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Anisotropic Motion of a Phase Interface

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Well-Posedness of the Initial Value Problem

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Introduction.

In a previous paper [AG] we formulated a mathematical model for the dynamics of a melting solid which places particular emphasis on the effect of surface phenomena. The purpose of this paper is to study the well posedness of the initial value problem which this model defines, as well as the asymptotic behaviour of its global solutions, whenever they exist.

1. The Model.

In the model we assumed that at time t the solid occupies a region $\Omega(t) \subset \mathbf{R}^2$ whose boundary $\partial\Omega(t)$ is a piecewise smooth (C^∞, say) curve, with a finite number of corners $P_1(t), \ldots, P_N(t)$. We derived two equations for the time evolution of $\Omega(t)$ (i.e. of its boundary). The first of these two equations is a relation between the normal velocity of any point Q on the boundary, and the orientation and curvature of the front $\partial\Omega(t)$ at this particular point Q, and time t. The other equation arises from the requirement that the *capillary force* be continuous at the corner points $P_1(t), \ldots, P_N(t)$.

To formulate the first law of motion, let Q be any point on the smooth part of $\partial\Omega(t)$. The angle which the normal to $\partial\Omega(t)$ at Q makes with the y - axis will be called θ; the curvature of $\partial\Omega(t)$ at Q will be denoted by K or k.

Assuming that the motion of the smooth part of the boundary is smooth in time, the normal velocity $V = V(Q,t)$ of $\partial\Omega(t)$ at Q and at time t is well defined, and the first equation of motion may be written as

$$g(\theta)K = \beta(\theta)V + F. \tag{1.1}$$

Here F is the relative free-energy density of the solid phase relative to the liquid phase and β and g are functions which come from the constitutive description of the interface. It follows from thermodynamical considerations that β, the *kinetic coefficient,* is nonnegative, and that $g(\theta)$ is given by

$$g(\theta) = f(\theta) + f''(\theta) \tag{1.2}$$

with $f(\theta)$ the *interfacial free energy.*

The other motive law requires that, at the corners $P_1(t), \ldots, P_N(t)$, the capillary force

$$\mathfrak{C}(\theta) = f(\theta)\mathfrak{T}(\theta) + f'(\theta)\mathfrak{N}(\theta) \tag{1.3}$$

be continuous, where

$$\mathfrak{N}(\theta) \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \mathfrak{T}(\theta) \stackrel{\text{def}}{=} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

(We shall use this notation throughout the paper.)

The relation $g(\theta)K = \beta(\theta)V + F$ is equivalent to a parabolic PDE whenever $g(\theta) \neq 0$; but this PDE is *backwards parabolic* if $g(\theta) < 0$. To obtain a well–posed initial–value problem we must therefore exclude domains $\Omega(t)$ whose boundaries contain points Q with $g(\theta(Q)) < 0$. If $g(\theta) > 0$ for all $\theta \in \mathbf{R}$, then this condition is vacuous, but if the interfacial free energy $f(\theta)$ is such that there are angles θ for which $g(\theta) < 0$, then we can only discuss domains $\Omega(t)$ for which $g(\theta(Q)) > 0$ at every point $Q \in \partial\Omega(t)$.

2. The Frank diagram and stable angles.

Throughout this paper we shall consider a fixed free-energy function $0 < f \in C^\infty(\mathbf{R}/2\pi\mathbf{Z})$. Associated with this function we have the *Frank diagram* $\mathcal{F}$, which is the locus of all points of the form

$$\mathcal{F}(\theta) = \left(\frac{\cos\theta}{f(\theta)}, \frac{\sin\theta}{f(\theta)}\right) = \frac{\mathfrak{N}(\theta)}{f(\theta)}$$

with $\theta \in \mathbf{R}$. The set of *stable angles* is defined by

$$\Theta = \{\theta \in \mathbf{R}/2\pi\mathbf{Z} | g(\theta) > 0\},$$

where $g(\theta) = f''(\theta) + f(\theta))$. It corresponds to the set of θ's for which the Frank diagram is convex at $\mathcal{F}(\theta)$; we refer to f as *stable* if $\Theta = \mathbf{R}/2\pi\mathbf{Z}$, so that $f(\theta) + f''(\theta) > 0$ for all θ.

In section 8 of [AG] we also defined the set of *globally stable angles*, Θ_{gs} to be the set of θ's for which $\mathcal{F}(\theta)$ is an extreme point of the Frank diagram (i.e. the set of points where $\mathcal{F}$ coincides with the boundary of its convex hull).

The capillary force $\mathfrak{C}(\theta)$ is given by

$$\begin{aligned}\mathfrak{C}(\theta) &= f(\theta)\mathfrak{T}(\theta) + f'(\theta)\mathfrak{N}(\theta) \\ &= -f(\theta)^2 \frac{d}{d\theta}\left\{\frac{\mathfrak{N}(\theta)}{f(\theta)}\right\},\end{aligned}$$

so that $\mathfrak{C}(\theta)$ points in the same direction as the tangent to the Frank diagram at $\mathcal{F}(\theta)$. In fact, the length of $\mathfrak{C}(\theta)$ is such that the tangent to the Frank diagram is given by

$$\{x \in \mathbf{R}^2 | x \times \mathfrak{C}(\theta) = 1\}$$

(where $x \times y = x_1y_2 - x_2y_1$ is the two-dimensional crossproduct.)

For each $k \geq 1$ and $\alpha \in (0,1)$ we define $\mathfrak{D}^{k,\alpha}$ to be the set of *all domains* $\Omega \subset \mathbf{R}^2$ *whose boundary* $\partial\Omega$ *is compact, piecewise* $h^{k,\alpha}$*, whose outward unit normal* $\mathfrak{n} = (\cos\theta, \sin\theta)$ *satisfies* $\theta \in \Theta$ *on all of* $\partial\Omega$*, and for which the capillary force* $\mathfrak{C}$ *is continuous on* $\partial\Omega$.

Here $h^{k,\alpha}$ means that the boundary is locally the graph of a function whose k-th derivative is "little-Hölder" continuous of exponent α.

Even though we require $\partial\Omega$ to be compact, the domain Ω itself may be unbounded. This allows Ω to be an "exterior domain."

We shall denote the set $\cap_{k\geq 2}\mathfrak{D}^{k,\alpha}$ of domains with piecewise C^∞ boundaries by $\mathfrak{D}^\infty$.

Since any $\Omega \in \mathfrak{D}^{k,\alpha}$ has a piecewise C^1 boundary, the continuity requirement of the capillary force is only relevant at the corners of $\partial\Omega$. Near such a corner $\partial\Omega$ will have two tangents, with unit normals $\mathfrak{n}_j = (\cos\theta_j, \sin\theta_j)$ $(j = 1,2)$; the continuity condition then says that $\mathfrak{C}(\theta_1) = \mathfrak{C}(\theta_2)$. In view of our interpretation of $\mathfrak{C}(\theta)$ as the tangent to the Frank diagram, we see that the capillary force will be continuous if and only if the tangents to $\mathcal{F}(\theta_1)$ and $\mathcal{F}(\theta_2)$ coincide at every corner (θ_1, θ_2); thus the allowable corners correspond to the *bitangents of the Frank diagram.*

Just as in [AG] we shall assume throughout this paper that the free energy $f(\theta)$ is regular (see [AG, section 8] for the precise definition of this term.) This implies, in particular, that the Frank diagram has at most a finite number of bitangents, and that for any corner (θ_1, θ_2) corresponding to such a bitangent one has $g(\theta_1), g(\theta_2) > 0$. It also implies that there are at most a finite number of corners (θ_1, θ_2); indeed, each corner corresponds to a bitangent, and the condition $g(\theta_1), g(\theta_2) > 0$ tells us that the curvature of the Frank diagram at $\mathcal{F}(\theta_j)$ $(j = 1,2)$ is nonzero, so that the Frank diagram intersects its bitangents in isolated points; in other words, the Frank diagram cannot have an entire line segment in common with one of its bitangents.

3. The main existence theorem.

A great part of the present paper will be devoted to the proof of the following theorem, which we had already announced in [AG] (see p.350, and p. 368).

Theorem 3.1. *Assume that the free energy $f(\theta)$ is regular, and that both $f(\theta)$ and $\beta(\theta)$ are positive smooth, i.e. C^∞ functions. Then, given any $\Omega_0 \in \mathfrak{D}^{2,\alpha}$ there exists a unique maximal family of regions $\Omega : [0, T_{\max}) \to \mathfrak{D}^{2,\alpha}$ whose boundary satisfies (1.1), and for which $\Omega(0) = \lim_{t\downarrow 0} \Omega(t) = \Omega_0$.*

For $0 < t < T_{\max}$ this solution is actually piecewise smooth, i.e. it satisfies $\Omega(t) \in \mathfrak{D}^\infty$.

If the solution only exists for a finite time, i.e. if $T_{\max} < \infty$, then one of the following must hold:

- $\mathbf{E}_1$ *The maximal curvature of $\partial\Omega(t)$ becomes infinite as $t \uparrow T_{\max}$,*
- $\mathbf{E}_2$ *The maximal curvature of $\partial\Omega(t)$ remains bounded, and the piecewise smooth curve $\partial\Omega(t)$ converges to some limit curve $\partial\Omega(T_{\max})$, but this limit curve has a self intersection,*
- $\mathbf{E}_3$ *The length of one of the smooth arcs in $\partial\Omega(t)$ tends to zero as $t \uparrow T_{\max}$.*

This theorem says that solutions exist for a short time, that they are smooth for all positive times, and that they can only become singular if the curvature blows up, the boundary develops a self intersection, or if an arc of the boundary vanishes. The latter can only happen if the free energy is not stable.

If the free energy is stable then a stronger version of this theorem was proved in [A1, part I]. The stronger conclusion still holds for the evolution equations we are considering, provided we add an extra hypothesis.

We say that a domain $\Omega \in \mathfrak{D}^{2,\alpha}$ has *convex corners* if each corner $P \in \partial\Omega$ has a neighborhood $B_\delta(P)$ such that $B_\delta(P) \cap \Omega$ is convex,

In [A1, part I] we introduced a quantity $\alpha_\delta(\Omega)$ which measures the variation of the tangent or unit normal to $\partial\Omega$. In our setting of domains in the flat, Euclidean plane we can define this quantity as follows:

$$\alpha_\delta(\Omega) \stackrel{\text{def}}{=} \sup \{ |\theta(P) - \theta(Q)| \mid P, Q \in \partial\Omega; \mathrm{dist}_{\partial\Omega}(P, Q) \le \delta \}.$$

Here $\mathrm{dist}_{\partial\Omega}(P, Q)$ denotes the distance between P and Q as measured along the curve. For any domain with C^1 boundary $\alpha_\delta(\Omega) \downarrow 0$ as $\delta \downarrow 0$ (this is equivalent to the uniform continuity of the unit normal); if one has a sequence of domains $\{\Omega_n\}$ with smooth boundaries, then $\alpha_\delta(\Omega_n) \le \alpha_*$ for some $\alpha_* < \pi$ and $\delta > 0$ implies that the curves $\partial\Omega_n$ are uniformly locally Lipschitz curves (see [A1, part I].)

The stronger version of theorem 3.1, analogous to the result in [A1, part I] is the following.

Theorem 3.2. *Let $\Omega : [0, T_{\max}) \to \mathfrak{D}^{2,\alpha}$ be a maximal solution of* (1.1), *and assume that $\Omega(t)$ is admissible and has convex corners for all* $0 < t <$

$T_{\max}$. *Then either the conditions* $\mathbf{E}_2$ *or* $\mathbf{E}_3$ *of the main existence theorem hold, or else*

$\mathbf{E}_4$ *For any* $\delta > 0$, $\limsup_{t\uparrow T_{\max}} \alpha_\delta(\Omega(t)) \geq \pi$.

Thus, if the curvature of $\partial\Omega(t)$ blows up, the boundary must loose its local graphlike character.

We refer to $\Omega \in \mathfrak{D}^{k,\alpha}$ as *admissible* if the outward normal $\mathfrak{n} = (\cos\theta, \in \theta)$ satisfies $\theta \in \Theta_{g,}$ on all of $\partial\Omega$. The difference between admissible domains with convex corners and arbitrary domains $\Omega \in \mathfrak{D}^{2,\alpha}$ will come up again in the chapter on the geometry of $\partial\Omega(t)$, where we shall prove the following theorem, which is known for stable free energies (cf. the work of Giga, Goto and Chen [GGC], as well as Evans and Spruck [ES], and also [A1]).

Theorem 3.3. **(Containment principle)** *Let* $\Omega_1, \Omega_2 : [0,T) \to \mathfrak{D}^{2,\alpha}$ *be admissible solutions of* (1.1) *with convex corners, for which the closure of* $\Omega_1(0)$ *is contained in* $\Omega_2(0)$. *Then the closure of* $\Omega_1(t)$ *is contained in* $\Omega_2(t)$ *for all* $0 < t < T$.

In addition we shall also show by a simple counterexample that this theorem does not hold if one of the two solutions $\Omega_{1,2}(t)$ is not admissible, even though the initial value problem remains well posed in this case (by theorem 3.1).

One would expect that the stronger existence theorem for admissible evolutions also breaks down for inadmissible evolutions, or for evolutions of domains without convex corners, i.e. one would expect that there exist smooth solutions $\Omega : [0, T_{\max}) \to \mathfrak{D}^{2,\alpha}$ of (1.1), with at least one concave corner, for which the curvature blows up, but for which $\alpha_\delta(\Omega(t))$ remains bounded from above by some $\alpha_* < \pi$. Unfortunately we have not been able to find an example that would prove this.

In general, a solution $\Omega : [0,T) \to \mathfrak{D}^{2,\alpha}$ which starts with convex corners only can develop a concave corner. However, if the initial domain $\Omega(0)$ is admissible, i.e. if all its tangents are strictly stable, then $\Omega(t)$ will also be admissible, and all of its corners will remain convex (see section 10.)

In the last two sections we consider the case of a smooth, strictly stable free energy, and we study the long time behaviour of a growing solution.

As we noted in [AG, section 6.1] there is a unique domain Ω_* (up to translation) that is stationary under the flow determined by (1.1); Ω_* is a dilation of the region

$$\Gamma(f) = \left\{ x \in \mathbf{R}^2 \mid x \cdot \mathfrak{N}(\theta) \leq f(\theta) \forall \theta \in \mathbf{R} \right\}, \tag{3.1}$$

the *Wulff region for* $f(\theta)$. In section 15 we discuss the stability of Ω_*: we show that it is an unstable steady state for (1.1), and we show that it has a one dimensional unstable manifold. Section 15 is named after its final result, the four node theorem. This theorem (15.4) asserts that for any solution $\Omega : [0, \infty) \to \mathfrak{D}^{2,\alpha}$ that converges to Ω_* the boundary $\partial\Omega(t)$ has at least four nodes, i.e. for each $t \geq 0$ the velocity V vanishes at at least four different points on $\partial\Omega(t)$.

Finally, we study the asymptotic behaviour of one of the two solutions on the unstable manifold of Ω_*. The solution that we consider is given by a smooth convex domain $\Omega_+(t)$ that expands to fill the entire plane; it has an asymptotic shape, which we identify as the Wulff region for $\beta(\theta)^{-1}$. Then, using the containment principle (theorem 3.3), we show that *any* solution $\Omega(t)$ which eventually fills up the whole plane has the same asymptotic shape as $\Omega_+(t)$. The precise result is

Theorem 3.4. *Let $\Omega : [0, \infty) \to \mathfrak{D}^{2,\alpha}$ be a solution of (1.1) with $\Omega(0) \supset \Omega_*$. Then $t^{-1}\Omega(t)$ converges as $t \to \infty$ to a dilation of the Wulff region for $\beta(\theta)^{-1}$.*

A result of this type was first established by H. M. Soner [So], who shows, using a maximum principle, that for $\Omega(0)$ of sufficiently large area, $a_1(t)\Omega_w \subset \Omega(t) \subset a_2(t)\Omega_w$ with Ω_w the Wulff region for $\beta(\theta)^{-1}$; Soner shows that the functions $a_{1,2}(t)$ have the same asymptotic growth rate, in the sense that $a_2(t) - a_1(t) = o(t^{-1})$ as $t \to \infty$. Our result is slighty stronger: we do not assume that the polar diagram of β be convex, as Soner does, and we show convergence of the corresponding support functions (cf. lemma 16.1).

Giga, Goto and Chen [GGC] and Soner [So] have formulated a theory of weak "viscosity" solutions for equations like (1.1), a theory applicable to our model if the free energy is strictly stable. A similar theory was simultaneously created by Evans and Spruck [ES], for the mean curvature flow, which arises when $F = 0, f(\theta) \equiv 1, \beta(\theta) \equiv 1$. The theories of Giga, Goto & Chen, Soner and Evans & Spruck are more general, in the sense that they apply to to the motion of n dimensional hypersurfaces of $\mathbf{R}^{n+1}$, but in their present state are applicable only to stable free energies.

Well-posedness.

In [A1] a short time existence and uniqueness theorem for initial value problems like (1.1) was proved. If the free energy function $f(\theta)$ is such that $g(\theta) > 0$ for all θ, then the results in [A1] may be used to obtain the short time existence of solutions to the initial value problem (1.1). See [AG, section 7.1] for a precise formulation of the resulting theorem. The purpose of this section is to show how one can adapt the arguments in [A1] to obtain a similar existence result for the initial value problem for a free-energy that is not stable. The precise result which we shall prove in sections 4, 5, 6, and 7 is stated more precisely as follows:

Theorem(local existence). *Let $\Omega_0 \in \mathfrak{D}^{2,\alpha}$ be a given domain. Then there is a $T = T(\alpha, \Omega_0) > 0$ and an evolving family of regions $\Omega(t)$, $(0 \le t < T)$ with $\Omega(0) = \Omega_0$, that satisfies (1.1). For positive t, $\partial\Omega(t)$ is actually C^∞ smooth. The solution is unique within the class of solutions $\Omega(t)$, $(0 \le t < T)$ with $\Omega(t) \in \mathfrak{D}^{2,\alpha}$ for $0 \le t < T$.*

In sections 8 and 9 we show how one can improve this theorem to obtain theorem 3.1.

4. An equivalent formulation of the initial value problem.

In appropriate coordinates equation (1.1) is equivalent to a scalar parabolic PDE. In this section we recall the arguments from section three of [A1] to see how this equation arises, and also to see what kind of boundary conditions arise at the corner points.

Let $\Omega(t) \in \mathfrak{D}^{k,\alpha}(0 \le t < T)$ be an evolving family of regions, whose boundary satisfies (1.1). If $\partial\Omega(t)$ has more than one component, then the evolution of each of these components is independent of the evolution of the others, at least as long as they do not touch each other. Therefore we may assume from here on, without loss of generality, that $\partial\Omega$ has only one component, and that $\Omega(t)$ is either the region inside or outside of $\partial\Omega(t)$.

If $X : (a, b) \times [0, T) \to \mathbf{R}^2$ is a parametrization of a smooth part of the boundary $\partial\Omega$, then the normal velocity $V(\xi, t)$ is given by

$$V(\xi,t) = \left\langle \frac{\partial X}{\partial t}, \mathfrak{n}(\xi,t) \right\rangle,$$

and the reader can verify that V is indeed defined independently of the parametrization $X(\xi, t)$ (cf. the "invariance theorem" of [AG, appendix B]). Thus the equation (1.1) may be rewritten as

$$\left\langle \frac{\partial X}{\partial t}, \mathfrak{n}(\xi,t) \right\rangle = \frac{g(\theta(\xi,t))K(\xi,t) - F}{\beta(\theta(\xi,t))} \tag{4.1}$$

with $\mathfrak{n}(\xi, t)$ the unit normal to $\partial\Omega(t)$ at $X(\xi, t)$.

