_i(x) G(V^i) \tag{21}$$

for every continuous function G, then the YOUNG measure associated with the sequence U^ε is the measurable family of all these ν_x.

For a general uniformly bounded sequence of functions U^ε taking their values in a closed subset K of R^p, there is a subsequence and a measurable family of ν_x which are probability measures on K such that for every continuous function F on R^p, one has

$$F(U^\varepsilon) \text{ converges weakly * to } f \text{ in } L^\infty(\Omega) \tag{22}$$

with the same initial data

$$u^0(x,0) = v(x), \frac{\partial u^0}{\partial t}(x,0) = w(x), \tag{9}$$

where ρ^{eff} and a^{eff} are independent of x, due to the periodicity hypothesis, and satisfy

$$0 < \rho_- \leq \rho^{eff} \leq \rho_+ < \infty \tag{10}$$

and

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}^{eff}\xi_i\xi_j \leq \beta|\xi|^2 \text{ for all } \xi \in R^N. \tag{11}$$

Of course, the effective density ρ^{eff} is the average of ρ,

$$\rho^{eff} = \frac{1}{|Y|}\int_Y \rho(y)dy, \tag{12}$$

but the effective acoustic tensor a^{eff} is not in general the average of a.

Because of the periodicity hypothesis, one can give a simple algorithm for computing the effective acoustic tensor a^{eff}: it requires solving N elliptic problems on the unit cell Y.

For each choice of a vector $\lambda \in R^N$, there is a unique Y-periodic function $z_\lambda \in H^1_{loc}(R^N)$ solution of

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial y_i}(a_{ij}(y)(\frac{\partial z_\lambda}{\partial y_j} + \lambda_j)) = 0 \tag{13}$$

and satisfying the normalization condition

$$\int_Y z_\lambda(y)dy = 0. \tag{14}$$

Then for $i = 1, \ldots, N$ one has

$$(a^{eff}\lambda)_i = \frac{1}{|Y|}\int_Y (\sum_{j=1}^{N} a_{ij}(y)(\frac{\partial z_\lambda}{\partial y_j} + \lambda_j))dy. \tag{15}$$

Repeating this computation for N linearly independent vectors λ determines a^{eff}, which is positive definite (and symmetric as $a(y)$ is symmetric almost everywhere in Y).

WEAK CONVERGENCE, HOMOGENIZATION, YOUNG MEASURES

In the preceding example with a periodic structure, averaging a function on a period εY should be considered analogous to making a macroscopic measurement in an experiment where something happens at a microscopic level; in our example one sees the microscopic level by looking at a length scale of the order of ε. In a nonperiodic situation, averaging is replaced by weak convergence: a sequence f^ε converges weakly to f^0 as ε tends to 0 if $\int f^\varepsilon(x)\varphi(x)dx \to \int f^0(x)\varphi(x)dx$ for a suitable class of functions φ : ε is usually a length (or time) scale and f^0 will be called the macroscopic quantity corresponding to the microscopic quantity f^ε. Of course, there are macroscopic quantities without microscopic analog: there is no function H such that if f^ε converges weakly to f^0 and $g^\varepsilon = (f^\varepsilon)^2$ converges weakly to $g^0 \geq (f^0)^2$ one can deduce that $h^\varepsilon = H(f^\varepsilon)$ converges weakly to $g^0 - (f^0)^2$. This is analog to the situation of a gas when the microscopic velocity is not equal to the macroscopic velocity: the averaged kinetic energy is more than the kinetic energy computed from the macroscopic velocity and the difference is then called the internal energy, usually related to temperature which only has a macroscopic meaning.

A similar formula for the corrector $B^{(2)}$ exists in the case of nonperiodic microstructures and uses H-measures, but at the moment there is no general formula for expressing the correctors $B^{(r)}$ with $r \geq 3$.

If Ω is an open subset of R^N and U^ϵ is a sequence converging to 0 in $(L^2(\Omega))^p$ weak, then after extraction of a subsequence one can define a hermitian nonnegative $p \times p$ matrix of Radon measures μ in (x, ξ), with $\xi \in S^{N-1}$ the unit sphere in R^N; μ is called the H-measure associated to the subsequence and it enables to compute the weak * limits of products of the type $L_1(U_i^\epsilon)\overline{L_2(U_j^\epsilon)}$ where L_1 and L_2 are some "pseudo-differential" operators of order 0, U^ϵ being extended by 0 outside Ω.

The class of symbols of these "pseudo-differential" operators of order zero have the form

$$s(x,\xi) = \sum_{n=1}^{\infty} a_n(\xi) b_n(x) \tag{31}$$

with $a_n \in C(S^{N-1})$, the space of continuous functions on the unit sphere and $b_n \in C_0(R^N)$, the space of continuous functions converging to 0 at infinity, with

$$\sum_{n=1}^{\infty} ||a_n||.||b_n|| < \infty \tag{32}$$

where the norms are sup norms.

