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CO–VOLUME METHODS FOR DEGENERATE PARABOLIC PROBLEMS

LISA A. BAUGHMAN AND NOEL J. WALKINGTON*

Abstract. A complementary volume (co–volume) technique is used to develop a physically appealing algorithm for the solution of degenerate parabolic problems, such as the Stefan problem. It is shown that, these algorithms give rise to a discrete semigroup theory that parallels the continuous problem. In particular, the discrete Stefan problem gives rise to nonlinear semigroups in both the discrete L^1 and H^{-1} spaces.

Key words. semigroup, co–volume, Stefan problem.
AMS(MOS) subject classifications. 65M12, 35K55.

1. Introduction. The classical Stefan problem models heat conduction in a material which may undergo a change in phase. In each of the phases, the balance of energy reduces to the classical heat conduction problem,

$$\frac{de}{dt} - k_i \Delta u = f, \qquad in\ \Omega_i$$

where e is the energy, u is the temperature and k_i the conductivity in phase i which currently occupies a region $\Omega_i \subset \Omega$. In each phase, u is typically an affine function of the energy e with slope c_i, the specific heat of the i^{th} phase; however, any monotone function of e may be accommodated. It is assumed that each phase change takes place at a known temperature. At an interface between two phases, the balance of energy requires the jump in the normal component of the flux, $\mathbf{q} = -k\nabla u$, to equal the product of the latent heat L associated with the phase change and the normal velocity of the interface. This problem can be compactly written as a distributional equation

$$\frac{de}{dt} - \Delta K(e) = f \qquad in\ \mathcal{D}'((0,T) \times \Omega), \tag{1}$$

where $K : \mathbb{R} \to \mathbb{R}$ is a monotone function. A typical form of K for the two phase problem is shown in Figure 1. It can be shown that each term in equation (1) is in $H^{-1}(\Omega)$, so that (1) may also be written in the form,

$$\int_\Omega \frac{de}{dt} v + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v, \qquad u = K(e), \tag{2}$$

which holds for all functions $v \in H_0^1(\Omega)$, provided the first term is interpreted as a dual pairing. The description of the problem is completed by specifying an initial condition for e and boundary conditions on u.

Note that by adding a constant to u we may always assume that $K(0) = 0$, and this will be assumed in the sequel. Equation (1) can be used to model several other degenerate diffusion problems, eg. diffusion through porous media, the only difference being the form of the monotone function K. For the Stefan problem, K is Lipschitz,

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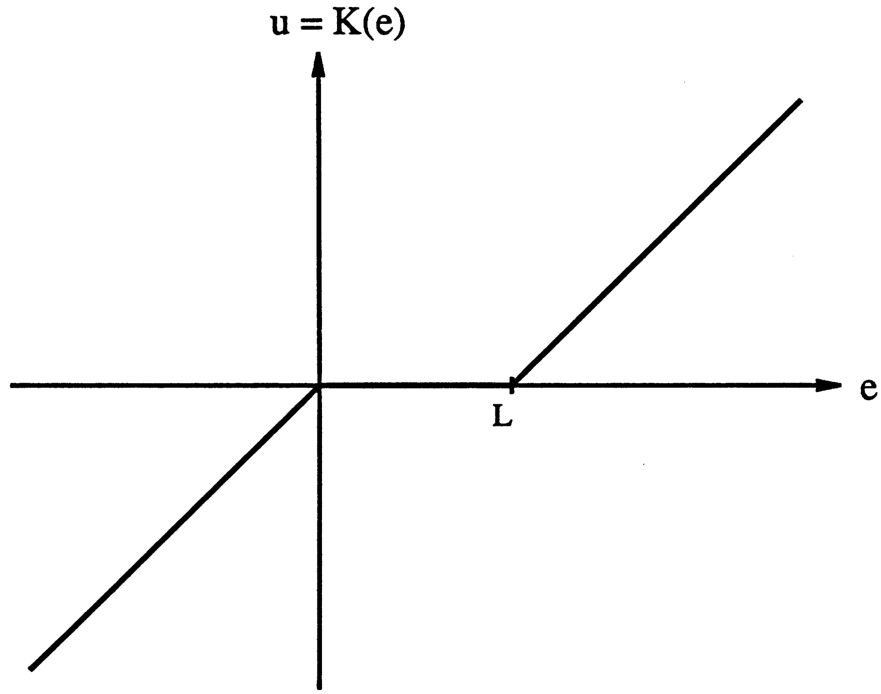


Figure 1. Energy Temperature Relationship.

and this frequently leads to estimates on the temperature u; however, this is not so for the porous medium equation. Below we will use terminology appropriate to to Stefan problem; however, all of the results are applicable to the porous medium equation provided the reciprocal of the Lipschitz constant for K is interpreted as zero. In order to minimize the technical detail, it will be assumed that Ω is a bounded Lipschitz domain, and that a zero boundary condition is specified for the temperature u.