Since $\partial\Omega(0)$ is a locally Lipschitz curve, there is an open neighbourhood $\mathcal{O} \supset \partial\Omega(0)$, and a diffeomorphism $\sigma : (\mathbf{R}/\mathbf{Z}) \times (-1, 1) \to \mathcal{O}$ for which $\sigma^{-1}(\partial\Omega(0))$ is the graph $\Gamma_{u_0} = \{(\xi, u_0(\xi)) \mid \xi \in \mathbf{R}/\mathbf{Z}\}$ of a locally Lipschitz function $u_0 : \mathbf{R}/\mathbf{Z} \to (-1, 1)$ (see figure 4.1). In fact, since $\partial\Omega(0) \in \mathfrak{D}^{k,\alpha}$, the function u_0 will be piecewise $h^{k,\alpha}$; it will be continuous, and all its derivatives up to order k will be little-Hölder continuous, except at a finite number of points $\xi_1, \ldots, \xi_N \in \mathbf{R}/\mathbf{Z}$, where they will have simple jump discontinuities. The points $P_j = \sigma(\xi_j, u_0(\xi_j))$ are then of course the corner points of $\partial\Omega(0)$.

Assuming that the boundary and its tangent move continuously, there will be a short time interval $0 \le t < T_1$ during which $\partial\Omega(t)$ can be similarly represented as the graph of a function $\xi \to u(\xi, t)$. For each fixed

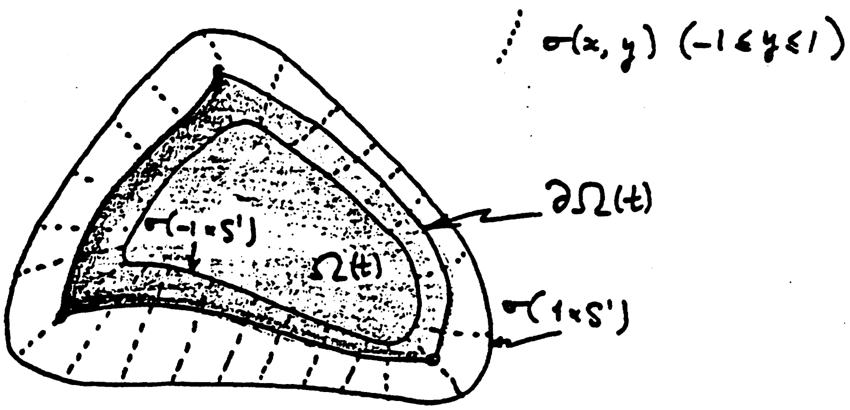

Figure 4.1

t the function $u(\cdot,t)$ will be piecewise $h^{k,\alpha}$, with singularities at N points $\xi_1(t),\ldots,\xi_N(t)$. We shall now show that (1.1) is equivalent to a scalar, quasilinear, parabolic PDE for the function $u(\xi,t)$. Even though one could compute the precise form of this equation (it involves first and second order derivatives of the diffeomorphism σ) we shall not do so. The only relevant fact which we shall need is that u satisfies such a parabolic PDE; this will allow us to conclude a local existence and regularity theorem.

To obtain the PDE, we define $\sigma_j(\xi,\eta)$ $(j = 1,2)$ to be the partial derivatives of σ. Let the pull back under σ of the Euclidean metric on $\mathbf{R}^2$ be

$$(ds)^2 = E(\xi,\eta)(d\xi)^2 + 2F(\xi,\eta)d\xi d\eta + G(\xi,\eta)(d\eta)^2$$

so that $E = \langle\sigma_1,\sigma_1\rangle$, $F = \langle\sigma_1,\sigma_2\rangle$ and $G = \langle\sigma_2,\sigma_2\rangle$ are smooth functions on $S^1\times[-1,1]$, which only depend on σ. Then the unit tangent and normal to $\partial\Omega(t)$ at $\sigma(\xi,u(\xi,t))$ are

$$\mathfrak{t}(\xi,t) = \frac{\sigma_1 + u_\xi\sigma_2}{\sqrt{E+2Fu_\xi+G(u_\xi)^2}}$$
$$\mathfrak{n}(\xi,t) = \frac{-(F+Gu_\xi)\sigma_1 + (E+Fu_\xi)\sigma_2}{\sqrt{(EG-F^2)(E+2Fu_\xi+G(u_\xi)^2)}},$$

and we can define the angle $\theta(\xi,t)$ by requiring $\mathfrak{N}(\theta(\xi,t)) = \mathfrak{n}(\xi,t)$. Furthermore, the normal velocity V will be given by

$$V(\xi,t) = \left\langle \frac{\partial\sigma(\xi,u(\xi,t))}{\partial t}, \mathfrak{n}(\xi,t)\right\rangle = A_0(\xi,u,u_\xi)\frac{\partial u}{\partial t}$$

with

$$\begin{aligned} A_0(\xi,\eta,p) &= \langle \sigma_2(\xi,\eta), \mathfrak{n} \rangle \\ &= \sqrt{\frac{EG-F^2}{E+2Fp+Gp^2}} \\ &> 0. \end{aligned}$$

The curvature K of $\partial\Omega(t)$ at $\sigma(\xi, u(\xi,t))$ can be written as

$$K(\xi,t) = A_1(\xi,u,u_\xi)u_{\xi\xi} + A_2(\xi,u,u_\xi),$$

where $A_1(\xi,\eta,p)$ and $A_2(\xi,\eta,p)$ are smooth functions of their arguments. Since we're only considering (ξ,η,p)'s for which the corresponding $g(\theta)$ is positive, $A_1(\xi,\eta,p)$ is (strictly) positive. Thus the relation $\beta(\theta)V = g(\theta)K + F$ is equivalent to the PDE

$$\frac{\partial u}{\partial t} = A_3(\xi,u,u_\xi)u_{\xi\xi} + A_4(\xi,u,u_\xi),$$

where $A_3 = A_1/A_0$ and $A_4 = A_2/A_0$ are again smooth functions, whenever they are defined, and where $A_3(\xi,\eta,p) > 0$.

The functions $A_j(\xi,\eta,p)$ are not necessarily defined for all (ξ,η,p). To find their domain, we introduce for any $(\xi,\eta,p) \in \mathbf{R}/\mathbf{Z} \times (-1,1) \times \mathbf{R}$ the vector $\mathfrak{v} = \sigma_1(\xi,\eta) + p\sigma_2(\xi,\eta)$. From this vector one can determine the unique angle $\vartheta(\xi,\eta,p) \in \mathbf{R}/2\pi\mathbf{Z}$ for which $\mathfrak{N}(\vartheta) = \mathfrak{v}/|\mathfrak{v}|$. Then $\vartheta : \mathbf{R}/\mathbf{Z} \times (-1,1) \times \mathbf{R} \to \mathbf{R}$ is a smooth function, and the domain of the A_j is given by

$$\Pi_\sigma = \vartheta^{-1}(\Theta) = \{(\xi,\eta,p) \in \mathbf{R}/\mathbf{Z} \times (-1,1) \times \mathbf{R} : \vartheta(\xi,\eta,p) \in \Theta\}.$$

This is the open subset of $\mathbf{R}/\mathbf{Z} \times (-1,1) \times \mathbf{R}$ on which the functions A_j are smooth.

The equation (4.1) will be satisfied at all (ξ,t), with the exception of the corner points. At each corner point $\sigma(\xi_j(t), u(\xi_j(t),t)) = P_j(t)$ our model requires the capillary force $\mathfrak{C}$ to be continuous, and we have seen that this means that the two tangents at $P_j(t)$ to $\partial\Omega(t)$ have prescribed directions $\theta_j^\pm \in \Theta$. Since $\vartheta(\xi,\eta,p)$ is an increasing function of p, there exist uniquely defined smooth functions $p_j^\pm : \mathbf{R}/\mathbf{Z} \times (-1,1) \to \mathbf{R}$ such that

$$\vartheta(\xi,\eta,p_j^\pm(\xi,\eta)) = \theta_j^\pm$$

holds. The capillary force will then only be continuous if one has

$$u_\xi(\xi_j(t) \pm 0, t) = p_j^\pm(\xi_j(t), u(\xi_j(t),t))$$

for $j = 1, \ldots, N$.

Thus we are led to consider the problem of finding a solution of the following parabolic equation,

$$\frac{\partial u}{\partial t} = \mathbf{f}(x, u, u_\xi, u_{\xi\xi}), \tag{4.2}$$

where $0 \le t \le T$ and $\xi \in \mathbf{R}/\mathbf{Z} \setminus \{\xi_1(t), \dots, \xi_n(t)\}$. The function $u(\xi, t)$ is assumed to be a classical solution away from the free boundaries $\xi = \xi_1(t), \dots, \xi = \xi_n(t)$, and to satisfy the following jump conditions at these boundaries:

$$\left.\begin{array}{c} u \text{ is continuous.} \\ u_\xi(\xi_j(t) \pm 0, t) = p_j^{\pm}(\xi_j(t), u(\xi_j(t), t)) \end{array}\right\} \tag{4.3}$$

At $t = 0$ we have the initial data

$$u(\xi, 0) = u_0(\xi) \text{ and } \xi_j(0) = \xi_{j,0}. \tag{4.4}$$

We shall assume that the variable ξ lives in the circle $S^1 = \mathbf{R}/\mathbf{Z}$, i.e. that u is a periodic function of ξ, with period one:

$$u(\xi, t) \equiv u(\xi + 1, t). \tag{4.5}$$

We also assume that any period interval contains a finite number of free boundary points. If there are N free boundary points, then we can number the $\xi_j(t)$ so that one has

$$\left.\begin{array}{c} \xi_{j+N}(t) \equiv \xi_j(t) + 1 \\ \xi_j(t) < \xi_{j+1}(t) \end{array}\right\} \tag{4.6}$$

To simplify our notation we shall write

$$j^2 u(\xi, t) = (u_\xi(\xi, t), u_\xi(\xi, t), u_{\xi\xi}(\xi, t)),$$

and sometimes omit the t, or even both the t and ξ variables. The symbol $j^2 u$ stands for the *two-jet* of the function $u(\cdot, t)$ at ξ, which is a name for all its ξ-derivatives up to second order, or, equivalently, its second order Taylor polynomial.

The two jet $j^2 u(\xi, t)$ belongs to the space $\mathbf{R}^3$, which we shall sometimes also write as J^2, if we want to emphasize that its elements are to be regarded as two-jets.

We shall prove a local existence and regularity result under the following hypotheses.

Hypotheses on f.

[$\mathbf{f}_1$] **f** *is a smooth (i.e., C^∞) function, defined on some open subset $\mathcal{O} \subset S^1 \times J^2$.*

[$\mathbf{f}_2$] *The* PDE*(4.2) is parabolic, i.e.* $\frac{\partial}{\partial q}\mathbf{f}(\xi, u, p, q) > 0$ *for all* $(\xi, u, p, q) \in \mathcal{O}$.

Hypotheses on p.

[p_1] *The $p_j^{\pm}$ are periodic in j and ξ, i.e.* $p_{j+N}^{\pm}(\xi,u) = p_j^{\pm}(\xi+1,u) = p_j^{\pm}(\xi,u)$.

[p_2] *The $p_j^{\pm}$ are smooth functions on $S^1 \times J^2$.*

Hypotheses on the Initial Data

[ID$_1$] *The $\xi_{j,0}$ satisfy* (4.6), *and $u_0(\xi)$ satisfies (4.3).*

[ID$_2$] *$u_0|[\xi_{j,0}, \xi_{j+1,0}]$ is a $h^{2,\alpha}$ function, for some $\alpha \in (0,1)$.*

[ID$_3$] *The graph of $\xi \to (u_0(\xi), u_{0,\xi}(\xi), u_{0,\xi\xi}(\xi))$ is contained in the domain $\mathcal{O}$ of* $\mathbf{f}$.

The theorem we'll prove is the following:

Theorem 4.1. *The initial value problem has a unique classical solution on some short time interval* $[0,T)$.

We'll prove this result by reducing the problem to an abstract parabolic initial value problem in the sense of DAPRATO AND GRISVARD, so that their results in [DPG] give local existence and uniqueness of the solution. The remarks in [A3] show that this approach actually gives C^∞ smoothness in time, and hence in space (by repeatedly using equation (4.2) to trade off time derivatives for space derivatives).

5. Proof of theorem 4.1.

We shall regard the $\xi_k(t)$ as dependent variables, so that we need an equation for $\xi_k'(t)$. By (4.3) we have

$$u(\xi_k(t)+0,t) = u(\xi_k(t)-0,t), \tag{5.1}$$

and, differentiating this relation with respect to time, we get

$$\xi_k'(t) = -\frac{u_t(\xi_k(t)+0,t) - u_t(\xi_k(t)-0,t)}{u_\xi(\xi_k(t)+0,t) - u_\xi(\xi_k(t)-0,t)}. \tag{5.2}$$

Using the PDE (4.2) and the jump condition (4.3), we find that $\xi_k'(t)$ depends on $\xi_k(t)$ and $j^2u(\xi_k(t)\pm 0,t)$. If we define

$$\mathfrak{X}_k : S^1 \times J^2 \times J^2 \longrightarrow \mathbf{R}$$

by

$$\mathfrak{X}_k(\xi, j_1, j_2) = -\frac{\mathbf{f}(\xi, j_1) - \mathbf{f}(\xi, j_2)}{p_k^+(\xi,u_1) - p_k^-(\xi,u_2)}, \tag{5.3}$$

(where $j_1 = (u_1, u_1', u_1'')$ and $j_2 = (u_2, u_2', u_2'')$) then we can rewrite (4.2) as

$$\xi_k'(t) = \mathfrak{X}_k(\xi_k(t), j^2u(\xi_k(t)+0,t), j^2u(\xi_k(t)-0,t)). \tag{5.4}$$

The next step in the proof is to introduce a new coordinate, ζ, which we define by

$$\xi_k(\zeta,t) = (1-\zeta)\xi_k(t) + \zeta\xi_{k+1}(t) \qquad (0 \le \zeta \le 1). \tag{5.5}$$

We also define a new set of functions $V_j(\zeta,t)$:

$$\begin{aligned} V_j(\zeta,t) =& u(\xi_j(\zeta,t),t) - [p_j^+(\zeta_j(t), u(\xi_j(t),t))(1-\zeta)+ \\ & p_{j+1}^-(\zeta_{j+1}(t), u(\xi_{j+1}(t),t))\zeta] \times \zeta(1-\zeta)\Delta_j(t), \end{aligned}$$

where

$$\Delta_j(t) \stackrel{\text{def}}{=} \xi_{j+1}(t) - \xi_j(t).$$

If the functions V_j and the free boundaries ξ_j are known, as functions of (ζ,t) and t, respectively, then one can reconstruct the original function $u(\xi,t)$ as follows:

$$\begin{aligned} u(\xi_j(\zeta,t),t) = V_j(\zeta,t) + [p_j^+(\xi_j, V_j(0))(1-\zeta)+ \\ p_{j+1}^-(\xi_{j+1}, V_j(1))\zeta] \times \zeta(1-\zeta)\Delta_j(t). \end{aligned} \tag{5.6}$$

This relation is easily obtained, if one realizes that

$$\begin{aligned} V_j(0,t) &= u(\xi_j(t),t) \\ V_j(1,t) &= u(\xi_{j+1}(t),t). \end{aligned}$$

In particular, continuity of u at $\xi_j(t)$ is equivalent to

$$V_{j-1}(1,t) = V_j(0,t) \tag{5.7}$$

We have chosen the functions $V_j(\zeta,t)$ in such a way that $u(\xi,t)$ satisfies the jump condition (4.3) if and only if

$$\frac{\partial V_j}{\partial \zeta}(0,t) = \frac{\partial V_j}{\partial \zeta}(1,t) = 0 \tag{5.8}$$

for all j. We can also reformulate the evolution law (4.4) for the $\xi_k(t)$ as follows

$$\xi_k'(t) = \mathfrak{X}_k(\xi_k(t), \mathcal{D}_k \cdot j^2 V_k(0,t), \mathcal{D}_{k-1} \cdot j^2 V_{k-1}(1,t)), \tag{5.9}$$

where $\mathcal{D}_k$ is the 3×3-matrix

$$\mathcal{D}_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Delta_j(t)^{-1} & 0 \\ 0 & 0 & \Delta_j(t)^{-2} \end{pmatrix}$$

which relates j^2u and j^2V_k via

$$j^2u(\xi_k(\zeta,t),t) = \mathcal{D}_k \cdot j^2V_k(\zeta,t)$$

(the substitution $\xi = \xi_k(\zeta,t)$ induces a linear map on the space of two jets J^2, whose matrix is given by $\mathcal{D}_k$.)

The righthand side of (5.9) is therefore a function of

$$\Xi \stackrel{\text{def}}{=} (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbf{R}^N$$

and

$$j^2\mathbf{V}(\xi,t) \stackrel{\text{def}}{=} (j^2V_1(\xi,t),\ldots,j^2V_N(\xi,t)) \in J^2 \otimes \mathbf{R}^N$$

evaluated at $\xi = 0$ and $\xi = 1$. We denote this function by

$$\mathfrak{Y}_k : \mathbf{R}^N \times (J^2 \otimes \mathbf{R}^N) \times (J^2 \otimes \mathbf{R}^N) \longrightarrow \mathbf{R},$$

so that (4.9) may finally be rewritten as

$$\xi_k'(t) = \mathfrak{Y}_k(\Xi(t), j^2\mathbf{V}(0,t), j^2\mathbf{V}(1,t)). \tag{5.10}$$

Next, we shall derive an equation for $\partial\mathbf{V}/\partial t$. Let

$$\begin{aligned} P_k(\Xi,\mathbf{V}(0),\mathbf{V}(1)) = [&p_k^+(\xi_k, V_k(0))(1-\zeta) \\ &+ p_{k+1}^-(\xi_{k+1}, \mathbf{V}_k(1))\zeta] \times \zeta(1-\zeta)\Delta_k, \end{aligned}$$

where $\Delta_k = \xi_{k+1} - \xi_k$; then (5.6) says that

$$u(\xi_k(\zeta,t),t) = V_k(\zeta,t) + P_k(\Xi(t),\mathbf{V}(0,t),\mathbf{V}(1,t),\zeta).$$

Differentiate this with respect to time and use

$$\frac{\partial\xi_k(\zeta,t)}{\partial t} = (1-\zeta)\mathfrak{Y}_k + \zeta\mathfrak{Y}_{k+1}$$

to get

$$u_t + u_\xi[(1-\zeta)\mathfrak{Y}_k + \zeta\mathfrak{Y}_{k+1}] = \frac{\partial V_k}{\partial t} + \mathcal{R}_k \tag{5.11}$$

in which $\mathcal{R}_k$ is some function of Ξ, $\mathbf{V}(0)$, $\mathbf{V}(1)$ and their time derivatives.

From (4.2) we know that

$$u_t = \mathbf{f}(\xi, j^2u) = \mathbf{f}(\xi_k(\zeta,t), \mathcal{D}_k \cdot j^2V_k).$$

It therefore follows from $u(\xi_k(t),t) = V_k(0,t)$ that the time derivatives $\mathbf{V}_t(0,t)$ and $\mathbf{V}_t(1,t)$ are functions of the two-jets of $\mathbf{V}$ at $\zeta = 0$ and $\zeta = 1$, and of Ξ.

If we combine this with (5.11), we discover that the time evolution of $\mathbf{V}$ is given by

$$\frac{\partial V_k}{\partial t} = \mathfrak{F}_k(\Xi, j^2\mathbf{V}(0,t), j^2\mathbf{V}(1,t); j^2\mathbf{V}(\zeta,t)) \tag{5.12}$$

for $k = 1, 2, \ldots, N$ and for some smooth functions

$$\mathfrak{F}_k : \mathbf{R}^N \times (J^2 \otimes \mathbf{R}^N)^3 \longrightarrow \mathbf{R}.$$

One could, with some effort, compute the explicit form of these functions $\mathfrak{F}_k$ $(1 \le k \le N)$, but all that we really need to know about them is that they are smooth, and that their derivative with respect to their last argument is strictly positive, i.e.