The standard operator S with symbol s is defined by

$$\mathbf{F}(Su)(\xi) = \sum_{n=1}^{\infty} a_n(\frac{\xi}{|\xi|}) F(b_n u)(\xi), \text{ almost everywhere in } \xi \in R^N, \text{ for } u \in L^2(R^N) \tag{33}$$

where $\mathbf{F}$ denotes the Fourier transform. A linear continuous operator L from $L^2(R^N)$ into itself is said to have symbol s if $L - S$ is a compact operator from $L^2(R^N)$ into itself.

With these notations, if L_1 and L_2 are operators with symbols s_1 and s_2, and

$$L_1(U_i^\epsilon)\overline{L_2(U_j^\epsilon)} \text{converges weakly to a measure } \nu, \tag{34}$$

one has

$$\langle \nu, \varphi \rangle = \langle \mu^{ij}, \varphi s_1 \overline{s_2} \rangle \text{ for every } \varphi \in C_c(\Omega), \tag{35}$$

the space of continuous functions with compact support in Ω.

An important consequence is the localization principle, which expresses how the H-measure is constrained by any differential information on the sequence U^ϵ: if A_{ij} are continuous functions in Ω and

$$\sum_{i=1}^{N} \sum_{j=1}^{p} \frac{\partial}{\partial x_i}(A_{ij} U_j^\epsilon) \to 0 \text{ in } H_{loc}^{-1}(\Omega) \tag{36}$$

then

$$\sum_{i=1}^{N} \sum_{j=1}^{p} \xi_i A_{ij}(x) \mu^{jk} = 0 \text{ for } k = 1, \ldots, p. \tag{37}$$

An early version of the theory, the compensated compactness theory, could only handle the case of constant coefficients and discuss the possible weak limits of quadratic quantities; H-measures give new results in other directions.

with

$$f(x) = (y_x, F) \text{ almost everywhere in } \quad \text{fi.} \tag{23}$$

What the YOUNG measures do for a mixture is to know what are the local proportions of all the materials used in that mixture.

SMALL AMPLITUDE HOMOGENIZATION I

YOUNG measures only see statistics and they cannot, except in dimension 1, help computing effective coefficients like the algorithm (13)-(14)-(15) for a periodic case. In a layered material, they lack the knowledge of an important geometric parameter, the direction of the layers. In order to compute some second order corrections in homogenization, I have introduced a new tool, which I naturally called *H*-measures[3], which is a measure in (x,£), where £ is a unit direction of an hyperplane. In the periodic case, they can be described by using the Fourier expansion of the coefficients: if we assume that

$$a_{ij}(y) = A_{ij} + \gamma b_{ij}(y) \tag{24}$$

where A is symmetric positive definite and 7 is small, then a^{e}^ is analytic in 7

$$a^{e}» = A + 7\wedge 5 \frac{1}{|MI|} \int_{JY} b(z)dz + y^2 B\wedge + y^3 B\wedge + \ldots\ldots, \tag{25}$$

as well as the functions $z\backslash$ solutions of (13)-(14), and therefore by an easy induction one can compute all the correctors B^ by using the Fourier coefficients of *b.*

In the case where Y is the unit cube, one has

$$b_m = \int_{JY} b\{z)e^{-2i*\{mz)}dz, \quad meZ^N \tag{26}$$

and

$$Kv) = \sum_{m\in Z^N} *me*'^{(m''}> \text{ in } L\backslash Y) \tag{27}$$

and an easy computation gives

$$B< =* \sum_{m\in Z^N\backslash 0} b\& \;'' \;\bullet \tag{28}$$

Using (26), the formula (28) can be expressed in terms of the 2-point correlation function

$$C(h) = \frac{1}{|Y|} \int_Y b(z + h)b(z)dz, \tag{29}$$

but it uses a singular integral with kernel

$$K(h) = \sum_{m\in Z^N\backslash 0} e^{2i\pi(m.h)} \frac{m \otimes m}{(Am.m)}. \tag{30}$$

For a sequence which is not periodic, one cannot define n-point correlation functions without the knowledge of a characteristic length. The i/-measures which I have introduced do not use any characteristic length in their definition, and therefore one cannot deduce from them the 2-point correlation function; however, there are situations where the complete knowledge of the 2-point correlation function is not necessary and where .//-measures contain all the desired information.

but not necessarily strongly in $L^2(\Omega)$, and such that $(u^\varepsilon)^2$ converges weakly to a function in $L^1(\Omega)$; for such a general sequence the H-measure need not be atomic in ξ.

If a scalar function u^ε is defined by

$$u^\varepsilon = \varepsilon^{-\frac{N}{2}} f(\frac{x-z}{\varepsilon}) \text{ with } z \in R^N \text{ and } f \in L^2(R^N) \tag{48}$$

then u^ε corresponds to the H-measure μ defined by

$$\langle \mu, \varphi \rangle = \int_{R^N} |\mathbf{F} f(\xi)|^2 \varphi(z, \frac{\xi}{|\xi|}) d\xi \tag{49}$$

for all continuous functions φ with compact support in $R^N \times S^{N-1}$. Such a sequence will be called a concentration effect at the point z, a more general concentration effect being a sequence u^ε converging weakly but not strongly to u^0 in $L^2(\Omega)$ and such that $(u^\varepsilon - u^0)^2$ converges weakly * to a measure which is singular with respect to the Lebesgue measure.

Of course, a general weakly converging sequence may show both oscillations and concentration effects.

For some partial differential equations of hyperbolic nature, one can measure in a quantitative way the propagation of oscillations and concentration effects, and this is done by deriving a partial differential equation in (x, ξ) for the H-measure μ. In the case of a first order scalar equation, let u^ε converge to 0 weakly in $L^2(\Omega)$, correspond to a H-measure μ and satisfy

$$\sum_{i=1}^{N} b_i(x) \frac{\partial u^\varepsilon}{\partial x_i} = f^\varepsilon \tag{50}$$

with f^ε converging strongly to 0 in $H^{-1}_{loc}(\Omega)$, the coefficients b_i being of class C^1, $i = 1, \ldots, N$. The localization principle implies then that μ satisfies

$$P(x, \xi)\mu = 0 \tag{51}$$

where

$$P(x, \xi) = \sum_{i=1}^{N} b_i(x)\xi_i. \tag{52}$$

In order to study the propagation properties for oscillations and concentration effects present in the sequence u^ε, we assume moreover that the coefficients b_i are real, and that f^ε converges weakly to 0 in $L^2(\Omega)$. Under these hypotheses, the H-measure μ satisfies the equation

$$\langle \mu, \{\varphi, P\} - \varphi div b \rangle = \langle 2Re\mu^{12}, \varphi \rangle \tag{53}$$

for all C^1 test functions φ with compact support in x, where $\{g, h\}$ denotes the Poisson bracket

$$\{g, h\} = \sum_{i=1}^{N} (\frac{\partial g}{\partial \xi_i} \frac{\partial h}{\partial x_i} - \frac{\partial g}{\partial x_i} \frac{\partial h}{\partial \xi_i}). \tag{54}$$

Equation (53) expresses a propagation effect along the bicharacteristics associated to $P(x, \xi)$

$$\frac{dx_i}{dt} = \frac{\partial P}{\partial \xi_i}; \frac{d\xi_i}{dt} = -\frac{\partial P}{\partial x_i} \text{ for } i = 1, \ldots, N, \tag{55}$$

the equation in ξ being homogeneous in ξ and inducing therefore an equation on the unit sphere. In the propagation equation (53), the source term μ^{12} corresponds to the H-measure associated to the sequence $(u^\varepsilon, f^\varepsilon)$, so that μ^{11} is μ; if $f^\varepsilon = Lu^\varepsilon$ where L has symbol s, then $\mu^{12} = \bar{s}\mu$, but if f^ε is nonlinear in u^ε it is not known yet how to describe what μ^{12} can be for a given μ.

Equation (53) can be supplemented with an initial condition for the H-measure μ.

We do not consider the case of periodic coefficients anymore. Let A be positive definite, and let $B^\varepsilon \rightharpoonup B^0$ in $L^\infty(\Omega; L(R^N, R^N))$ weak * and assume that $B^\varepsilon - B^0$ corresponds to a H-measure μ. For γ small and $f \in H^{-1}(\Omega)$, we solve

$$-div((A+\gamma B^\varepsilon)grad u^\varepsilon) = f \text{ in } \Omega \text{ with } u^\varepsilon \in H_0^1(\Omega). \tag{38}$$

After extraction of a subsequence (independent of f), u^η converges weakly to u^0 solution of

$$-div((A^{eff}(x;\gamma)grad u^0) = f \text{ in } \Omega \text{ with } u^0 \in H_0^1(\Omega), \tag{39}$$

where A^{eff} is analytic in γ

$$A^{eff}(x;\gamma) = A + \gamma B^0 - \gamma^2 M + O(\gamma^3) \tag{40}$$

and the correction M can be computed from the H-measure μ: for $i,\ j = 1, \dots,\ N$ and $\varphi \in C_c(\Omega)$, one has

$$\int_\Omega M_{ij}(x)\varphi(x)dx = \sum_{k,l=1}^N \langle \mu^{ik,lj}, \varphi(x)\frac{\xi_k \xi_l}{(A\xi.\xi)}\rangle. \tag{41}$$