Lemma 5.1. *For any $k = 1, \ldots, N$ one has*

$$\frac{\partial}{\partial u''_\zeta} \mathfrak{F}_k(\Xi, j_0, j_1, j_\zeta) > 0.$$

Here we have used the notation $j = (u, u', u'')$ for two-jets. The lemma follows after a lengthy computation from our hypothesis

$$\frac{\partial \mathrm{f}(x, u, p, q)}{\partial q} > 0.$$

6. Interlude on abstract parabolic equations.

In their paper [DPG] DaPrato and Grisvard showed how, using the theory of analytic semigroups and interpolation spaces, one can prove the existence of short term solutions of a large class of initial value problems of parabolic nature. Their results, which we shall use below, may be summarised as follows.

Let $X_1 \subset X_0$ be pair of Banach spaces, where the inclusion is dense and continuous, and let $\mathcal{O} \subset X_1$ be some open subset (in the topology of X_1). Then for any Fréchet differentiable map $\Phi : \mathcal{O} \to X_0$ DaPrato and Grisvard consider the following initial value problem:

$$\left.\begin{aligned} &x'(t) = \Phi(x(t)) \qquad (0 \le t \le T) \\ &x(0) = x_0 \end{aligned}\right\} \tag{IVP}$$

where $x_0 \in \mathcal{O}$ is prescribed, and the solution $x : [0, T] \to X_1$ should be (at least) a ***strict solution***, i.e. it should be continuous as an X_1 valued function and continuously differentiable as an X_0 valued function.

In general (*IVP*) need not have any solution at all, however small one chooses the time interval T; indeed, without any further restrictions or assumptions on Φ the *IVP* is so general that it includes (say) the backwards heat equation (take $X_0 = L_2(\mathbf{R}), X_1 = H^2(\mathbf{R})$ and $(\Phi x)(t) = -x''(t)$.)

To overcome this difficulty, DaPrato and Grisvard introduce the following condition on the Fréchet derivative of Φ at the initial value x_0:

Condition MR. *There exist Banach spaces $Y_1 \subset Y_0$ for which one has the following continuous and dense inclusions*

$$X_1 \subset Y_1 \subset X_0 \subset Y_0,$$

and such that X_0 is a continous interpolation space of the pair (Y_1, Y_0).

The linear operator $A = d\Phi(x_0) : X_1 \to X_0$ generates an analytic semigroup on X_0. Moreover, it extends to a bounded operator

$$A' : Y_1 \to Y_0$$

and this extension A' generates an analytic semigroup on Y_0.

They proved that this condition implies the existence of a strict solution $x(t)$ to the initial value problem. At the heart of their proof, lies the (nontrivial) observation that the condition MR implies the solvability of the linearised version of (*IVP*), i.e. of

$$\begin{aligned} &\xi'(t) = d\Phi(x_0)\xi(t) + f(t) \qquad (0 \le t \le T) \\ &\xi(0) = \xi_0 \in X_1 \end{aligned}$$

for arbitrary $f \in C([0,T]; X_0)$.

Given the solvability of the linear problem one can use a very standard contraction mapping argument to solve the nonlinear problem, on a short enough time interval $[0,T]$. As was pointed out in [A3] this allows one to prove smooth dependence of the solution on initial data, as well as higher regularity of the solution for $t > 0$ (i.e. the smoothing property of the parabolic equation). All this may be summarized as follows.

Theorem 6.1. *If $\Phi : \mathcal{O} \to X_0$ is an infinitely differentiable map whose derivative $d\Phi(x_0)$ satisfies the condition* MR *for any $x_0 \in \mathcal{O}$, then (IVP) generates a C^∞ smooth local semiflow on $\mathcal{O}$.*

7. A nonlinear semiflow on a Banach space.

The equations (5.10), (5.12), together with the boundary conditions (5.7) and (5.8)

$$\begin{aligned}
V_{j-1}(1,t) &= V_j(0,t)\\
\frac{\partial V_j}{\partial \zeta}(0,t) &= \frac{\partial V_j}{\partial \zeta}(1,t) = 0\\
\xi_k'(t) &= \mathfrak{Y}_k(\Xi(t), j^2\mathbf{V}(0,t), j^2\mathbf{V}(1,t))\\
\frac{\partial V_k}{\partial t} &= \mathfrak{F}_k(\Xi, j^2\mathbf{V}(0,t), j^2\mathbf{V}(1,t); j^2\mathbf{V}(\zeta,t))
\end{aligned}$$

are equivalent to the original problem (4.2). We'll show that the nonlinear analytic semigroup approach works for this system. We begin by introducing a few Banach spaces.

Let $0 < \alpha < 1$ be given, and let

$$H_\alpha = h^\alpha([0,1])$$

denote the "little Hölder" space of exponent α, i.e. the closure of smooth functions in the usual Hölder space $C^\alpha([0,1])$. H_α is a proper subspace of $C^\alpha([0,1])$. As usual, we shall give it the same norm.

Define

$$F_0^\alpha = \overbrace{H_\alpha \oplus \cdots \oplus H_\alpha}^{N terms}$$

and let $E_0^\alpha \subset F_0^\alpha$ be the closed subspace consisting of all those $(V_1, \cdots, V_N)$ which satisfy

$$V_j(1) = V_{j+1}(0)$$

for all j, with $V_{j+N} = V_j$ implicitely understood.

We let

$$H_{2,\alpha} = \{f \in h^{2,\alpha}([0,1]) \mid f'(0) = f'(1) = 0\}$$

and define

$$E_1^\alpha = \{\mathbf{V} \in E_0^\alpha \mid V_1, \cdots, V_N \in H_{2,\alpha}\}.$$

Then the functions $\mathfrak{Y}_1, \cdots, \mathfrak{Y}_N$ and $\mathfrak{F}_1, \cdots, \mathfrak{F}_N$ define a smooth map

$$\begin{aligned}
\Phi : \mathbf{R}^N \oplus E_1^\alpha &\to \mathbf{R}^N \oplus E_0^\alpha\\
(\Xi, \mathbf{V}) &\mapsto (\mathfrak{Y}_1, \cdots, \mathfrak{Y}_N, \mathfrak{F}_1, \cdots, \mathfrak{F}_N)
\end{aligned}$$

and our initial value problem is equivalent with the following "abstract parabolic equation":

$$\left.\begin{aligned}
x'(t) &= \Phi(x(t))\\
x(0) &= (\xi_1(0), \cdots, \xi_N(0), V_1(\cdot,0), \cdots, V_N(\cdot,0))
\end{aligned}\right\} \qquad (7.1)$$

By the results of DaPrato and Grisvard [DPG] one can immediately conclude the existence of a strict solution

$$x \in C^1([0,t_0];\mathbf{R}^N \oplus E_0) \cap C^0([0,t_0];\mathbf{R}^N \oplus E_1)$$

once one has verified that the Fréchet derivative $d\Phi(x_0) : \mathbf{R}^N \oplus E_1^\alpha \to \mathbf{R}^N \oplus E_0^\alpha$ satisfies the following condition:

Condition MR. *The linear operator $d\Phi(x_0)$ generates an analytic semigroup on $\mathbf{R}^N \oplus E_0^\alpha$. Moreover, it extends to a bounded operator*

$$A_{\alpha'} : \mathbf{R}^N \oplus E_1^{\alpha'} \to \mathbf{R}^N \oplus E_0^{\alpha'}$$

for some $\alpha' \in (0,\alpha)$, and this extension $A_{\alpha'}$ also generates an analytic semigroup on $\mathbf{R}^N \oplus E_0^{\alpha'}$.

To verify this condition one can use the methods which we used in [A5]: we shall outline the procedure, and leave some of the details to the reader.

First, one splits the linear operator $d\Phi(x_0)$ into four matrix components,

$$d\Phi(x_0) = \begin{pmatrix} \mathcal{M} & \mathcal{N} \\ \mathcal{P} & \mathcal{Q} \end{pmatrix} : \mathbf{R}^N \oplus E_1^\alpha \to \mathbf{R}^N \oplus E_0^\alpha,$$

where $\mathcal{M} \in \mathfrak{L}(\mathbf{R}^N,\mathbf{R}^N)$, $\mathcal{N} \in \mathfrak{L}(E_1^\alpha,\mathbf{R}^N)$, $\mathcal{P} \in \mathfrak{L}(\mathbf{R}^N,E_0^\alpha)$ and $\mathcal{Q} \in \mathfrak{L}(E_1^\alpha,E_0^\alpha)$.

The extension lemma in [A5] says that, since $d\Phi(x_0)$ is a finite dimensional extension of $\mathcal{Q}$, the condition MR for $d\Phi(x_0)$ is equivalent to the the condition MR for $\mathcal{Q}$. In other words, it suffices to prove that $\mathcal{Q}$ extends to an operator $\mathcal{Q} : E_1^{\alpha'} \to E_0^{\alpha'}$, for some $0 < \alpha' < \alpha$, and that this extension generates an analytic semigroup on $E_0^{\alpha'}$.

To analyze the operator $\mathcal{Q}$ we observe that it is the Fréchet derivative of the map

$$\Psi^\alpha : \mathbf{V} \in E_1^\alpha \longrightarrow (\mathfrak{F}_1(\cdots),\ldots,\mathfrak{F}_N(\cdots)) \in E_0^\alpha.$$

This map is well defined for any $\alpha \in (0,1)$, so that its derivative $\mathcal{Q} = d\Psi^\alpha(x_0)$ at any x_0 extends to a bounded linear operator $d\Psi^{\alpha'}(x_0)$ from $E_1^{\alpha'}$ to $E_0^{\alpha'}$, for any $\alpha' \in (0,\alpha)$. Therefore it remains to show that $\mathcal{Q}$ and its extension $d\Psi^{\alpha'}(x_0)$ generate an analytic semigroups on E_0^α and $E_0^{\alpha'}$, respectively. Below we prove that $d\Psi^\alpha(x_0)$ does indeed generate an analytic semigroup; the proof can be carried through verbatim for $d\Psi^{\alpha'}(x_0)$, simply by replacing α by α', wherever it occurs.

From (5.12) it is clear that the map Ψ^α extends to a map from F_1^α to F_0^α, simply by using the same formula, given in (5.12), to evaluate $(\mathfrak{F}_1,\ldots,\mathfrak{F}_N)$ (recall that we had defined $F_1^\alpha = H_{2,\alpha} \oplus \cdots \oplus H_{2,\alpha}$.) The operator $\mathcal{Q}$ therefore also extends to a linear map from F_1^α to F_0^α; we'll denote this map by Q'. Since $F_k^\alpha = E_k^\alpha \oplus L$, where L is the N dimensional linear subspace of F_1^α spanned by the functions

$$(0, 0, \ldots, \overset{k}{\mathbf{1}}, \ldots, 0)$$

with $1 \le k \le N$ ($\mathbf{1}$ is the constant function whose value is 1), we can again apply the extension lemma from [A5]. The conclusion is that Q generates an analytic semigroup on E_0^α if and only if Q' generates one on F_0^α.

On F_1^α the operator Q' can be represented by an $N \times N$ matrix of operators,

$$Q' = (Q_{jk})_{1 \le j,k \le N}$$

where

$$Q_{jk} : H_{2,\alpha} \to H_{2,\alpha}$$

is the Fréchet derivative of $\mathfrak{F}_j(\Xi, j^2\mathbf{V}(0), j^2\mathbf{V}(1); j^2V_j)$ with respect to $V_k \in H_{2,\alpha}$.

If one keeps in mind that evaluation of $j^2\mathbf{V}$ at either $\zeta = 0$ or $\zeta = 1$ is a linear functional on F_1^α, and that any linear operator of the form

$$w \in H_{2,\alpha} \to \sum_{i=0}^{2} f_i(\zeta)\partial_\zeta^i w(0)$$

with $f_0, f_1, f_2 \in H_\bullet$ has finite rank and therefore is compact, then one realizes that the off-diagonal elements Q_{jk} ($j \neq k$) in the matrix of Q' are compact operators, so that Q is a compact perturbation of the diagonal matrix

$$Q'' = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & L_N \end{pmatrix}$$

where each diagonal element $L_j : H_{2,\alpha} \to H_\alpha$ is a standard Sturm-Liouville operator.

Since each L_j generates an analytic semigroup, the operator Q'' also generates an analytic semigroup; since Q' is a compact perturbation of Q'', so do Q', and (by our earlier remarks involving the extension lemma) Q.

Thus we have verified that $d\Phi(x_0)$ satisfies condition MR and the existence of a short term solution to the initial value problem follows.

8. The topology and the Semiflow on $\mathfrak{D}^{2,\alpha}$.

In the last four sections we showed how one can parametrize a part of $\mathfrak{D}^{2,\alpha}$ by an open subset of a Banach space. We recall how this parametrization came about.

First one chooses a diffeomorphism $\sigma : S^1 \times [-1,1] \hookrightarrow \mathbf{R}^2$. Given this diffeomorphism one can assign a domain Ω^u to any periodic piecewise $h^{2,\alpha}$ function $y = u(x) = u(x+1)$ with $\sup|u(x)| < 1$, by letting $\partial\Omega^u$ be the image under σ of the graph of u, i.e. $\{\sigma(x,u(x)) : x \in S^1\}$; the graph of u has a natural orientation ("from left to right"), which gives us an orientation for $\partial\Omega^u$, and from which we can tell on which side of $\partial\Omega^u$ we can find Ω^u.

Next, one denotes the singular points of $u(x)$ by $\xi_1, \ldots, \xi_N$, and one introduces N functions $V_1, \ldots, V_N \in h^{2,\alpha}([0,1])$ by the transformation (5.6).

Conversely, there is a $\delta = \delta(\sigma) > 0$, which only depends on the diffeomorphism σ such that any $(\Xi, \mathbf{V}) \in \mathbf{R}^N \oplus E_1^\alpha$ with $|V_j(\zeta)| < \delta$ for $1 \le j \le N, \zeta \in [0,1]$ defines a periodic piecewise $h^{2,\alpha}$ function $u(x)$ with $\sup|u| < 1$. In this way we get a transformation $\varphi_\alpha^\sigma : \mathcal{O}_\sigma \hookrightarrow \mathfrak{D}^{2,\alpha}$, where $\mathcal{O}_\sigma = \{(\Xi, \mathbf{V}) \in \mathbf{R}^N \oplus E_1^\alpha : \sup|V_j| < \delta\}$. We denotes its range by

$$\mathfrak{D}^{2,\alpha}(\sigma) = \varphi_\alpha^\sigma(\mathcal{O}_\sigma).$$

We leave the straightforward proof of the following lemma to the reader.

Lemma 8.1. *Let two diffeomorphisms $\sigma_{1,2} : S^1 \times [-1,1] \hookrightarrow \mathbf{R}^2$, for which $\mathfrak{D}^{2,\alpha}(\sigma_1) \cap \mathfrak{D}^{2,\alpha}(\sigma_2)$ is nonempty be given. Then the transition map $(\varphi_\alpha^{\sigma_1})^{-1} \circ \varphi_\alpha^{\sigma_2}$ is continuous.*

We define a topology on $\mathfrak{D}^{2,\alpha}$, by requiring that all $\mathfrak{D}^{2,\alpha}(\sigma)$'s are open subsets of $\mathfrak{D}^{2,\alpha}$, and that the φ_α^σ's are homeomorphisms. Thus $\mathfrak{D}_N^{2,\alpha} = \{\Omega \in \mathfrak{D}^{2,\alpha} : \Omega \text{ has } N \text{ corners}\}$ is a *topological* Banach manifold, modelled on $\mathbf{R}^N \oplus E_1^\alpha$; it turns out that the transition maps $(\varphi_\alpha^{\sigma_1})^{-1} \circ \varphi_\alpha^{\sigma_2}$ are, in general, not Fréchet differentiable, so that $\mathfrak{D}^{2,\alpha}$ is not a differentiable Banach manifold.

The local existence theorem which was derived in the previous sections implies that (1.1) defines a continuous local semiflow on $\mathfrak{D}^{2,\alpha}$.

Lemma 8.2. *Let $\Omega : [0, T_{\max}) \to \mathfrak{D}^{2,\alpha}$ be a maximal solution in $\mathfrak{D}^{2,\alpha}$. Then $\Omega(t) \in \mathfrak{D}^\infty$ for all $0 < t < T_{\max}$.*

Proof. Our local existence theorem provides us with a solution of (1.1) starting at any $\Omega \in \mathfrak{D}^{2,\alpha}$, which exists at least for a certain time, which we shall denote by $T_\alpha(\Omega)$; $T_\alpha(\Omega)$ is a lower semicontinuous function on $\mathfrak{D}^{2,\alpha}$. The smoothness of the local solution implies that $\Omega(t') \in \mathfrak{D}^\infty$ for all $t < t' < t + T_\alpha(\Omega(t))$. But the lower semicontinuity of T_α ensures that one has $t' < t < t' + T_\alpha(\Omega(t'))$ for any $0 < t < T_{\max}$ and t', provided one chooses $t' < t$ close enough to t. Thus $\Omega(t) \in \mathfrak{D}^\infty$ for $0 < t < T_{\max}$.

Q. E. D.

Lemma 8.3. *Let $\{\Omega_n\}_{n=1,2,\dots} \subset \mathfrak{D}_N^{2,\alpha}$ be a sequence of domains, and let $\gamma_n^1, \dots, \gamma_n^N \subset \partial\Omega_n$ be the smooth arcs in the boundary of Ω_n. Assume that the following holds:*

1. *There is a $\delta > 0$ such that $\delta \le \text{length}(\gamma_n^j) \le \delta^{-1}$ for all n, j.*
2. *The curvatures of the γ_n^j are uniformly α-Hölder continuous,*
3. *The $\partial\Omega_n$ are contained in some fixed (large) neighborhood of the origin,*
4. *The sequence of curves $\partial\Omega_n$ does not "develop a self intersection," i.e. there is a constant $\lambda > 0$ such tha for any pair of points $P, Q \in \partial\Omega_n$ one has $d_{\mathbf{R}^2}(P,Q) \ge \lambda d_{\partial\Omega_n}(P,Q)$, where $d_{\mathbf{R}^2}$ and $d_{\partial\Omega_n}$ are the Euclidean distance and the distance along the boundary, respectively.*

Then the sequence $\{\Omega_n\}_{n\ge 1}$ is precompact in $\mathfrak{D}^{2,\beta}$ for any $0 < \beta < \alpha$.

We shall only sketch the proof of this theorem, and leave the details to the reader.

One can extract a subsequence such that the sequences of sets $\{\partial\Omega_n\}$ and $\{\Omega_n\}$ both converge in the Hausdorff metric on compact subsets of the plane. Choose such a subsequence, and denote it by Ω_n again Using the uniform Hölder continuity of the curvatures one can show that the arclength parametrizations of the γ_n^j converge in $C^{2,\beta}$ for any $0 < \beta < \alpha$. The fourth condition guarantees that the limit of the $\partial\Omega_n$'s is again a $C^{2,\alpha}$ curve, which is the boundary of the limit of the Ω_n's. Since the $\partial\Omega_n$'s converge in $C^{2,\beta}$, their tangents certainly converge, and for large enough n all Ω_n's lie in some common $\mathfrak{D}^{2,\alpha}(\sigma)$. If $V_1^n, \dots, V_N^n, \xi_1^n, \dots, \xi_N^n$ are the data representing the Ω_n, then one can deduce from the $C^{2,\beta}$ convergence of the γ_n^j that the $V_1^n, \dots, V_N^n, \xi_1^n, \dots, \xi_N^n$ also converge in the appropriate topology.

Given this lemma, we can easily prove the following weaker version of the main existence theorem.

Lemma 8.4. *Let $\Omega : [0, T_{\max}) \to \mathfrak{D}^{2,\alpha}$ be a maximal solution of* (1.1), *with $T_{\max} < \infty$. Then either*

$\mathbf{E}_1'$ *The curvatures of $\partial\Omega(t)$ are not uniformly α-Hölder continuous,*

holds, or else one of the two conditions $\mathbf{E}_2$ or $\mathbf{E}_3$ of theorem 3.1, must occur.

Proof. Assume that none of the three conditions $\mathbf{E}_1', \mathbf{E}_2$ or $\mathbf{E}_3$ hold. Then our compactness lemma (8.3) implies that $\Omega([0, T_{\max}))$ is precompact in $\mathfrak{D}^{2,\beta}$ for any $0 < \beta < \alpha$. Since (1.1) defines a continuous local semiflow on $\mathfrak{D}^{2,\beta}$ the maximal solution in $\mathfrak{D}^{2,\beta}$ starting at $\Omega(0)$ must exist longer than $T_{\max}$; assume it is defined for $0 \le t < T'$, where $T' > T_{\max}$. By lemma (8.2) the extended solution $\Omega(t)$ must be smooth for all $t < T'$, and hence constitutes a solution in $\mathfrak{D}^{2,\alpha}$. Hence our original solution wasn't maximal in $\mathfrak{D}^{2,\alpha}$ after all.

Q. E. D.

9. Lipschitz continuity of bounded curvature.

In this section we shall complete the proof of the main theorem. We shall consider a maximal solution $\Omega : [0, T_{\max}) \to \mathfrak{D}^{k,\alpha}$ of (1.1) whose life span $T_{\max}$ is finite, and for which none of the three conditions $\mathbf{E}_1, \mathbf{E}_2$ or $\mathbf{E}_3$ of the main theorem occur. As we shall see, the curvature of $\partial\Omega(t)$ is uniformly Hölder continuous in this situation, which contradicts lemma 8.4 and therefore shows that theorem 3.1 must hold.

Instead of showing Hölder continuity of K, we shall show that K_s remains bounded, so that K is uniformly Lipschitz continuous, and therefore certainly uniformly Hölder continuous. We shall obtain our bound on K_s by means of a "blow-up argument." To reach a contradiction we assume that K_s is not bounded, and we choose a sequence $t_n \uparrow T_{\max}$, as well as a sequence of points $P_n \in \partial\Omega(t_n)$, for which

$$|K_s(P,t)| \leq |K_s(P_n, t_n)| \text{ for all } t \leq t_n, P \in \partial\Omega(t), \tag{9.1}$$

while $|K_s(P_n, t_n)| \uparrow \infty$. The boundary $\partial\Omega(t)$ of the family of domains $\Omega(t)$ consists of a finite number of parametrized arcs $\mathfrak{r}^j(p,t)$ $(j = 1, 2, \cdots, N)$; we may assume that all the points P_n lie on the same arc, and we shall denote the parametrization of this arc by $\mathfrak{r}(p,t)$, where $P(t) \leq p \leq Q(t)$ for $0 \leq t < T_{\max}$.

Define

$$\epsilon_n = |K_s(P_n, t_n)|^{-1},$$

and let $\Omega_n(t)$ be the domain given by

$$\Omega_n(t) = \phi_n(\Omega(t_n + \epsilon_n^2 t)), \qquad \left(\frac{-t_n}{\epsilon_n^2} \leq t \leq 0\right)$$

where the affine transformation ϕ_n is given by $\phi_n(x) = (x - P_n)/\epsilon_n$, i.e. by translating P_n to the origin, and magnifying by ϵ_n^{-1}. The part of $\partial\Omega_n(t)$ which at $t = 0$ contains the origin is parametrized by $\mathfrak{r}_n(p,t) = \epsilon_n^{-1}\mathfrak{r}(p, t_n + \epsilon_n^2 t)$. One easily computes that the curvature K_n and velocity V_n of the rescaled boundaries $\partial\Omega_n(t)$ satisfy the following estimates

$$|K_n| + |K_{n,s}| + |V_n| \leq C\epsilon_n, \tag{9.2}$$

$$|K_{n,s}| = \epsilon_n \text{ at } P_n, t = 0, \tag{9.3}$$

as well as

$$V_n = \frac{g(\theta_n)K_n - \epsilon_n F}{\beta(\theta_n)}. \tag{9.4}$$

At this point we must consider two different cases, depending on the distance d_n along the boundary $\partial\Omega_n(0)$ of the point P_n to the end of the arc $\mathfrak{r}_n([P(t_n), Q(t_n)], 0)$. Either d_n remains bounded, or else we may assume that $d_n \uparrow \infty$ after passing to a subsequence, if necessary; we shall refer to these cases as blow up at a corner point, and interior blow up.

Case 1–Interior blow up. For each n we choose Euclidean coordinates in which the normal to $\partial\Omega(0)$ at the origin is vertical, i.e. in which $\partial\Omega_n(0)$ is tangent to the x-axis.

Let $\tilde{\gamma}_n$ be the section of $\partial\Omega_n(0)$ through the origin which extends a distance $2\epsilon_n^{-1/2}$ in both directions. Then $|K_n| \le C\epsilon_n$ implies $|\theta_n(s) - \theta_n(0)| \le C\epsilon_n|s|$, where s denotes arclength along $\tilde{\gamma}_n$, measured from the origin. We have chosen our coordinates such that $\theta_n(0) = -\pi/2$. If $(x_n(s), y_n(s))$ is an arclength parametrization of $\tilde{\gamma}_n$, we have

$$\left|\frac{dx_n}{ds}\right| = |\sin\theta_n(s)| = |\cos(\theta_n(s) - \theta_n(0))| \ge \frac{1}{2},$$
$$\left|\frac{dy_n}{ds}\right| = |\cos\theta_n(s)| = |\sin(\theta_n(s) - \theta_n(0))| \le C\epsilon_n|s| \le 2C\epsilon_n|x_n(s)|.$$

It follows that $\tilde{\gamma}_n$ is the graph of a function $y = u^n(x)$, which is defined on an interval containing $\{|x| \le \epsilon_n^{-1/2}\}$, and on which u^n satisfies

$$|u^n_x| \le 4C\epsilon_n|x| \qquad (\le 4C\sqrt{\epsilon_n}),$$
$$|u^n_{xx}| = |K_n|\left(1 + (u^n_x)^2\right)^{3/2} \le 2C\epsilon_n,$$
$$|u^n| \le C\epsilon_n x^2 \qquad (\le C)$$

for large enough n.

Let γ_n be the subarc of $\tilde{\gamma}_n$ on which $|x| \le \epsilon_n^{-1/2}$, and let $\gamma_n(t)$ be the subarc of $\partial\Omega_n(t)$ whose endpoints evolve normal to $\partial\Omega_n(t)$, and which at t=0 coincides with γ_n. The normal time derivative of the tangent is given by $\theta_t = V_s$ (see (10.1)). Using $V = \Phi(\theta)K - \Psi(\theta)$, with $\Phi(\theta) = g(\theta)/\beta(\theta), \Psi(\theta) = F/\beta(\theta)$, one finds that

$$|\theta_t| = |V_s| \le c_1 C\epsilon_n,$$

for some constant c_1 which only depends on the L_∞ norm of the derivatives of Φ, Ψ, i.e. of f and β, with respect to θ – in particular, c_1 does not depend on n.

It follows that if $-\epsilon^{-1/2} \le t \le 0$, then $\gamma_n(t)$ is still a graph $y = u^n(x,t)$. We get the estimate

$$|u^n_x| \le 4C\sqrt{\epsilon_n} + c_1 C\epsilon_n|t| \le (4 + c_1)C\sqrt{\epsilon_n},$$

and hence

$$|u^n_{xx}| = |K_n|\left(1 + (u^n_x)^2\right)^{3/2} \le 2C\epsilon_n$$

for large n.

Since $|V_n| \le C\epsilon_n$ the endpoints of $\gamma_n(t)$ cannot move farther than $C\epsilon_n^{-1/2}$ during the time interval $-\epsilon_n^{-1/2} \le t \le 0$, so that all the $u^n(x,t)$'s are defined when $|x| \le \epsilon_n^{-1/2} - 1$, and $-\epsilon_n^{-1/2} \le t \le 0$.

Using

$$u_{xxx} = \left(K\left(1+u_x^2\right)^{3/2}\right)_x = \left(1+u_x^2\right)^2 K_s + 3u_x u_{xx}\left(1+u_x^2\right)^{1/2} K$$

and $|K_s| \le C\epsilon_n$ one finds

$$|u^n_{xxx}| \le c_2 C\epsilon_n;$$

from $u_t = \left(1+u_x^2\right)^{1/2} V$ one obtains

$$|u^n_t| \le 2C\epsilon_n.$$

At the origin we had $|K_{n,s}| = \epsilon_n$. From $u^n_x(0) = 0$ it follows that at the origin, and at $t = 0$ we have

$$K = \frac{u_{xx}}{(1+u_x^2)^{3/2}} \Rightarrow K_{n,s} = u^n_{xxx}(0) = \pm\epsilon_n. \tag{9.5}$$

Due to (9.4), each u^n satisfies a quasilinear PDE of the form

$$u_t = Q_n(u_x)u_{xx} + \epsilon_n B_n(u_x) \tag{9.6}$$

where the Q_n and B_n are smooth functions, which converge as $n \to \infty$. (See [AG, section 9.4] for the derivation of this equation.)

As $n \to \infty$ the u^n's tend to zero; instead we must consider $v^n = u^n/\epsilon_n$. The v^n's satisfy $v^n(0,0) = v^n_x(0,0) = 0$; they also satisfy the estimate $|v_{xx}|, |v^n_t| \le C$ as well as the equation

$$v_t = Q_n(u_x)v_{xx} + B_n(u_x). \tag{9.7}$$

Since (9.7) is a quasilinear uniformly parabolic PDE, the bounds $|v^n_{xx}| + |v^n_t| \le C$ imply that all derivatives of the v^n's are uniformly bounded on compact subsets of $(-L+1, L-1) \times (-L, 0]$. Therefore we can extract a convergent subsequence, whose limit $v(x,t)$ will satisfy

$$|v_{xx}| + |v_t| \le 4C, \tag{9.8}$$

$$v_t = Q_* v_{xx} + B_*, \tag{9.9}$$

for some constants Q_* and B_*, and

$$v_{xxx}(0,0) = \pm 1 \tag{9.10}$$

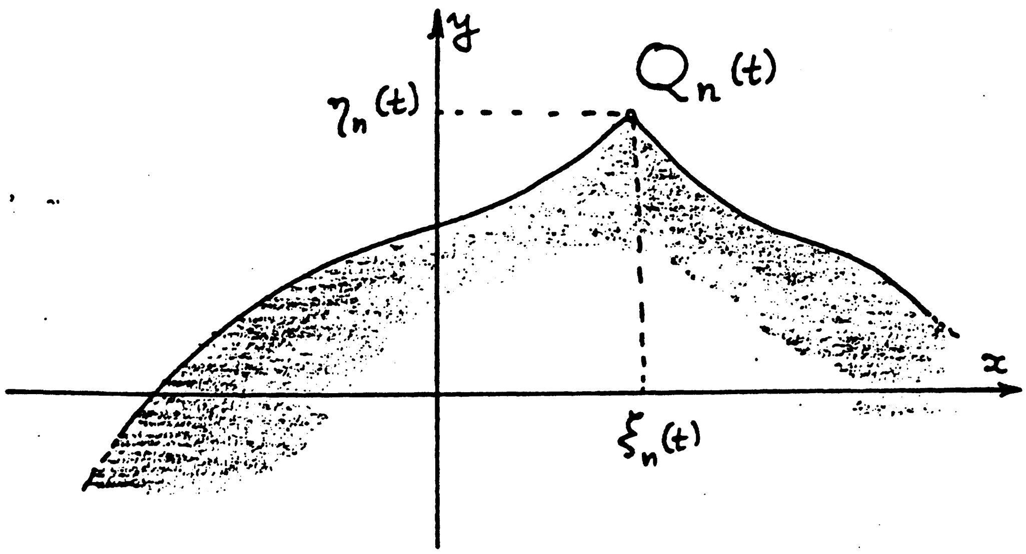

Figure 9.1

It follows from $v^n_{xxx}(0,0) = \pm 1$ for all n, that $v_{xxx}(0,0) = \pm 1$, so that $v_{xx}(x,t)$ is not constant. On the other hand, if one differentiates (9.10) twice with respect to x, then one finds that $w = v_{xx}$ is a bounded solution of $w_t = Q_* w_{xx}$ on $\{(x,t) : x \in \mathbf{R}, -\infty < t \leq 0\}$. But such a solution must be constant! The contradiction shows that K_s cannot blow up in the interior of $\partial\Omega(t)$, i.e. away from the corners.

Case 2–Blow up at a corner. Assume that the distances d_n remain bounded; let $Q_n(t)$ be the corner point of $\partial\Omega_n(t)$ which is the closest to the origin. Since the limit curve doesn't develop self intersections, and since the lengths of the arcs in $\partial\Omega(t)$ are assumed to be bounded from below, the point $Q_n(t)$ will be uniquely determined for large enough n, and for all t in the time interval $-\epsilon_n^{-1/2} \leq t \leq 0$. In fact, given any $R > 0$, $Q_n(t)$ will be the only corner in a disk of radius R centered at the origin.

For a general solution of (1.1) the velocity of any corner is bounded by *const* $\times K$, where K is the maximum of the two curvatures at the corner in question. Thus we find that the velocity of our corner $Q_n(t)$ is bounded by $C\epsilon_n$.

At the corner $Q_n(t)$ two arcs meet. The unit normals to these two arcs at the corner $Q_n(t)$ are prescribed by $\mathbf{n} = \mathfrak{N}(\theta_{1,2})$, where $\{\theta_1, \theta_2\}$ corresponds to a bitangent to the Frank diagram. By rotating our coordinate system, if necessary, we can arrange that neither of the tangents to the two arcs is vertical at the corner, and that one of them points to the right, and one to the left (as in figure 9.1).

Let $\gamma_n(t)$ be the union of the two arc-segments of length $-\epsilon_n^{-1/2}$ emanating from $Q_n(t)$. As in the previous case it follows from the curvature and velocity bounds $|K_n| + |V_n| \le C\epsilon_n$ that for $-\epsilon_n^{-1/2} \le t \le 0$ $\gamma_n(t)$ is the graph of a function $y = u^n(x,t)$, which is defined for $|x| \le c_3\epsilon_n^{-1/2}$, for some $c_3 > 0$. Again, these u^n's are solutions of an equation like (9.6), and they satisfy

$$|u^n_{xxx}| + |u^n_{xx}| + |u^n_t| \le c_4 C\epsilon_n. \tag{9.11}$$

If $(\xi_n(t), \eta_n(t))$ are the coordinates of the corner point $Q_n(t)$, then we also have

$$u^n_x(\xi_n(t), t) = p_\pm$$

where $p_\pm = -\cot\theta_{1,2}$.

Since $|Q'_n(t)| \le C\epsilon_n$ we also have $|\xi'_n(t)|, |\eta'_n(t)| \le C\epsilon_n$.

We now introduce the functions $U^n(x,t) = u^n(\xi_n(t) + x, t)$ and

$$v^n(x,t) = \frac{u^n_x(\xi_n(t) + x, t) - p(x)}{\epsilon_n}$$

where $p(x) = p_+$ if $x > 0$, and p_- if $x < 0$. The U^n satisfy the following estimates:

$$|U^n_{xx}| = |u^n_{xx}| \le 2C\epsilon_n,$$
$$|U^n_{xxx}| = |u^n_{xxx}| \le 2C\epsilon_n,$$
$$|U^n_t| = |u^n_t| + |u^n_x| \cdot |\xi'_n| \le c_5\sqrt{\epsilon_n}C\epsilon_n + 2C\epsilon_n \le 3C\epsilon_n,$$

if n is large enough.

A short computation shows that the v^n satisfy the following PDE

$$v^n_t = Q(U^n_x)v^n_{xx} + (Q'(U^n_x)U^n_{xx} + \epsilon_n B'(U^n_x) + \xi'_n(t))v^n_x \tag{9.12}$$

while they also satisfy $v^n(0,t) \equiv 0$. The bounds (9.11) imply that v^n_x and v^n_{xx} are uniformly bounded. Moreover, we have $v^n_{xx}(-\xi_n(0), 0) = \epsilon_n^{-1}u^n_{xxx}(0,0) = \pm 1$.

By interpolation the two estimates for $|U^n_t|$ and $|U^n_{xxx}|$ imply that U^n_{xx} and U^n_x are Hölder continuous in time, of exponents $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Therefore the quantities $Q(U^n_x), Q'(U^n_x)$ and $B'(U^n_x)$ are uniformly Hölder continuous functions of x and t, except at $x = 0$; more precisely, they are Hölder continuous on $[-c_3\epsilon_n^{-1/2}, 0) \times [-\epsilon_n^{-1/2}, 0]$ and on $(0, c_3\epsilon_n^{-1/2}] \times [-\epsilon_n^{-1/2}, 0]$.

Finally, the $\xi'_n(t)$ are also uniformly Hölder continuous of exponent $\frac{1}{3}$, since they are given by

$$\xi'_n(t) = -\frac{[Q(U^n_x)U^n_{xx} + \epsilon_n B(U^n_x)]_{x=0}}{p_+ - p_-},$$

in which $[f]_{x=0}$ denotes the jump in f at $x = 0$, i.e. $f(0+,t) - f(0-,t)$.

Thus the v^n's are solutions of uniformly parabolic linear equations, whose coefficients are uniformly Hölder continuous. By the classical interior Schauder estimates for such equations both v_t^n and v_{xx}^n are uniformly Hölder continuous, and we can extract a subsequence for which v^n, v_x^n, v_{xx}^n and v_t^n converge uniformly on bounded sets. Keeping in mind that $|\xi_n'(t)| + |U_{xx}^n| \le C\epsilon_n$, we conclude that the limit of our subsequence is a bounded solution of

$$v_t = Q_* v_{xx}, \qquad v(0,t) \equiv 0.$$

After passing to a further subsequence we may assume that the $\xi_n(t)$ also converge uniformly, say to some ξ_*. Then we have $v_{xx}(-\xi_*, 0) = \pm 1$.

Thus we have a bounded solution of $v_t = Q_* v_{xx}$ with $v(0,t) \equiv 0$, and $v_{xx}(-\xi_*, 0) = \pm 1$. But this is impossible, since any bounded solution with zero boundary data must be a constant, zero in fact. Thus we have a contradiction in the second case as well.

Q. E. D.

Geometry of the Interface.

10. Inflection points.

In [AG, section 7.3] we gave a heuristic argument which showed that for a smoothly evolving interface "the number of fingers cannot increase with time." Here we'll prove this result for the case of stable free energy, and also for the case of unstable free energy.

In addition we'll also prove the following result.

Theorem 10.1. *Let $\Omega : [0,T) \to \mathfrak{D}^{k,\alpha}$ be a solution of (1.1). Then for any given line ℓ the number of tangents to $\partial\Omega(t)$ which are parallel to ℓ is finite (not counting facets); this number drops whenever such a tangent has third or higher order contact with $\partial\Omega(t)$, i.e. whenever the corresponding point of tangency is an inflection point.*

Let $\mathfrak{r}(p,t)$ $(P(t) \le p \le Q(t), 0 < t < T)$ be an evolving curve in the sense of section [AG, section 2.2], so that $\mathfrak{r}_t$ is always perpendicular to the curve. In this section we shall regard all quantities as functions of p and t, so that the subscript $(\cdots)_t$ denotes differentiation in the direction $p = const$, i.e. the normal time-derivative (in [AG, section 2.2] we called this $(\cdots)^\circ$.)