In the particular case of a mixture of isotropic materials, with $A = a_0(x)I$ and $B^\varepsilon(x) = b^\varepsilon(x)I$, with

$$b^\varepsilon \rightharpoonup b_0 \text{ and } (b^\varepsilon - b_0)^2 \rightharpoonup \beta^2 \text{ in } L^\infty(\Omega) \text{ weak } *, \tag{42}$$

one deduces

$$Trace(M) = \frac{\beta^2}{a_0,} \tag{43}$$

and therefore if the effective material is isotropic, i.e. $A^{eff}(x;\gamma) = a^{eff}(x;\gamma)I$, or only isotropic at order 2 in γ, i.e. if $M(x) = m(x)I$, then one has

$$a^{eff} = a_0 + \gamma b_0 - \gamma^2 \frac{\beta^2}{N a_0} + O(\gamma^3). \tag{44}$$

This latter result was previously known under additional hypotheses of symmetry.

We will discuss later a similar result for isotropic linearized elasticity.

PROPAGATION OF OSCILLATIONS AND CONCENTRATION EFFECTS I

H-measures can be used to describe both oscillations and concentration effects.

If a scalar function u^ε is defined by

$$u^\varepsilon(x) = v(x, \frac{x}{\varepsilon}) \tag{45}$$

with $v(x,y)$ periodic in y with average 0 (and smooth enough), the period Y being the unit cube for simplification, and if the Fourier expansion of v in y is

$$v(x,y) = \sum_{m \in Z^N \setminus 0} v_m(x) e^{2i\pi(m.y)}, \tag{46}$$

then u^ε corresponds to the H-measure μ defined by

$$\langle \mu, \varphi \rangle = \sum_{m \in Z^N \setminus 0} \int_\Omega |v_m(x)|^2 \varphi(x, \frac{m}{|m|}) dx \tag{47}$$

for all continuous functions φ with compact support in $\Omega \times S^{N-1}$. Such a sequence will be called a periodically modulated oscillating sequence, a more general oscillating sequence being a sequence u^ε converging weakly and is here related to the linearized strain tensor e^ϵ defined by

$$e_{ij}^\epsilon = \frac{1}{2}\left(\frac{\partial u_i^\epsilon}{\partial x_j} + \frac{\partial u_j^\epsilon}{dx_i}\right) \text{ for } i,j = 1,\dots,\ N, \tag{63}$$

where $u^\epsilon(x)$ denotes the displacement of the point x. The particular constitutive relation corresponding to an isotropic material has the form

$$\sigma_{ij}^\epsilon = 2\mu^\epsilon e_{ij}^\epsilon + \lambda^\epsilon \delta_{ij} \sum_{k=1}^{N} e_{kk}^\epsilon \tag{64}$$

and the hypothesis of small amplitude means that

$$\mu^\epsilon = \mu_0 + \gamma\mu_1^\epsilon; \lambda^\epsilon = \lambda_0 + \gamma\lambda_1^\epsilon, \tag{65}$$

where for simplification we assume that μ_1^ϵ and λ_1^ϵ converge to 0 in $L^\infty(\Omega)$ weak *. Of course, γ is assumed small enough so that μ^ϵ and λ^ϵ uniformly satisfy the usual ellipticity condition required for applying the theory of homogenization, i.e. $\mu^\epsilon > 0$ and $2\mu^\epsilon + N\lambda^\epsilon > 0$. The homogenized equation will have the same form, but may correspond to a general anisotropic material with constitutive relation

$$\sigma_{ij} = \sum_{k,l=1}^{N} C_{ijkl}^{eff} e_{kl} \tag{66}$$

with the usual symmetries in $ijkl$. Of course the effective elasticity tensor C^{eff} is analytic in γ

$$C_{ijkl}^{eff} = \mu_0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda_0\delta_{ij}\delta_{kl} - \gamma^2 D_{ijkl} + O(\gamma^3) \tag{67}$$

and the coefficients D_{ijkl} can be expressed in terms of the H-measure ν associated to the sequence $(\mu_1^\epsilon, \lambda_1^\epsilon)$. The formulae involve the moments of order 4 of ν^{11}, the moments of order 2 of ν^{12} and the moment of order 0 of ν^{22}. The contribution of ν^{11} to D_{ijkl} is obtained by integrating

$$\frac{(\delta_{ik}\xi_j\xi_l + \delta_{il}\xi_j\xi_k + \delta_{jk}\xi_i\xi_l + \delta_{jl}\xi_i\xi_k)}{\mu_0} - 4\frac{\xi_i\xi_j\xi_k\xi_l(\mu_0+\lambda_0)}{\mu_0(2\mu_0+\lambda_0)}, \tag{68}$$

the contribution of ν^{12} to D_{ijkl} is obtained by integrating

$$2\frac{(\delta_{kl}\xi_i\xi_j + \delta_{ij}\xi_k\xi_l)}{(2\mu_0+\lambda_0)}, \tag{69}$$