If $\theta(p,t)$ is the angle between the tangent $\mathfrak{r}_p(p,t)$ and some fixed direction, then one of the transport identities ([AG, section 2.18]) says that $\theta_t = -V_s$, where $V(p,t)$ is the normal velocity of $\mathfrak{r}$, and V_s is its derivative with respect to arclength. To eliminate the arclength from this relation, one introduces

$$J(p,t) \stackrel{\text{def}}{=} |\mathfrak{r}_p(p,t)| = \frac{\partial s(p,t)}{\partial p},$$

and writes V_s as $V_p(p,t)/J(p,t)$.

Using our first law of motion (1.1), which says $g(\theta)K = \beta(\theta)V + F$, as well as the relation $K = \theta_s = \theta_p(p,t)/J(p,t)$, we can eliminate both V and K, and we get

$$\theta_t = \frac{1}{J(p,t)} \frac{\partial}{\partial p} \left(\frac{g(\theta)}{J(p,t)\beta(\theta)} \frac{\partial \theta}{\partial p} + \frac{F}{\beta(\theta)} \right) \tag{10.1}$$

which has the form

$$\theta_t = a(p,t)\theta_{pp} + b(p,t)\theta_p \tag{10.2}$$

where

$$a(p,t) = \frac{g(\theta)}{J(p,t)^2 \beta(\theta)} > 0,$$

and $b(p,t)$ is some other expression involving $\theta(p,t)$ and possibly $\theta_p(p,t)$. Its precise form doesn't matter: what matters is that both $a(p,t)$ and $b(p,t)$ are well defined smooth functions whenever one has a smoothly evolving interface, and that differentiating (10.2) with respect to p shows that $\kappa = \theta_p$ also satisfies a linear parabolic PDE:

$$\kappa_t = a(p,t)\kappa_{pp} + (b(p,t) + a_p(p,t))\kappa_p + b_p(p,t)\kappa \tag{10.3}$$

Since $K = \theta_s = \kappa(p,t)/J(p,t)$, the inflection points of the interface, i.e. the zeroes of K, correspond to the zeroes of $\kappa = \theta_p$.

The case in which the free energy is stable. If our evolving interface is a simple closed curve then $\mathfrak{r}(p,t)$ will be a periodic function of p, whose period is independent of time – we may assume this period is 1. All other quantities (θ, V and K) will also be periodic with period 1. Since our solution is smooth, i.e. C^∞, it follows immediately from the results in [A2] and the fact that θ_p satisfies (10.3) that *the number of zeroes of* $\theta_p(\cdot t)$ *is a finite and nonincreasing function of* $t \in (0,T)$, and that, except at a discrete set of times $\{0 < \ldots < t_n < t_{n+1} < \ldots\}$ (which may accumulate at $t = 0$), *all zeroes of* $\theta_p(\cdot t)$ *are simple.* In other words, the number of inflection points is always finite, and does not increase with time.

Instead of applying the results from [A2] to θ_p we could have applied them directly to θ itself; the conclusion would be that $\theta(\cdot,t)$ only has a finite number of zeroes, and that this number drops whenever one of those zeroes is degenerate. Since zeroes of $\theta(\cdot,t)$ correspond to the vertical tangents to $\partial\Omega(t)$, this proves the theorem, in the case that the line ℓ is vertical. To reach the same conclusion for any other line, we could rotate our coordinate system so that this line becomes vertical; alternatively, we could observe that $\theta(p,t) - \vartheta$ is a solution of (10.2) for any constant ϑ, if $\theta(p,t)$ is a solution, and apply the foregoing arguments to $\theta(p,t) - \vartheta$.

The case in which the free energy is unstable. Let $\mathfrak{r}(p,t)$ be an evolving arc both of whose endpoints are one leg of a corner, and for which $g(\theta(p,t)) > 0$ holds for all $P(t) \le p \le Q(t)$ and $0 < t < T$. The angles $\theta(P(t),t)$ and $\theta(Q(t),t)$ then belong to a finite set determined by the Frank diagram, and hence they are constant, say θ_P and θ_Q, respectively. The results from [A2] only apply to solutions of equations like (10.2) and (10.3) if the solutions are defined on rectangles $[P,Q] \times (0,T)$ (with P and Q independent of time). Fortunately this is no problem, since we can introduce a new variable $p' = (p - P(t))/(Q(t) - P(t))$, running between 0 and 1, and verify that in the new (p',t) coordinates θ still satisfies an equation like (10.2) while $\theta_{p'}$ still satisfies one like (10.3).

Observe again that for any ϑ the function $\theta(p,t) - \vartheta$ satisfies (10.2), and either vanishes identically at $p = P(t)$ or $p = Q(t)$ (this happens when ϑ and θ_P or θ_Q coincide), or else $\theta(P(t),t) - \vartheta \ne 0$ and $\theta(Q(t),t) - \vartheta \ne 0$ for *all* $t \in (0,T)$. This implies that $\theta(p,t) - \vartheta$ will only have a finite number of zeroes for each $0 < t < T$, and that, except at a discrete set of times, these zeroes will be simple, i.e. $\theta_p \ne 0$. The statement in the theorem is nothing but the geometrical interpretation of these statements.

If we specialize the foregoing argument to the case $\theta = \theta_P$ or $\theta = \theta_Q$, we reach the following conclusion.

Lemma 10.2. *The curvature at a corner point can only vanish at a discrete set of times $\{0 < \ldots < t_n < t_{n+1} < \ldots\}$. This sequence may accumulate at $t = 0$, but since the number of zeroes of either $\theta - \theta_P$ or $\theta - \theta_Q$ must go down at each t_n, the sequence cannot have any other accumulation points.*

If we assume that our initial interface is admissible, i.e. if we assume that all angles $\theta(p,0)$ are globally stable, then we must have $\theta_P \le \theta(p,0) \le \theta_Q$ (or $\theta_Q \le \theta(p,0) \le \theta_P$). From the maximum principle it follows that $\theta_P \le \theta(p,t) \le \theta_Q$ for all $t > 0$, so that the interface is admissible for all positive times: *an initially admissible domain $\Omega(0)$ remains admissible.*

Admissibility of the interface implies that $\theta(p,t) - \theta_P \ge 0$, so that the Hopf boundary point lemma for parabolic equations (see [PW]) tells us that either $\theta_p(p,t) \ne 0$ for all $0 < t < T$ or $\theta(p,t) \equiv \theta_P$. Thus *for an admissible interface the curvatures at a corner point never vanish, except on facets.*

A consequence of this is that *if $\Omega(t)$ $(0 < t < T)$ evolves according to (1.1), is admissible, and has convex corners, then it has strictly convex corners.*

If we assume that the curvatures at the endpoints of our evolving arc are nonzero (as is the case for an admissibly evolving interface). Then $\theta_{p'}(p',t)$ satisfies an equation like (10.3) on the rectangle $[0,1] \times (0,T)$, and does not vanish on the sides $[0,1] \times (0,T)$ of this rectangle. Moreover the zeroes of $\theta_{p'}$ correspond to those of θ_p and K, and hence to inflection points. By theorem D of [A2] this implies:

Lemma 10.3. *The number of inflection points on the arc $\mathfrak{r}(p,t)$ is finite and nonincreasing in time, while except at a discrete set of times all inflection points of the arc are simple.*

The tangent to an inflection point. Let $Q(t) \in \partial\Omega(t)$ be a nondegenerate inflection point for $t_0 < t < t_1$, and let $\vartheta(t)$ be the angle of the normal to $\partial\Omega(t)$ at $Q(t)$. If $\partial\Omega(t)$ is parametrized by $\mathfrak{r}(p,t)$, then there is a smooth function $q(t)$ $(t_0 < t < t_1)$ for which $Q(t) = \mathfrak{r}(q(t),t)$, and

$$k(q(t),t) = 0, \text{ i.e. } \frac{\partial\theta}{\partial p}(q(t),t) = 0,$$

while

$$\vartheta(t) = \theta(q(t),t).$$

Using $k = 0$ at the inflection point one finds:

$$\begin{aligned}\vartheta'(t) &= \theta_p(q(t),t)q'(t) + \theta_t(q(t),t)\\ &= \theta_t(q(t),t)\\ &= \frac{\partial V}{\partial s}(q(t),t).\end{aligned}$$

If $\Omega(t)$ evolves according to (1.1), and the inflection point is nondegenerate, as we have assumed, then we get

$$\begin{aligned}\vartheta'(t) &= \frac{\partial}{\partial s}\left(\frac{g(\theta)k - F}{\beta(\theta)}\right)\\ &= \frac{g(\theta)}{\beta(\theta)}k_s(Q(t))\\ &\neq 0.\end{aligned}$$

Therefore we have the following observation.

Lemma 10.4. *The normal angle of the tangent of a nondegenerate inflection point of a smooth arc which satisfies (1.1) is a strictly monotone function of time.*

11. The evolution of convex pieces.

Let $\Omega : [0,T) \to \mathfrak{D}^{2,\alpha}$ be a classical solution of (1.1) with convex corners, and consider a family of arcs $\Gamma(t) \subset \partial\Omega(t)$. We shall assume that the arc $\Gamma(t)$ contains no inflection points or facets, and that the end points of $\Gamma(t)$ are either inflection points, or corners at which $\Gamma(t)$ meets a facet. On each segment of $\Gamma(t)$ between corners, the angle (or Gaußmap) $\theta : \Gamma(t) \to \mathbf{R}$ is one to one, its range being an interval. Our hypothesis that $\Omega(t)$ has convex corners implies that the intervals one gets this way are disjoint. We shall denote them by:

$$J^t = \bigcup_{j=0}^{n} J_j^t = (\theta_0(t), \theta_1) \cup (\theta_2, \theta_3) \cup \ldots \cup (\theta_{2n}, \theta_{2n+1}(t)),$$

where $\theta_j < \theta_{j+1}$ and $J_j^t = (\theta_{2j}, \theta_{2j+1})$. The θ_j with $1 \le j \le 2n$ do not depend on time, and if either of the endpoints is a corner connected to a facet, then the corresponding $\theta_0(t)$ or $\theta_{2n+1}(t)$ is also independent of time; otherwise, if the endpoint in question is an inflection point, then one has $\theta_0'(t) > 0$ or $\theta_{2n+1}'(t) < 0$ respectively.

Define

$$Q_T = \bigcup_{0<t<T} J^t \times \{t\}; \qquad S_T = \bigcup_{0<t<T} \{\theta_0(t), \theta_{2n+1}(t)\} \times \{t\}.$$

Since the angle θ is a good coordinate on the arc, we have a parametrization $X(\theta, t)$ of the arc in θ, t coordinates. The support function of the arc is given by $p(\theta,t) = \langle \mathfrak{N}(\theta,t), X(\theta,t)\rangle$, and one can recover the parametrisation from the support function via $X(\theta,t) = p(\theta,t)\cdot\mathfrak{N}(\theta) - p_\theta(\theta,t)\cdot\mathfrak{T}(\theta)$. It is well known that the curvature as a function of (θ,t) is given by $k = -(p_{\theta\theta}+p)^{-1}$. In [AG] we showed that $k(\theta,t)$ satisfies the following equation:

$$\frac{\partial k}{\partial t} = k^2\left[\frac{\partial^2 v}{\partial \theta^2} + v\right], \tag{11.1}$$

where $v(\theta,t) = p_t(\theta,t) = \langle X_t, \mathfrak{N}\rangle$ is the normal velocity of the arc (see equation (2.23) of [AG]). Combined with the first equation of motion $\beta(\theta)V + F = g(\theta)k$, this equation gives a quasilinear parabolic PDE for the curvature k, or, equivalently, for the velocity $v(\theta,t) = V(\theta,k)$. This equation is:

$$\frac{\partial v}{\partial t} = \frac{(\beta(\theta)v + F)^2}{g(\theta)\beta(\theta)}\left[\frac{\partial^2 v}{\partial \theta^2} + v\right]. \tag{11.2}$$

If we are dealing with a convex simple closed curve, then $k(\theta,t)$ is defined and nonzero for all $\theta \in \mathbf{R}/2\pi\mathbf{Z}$; if, in addition, the free energy is strictly stable for all angles, then $g(\theta) > 0$ for all θ, and the equation (11.2) with periodic boundary conditions completely determines the evolution of $v(\theta,t)$, and hence of the domain $\Omega(t)$.

The continuity of the parametrization $X(\theta,t)$ across the gap $[\theta_{2j-1},\theta_{2j}]$ in the angle domain J^t implies a set of boundary conditions for $v(\theta,t)$, one for each gap. Recalling the relation between X and p, we find that the condition $X(\theta_{2j-1},t) = X(\theta_{2j},t)$ is equivalent to

$$(p\mathfrak{N} - p_\theta\mathfrak{T})(\theta_{2j-1},t) = (p\mathfrak{N} - p_\theta\mathfrak{T})(\theta_{2j},t).$$

If one differentiates this with respect to time, and uses $v = p_t = \langle X_t, \mathfrak{N} \rangle$, then one finds:

$$(v\mathfrak{N} - v_\theta \mathfrak{T})(\theta_{2j-1}, t) = (v\mathfrak{N} - v_\theta \mathfrak{T})(\theta_{2j}, t). \tag{11.3}$$

One can rewrite this as follows:

$$\begin{pmatrix} v(\theta_{2j-1}, t) \\ v_\theta(\theta_{2j-1}, t) \end{pmatrix} = \mathcal{R}(\theta_{2j} - \theta_{2j-1}) \cdot \begin{pmatrix} v(\theta_{2j}, t) \\ v_\theta(\theta_{2j}, t) \end{pmatrix}, \tag{11.4}$$

where

$$\mathcal{R}(\alpha) \stackrel{\text{def}}{=} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

If one of the ends of $\Gamma(t)$ is a corner whose other leg is a facet, then we get an additional boundary condition for v at this end.

Let $\theta_0(t) = \theta_0$ be such an end, and let θ_{-1} be the other angle of the corresponding corner. The normal velocity of the adjacent facet is a constant, namely $-F/\beta(\theta_{-1})$. Since $X(\theta_0, t)$ must lie on this facet, we have:

$$\langle \mathfrak{N}(\theta_{-1}), X_t(\theta_0, t) \rangle = \frac{-F}{\beta(\theta_{-1})},$$

which implies

$$v_\theta(\theta_0, t) = \cot(\theta_0 - \theta_{-1}) v(\theta_0, t) - \frac{F}{\beta(\theta_{-1}) \sin(\theta_0 - \theta_{-1})}. \tag{11.5}$$

12. A maximum principle.

Define $J^t, \bar{Q}_T$ and S_T as in the previous section, and let $a, b, c : \bar{Q}_T \to \mathbf{R}$ be given bounded functions, with $a \geq \delta > 0$. Consider a classical solution $u \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ of

$$u_t \geq a(\theta, t) u_{\theta\theta} + b(\theta, t) u_\theta + c(\theta, t) u, \tag{12.1}$$

which also satisfies the following boundary conditions:

$$\begin{pmatrix} u(\theta_{2j-1}, t) \\ u_\theta(\theta_{2j-1}, t) \end{pmatrix} = \mathcal{A}_j \cdot \begin{pmatrix} u(\theta_{2j}, t) \\ u_\theta(\theta_{2j}, t) \end{pmatrix} \tag{12.2}$$

where $\mathcal{A}_j = \begin{pmatrix} p_j & q_j \\ r_j & s_j \end{pmatrix}$, and $\det(\mathcal{A}_j) > 0, q_j < 0$.

The following theorem is a variation on the classical "strong" maximum principle; the proof of our theorem uses many of the same ideas as the proof of the classical theorem, which the reader can find in the book of M. Protter and H. Weinberger [PW].

Theorem 12.1. *If $u \geq 0$ on S_T, then $u \geq 0$ on Q_T. Moreover, either u vanishes everywhere in Q_T, or u is strictly positive in $\bar{Q}_T \setminus S_T$.*

Proof. We shall first prove the weaker conclusion "$u \geq 0$ on Q_T" and then show that it implies the second statement in the theorem.

By considering $e^{-\lambda t}u(\theta, t)$ instead of u we may assume that $c(\theta, t) \leq 0$.

We may assume that the inequality in (12.1) is strict. Indeed, let $\psi \in C^2(\mathbb{R})$ be any function which satisfies the conditions (12.2); then for sufficiently large $\mu > 0$ the function $w = e^{\mu t}\psi(\theta)$ will satisfy the strict form of (12.1), and one can apply the arguments given below to $u + \epsilon w$, for any $\epsilon > 0$; these will show that $u + \epsilon w \geq 0$ for all positive ϵ, and hence that $u \geq 0$ on Q_T. By the same arguments we may also assume that $u > 0$ on S_T, instead of merely $u \geq 0$ on S_T.

To reach a contradiction suppose that $u(\theta, t) < 0$ for some $(\theta, t) \in Q_T$. Then there exists a smallest $t_* \geq 0$ such that $u(\theta_*, t_*) \leq 0$ for some $\theta_* \in J^{t_*}$. Since $u > 0$ on S_T, we have $t_* > 0$, and by the classical maximum principle for parabolic PDE's θ_* cannot be an interior point, so that it must be one of the θ_j's (with $1 \leq j \leq 2n$).

First consider the case in which $j = 2k$ is even. The Hopf boundary point lemma implies that $u_\theta(\theta_{2k}, t_*) > 0$. If one combines this with the boundary condition (12.2), one finds

$$\begin{aligned} u(\theta_{2k-1}, t_*) &= p_k u(\theta_{2k}, t_*) + q_k u_\theta(\theta_{2k}, t_*) \\ &= q_k u_\theta(\theta_{2k}, t_*) \\ &< 0. \end{aligned} \tag{12.3}$$

This is a contradiction, since $u \geq 0$ for $t \leq t_*$.

Given the observation that $\mathcal{A}_j^{-1} = \begin{pmatrix} p_j' & q_j' \\ r_j' & s_j' \end{pmatrix}$ with

$$q_j' = \frac{-q_j}{\det(\mathcal{A}_j)} > 0,$$

one can deal with the other case, in which j is odd, in the same way as the case in which j is even.

The second statement in the theorem may be proved as follows. Assume that the solution is not identically zero. By the ordinary strong maximum principle we must have $u > 0$ everywhere in Q_T. Thus if u vanishes somewhere in $\bar{Q}_T \setminus S_T$, then there is a $t_* > 0$ and a θ_j $(1 \leq j \leq 2n)$ for which $u(\theta_j, t_*) = 0$. But now the Hopf boundary point lemma and the calculation (12.3) lead us to the same contradiction we had above. Therefore u must be strictly positive if it doesn't vanish everywhere.

Q. E. D.

Addendum 12.2. *If $\theta_0(t) \equiv \theta_0$ and $\theta_{2n+1}(t) \equiv \theta_{2n+1}$ are constant, and if one assumes that $u(\theta, t)$ satisfies*

$$u_\theta(\theta_0, t) \le m_0(t) u(\theta_0, t), \qquad u_\theta(\theta_{2n+1}, t) \ge m_1(t) u(\theta_{2n+1}, t) \tag{12.4}$$

for $0 < t \le T$ and for certain continuous functions $m_j : [0, T) \to \mathbf{R}$, then the conclusion of theorem 12.1 still holds.

Proof. We shall show that the boundary condition (12.4) implies that $u \ge 0$ on $\bar{Q}_T$; strict positivity in the interior of Q_T then follows from theorem 12.1.

By replacing u with $u + \epsilon w$ for arbitrary small $\epsilon > 0$, and for some $w = e^{\mu t}\psi(\theta)$, we may assume, as in the proof of the previous theorem, that $u > 0$ at $t = 0$.

If there were some (θ, t) such that $u(\theta, t) \le 0$, then one could choose a minimal $t_* > 0$, for which there exists a θ_* with $u(\theta_*, t_*) = 0$. The proof of theorem 12.1 shows that θ_* cannot be an interior point or one of the $\theta_1, \ldots, \theta_{2n}$, so that θ_* must be either θ_0 or θ_{2n+1}; we shall assume the former.

Since $u > 0$ for $t < t_*$ it follows from the Hopf boundary point lemma that $u_\theta(\theta_0, t_*) > 0$. On the other hand the boundary condition (12.4) implies that $u_\theta(\theta_0, t_*) \le m_0(t_*) u(\theta_0, t_*) = 0$. Thus we have a contradiction which shows that $u \ge 0$ on $\bar{Q}_T$.

Q. E. D.

13. Containment.

In this section we shall prove the inclusion theorem 3.3 of Section 3, and also describe the counterexample that was mentioned there.

Let $\Omega_{1,2} : [0, T) \to \mathfrak{D}^{2,\alpha}$ be two *admissible* classical solutions of (1.1) which have *convex corners*, for which $\bar{\Omega}_1(0) \subset \mathrm{Int}(\Omega_2(0))$, and assume that there is some $t_* > 0$ for which the boundaries of $\Omega_1(t)$ and $\Omega_2(t)$ meet. We may assume that t_* is the smallest $t > 0$ for which this happens. Let P be some point in the intersection $\partial\Omega_1(t_*) \cap \partial\Omega_2(t_*)$. By the strong maximum principle P must be a common corner point of $\partial\Omega_1(t_*)$ and $\partial\Omega_2(t_*)$. Denote the angles corresponding to this corner by $\alpha < \beta$, and let $p_{1,2}(\theta, t)$ be the support functions of $\partial\Omega_{1,2}(t)$ near (P, t_*).

Since curvatures at convex corners of admissible solutions never vanish (see section 10) the angle θ will be a good coordinate on both curves $\partial\Omega_{1,2}(t)$ in a neighborhood of P and for t close to t_*. Thus, if we chose $\epsilon > 0$ small enough, then the $p_{1,2}(\theta, t)$ are well defined on $\bar{Q} = \bar{J} \times [t_* - \epsilon, t_*]$, where $J = (\alpha - \epsilon, \alpha) \cup (\beta, \beta + \epsilon)$.

The support functions satisfy $p_t = -(p_{\theta\theta} + p)^{-1}$, as well as the jump condition

$$\begin{pmatrix} p(\beta, t) \\ p_\theta(\beta, t) \end{pmatrix} = \mathcal{R}(\beta - \alpha) \cdot \begin{pmatrix} p(\alpha, t) \\ p_\theta(\alpha, t) \end{pmatrix}.$$

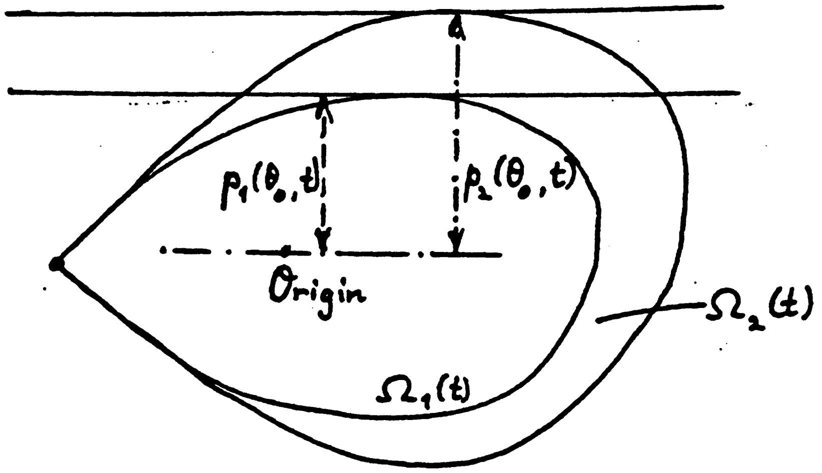

Figure 13.1

Moreover, it is geometrically evident that $p_2(\theta,t) > p_1(\theta,t)$ on $\bar{Q}$, except at (α, t_*) and (β, t_*) (see figure 13.1).

The difference $w(\theta,t) = p_2(\theta,t) - p_1(\theta,t)$ then satisfies a linear parabolic equation, $w_t = a(\theta,t)(w_{\theta\theta} + w)$, for some smooth $a(\theta,t) > 0$. Since $\Omega_1(t_*)$ and $\Omega_2(t_*)$ are tangent at the corner point P we have $w(\alpha, t_*) = 0$ and $w(\beta, t_*) = 0$. On the other hand we have just observed that $w(\theta,t) > 0$ at all other points in $\bar{Q}$, so that the maximum principle of the previous section tells us that both $w(\alpha, t_*) > 0$ and $w(\beta, t_*) > 0$, which is a contradiction. Therefore the closed domain $\bar{\Omega}_1(t)$ must be contained in the open domain $\Omega_2(t)$ for all times t at which both are defined.

The counterexample. Assume the Frank diagram is as in figure 13.2: it has two inflection points, and only one bitangent. This bitangent meets the Frank diagram at the angles $\{\theta_1, \theta_2\}$, and these angles correspond to the only allowable corner which any $\Omega \in \mathfrak{D}^{2,\alpha}$ may have.

We let $\Omega_1 \in \mathfrak{D}^{2,\alpha}$ be a convex domain whose boundary has strictly negative curvature; this domain must have exactly one corner, P.

For $\Omega_2 \in \mathfrak{D}^{2,\alpha}$ we choose a domain which contains Ω_2, and whose boundary also contains the point P. However, we assume that P is a smooth point of $\partial\Omega_2$, which we can do since we allow the tangent angle $\theta_{\Omega_2}(P)$ to be stable, *but not strictly stable.* The reader can easily verify that we can choose Ω_2 so that the curvature of $\partial\Omega_2$ at P is any prescribed value. These means that we can prescribe the normal velocity of $\partial\Omega_2$ at P, and in particular that we can choose Ω_2 so that the corner $P(t)$ of $\Omega_1(t)$ will pierce through $\partial\Omega_2(t)$ as both domains evolve according to (1.1) (see

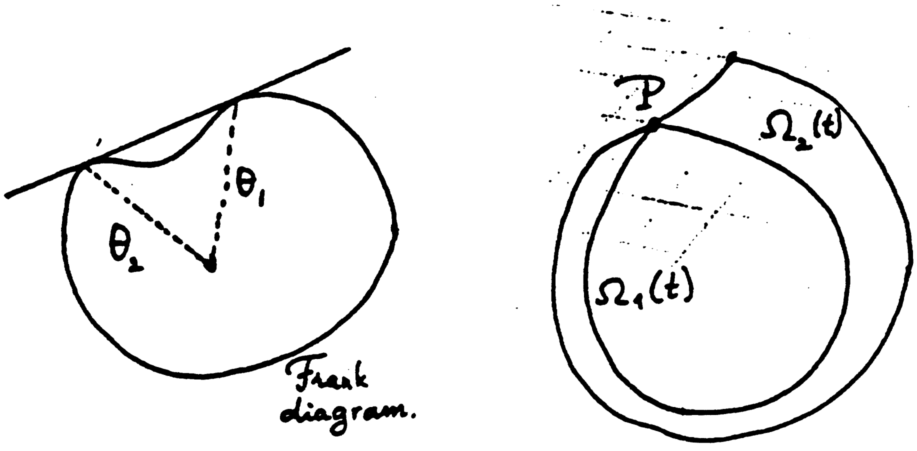

Figure 13.2, 13.3

figure 13.3.)

14. Proof of theorem 3.2.

Let $\Omega : [0,T) \to \mathfrak{D}^{2,\alpha}$ be a maximal classical solution of (1.1) with $T < \infty$, and assume that neither of the two conditions $\mathbf{E}_2$ or $\mathbf{E}_3$ of our main existence theorem 3.1 occur. Assume in addition that $\Omega(t)$ only has convex corners. Then theorem 3.2 asserts that for any $\delta > 0$ there is a $\tau_\delta > 0$ such that at $t = \tau_\delta$ one has $\alpha_\delta(\partial\Omega(\tau_\delta) > \pi - \delta$.

In this section we shall prove this, arguing by contradiction; thus we shall assume that there is some $\delta > 0$ such that $\alpha_\delta(\partial\Omega(t) \le \pi - \delta$, for any $0 < t < T$. Using a blow up argument we shall show that this is impossible.

Choose sequences $t_n \uparrow T$ and $P_n \in \partial\Omega(t_n)$ such that $|K(P,t)| \le |k(P_n, t_n)|$ for all $0 \le t \le t_n$ and $P_n \in \partial\Omega(t_n)$, and define the following "blow ups" of $\Omega(t)$:

$$\Omega_n(t) = \phi_n(\Omega(t_n + \epsilon_n^2 t)),$$

where

$$\epsilon_n = |k(P_n, t_n)|^{-1}, \qquad \phi_n(x) = \frac{x - P_n}{\epsilon_n}.$$

Then $\Omega_n : \left[-t_n/\epsilon_n^2, (T - t_n)/\epsilon_n^2\right) \to \mathfrak{D}^{2,\alpha}$ is a solution of

$$\beta(\theta)V = g(\theta)K - \epsilon_n F, \qquad (1.1)'$$

while the capillary force $\mathfrak{C}$ is still continuous on $\partial\Omega_n(t)$. Moreover, the curvatures of $\partial\Omega_n(t)$ for $t \le 0$ are uniformly bounded by $|K_{\partial\Omega_n(t)}| \le 1$;

the arguments of section nine then show that the $K_{\partial\Omega_n(t)}$'s are uniformly Lipschitz, and in fact that all derivatives of the curvature (with respect to arclength) are uniformly bounded for $t \le 0$. After passing to a subsequence, if necessary, we may assume that the $\partial\Omega_n(t)$ converge to some limit family $\Omega_\infty(t)$ $(-\infty < t \le 0)$ of domains (the "convergence" here is to be understood as follows: for any $t_0, R > 0$ the curve $\partial\Omega_n(t) \cap B_R(0)$ converges in the Hausdorff metric on compact sets in the plane to $\partial\Omega_\infty \cap B_R(0)$, uniformly on the bounded time interval $-t_0 \le t \le 0$.)

The limit family is a classical solution of

$$\beta(\theta)V = g(\theta)K,$$

it satisfies

$$|K_{\partial\Omega_\infty}(P,t)| \le 1 \text{ for all } P \in \partial\Omega_\infty(t), t \le 0;$$

moreover the origin lies on $\partial\Omega_\infty(0)$, and

$$|K_{\partial\Omega_\infty}(O,0)| = 1.$$

Since $\partial\Omega(t)$ does not shrink to a point (by assumption–otherwise $\mathbf{E}_3$ would occur), the blow up domain $\Omega_\infty(t)$ can never be a bounded domain.

Lemma 14.1. *$\Omega_\infty(t)$ is convex.*

Proof. We shall show that $\partial\Omega_\infty(t)$ contains no inflection points, and that all its corners are convex; the lemma follows easily from these two facts.

Let $Q(t_0) \in \partial\Omega_\infty(t_0)$ be an inflection point. Since inflection points are always nondegenerate, except at a discrete set of times, by theorem 10.3 we may assume that $Q(t_0)$ is nondegenerate; in particular, $\partial\Omega_\infty(t)$ has a nondegenerate inflection point $Q(t)$ near $Q(t_0)$ for t close to t_0, and the motion of this point is smooth in time. Let $\alpha(t)$ be the normal angle of the tangent to $\partial\Omega_\infty(t)$ at $Q(t)$. By lemma 10.4 it is a strictly monotone function of time.

Our original domain $\Omega(t)$ has a finite number of inflection points in its boundary, and this number does not increase with time, so there must be a $t_1 < T$ such that this number is constant when $t_1 < t < T$. Let $Q_1(t), \ldots, Q_m(t)$ be the inflection points of $\partial\Omega(t)$, and denote their normal angles by $\alpha_1(t), \ldots, \alpha_m(t)$. These angles are bounded monotone functions of time (lemma 10.4 again), so they have limits $\lim_{t\uparrow T} \alpha_j(t) = \alpha(T)$.

After passing to a subsequence once more, if necessary, the inflection point $Q(t)$ of the blown up domain can be written as the limit of $\phi_n(Q_j(t_n + \epsilon_n^2 t))$'s, for some fixed $j \in \{1, \ldots, m\}$. Since the curves $\partial\Omega_n(t)$ have bounded curvatures, their tangents will also converge, and hence we find that $\alpha(t) = \alpha_j(T)$ for all t near t_0. But we have just observed that $\alpha(t)$ cannot be constant, so we have a contradiction; therefore there cannot be any inflection points in $\partial\Omega_\infty(t)$.

Next we consider the corner points of $\Omega_\infty(t)$; let $Q(t)$ be one of them, and denote the corner points of $\Omega(t)$ by $P_1(t), \ldots, P_k(t)$. After taking yet another subsequence we may assume that $Q(t)$ is the limit of $Q_n(t) = \phi_n(P_j(t_n + \epsilon_n^2 t))$ for some fixed j.

Consider the tangent cone $\Gamma_n(t)$ to $\Omega_n(t)$ at $Q_n(t)$:

$$\Gamma_n(t) = \bigcap_{\rho>0} \left\{ \frac{q - Q_n(t)}{|q - Q_n(t)|} \mid q \in B_\rho(Q_n(t)) \cap \Omega_n(t) \right\}$$

Since any bounded part of any smooth component of $\partial\Omega_n(t)$ converges in C^∞ to the corresponding component of $\partial\Omega_\infty(t)$, the tangent cones $\Gamma_n(t)$ will converge to the tangent cone $\Gamma_\infty(t)$ of $\Omega_\infty(t)$ at $Q(t)$; since each $\Gamma_n(t)$ is convex, $\Gamma_\infty(t)$ is also convex.

If both components of $\partial\Omega_\infty(t)$ which meet at $Q(t)$ are *facets*, then near $Q(t)$ the domain $\Omega_\infty(t)$ coincides with its tangent cone, and hence it must be convex.

Assume therefore that at least one of the two arcs which meet at $Q(t_0)$, say $\gamma(t_0)$, is not a facet, and denote its curvature at $Q(t_0)$ by $\kappa(t_0)$. Clearly we have $\kappa(t_0) \leq 0$, for if $\kappa(t_0) > 0$, then the curvature at $Q_n(t_0)$ of the corresponding arc in $\partial\Omega_n(t_0)$ would also be positive, and $Q_n(t_0)$ would not be a convex corner of $\Omega_n(t_0)$.

Suppose that $\kappa(t_0) = 0$. By lemma 10.2 the number of points on $\partial\Omega_\infty(t)$ whose normal angle θ equals θ_Q must drop as t increases beyond t_0 (here θ_Q is the normal angle of $\gamma(t)$ at $Q(t)$.) In particular, for $t < t_0$ there must have been at least one other point on $\gamma(t)$, besides $Q(t)$, at which $\theta = \theta_Q$, and between this point and $Q(t)$ there must have been an inflection point. This cannot be, however, since we have just shown that $\partial\Omega_\infty(t)$ doesn't have inflection points.

Thus $\kappa(t_0) < 0$ and $Q(t_0)$ must be a convex corner point.

Q. E. D.

So we see that the blown up domain is convex, and its boundary consists of a finite number of smooth arcs $\gamma_0(t), \ldots, \gamma_n(t)$. We may order these arcs so that the end point of $\gamma_i(t)$ is the begin point of $\gamma_{i+1}(t)$. If one of these arcs, say $\gamma_j(t)$, is a facet, then it cannot contain more than one corner point; should both of its end points be corner points of $\Omega_\infty(t)$, then both arcs adjacent to $\gamma_j(t)$ would have the same normal angles at the corner points of $\gamma_j(t)$, and one of these two corner points would not be convex.

Thus if $\partial\Omega_\infty(t)$ contains a facet, this facet must be either $\gamma_0(t)$ or $\gamma_n(t)$.

Just as in section 10, the range of the normal angle $\theta : \partial\Omega_\infty(t) \to \mathbf{R}$ is the disjoint union of intervals $J^t = (\theta_0(t), \theta_1) \cup (\theta_2, \theta_3) \cup \ldots \cup (\theta_{2n}, \theta_{2n+1}(t))$. In the present situation we can also prove the following.

Lemma 14.2. *$\theta_0(t)$ and $\theta_{2n+1}(t)$ are independent of t, while $\theta_{2n+1} - \theta_0 \leq \pi - \delta$.*

Proof. Our hypothesis $\alpha_\delta(\partial\Omega(t)) \le \pi - \delta$ implies that the rescaled domains satisfy $\alpha_{\delta/\epsilon_n}(\partial\Omega_n(t) \le \pi - \delta$, and after taking the limit this becomes $\alpha_R(\partial\Omega_\infty(t) \le \pi - \delta$ for any $R > 0, t \le 0$. In particular we get $\theta_{2n+1} - \theta_0 \le \pi - \delta$.

If $\gamma_0(t)$ is a facet, then θ_0 and θ_1 coincide, and $\theta_0(t)$ is constant.

If $\gamma_0(t)$ is not a facet, then it is an unbounded convex curve whose asymptotic direction has normal angle $\theta_0(t)$. The curvature of $\partial\Omega_\infty(t)$ is bounded by $|K| \le 1$, so the normal velocity of $\partial\Omega_\infty(t)$ is also bounded:

$$|V| = \left|\frac{g(\theta)}{\beta(\theta))}K\right| \le \sup_\theta \left|\frac{g(\theta)}{\beta(\theta))}\right|\} < \infty.$$

This prevents the asymptotic direction $\theta_0(t)$ to change, since such a change would require $\gamma_0(t)$ to "rotate."

In either case $\theta_0(t)$ does not depend on t, while the same arguments can be used to show that $\theta_{2n+1}(t)$ doesn't change either.

Q. E. D.

As we observed in section 11, the normal velocity $v(\theta, t)$ of $\partial\Omega_\infty(t)$ satisfies

$$v_t = \frac{\beta(\theta)}{g(\theta)} v^2 (v_{\theta\theta} + v) \tag{14.1}$$

for $\theta \in J$, and $t \le 0$ (see (11.2).) Across the gaps in the domain J, the velocity v satisfies the boundary conditions (11.4). If the arc $\gamma_0(t)$ is a facet, then θ_0 and θ_1 coincide, so that the open interval (θ_0, θ_1) is empty, and v_θ is only defined on $J = (\theta_2, \theta_3) \cup \ldots \cup (\theta_{2n}, \theta_{2n+1})$; in this case the first boundary condition of the form (11.4) must be replaced by (11.5), i.e. by

$$v_\theta(\theta_2, t) = \cot(\theta_2 - \theta_1) \cdot v(\theta_2, t). \tag{14.2}$$

If $\gamma_n(t)$ is a facet, then θ_{2n} and θ_{2N+1} coincide, and we get the following boundary condition for v at $\theta = \theta_{2n-1}$:

$$v_\theta(\theta_{2n-1}, t) = \cot(\theta_{2n} - \theta_{2n-2}) \cdot v(\theta_{2n-1}, t). \tag{14.3}$$

If one of the end arcs of $\partial\Omega_\infty(t)$ is not a facet, then the following lemma gives us the boundary behaviour of v at $\theta = \theta_0$ or $\theta = \theta_{2n+1}$.

Lemma 14.3. *If $\gamma_0(t)$ is not a facet, then $\lim_{\theta\downarrow\theta_0} v(\theta, t) = 0$, uniformly on bounded time intervals. Likewise, if $\gamma_n(t)$ is not a facet, then $\lim_{\theta\uparrow\theta_{2n+1}} v(\theta, t) = 0$, uniformly on bounded time intervals.*

Proof. Without loss of generality we may assume that $\theta_0 = -\frac{\pi}{2}$, and that $\gamma_0(t)$ is not a facet; the other case is proved in the same way.