and the contribution of ν^{22} to D_{ijkl} is obtained by integrating

$$\frac{\delta_{ij}\delta_{kl}}{(2\mu_0+\lambda_0)}. \tag{70}$$

In the very special case where the effective material is isotropic, or simply isotropic at order 2 in γ, i.e. if D_{ijkl} has the form

$$D_{ijkl} = M(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \Lambda\delta_{ij}\delta_{kl}, \tag{71}$$

then M and Λ can be computed using only the moments of order 0 of ν, i.e. from the weak * limits of the quantities $(\mu_1^\epsilon)^2$, $\mu_1^\epsilon\lambda_1^\epsilon$ and $(\lambda_1^\epsilon)^2$: one finds

$$M = \frac{4(N+1)\mu_0 + 2N\lambda_0}{N(N+2)\mu_0(2\mu_0+\lambda_0)} weak * lim(\mu_1^\epsilon)^2 \tag{72}$$

and

$$2M + N\Lambda = \frac{1}{N(2\mu_0+\lambda_0)} weak * lim(2\mu_1^\epsilon + N\lambda_1^\epsilon)^2. \tag{73}$$

PROPAGATION OF OSCILLATIONS AND CONCENTRATION EFFECTS II

Let us consider now the question of propagation of oscillations and concentration effects for a wave equation. For this we consider a sequence u^ε converging weakly to 0 in $H^1(R^N \times (0,T))$ and satisfying

$$\frac{\partial}{\partial t}(\rho(x)\frac{\partial u^\varepsilon}{\partial t}) - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u^\varepsilon}{\partial x_j}) = f^\varepsilon \tag{56}$$

with $grad u^\varepsilon$ corresponding to a H-measure μ; the localization principle implies that μ has the form

$$\mu^{ij} = \xi_i\xi_j\nu \text{ for } i,j = 1,\dots,\ N \tag{57}$$

with a nonnegative measure ν. Assuming that the functions ρ and a_{ij}, $i,\ j = 1,\dots,\ N$ are continuous and that f^ε converges strongly to 0 in H^{-1}_{loc}, the localization principle implies that ν satisfies

$$Q(x,\xi)\nu = 0 \tag{58}$$

with

$$Q(x,\xi) = \rho(x)\xi_0^2 - \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \tag{59}$$

where, as usual, t is replaced by x_0 with dual variable ξ_0. Notice that (58) is a way to describe the principle of equipartition of energy.

If we assume now that ρ is real positive and of class C^1, that the acoustic tensor a is real symmetric positive definite and of class C^1, then ν satisfies the propagation equation

$$\langle \nu, \{\varphi, Q\}\rangle = \langle 2Re\nu^{12}, \varphi\rangle \tag{60}$$

for all C^1 test functions φ with compact support in x. In (60), ν^{12} corresponds to some components of the H-measure associated to $(grad u^\varepsilon, f^\varepsilon)$. Equation (60) expresses a propagation effect along the classical light rays, which are the bicharacteristics associated to $Q(x,\xi)$,

$$\frac{dx_i}{d\tau} = \frac{\partial Q}{\partial \xi_i}; \frac{d\xi_i}{d\tau} = -\frac{\partial Q}{\partial x_i} \text{ for } i = 1,\dots,\ N, \tag{61}$$

the equation in ξ being homogeneous in ξ and inducing therefore an equation on the unit sphere. This result gives a mathematical framework for what is meant by a light beam at a point x_0 pointing in a direction ξ_0. The H-measure is neither a solution of the wave equation, nor a formal asymptotic solution for high frequency: it describes, in the limit of infinite frequency, the way to decide where the energy goes for any oscillating sequence of initial data with finite energy.

Equation (60) can be supplemented with an initial condition for the H-measure ν.

SMALL AMPLITUDE HOMOGENIZATION III

Once the framework of an application of H-measures has been developed, it can be generalized to various equations or systems at the only expense of having to perform some often tedious computations of linear algebra. Let us describe for example the question of small amplitude homogenization for the system of linearized elasticity in the case of a mixture of isotropic materials[4]. The stress tensor σ^ε satisfies the equilibrium equation

$$\sum_{j=1}^{N} \frac{\partial \sigma^\varepsilon_{ij}}{\partial x_j} = f_i \text{ for } i = 1,\dots,\ N, \tag{62}$$

where u^0 is the solution of

$$-div(gradu^0 + m^0\chi_\Omega) = 0 \text{ in } R^3. \tag{81}$$

By following the construction of ν and μ, one can put a topology on X that makes it compact and renders J_1 continuous, so that J_1 does attain its minimum on X. The initial problem is imbedded into this new one and corresponds to each ν_x being a Dirac mass and μ being 0. Of course, the preceding result is only a change of language for recasting the problem.

A precise description of X is not yet available, but a partial result shows that for a given YOUNG measure ν, there is a pair (ν, μ) belonging to X such that $\sum_{i,j=1}^{3} \langle \mu^{ij}, \xi_i\xi_j \rangle = 0$. This gives rise to a new relaxed problem defined on the set Y of all YOUNG measures, where one defines a functional J_2 by the formula

$$J_2(\nu) = \int_{R^3} |gradu^0|^2 dx + \int_\Omega \langle \nu_x, \varphi \rangle dx - \int_\Omega H_0.m^0 dx, \tag{82}$$

with m^0 and u^0 defined by (79) and (81). If Y is equipped with the weak * topology, then Y is compact and J_2 is lower semi-continuous and does attain its minimum. The initial problem is imbedded into this new one and corresponds to each ν_x being a Dirac mass.

We finally define Z to be the convex set of functions m^0 satisfying

$$|m^0(x)| \leq 1 \text{ almost everywhere in } \Omega, \tag{83}$$

and define the functional J_3 by

$$J_3(m^0) = \int_{R^3} |gradu^0|^2 dx + \int_\Omega (\psi(m^0) - H_0.m^0) dx, \tag{84}$$

where u^0 is defined by (81) and where ψ is the convex function defined on the unit ball by

$$\psi(m) = Inf_\nu \langle \nu, \varphi \rangle \text{ for all probability measures } \nu \text{ on } S^2 \text{ with center of mass m.} \tag{85}$$

If we equip Z with the weak * topology, then Z is compact and J_3 is lower semi-continuous and does attain its minimum. The initial problem is imbedded into this new one and corresponds to m^0 taking almost everywhere its values on the unit sphere.

If no solution of this last problem satisfies $|m^0(x)| = 1$ almost everywhere, then there are no classical solution mimimizing J, and minimizing sequences tend to create somewhere in Ω some tiny magnetic domains, the statistics of orientations for m being described by the YOUNG measure ν satisfying (85); the H-measure μ, sees another kind of information, like the orientations of the walls of these magnetic domains. Of course, having neglected the exchange energy, there is nothing to limit the size of the magnetic domains in this simplified model.

NONLOCAL EFFECTS INDUCED BY HOMOGENIZATION

For explaining some strange rules invented by physicists, like absorption and spontaneous emission of particles, it is important to realize that effective equations may contain nonlocal terms in space or/and time even when the microscopic level is described by classical partial differential equations; this phenomenon seems actually quite usual when dealing with hyperbolic equations. As a typical example, we consider the following problem, which has been studied by Youcef AMIRAT, Kamal HAMDACHE and Abdelhamid ZIANI[8], and by myself[9]:

$$\frac{\partial u^\epsilon}{\partial t} + a^\epsilon(y)\frac{\partial u^\epsilon}{\partial x} = f(x,y,t); u^\epsilon(x,y,0) = v(x,y), \tag{86}$$

where the sequence a^ϵ converges to a^0 in L^∞ weak *. If we assume that a^ϵ satisfies

$$0 < \alpha \leq a^\epsilon(y) \leq \beta < \infty \text{ almost everywhere} \tag{87}$$

A PROBLEM MIXING YOUNG MEASURES AND H-MEASURES

The solutions of many important problems seem to require the use of a mathematical object, yet to be developed, which will encompass both the YOUNG measures and the fT-measures. Partial results about the relations between YOUNG measures and i/-measures have been obtained with FYangois MURAT[5] and we can see how they can be used on an example.

We consider the model of micromagnetics of William BROWN[6], for a crystal occupying a bounded open domain ft of J?[3], as studied recently by Richard JAMES *k* David KINDERLEHRER[7]. After normalization, we consider the equation

$$—div(gradu + mxn) = 0 \text{ in } .ft^3 \quad (74)$$

where xn is the characteristic function of ft and m satisfies the constraint

$$|m(x)| = 1 \text{ almost everywhere in } ft, \quad (75)$$

and we seek m minimizing the quantity *J(m)* defined by

$$J(m)= [_{JR*} \backslash gradu\backslash^2 dx + [_{Jn} (<p(m) - H_0.m)dx. \quad (76)$$

In that model the magnetic field *H* is *gradu* and the magnetic induction field *B* is *H* + m, m corresponding to a spin effect, *<p* is an anisotropic energy due to the crystalline nature of the body and H_o is an applied magnetic field.

An exchange energy, usually taken to be quadratic in *gradm,* has been neglected.

The mathematical difficulty comes from the fact that the functional *J* is not lower semicontinuous for the natural topology for m, the $(L^{\circ\circ}(ft))^3$ weak * topology. Minimizing sequences might then develop oscillations, and this is in qualitative agreement with the experimentally observed formation of small magnetic domains, although there are still some quantitative discrepancies and it is not clear yet how good this model is. A better understanding of the relaxation of the functional *J* might shed some light upon this question.