Given any $t_0 < 0$, we can find a $\xi < 0$ such that for $t_0 \le t \le 0$ the part of $\gamma_0(t)$ in the half plane $\{x \le \xi\}$ is given by the graph $y = h(x, t)$ of a smooth function h. Our assumptions imply that *i* $h_{xx} > 0$ ($\Omega_\infty(t)$ is convex),

ii $h_x(x,t) \to 0$ as $x \downarrow -\infty$, (since $\theta_0 = -\frac{\pi}{2}$), and hence $h_x(x,t) > 0$.

Moreover, h is a classical solution of a parabolic PDE of the form

$$h_t = a(h_x)h_{xx}.$$

By choosing ξ larger negative, we can ensure that $h_x < 1$ for $x \le \xi, t \in [t_0, 0]$; the boundedness of the curvature then implies that $h_{xx} = K(1 + h_x^2)^{3/2}$ is also bounded uniformly. Together with classical parabolic estimates this then implies that all derivatives of h are uniformly bounded, and hence, by interpolation, that $h_{xx}(x,t) \to 0$ as $x \downarrow 0$, uniformly in $t \in [t_0, 0]$.

Now let $\epsilon > 0$ be given. We shall find a $\delta > 0$ such that $|K| < \epsilon$ whenever $\theta < \delta$, and $t \in [t_0, t]$, which is what is claimed in the lemma.

First choose $\xi_\epsilon < \xi$ such that $h_{xx}(x,t) < \epsilon$ if $x \le \xi_\epsilon$ and $t_0 \le t \le 0$; then choose δ as follows:

$$\tan\delta = \inf_{t_0 \le t \le 0} h_x(\xi_\epsilon, t).$$

If $Q \in \gamma_0(t)$ is any point with $\theta(Q) < \delta$, then the x coordinate of Q must satisfy $x < \xi_\epsilon$, and hence the curvature of $\partial\Omega_\infty(t)$ at Q is bounded by

$$|k| = \left| \frac{h_{xx}}{(1 + h_x^2)^{3/2}} \right| \le |h_{xx}| < \epsilon.$$

Q. E. D.

To complete the proof of theorem 3.2 we shall construct a supersolution of (11.2), and use the maximum principle of section 12 to conclude that our blown up solution $v(\theta, t)$ must vanish. This furnishes us with a contradiction, since we have, by construction, $|v| = |g(\theta)/\beta(\theta)k| = g(\theta)/\beta(\theta) \ne 0$, at $t = 0$, at the origin. The contradiction shows that blow up cannot occur under the hypotheses of theorem 3.2.

We must consider three (slightly) different cases, depending on whether the ends of $\partial\Omega_\infty(t)$ are curved or flat.

First we deal with the case in which neither $\gamma_0(t)$ nor $\gamma_n(t)$ are facets.

Assume that $J \subset [\delta/2, \pi - \delta/2]$, and let $W(\theta) = W(\lambda, \theta)$ be the solution on J of

$$W''(\theta) + W(\theta) = \frac{-\lambda}{W(\theta)}, \qquad (\theta \in J) \tag{14.4}$$

$$W(\theta_0) = \sin\theta_0, \qquad W'(\theta_0) = \cos\theta_0, \tag{14.5}$$

$$\begin{pmatrix} W(\theta_{2j}) \\ W'(\theta_{2j}) \end{pmatrix} = \mathcal{R}(\theta_{2j-1} - \theta_{2j}) \begin{pmatrix} W(\theta_{2j-1}) \\ W'(\theta_{2j-1}) \end{pmatrix}. \tag{14.6}$$

For $\lambda = 0$ the solution is given by $W(\theta) = \sin\theta$, and by continuous dependence on parameters $W(\lambda, \theta)$ is defined and strictly positive on $\bar{J}$ for small enough $\lambda > 0$; we fix such a λ.

If we choose $\mu > 0$ large enough, then

$$w(\theta, t) = w(\lambda, \mu; \theta, t) = \mu \frac{W(\lambda, \theta)}{\sqrt{t}}$$

is a super solution of (14.1). Indeed, if $\mu > \sup_\theta \sqrt{g(\theta)/2\lambda\beta(\theta)}$ then we have:

$$\begin{aligned} w_t - \frac{\beta(\theta)}{g(\theta)} w^2(w_{\theta\theta} + w) &= \left(\lambda\mu^2 \frac{\beta(\theta)}{g(\theta)} - \frac{1}{2}\right) \frac{\mu W(\theta)}{t^{3/2}} \\ &\geq 0. \end{aligned}$$

Now compare $w(\theta, t+\tau)$ and $v(\theta, t)$ for some given $\tau > 0$ on the time interval $-\tau < t \leq 0$. As $t \downarrow -\tau$ we certainly have $w(\theta, t+\tau) > v(\theta, t)$, and as $\theta \downarrow \theta_0$ or $\theta \uparrow \theta_{2n+1}$ lemma 14.3 tells us that we also have $w(\theta, t+\tau) > v(\theta, t)$; the difference $w(\theta, t+\tau) - v(\theta, t)$ satisfies a linear parabolic inequality like (12.1), as well as the boundary conditions (12.2), so that our maximum principle of theorem 12.1 implies $w(\theta, t+\tau) > v(\theta, t)$ for all $\theta \in J$ and $t \in (-\tau, 0]$. By fixing t and letting $\tau \uparrow \infty$ we get $v(\theta, t) = 0$ for all θ, t.

Next we consider the case in which $\gamma_0(t)$ is a facet, but $\gamma_n(t)$ isn't. In this case $\theta_0 = \theta_1$, and we define W to be the solution of (14.4) and (14.6), but now with the inital conditions

$$W(\theta_0) = \sin(\theta_2 - \theta_1), \qquad W'(\theta_0) = \cos(\theta_2 - \theta_1). \tag{14.7}$$

Just as before we define $w = \mu W t^{-1/2}$, and one verifies that for large enough μ and small enough λ w is a super solution of (14.1). Moreover, w satisfies the same linear boundary condition (14.2) as v, so that we may apply the maximum principle contained in the addendum 12.2, to conclude that $w(\theta, t+\tau) > v(\theta, t)$ for $\theta \in J$ and $t \in (-\tau, 0]$; letting $\tau \uparrow \infty$ we again find that v must vanish for all θ, t.

The case in which $\gamma_n(t)$ is a facet, but $\gamma_0(t)$ isn't can be dealt with in the same way.

The last case which we have to consider occurs when both $\gamma_0(t)$ and $\gamma_n(t)$ are facets, so that $\theta_0 = \theta_1$ and $\theta_{2n} = \theta_{2n+1}$. In this case we define $w(\theta, t)$ to be the same as above, in the case of one facet, and we observe that at $\theta = \theta_{2n-1}$ we have for small enough $\lambda > 0$:

$$w_\theta(\theta_{2n-1}, t) \geq \cot(\theta_{2n} - \theta_{2n-1}) w(\theta_{2n-1}, t);$$

indeed, recall that as $\lambda \downarrow 0$ we have

$$\begin{aligned} \frac{w_\theta}{w} = \frac{W_\theta}{W} &\to \frac{\cos(\theta_{2n-1} - \theta_1)}{\sin(\theta_{2n-1} - \theta_1)} \\ &= -\cot(\pi + \theta_1 - \theta_{2n-1}) \\ &\geq -\cot(\theta_{2n} - \theta_{2n-1}), \end{aligned}$$

since $\theta_{2n} - \theta_1 = \theta_{2n+1} - \theta_0 < \pi$.

Therefore we may apply the maximum principle of addendum 12.2 again, and conclude that v vanishes.

Q. E. D.

15. The four node theorem.

In this section we consider the unique steady state which exists if $F < 0$, and we study the linearised flow near this steady state. In particular we prove that any initial value Ω_0 which lies on the stable manifold of Ω_* has at least four nodes.

> *From here on we shall assume that the free energy is strictly stable, i.e. that $f''(\theta) + f(\theta) > 0$ for all angles $\theta \in \mathbf{R}/2\pi\mathbf{Z}$.*

As in section 11, we can represent any C^2 convex region up to a translation by its curvature k as a function of the angle θ, or by the equivalent quantity $v = V(\theta, k) = (g(\theta)k - F)/\beta(\theta)$. We recall from (11.2) that the evolution of convex domains is then described by the following initial value problem

$$\left.\begin{aligned} &\frac{\partial v}{\partial t} = \frac{(\beta(\theta)v + F)^2}{g(\theta)\beta(\theta)} \left[\frac{\partial^2 v}{\partial \theta^2} + v\right] \\ &v(\theta + 2\pi, t) = v(\theta, t) \qquad (\forall \theta, t) \\ &v(\theta, 0) = v_0(\theta) \qquad (\forall \theta) \end{aligned}\right\} \tag{15.1}$$

Since the curvature of the boundary of a convex domain is negative (in our conventions), we shall only consider solutions of (15.1) which satisfy $v(\theta, t) < -F/\beta(\theta)$ (this condition is equivalent to $k < 0$). Moreover, we shall only consider those v's which correspond to closed curves; recall that a function $k(\theta)$ determines a closed curve if and only if the first Fourier coefficient $\int_0^{2\pi} e^{i\theta} k(\theta)^{-1} d\theta$ of $1/k(\theta)$ vanishes. This prompts us to define the following spaces:

$$\mathcal{O}^\alpha = \left\{ v \in h^\alpha(\mathbf{R}/2\pi\mathbf{Z}) \,\middle|\, v(\theta) < \frac{-F}{\beta(\theta)} \text{ for } \theta \in \mathbf{R}/2\pi\mathbf{Z} \right\},$$
$$\mathcal{X}^\alpha = \left\{ v \in \mathcal{O}^\alpha \,\middle|\, \int_0^{2\pi} e^{i\theta} \frac{f''(\theta) + f(\theta)}{\beta(\theta)v(\theta) + F} d\theta = 0 \right\},$$

for $\alpha \in (0, 1)$, and

$$\mathcal{O}^{2,\alpha} = \mathcal{O}^\alpha \cap h^{2,\alpha}(\mathbf{R}/2\pi\mathbf{Z}), \qquad \mathcal{X}^{2,\alpha} = \mathcal{X}^\alpha \cap h^{2,\alpha}(\mathbf{R}/2\pi\mathbf{Z}).$$

The reader can easily verify that $\mathcal{X}^\alpha$ is an analytic submanifold of $\mathcal{O}^\alpha$ of codimension two.

Theorem 15.1. *The initial value problem* (15.1) *generates an analytic local semiflow on $\mathcal{O}^\alpha$, which leaves X^α invariant.*

Proof. The initial value problem (15.1) is quasilinear, of the form $u'(t) = \mathcal{A}(u(t))u(t)$, where the operator $\mathcal{A}(u) : h^{2,\beta}(\mathbf{R}/2\pi\mathbf{Z}) \to h^\beta(\mathbf{R}/2\pi\mathbf{Z})$ (with $0 < \beta \le \alpha$) is given by

$$(\mathcal{A}(u)\cdot w)(\theta) = \frac{(\beta(\theta)u(\theta)+F)^2}{g(\theta)\beta(\theta)}(w''(\theta)+w(\theta)).$$

For any $u \in \mathcal{O}^\alpha$ this operator with domain $\mathrm{dom}(\mathcal{A}) = h^{2,\beta}(\mathbf{R}/2\pi\mathbf{Z})$ generates an analytic semigroup on $h^\beta(\mathbf{R}/2\pi\mathbf{Z})$. This allows us to apply theorems 2.11 and 2.12 of [A3], and conclude that (15.1) generates a local analytic semiflow on $\mathcal{O}^\alpha$.

To show that the submanifold X^α is invariant under ϕ^t, one observes that along any solution of (15.1) one has $k_t = k^2(v_{\theta\theta}+v)$, and hence

$$\frac{d}{dt}\int_0^{2\pi}\frac{e^{i\theta}}{k(\theta,t)}\,d\theta = -\int_0^{2\pi} e^{i\theta}(v_{\theta\theta}+v)\,d\theta = 0.$$

Q. E. D.

One can determine all the fixed points of the local semiflow ϕ^t; they are exactly the solutions $v(\theta)$ of $v''(\theta)+v(\theta) = 0$, i.e. they are given by $v(\theta) = V\cos(\theta-\alpha)$ for some $\alpha \in \mathbf{R}/2\pi\mathbf{Z}$ and $V \ge 0$. Clearly these functions can only belong to the space $\mathcal{O}^\alpha$ if $F < 0$; conversely, if $F < 0$, then there exist $V_\pm(\alpha) > 0$ for every α such that $v(\theta) = V\cos(\theta-\alpha)$ belongs to $\mathcal{O}^\alpha$ if and only if $-V_-(\alpha) < V < V_+(\alpha)$.

Therefore we shall assume throughout this section that $F < 0$.

Lemma 15.2. *If $F < 0$, then $v = 0$ is the only fixed point of the semiflow which lies in X^α.*

Proof. (cf. [AG, section 6.3]) First we note that $v \equiv 0$ satisfies the closing condition, since

$$\frac{1}{F}\int_0^{2\pi} e^{i\theta}(f''(\theta)+f(\theta))d\theta = 0.$$

Thus $v = 0$ does indeed belong to X^α. To show that $v = 0$ is the only candidate solution which corresponds to a closed curve, we consider

$$D(\lambda) = \int_0^{2\pi}\frac{\cos(\theta-\alpha)}{F+\lambda\cos(\theta-\alpha)}(f''(\theta)+f(\theta))d\theta.$$

If $v = \lambda\cos(\theta-\alpha)$ satisfies the closing condition, then $D(\lambda)$ must vanish. By differentiating under the integral one easily verifies that $D'(\lambda) < 0$, which shows that $D(\lambda)$ can only vanish for one value of λ; we have just seen that this value is $\lambda = 0$.

Q. E. D.

This lemma tells us that there is exactly one domain Ω (up to translation) which does not change under the evolution prescribed by (1.1). We can remove the ambiguity in the definition of Ω by requiring the origin to be the center of mass of this domain; we shall denote the resulting domain by Ω_*.

Lemma 15.3. *The fixed point $v = 0$ is a hyperbolic fixed point for the restricted semiflow, $\phi^t|_{X^\alpha}$. Its unstable manifold is one dimensional.*

Proof. To begin with, $v \equiv 0$ is a fixed point of the semiflow ϕ^t on $\mathcal{O}^\alpha$, so that the linearization of the semiflow at 0 is a one parameter semigroup $d\phi^t(0) = e^{tA}$ on the Banach space $h^\alpha(\mathbf{R}/2\pi\mathbf{Z})$; this semigroup is analytic, and its generator is obtained by linearizing (15.1) at 0. The operator one gets is of Sturm-Liouville type:

$$A = \frac{F^2}{\beta(\theta)g(\theta)}\left\{\frac{d^2}{d\theta^2}+1\right\},$$

with $\mathrm{dom}(A) = h^{2,\alpha}(\mathbf{R}/2\pi\mathbf{Z})$. The spectrum of A consists of eigenvalues which may be ordered as $\lambda_0 > \lambda_1 \geq \lambda_2 > \lambda_3 \geq \lambda_4 > \lambda_5\cdots$, and the eigenfunctions corresponding to the pair of eigenvalues $\{\lambda_{2j-1}, \lambda_{2j}\}$ have exactly $2j$ simple zeroes in $\mathbf{R}/2\pi\mathbf{Z}$; the eigenfunction corresponding to the first eigenvalue, λ_0 is positive.

By inspection one finds that 0 is a double eigenvalue of A, with eigenfunctions $\sin\theta$ and $\cos\theta$. Since these eigenfunctions have two zeroes, their eigenvalues must be λ_1 and λ_2. In other words, we have found that $\lambda_1 = \lambda_2 = 0$, so that $\lambda_0 > 0 > \lambda_3 \geq \lambda_4 > \cdots$.

To see whether 0 is a hyperbolic fixed point of $\phi^t|_{X^\alpha}$, we must consider the linearization of the semiflow on X^α, i.e. the restriction of $e^{tA} = d\phi^t(0)$ to the tangent space $T_0(X^\alpha)$ of X^α at 0. Since X^α is invariant under ϕ^t, the tangent space $T_0(X^\alpha)$ is invariant under e^{tA}, so that the restriction of e^{tA} does indeed define a one parameter semigroup on $T_0(X^\alpha)$.

By linearizing the defining equations of X^α one finds that the tangent space $T_0(X^\alpha)$ is given by

$$T_0(X^\alpha) = \left\{ v \in h^\alpha(\mathbf{R}/2\pi\mathbf{Z}) \;\middle|\; \int_0^{2\pi} e^{i\theta}\frac{g(\theta)}{\beta(\theta)^2}v(\theta)\,d\theta = 0 \right\}.$$

This space does not contain any of the eigenfunctions of A which have eigenvalue 0, i.e. it does not contain $\sin(\theta-\alpha)$ for any $\alpha \in \mathbf{R}$. Since $T_0(X^\alpha)$ has codimension two in $h^\alpha(\mathbf{R}/2\pi\mathbf{Z})$, we can write $h^\alpha(\mathbf{R}/2\pi\mathbf{Z})$ as the direct sum $h^\alpha(\mathbf{R}/2\pi\mathbf{Z}) = T_0(X^\alpha) \oplus Z$, where Z is the subspace of $h^\alpha(\mathbf{R}/2\pi\mathbf{Z})$ spanned by $\sin\theta$ and $\cos\theta$. This splitting is invariant under e^{tA}, and Z is exactly $\mathrm{kern}(A)$, so that the spectrum of A restricted to $T_0(X^\alpha)$ consists of all eigenvalues of A, except 0. Hence $v \equiv 0$ is indeed a hyperbolic fixed point of ϕ^t, with a one-dimensional unstable manifold.

Q. E. D.

Recall that we had defined a *node* on the boundary $\partial\Omega$ of a domain $\Omega \in \mathfrak{D}^{2,\alpha}$ to be a point P at which the normal velocity V vanishes, i.e. a point where $g(\theta)K - F = 0$. In [A1, part 2] it was shown that the number of nodes of a family of domains $\Omega(t)$ which evolves according to (1.1) does not increase with time.

Theorem 15.4. *Consider a domain $\Omega_0 \in \mathfrak{D}^{2,\alpha}$ whose corresponding solution $\Omega(t)$ to (1.1) exists for all positive times, and for which $\Omega(t)$ converges to Ω_* as $t \to \infty$. Then Ω_0 has at least four nodes.*

Proof. We assume that $\Omega(t)$ converges in C^2 to Ω_*. The convexity of Ω_* then implies that $\Omega(t)$ is convex for sufficiently large t. We shall show that if t is large enough, then $\Omega(t)$ has four nodes, and since the number of nodes is nondecreasing this will establish that Ω_0 also has at least four nodes.

For large t the domain $\Omega(t)$ is convex, and we may represent it by its normal velocity function $v(\theta, t)$, which is a solution of the linear parabolic PDE

$$v_t = a(\theta,t)(v_{\theta\theta} + v),$$

where $a(\theta,t) = (\beta(\theta)v(\theta,t) + F)^2/\beta(\theta)g(\theta)$.

Since $\Omega(t) \to \Omega_*$, the velocity v decays to zero, and since $v = 0$ is a hyperbolic fixed point of our semiflow ϕ^t, the velocity will decay at an exponential rate. It follows from the results in [He2] that $v(\theta,t) = Ce^{\lambda_j t}w(\theta)(1+o(1))$ as $t \to \infty$, where $w(\theta)$ is some eigenfunction of A, with eigenvalue λ_j, and C is some non-zero constant. Since v tends to zero, the eigenvalue λ_j must be negative, so that $j \geq 3$ and so that $w(\theta)$ has at least four simple zeroes. Thus, for large enough t, the velocity $v(\theta,t)$ will have at least four zeroes.