If a sequence m^e converges to m° in $(L^{\circ\circ}(ft))^3$ weak *, the computation of the limit of $<p(m^£)$ requires more than the weak * limit of $m^€$ (as *tp* is not affine) and can be obtained from the YOUNG measure *v* associated to a subsequence; on the other hand the computation of the limit of $\backslash gradu^€|^2$ cannot be computed from the YOUNG measure *v* alone, but can be computed from the i/-measure /i associated to a subsequence of m^e — m°. One has

$$/ <p\{m^€)xl>\{x)dx -> / (v_g,<p)1>(z)dz \quad (77)$$

and

$$\int_{R^3} \backslash gradu'\backslash^2 iKx)dx-* f \backslash gradu^\circ\backslash\wedge(x)dx +]T^3 (\wedge,V(*K\wedge> \quad (78)$$

for every bounded continuous function *tp,* where tx° denotes the solution corresponding to m°, which only satisfies $|m°(x)| \leq 1$ for almost every *x* £ ft, as m°(x) is the center of mass of the probability v_x which lives on the sphere S^2, i.e.

$$m\wedge(y) = (y_{yi}xi) \text{ almost everywhere in ft for f = 1,2,3.} \quad (79)$$

The crucial question is then to understand what relations link the YOUNG measure *v* and the .ff-measure /i. Without answering this question, we can only describe an abstract relaxation problem where we seek to minimize the functional *J* defined in the following way. We let *X* be the space of all pairs (i/, *p)* for which there exists a sequence m^e satisfying the constraint (75), such that $m^€$ defines the YOUNG measure */, and such that m^e — m° defines the H-measure ji, where m° is defined by (79). We define the functional *J* on *X* by the formula

$$J_1(\nu,\mu) = I_{JR*} \backslash gradu^\circ\backslash^2 dx + \sum^3_{jZ_L} <//',\&£;) + _{Jet} l(v_x,<p)dx - f_{Jft} H*.m^\circ dx, \quad (80)$$

With these notations one has

$$Z_1(x,t) = -2\int_0^t M^*(x,s,t)R^*(x,s,t)ds \tag{99}$$

$$Z_2(x,t) = a^0(x,t)\int_0^t\int_0^t M^*(x,s,\sigma)R^*(x,s,t)R^*(x,\sigma,t)dsd\sigma. \tag{100}$$

In the equation for U_3 appears the function M_3 such that $b_\epsilon(x,t_1)b_\epsilon(x,t_2)b_\epsilon(x,t_3)$ converges in L^∞ weak * to $M_3(x,t_1,t_2,t_3)$, and so on. A complete analysis would probably lead to similar procedures than that followed by theoretical physicists when they deal with their beloved diagrams.

QUASI-CRYSTALS

By quickly cooling some Al-Mn alloys, experimental physicists[11] discovered in 1984 a resulting material whose X-ray diffraction pattern showed an unexpected five-fold symmetry or icosahedral symmetry. Tiling games following the work of Roger PENROSE have often been played by theoretical physicists to generate average five fold symmetries, but they cannot cast any light upon what could have happened inside the material to create this strange observation.

If one submits a material to a combination of heat and stress, the material will change its microstructure in order to adapt itself to these new constraints: this is in essence what the blacksmith's art is about. If the different components of a mixture are allowed to rearrange themselves locally in order to optimize some criterium, one should find the optimal configuration by studying all the effective coefficients corresponding to given proportions and then optimize the criterium on this set of effective coefficients. Unfortunately, even for two-component mixtures, there is not yet a complete description of such a set of effective coefficients.

A reasonable guess is that optimal effective coefficients are usually on the boundary of the set of effective coefficients. In the case of small variations of elastic properties (and using the oversimplification of linearized elasticity), we have seen that H-measures can be used for computing a better approximation of the effective coefficients and that the formula used the set of moments of order 4 of a nonnegative measure on the sphere S^{N-1}. Even if thin ribbons have often been considered, they should be first thought as three-dimensional bodies in order to understand what happens inside them and so we should consider the case $N = 3$, although many tiling games have been played in the plane.

Of course, the preceding analysis would be useful if it was true that the results of X-ray diffraction experiments were connected to H-measures; strictly speaking it cannot be so as H-measures are defined without using any characteristic length, while for X-ray diffraction experiments it is important to select a wavelength related to the characteristic atomic distances. A variant of H-measures using a characteristic length has been introduced by Patrick GERARD[12] under the name of semi-classical measures, and it may be more appropriate for that question. Nevertheless, it might be useful to understand the structure of the set of moments of order 4 of nonnegative measures on the sphere S^2, as it appeared in the formula for computing second order effects, and check if five fold symmetry or icosahedral symmetry can indeed be related to the structure of this set of moments, which I have studied then with Gilles FRANCFORT and François MURAT[13].