Q. E. D.

16. Large time asymptotics.

In this section we consider a solution $\Omega(t)$ which encloses the steady state Ω_*, and which will fill up the whole plane $\mathbf{R}^2$ as $t \to \infty$. We show that the domain $\Omega(t)$, rescaled so that its diameter becomes $O(1)$, converges to a fixed shape, namely, the Wulff region of $1/\beta(\theta)$.

In the previous section we have seen that $v \equiv 0$ is a hyperbolic fixed point of the semiflow ϕ^t of X^α, and that its unstable manifold is one dimensional. This unstable manifold consists of two trajectories; we shall denote the corresponding solutions to the PDE (15.1) by $v^\pm(\theta,t)$. These solutions are only well defined up to a time translation, i.e. for any ϑ the functions $v^\pm(\theta, t+\vartheta)$ would have been acceptable solutions representing the trajectories on the unstable manifold. We shall assume that we have chosen a particular pair of solutions from these time translates.

The solutions $v^{\pm}(\theta,t)$ are in principle only defined for t sufficiently small, i.e. for $-\infty < t < t_{\pm}$. As $t \to -\infty$ the $v_{\pm}$ will decay exponentially:

$$v^{\pm}(\theta,t) = \pm C_{\pm} e^{\lambda_1 t} w_1(\theta)(1 + o(1)),$$

as $t \downarrow -\infty$ for certain positive constants C_+ and C_-; λ_1 and $w_1(\theta)$ are the principal eigenvalue and function of the linear operator A which we encountered in the previous paragraph. In particular, we may assume that $w_1(\theta) > 0$ for all θ. This implies that both $v^+(\theta,t)$ and $v_t^+(\theta,t)$ are positive for all θ, t. Likewise v^- and v_t^- are negative for all θ and t. In what follows we shall study the behaviour of the positive solution.

We leave the precise behaviour of the negative solution as a (partially) open problem. The negative solution corresponds to a solution of (1.1) which is convex, which shrinks, and which does not exist for all time. Indeed, if $m(t) = \max_\theta v(\theta,t)$, then it follows from (15.1) that $m'(t) \le Cm(t)^3$, where $C = \min_\theta \beta(\theta)/g(\theta)$; by integrating this differential inequality one sees that $m(t) \downarrow -\infty$ in finite time if $m(t) < 0$ for some $t \in \mathbf{R}$. Thus the solution to (1.1) corresponding to v_- becomes singular in finite time. It is not clear whether it will shrink to a point, and what its asymptotic shape will be. However, under an extra condition on the coefficient $\beta(\theta)$ M. Gage has shown that the solution is approximately self similar ([Ga]).

Concerning the positive solution v_+ we shall prove the following.

Lemma 16.1. *$v_+(\theta,t)$ is defined for all positive times, and as t tends to infinity, $v_+(\theta,t)$ converges monotonically to $v^\infty(\theta)$, where v^∞ is defined by*

$$v^\infty(\theta) = \sup_{\mathfrak{z}\in\mathbf{R}^2} \{ \mathfrak{z}\cdot\mathfrak{N}(\theta) \mid \forall_{\alpha\in\mathbf{R}}\, \mathfrak{z}\cdot\mathfrak{N}(\alpha) < h(\alpha) \}, \qquad h(\alpha) = \frac{-F}{\beta(\alpha)}.$$

Moreover $v^\infty(\theta)$ is the support function of $|F|^2\Gamma(\beta^{-1})$, where $\Gamma(\beta^{-1})$ is the Wulff region for β^{-1} defined by (3.1) with $f = \beta^{-1}$.

Proof. We first show that $v^\infty(\theta)$ is the support function of $|F|^2\Gamma(\beta^{-1})$. Let $\Lambda(\theta) = \{ \mathfrak{z} \in \mathbf{R}^2 \mid \mathfrak{z}\cdot\mathfrak{N}(\theta) < h(\theta) \}$. Then $\Gamma(h)$ is the closure of the intersection over all θ of the halfspaces $\Lambda(\theta)$, and

$$v^\infty(\theta) = \sup_{\mathfrak{z}\in\mathbf{R}^2} \{ \mathfrak{z}\cdot\mathfrak{N}(\theta \mid \mathfrak{z} \in \Gamma(h)^\circ \};$$

thus, since $\Gamma(h)$ is convex, $v^\infty(\theta)$ represents the support function of $\Gamma(h)$.

Next, for as long as $v^+(\theta,t)$ is defined it represents a solution of (1.1) which is convex, and whose curvature is bounded by $F/g(\theta) < k(\theta,t) < 0$. By our main existence theorem 3.1 this solution cannot become singular in finite time, so that $v^+(\theta,t)$ is defined for all $t \in \mathbf{R}$.

The velocity $v(\theta,t)$ is an increasing function of time, and it is bounded from above by $0 < v(\theta,t) < -F/\beta(\theta)$; therefore it must converge to some limit function $\bar{v}(\theta)$, which is also bounded by $0 < \bar{v}(\theta) \le -F/\beta(\theta)$.

It also follows from $v_t > 0$ and (15.1) that $v_{\theta\theta} \ge -M$, where $M = \sup_{\theta\in\mathbf{R}} -F/\beta(\theta)$. From this we may conclude that $|v_\theta| \le 2\pi M$. Indeed, for any given $t \in \mathbf{R}$ there must exist a $\theta_0 \in [0, 2\pi]$ at which $v_\theta(\theta_0, t) = 0$; for any other $\theta \in (\theta_0, \theta_0 + 2\pi)$ we then have

$$\begin{aligned} v_\theta(\theta,t) &= \int_{\theta_0}^{\theta} v_{\theta\theta}(\vartheta,t)\,d\vartheta \ge -2\pi M \\ &= -\int_{\theta}^{\theta_0+2\pi} v_{\theta\theta}(\vartheta,t)\,d\vartheta \le 2\pi M. \end{aligned}$$

So the $v(\theta,t)$'s are uniformly Lipschitz continuous, and they must converge uniformly to $\bar{v}$; this limit must also be Lipschitz continuous, with constant $2\pi M$. Moreover, in the sense of distributions we have

$$\bar{v}'' + \bar{v} \ge 0.$$

To complete the proof we must show that $\bar{v} = v^\infty$.

Clearly we have $\bar{v}(\theta) \le -F/\beta(\theta)$ for all $\theta \in \mathbf{R}$. Assume that for some θ_0 this inequality is strict. Then, by continuity, there is an $\epsilon > 0$ such that $\bar{v}(\theta) < -F/\beta(\theta) - \epsilon$ for $|\theta - \theta_0| < \epsilon$, and hence also $v(\theta,t) < -F/\beta(\theta) - \epsilon$ for $|\theta - \theta_0| < \epsilon$ and all $t \in \mathbf{R}$. On the strip $S = (\theta_0 - \epsilon, \theta_0 + \epsilon) \times \mathbf{R}$ we then have a bounded increasing solution of the *uniformly* parabolic PDE $v_t = (\beta v + F)^2/\beta g(v_{\theta\theta} + v)$, and it follows from the Schauder estimates for such solutions that all derivatives $\partial_\theta^n v(\theta,t)$ are bounded, and uniformly convergent on any subinterval $(\theta_0 - \epsilon', \theta_0 + \epsilon')$ with $\epsilon' < \epsilon$, as $t \uparrow \infty$. The limit of $\bar{v}(\theta,t)$ must be a smooth equilibrium of the PDE, i.e. it must satisfy $\bar{v}'' + \bar{v} = 0$.

So far we have found that the limit $\bar{v}$ satisfies $0 < \bar{v}(\theta) \le -F/\beta(\theta)$, is Lipschitz continuous, and on the set of θ's where $\bar{v}(\theta) < -F/\beta(\theta)$ holds, $\bar{v}$ is a smooth solution of $\bar{v}'' + \bar{v} = 0$; i.e. it is of the form $A\sin(\theta - \alpha)$. Since $\bar{v}$ is positive, $\bar{v}(\theta) < -F/\beta(\theta)$ cannot hold on any interval of length π or more; on such an interval $\bar{v}$ would coincide with $A\sin(\theta - \alpha)$, which has a zero in any interval of length $\ge \pi$.

The following two lemmas imply that v^∞ and $\bar{v}$ coincide, and hence they complete the proof of Lemma (16.1).

Lemma 16.2. $\bar{v} \le v^\infty$.

Proof. Choose A, α such that $A\sin(\theta_0 - \alpha) = v^\infty(\theta_0)$ and $\sin(\theta - \alpha) \le -F/\beta(\theta)$ for all $\theta \in (\alpha, \alpha + \pi)$. We claim that there exist $\theta_1, \theta_2 \in (\alpha, \alpha + \pi)$ such that

(*i*) $\theta_1 < \theta_0 < \theta_2$, and (*ii*) $A\sin(\theta_j - \alpha) = \frac{-F}{\beta(\theta_j)}$ for $j = 1, 2$.

To find θ_1, we choose P and Q such that $A\sin(\theta - \alpha) = w(\theta - \theta_0)$, where $w(\theta) = P\sin\theta + Q\cos\theta$. Suppose that $w(\theta - \theta_0) < -F/\beta(\theta)$ for all $\alpha \le \theta \le \theta_0$; then, for any $\tilde{P} > P$ sufficiently close to P, we have $\tilde{w}(\theta - \theta_0) < -F/\beta(\theta)$ for all $\theta \in (\alpha, \alpha + \pi)$, while $\tilde{w}(\theta_0) = w(\theta_0)$ (here $\tilde{w}(\theta) = \tilde{P}\sin\theta + Q\cos\theta$.) Thus for sufficiently small $\epsilon > 0$ we also have $(1 + \epsilon)\tilde{w}(\theta) \le -F/\beta(\theta)$. But $(1 + \epsilon)\tilde{w}(\theta)$ is of the form $\tilde{A}\sin(\theta - \tilde{\alpha})$, so that

$$v^{\infty}(\theta_0) \ge (1 + \epsilon)\tilde{w}(\theta_0) > w(\theta_0) = v^{\infty}(\theta_0).$$

The contradiction shows that θ_1 must exist after all; a similar argument shows that θ_2 exists as well.

Define $w(\theta) = A\sin(\theta - \alpha)$. Comparing w with our solution $v^+(\theta, t)$ of (15.1), we find that $v^+(\theta_0, t) < w(\theta_0), v^+(\theta', t) < w(\theta')$ for all $t \in \mathbf{R}$, and, since $v^+(\theta, t) \downarrow 0$ uniformly as $t \downarrow -\infty$, we also find that $v^+(\theta, t) < w(\theta)$ for all $\theta_0 < \theta < \theta'$, if t is small enough. By the maximum principle we then get $v^+(\theta, t) < w(\theta)$ for all $\theta_0 < \theta < \theta'$ and all $t \in \mathbf{R}$. Taking the limit $t \uparrow \infty$ this shows that $\bar{v}(\theta_0) \le w(\theta_0) = v^{\infty}(\theta_0)$. Since θ_0 was arbitrary, we have proved that $\bar{v} \le v^{\infty}$.

Q. E. D.

Lemma 16.3. $v^{\infty} \le \bar{v}$.

Proof. Let $A\sin(\theta - \alpha) \le -F/\beta(\theta)$ for $\alpha < \theta < \alpha + \pi$. We shall show that $A\sin(\theta - \alpha) \le \bar{v}$ on the same interval; since v^{∞} is defined as the supremum of all such $A\sin(\theta - \alpha)$'s the Lemma follows from this.

Let $\tilde{A} < A$ be the largest $\tilde{A}$ for which $\tilde{A}\sin(\theta - \alpha) \le \bar{v}(\theta)$ on $(\alpha, \alpha + \pi)$, and assume that $\tilde{A} < A$.

For some $\theta_0 \in (\alpha, \alpha + \pi)$ one will have $\tilde{A}\sin(\theta_0 - \alpha) = \bar{v}(\theta_0)$. At this value of θ one will also have $\bar{v}(\theta_0) = \tilde{A}\sin(\theta_0 - \alpha) = A\sin(\theta_0 - \alpha) < -F/\beta(\theta_0)$. Let (θ_1, θ_2) be the maximal interval containing θ_0 on which $\bar{v}(\theta) < -F/\beta(\theta)$; then $\bar{v}$ is a smooth solution of $v'' + v = 0$ throughout this interval which satisfies both $\bar{v}(\theta_0) = \tilde{A}\sin(\theta_0 - \alpha)$, and $\tilde{A}\sin(\theta - \alpha) \le \bar{v}(\theta)$ on (θ_1, θ_2), so that it must coincide with $\tilde{A}\sin(\theta - \alpha)$ on $[\theta_1, \theta_2]$. In particular it follows from the positivity of $\bar{v}$ that $[\theta_1, \theta_2] \subset (\alpha, \alpha + \pi)$.

All this leads us to the following contradiction: at the end points of the interval (θ_1, θ_2) we have

$$\frac{-F}{\beta(\theta_j)} = \bar{v}(\theta_j) = \tilde{A}\sin(\theta_j - \alpha) < \frac{-F}{\beta(\theta_j)},$$

which is absurd. Thus $\tilde{A} = A$, i.e. $A\sin(\theta - \alpha) \le \bar{v}(\theta)$ on $(\alpha, \alpha + \pi)$.

Q. E. D.

Lemma 16.4. *v^∞ is a C^1 function.*

Proof. In the sense of distributions we already know that $v^\infty_{\theta\theta} \geq -v^\infty \geq -M$, so that v^∞_θ is of bounded variation; in particular, v^∞ has left and right hand limits, $v^\infty(\theta \pm 0)$, at each $\theta \in \mathbf{R}/2\pi\mathbf{Z}$.

Fix any $\theta_0 \in [0, 2\pi)$. If $v^\infty(\theta_0) < -F/\beta(\theta_0)$, then $v^\infty(\theta)$ is of the form $A\sin(\theta - \alpha)$ near θ_0, and its derivative is clearly continuous at θ_0. So assume that $v^\infty(\theta_0) = -F/\beta(\theta_0)$.

Suppose that the left and right hand limits of v^∞_θ at θ_0 are different. Since $v^\infty_{\theta\theta} \geq -M$, we must have $v^\infty_\theta(\theta_0 - 0) < v^\infty_\theta(\theta_0 + 0)$. Choose $p \in \mathbf{R}$ and $\delta > 0$ such that

$$v^\infty_\theta(\theta_0 - 0) < p - \delta < p + \delta < v^\infty_\theta(\theta_0 + 0).$$

Then there is an $\epsilon > 0$ for which $v^\infty_\theta(\theta) < p - \delta$ holds when $\theta \in (\theta - \epsilon, \theta_0)$, while $v^\infty_\theta(\theta) > p + \delta$ when $\theta \in (\theta_0, \theta_0 + \epsilon)$. This implies that

$$v^\infty(\theta) \geq v^\infty(\theta_0) + p(\theta - \theta_0) + \delta|\theta - \theta_0|$$

on the interval $(\theta_0 - \epsilon, \theta_0 + \epsilon)$. On the other hand $v^\infty(\theta) \leq -F/\beta(\theta)$, with equality at $\theta = \theta_0$, which is impossible since $-F/\beta(\theta)$ is a smooth function.

Q. E. D.

Lemma 16.5. *$v^+_\theta(\theta, t)$ converges uniformly to $v^\infty_\theta(\theta)$, as $t \uparrow \infty$.*

Proof. Let $\delta > 0$ be given. We have just shown that v^∞ is C^1, so there exists an $\epsilon = \epsilon(\delta) > 0$ for which

$$v^\infty(\theta) \leq v^\infty(\theta_0) + v^\infty_\theta(\theta_0)(\theta - \theta_0) + \delta|\theta - \theta_0|$$

holds for any $\theta, \theta_0 \in \mathbf{R}/2\pi\mathbf{Z}$ with $|\theta - \theta_0| \leq \epsilon(\delta)$. We can choose $\epsilon(\delta)$ in such a way that it is a continuous function of δ, which vanishes when $\delta = 0$.

Now choose t_δ so that $v^+(\theta, t) \geq v^\infty(\theta) - \epsilon^2$ for all θ, and all $t \geq t_\delta$.

Since $v^+_{\theta\theta} \geq -v^+ \geq -M$, we have the following inequalities for $t \geq t_\delta$:

$$\begin{aligned} v^+(\theta \pm \epsilon, t) &\geq v^+(\theta) \pm v^+(\theta)\epsilon - \frac{M\epsilon^2}{2} \\ &\geq v^\infty(\theta) \pm v^+(\theta)\epsilon - \frac{M\epsilon^2}{2} - \epsilon^2. \end{aligned}$$

On the other hand we also have

$$v^+(\theta \pm \epsilon, t) \leq v^\infty(\theta \pm \epsilon) \leq v^\infty(\theta) \pm v^\infty_\theta(\theta)\epsilon + \delta\epsilon.$$

Subtracting these inequalities we find:

$$\left|v^\infty_\theta(\theta) - v^+_\theta(\theta, t)\right| \leq \delta + \left(\frac{M}{2} + 1\right)\epsilon(\delta).$$

for all θ and all large enough t.

Q. E. D.

These lemmas allow us to prove the main result of this section. Recall that we had denoted the domain corresponding to $v^+(\theta,t)$ by $\Omega_+(t)$, and let $p^+(\theta,t)$ be its support function. Then we have $p_t^+ = v^+$, so that

$$p^+(\theta,t) = p^+(\theta,0) + \int_0^t v^+(\theta,\tau)\,d\tau.$$

By dividing the equation by t on both sides, and letting t tend to infinity we find the following:

$$\lim_{t\to\infty} \frac{p^+(\theta,t)}{t} = v^\infty(\theta),$$

where the convergence is in $C^1(\mathbf{R}/2\pi\mathbf{Z})$. Since the boundary of $\Omega_+(t)$ is parametrized by

$$\mathfrak{X}(\theta,t) = p^+(\theta,t)\mathfrak{N}(\theta) - p_\theta^+(\theta,t)\mathfrak{T}(\theta),$$

we also get convergence of the parametrization:

$$\lim_{t\to\infty} \frac{\mathfrak{X}(\theta,t)}{t} = \mathfrak{X}^\infty(\theta),$$

where $\mathfrak{X}^\infty(\theta) = v^\infty(\theta)\mathfrak{N}(\theta) - v_\theta^\infty(\theta)\mathfrak{T}(\theta)$; the convergence is uniform in θ.

Now consider any solution $\Omega(t)$ of (1.1) which exists for all time, and whose initial value $\Omega(0)$ encloses the steady state Ω_*. Then if t_0 is sufficiently large, $\Omega(0)$ must be contained in $\Omega_+(t_0)$, while $\Omega(0)$ must contain $\Omega_+(-t_1)$ for sufficiently large $t_1 > 0$. By the containment property we get $\Omega_+(t-t_1) \subset \Omega(t) \subset \Omega_+(t+t_0)$ for all $t \geq 0$. The velocity of the boundary of $\Omega_+(t)$ is given by $v^+(\theta,t)$, so that it is uniformly bounded. Therefore $\Omega_+(t+t_0)$ is contained in an R neighborhood of $\Omega_+(t-t_1)$, where $R = (t_0+t_1)\sup v^+(\theta,t)$. Since $\Omega_+(t+t_0)$ certainly contains $\Omega_+(t-t_1)$, the Hausdorff distance between $\Omega(t)$ and $\Omega_+(t)$ is bounded by R. If we shrink the domains by dilating them by a factor t^{-1}, their Hausdorff distance becomes at most R/t, and the convergence of $\Omega_+(t)$ implies the following theorem.

Theorem 16.6. *Let $\Omega : [0,\infty) \to \mathfrak{D}^{2,\alpha}$ be a solution of (1.1) for which $\bar{\Omega}_* \subset \Omega(0)$. Then the domain $\hat{\Omega}(t)$ obtained from $\Omega(t)$ by shrinking it by a factor t^{-1} converges to the domain Ω^∞ whose support function is $v^\infty(\theta)$.*

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