In dimension 3, there are 15 moments of order 4, so the set of moments of order 4 of nonnegative measures on S^2 is a closed convex cone of R^{15}. One can show that points on the boundary of this convex cone can be obtained as moments of at least one measure which is combination of at most 5 Dirac masses, while a minimum of 6 is required for points in the interior.

The 15 moments of an isotropic distribution do not correspond to a boundary point. If a nonnegative measure has this list of moments and is a combination of only 6 Dirac masses, then the 6 points (and their antipodes) must be the vertices of a regular icosaedron, a geometry which had already been used by Gilles FRANCFORT and François MURAT[14] for constructing isotropic mixtures. On the other hand, the 15 moments of a transversally isotropic distribution may correspond to a boundary point. In such a case, if a nonnegative measure has this list of moments and is a combination of only 5 Dirac masses, then the 5 points (and their antipodes) must be the vertices of two regular pentagons.

and if the sequence a^ϵ defines a YOUNG measure ν, then one can characterize the effective equation which the weak * limit u^0 of the sequence u^ϵ must satisfy (there is indeed only one such equation if one restricts attention to linear convolution equations in x and t). Another nonnegative measure π will appear in the equation and it is obtained from ν through a nonlinear transformation. With data which are measurable and bounded, the effective equation is

$$\frac{\partial u^0}{\partial t} + a^0(y)\frac{\partial u^0}{\partial x} + M = f(x,y,t); u^0(x,y,0) = v(x,y) \tag{88}$$

where the nonlocal term M has the form

$$M(x,y,t) = -\int_0^t \int_{[\alpha,\beta]} \frac{\partial^2 u^0(x-\lambda(t-s),y,s)}{\partial x^2} d\pi_y(\lambda) ds. \tag{89}$$

One sees that a transport equation with velocity constant along the flow but fluctuating in another direction induces a nonlocal effect in space and time; of course the effective equation does possess the finite propagation speed property.

More general questions and in particular nonlinear effects should be understood in order to explain turbulence effects for example.

Quite intricate hierarchies of corrections can occur in nonlinear situations, as can be seen with the following example[10], related to the equation

$$\frac{\partial u^\epsilon}{\partial t} + (a^0(x,t) + \gamma b^\epsilon(x,t))(u^\epsilon)^2 = f(x,t); u^\epsilon(x,0) = v(x). \tag{90}$$

Let us assume that a^0 and b^ϵ are uniformly bounded measurable functions, that

$$0 < \alpha \leq a^0(x,t) \text{ almost everywhere,} \tag{91}$$

$$b^\epsilon \text{ is uniformly equicontinuous in t and converges weakly * to 0,} \tag{92}$$

and that the functions f and v are nonnegative, measurable and bounded. Then when the parameter γ is small, the solutions u^ϵ stay nonnegative and are defined for all t. After extracting a subsequence one has

$$u^\epsilon \text{ converges to } U_0 + \gamma^2 U_2 + \gamma^3 U_3 + O(\gamma^4) \text{ in } L^\infty \; weak * \tag{93}$$

where U_0 is the solution of

$$\frac{\partial U_0}{\partial t} + a^0(x,t)(U_0)^2 = f(x,t); U_0(x,0) = v(x), \tag{94}$$

and U_2 is the solution of

$$\frac{\partial U_2}{\partial t} + 2a^0(x,t)U_0U_2 + Z_1 + Z_2 = 0; U_2(x,0) = 0, \tag{95}$$

where the nonlinear memory effects Z_1 and Z_2 are defined in the following way. One first extracts a subsequence such that

$$b^\epsilon(x,s)b^\epsilon(x,t) \text{ converges to } M_2(x,s,t) \text{ in } L^\infty \; weak*, \tag{96}$$

and then one defines M* and R* by the formulae

$$M^*(x,s,t) = M_2(x,s,t)U_0(x,s)U_0(x,t) \tag{97}$$

$$R^*(x,s,t) = U_0(x,s)exp(-2\int_s^t a^0(x,\tau)U_0(x,\tau)d\tau). \tag{98}$$

At the moment, there is no obvious reason why only //-measures with the least number of Dirac masses for a given list of moments would play a role in such a problem. Understanding the correction in 7^3, or introducing a characteristic length in the definition of //-measures, could shed some light on this question.

CONCLUSION

YOUNG measures is a simple mathematical tool for describing questions of local statistics, too simple for solving the important questions of homogenization which are so crucial for understanding Physics. The introduction of //-measures is just a step forward in the construction of new mathematical tools for understanding more of these questions of Physics; //-measures appear quite useful for the computation of many quadratic corrections, but a better mathematical tool still has to be developed.

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