The existence and uniqueness theory for the Stefan problem is well understood; however, the numerical analysis for such problems is crude and gives sub–optimal results, [17, 8, 18]. Part of the problem is the lack of regularity of the solution which may be summarized as follows.

- If the initial temperature satisfies $u_0 = K(e_0) \in H^1_0(\Omega)$ and $f \in H^1[0,T,L^2(\Omega)]$, then $e \in L^\infty[0,T,L^2(\Omega)] \cap H^1[0,T;H^{-1}(\Omega)]$.
- If the initial temperature $u_0 = K(e_0)$, satisfies $\Delta u_0 \in L^1(\Omega)$. and $f \in Lip[0,T;L^1(\Omega)]$ then $e \in Lip[0,T;L^1(\Omega)]$,
- If $e_0 \in L^\infty(\Omega)$ and $f \in L^\infty[0,T;L^\infty(\Omega)]$, then $e \in L^\infty[0,T;L^\infty(\Omega)]$.
- If $e_0 \in L^2(\Omega)$, $f \in L^2[0,T;L^2(\Omega)]$, then $u \in L^2[0,T;H^1_0(\Omega)]$.
- If $u_0 = K(e_0) \in H^1_0(\Omega)$, $f \in H^1[0,T,H^1_0(\Omega)]$ and K is Lipschitz, then $u \in H^1[0,T;H^1_0(\Omega)]$.

These estimates are optimal, since any problem involving a phase change across a sharp interface will have a jump in the energy and temperature gradient. Other regularity results are known, for example Caffarelli and Friedman [6] discuss continuity of the temperature, and Magenes et. al. [13] have recently shown that the energy lies in a Nikolskiĭ space under suitable hypotheses on the data. Given the above regularity of the solution, it would be reasonable to expect first order rates of convergence for approximate solutions of the energy in $L^2[0,T;H^{-1}(\Omega)]$ and the temperature in $L^2[0,T;L^2(\Omega)]$ when K is Lipschitz. Numerical evidence suggests that these are indeed the correct rates; however, the analysis typically predicts rates of one half the expected values. Small improvements over rates of one half can be achieved by adding ϵ times the identity to K so that both K and its inverse are Lipschitz. Choosing ϵ as some power of the mesh size h can lead to rates of order $2/3$ is space and $1/2$ in time [9]. Under suitable conditions on the data, it can be shown that all of the phase change interfaces are sharp, and in this instance a different choice of ϵ will lead to a first order rate in h and order $2/3$ in time [16].

In this paper we discuss a numerical method, the co–volume scheme, for the solution of the Stefan problem. Co–volume schemes have been proposed in the past for linear elliptic problems [12], and, more recently, have been used very effectively by Nicolaides [14] to obtain stable low order schemes for problems in incompressible fluid mechanics. We establish that these techniques lead to numerical schemes with many desirable properties when used for the Stefan problem. For example, the discrete numerical scheme mimics the continuous problem in that it generates contraction semigroups in both the discrete $L^1(\Omega)$ and $H^{-1}(\Omega)$ spaces. Moreover, the proofs of such properties mimic the continuous proofs, and are frequently simpler since at the discrete level many of the technical issues disappear. Another desirable property is that the co–volume algorithm does not require the projection of a nonlinear function onto a finite element space. Most traditional schemes approximate the energy–temperature relationship, $u = K(e)$, by projecting $K(e_h)$ onto a finite element subspace (e_h being the approximate energy). Such projections are implemented using numerical quadrature, which is subject to large errors when K is not smooth. While we can not show that the rates of convergence are better for the co–volume scheme than traditional schemes, the simplicity of the method leads to a very efficient algorithm. Indeed, the explicit scheme discussed below will execute as quickly as a linear heat conduction code.

In the next section, we introduce the co–volume discretization and the corresponding discrete spaces of functions. Section 3 is reveals how the discrete scheme mimics the continuous problem and Section 4 establishes convergence of the discrete energy in both $L^1(\Omega)$ and $H^{-1}(\Omega)$. Finally, we present some numerical examples in Section 5.

2. Co–Volume Discretizations. The co–volume discretization is most easily described for a triangulation $\mathcal{T}_h$ of plane domains $\Omega \subset \mathbb{R}^2$. We assume that Ω is polygonal in order to avoid dealing with the errors introduced when $\Omega \neq \cup_{T\in\mathcal{T}_h} T$. Given $\mathcal{T}_h$, a dual (non–triangular) mesh is constructed with vertices corresponding to the circumcenters of the triangles, and edges joining the circumcenters of triangles that have an adjacent side. With each interior vertex of the original triangulation, there is an associated (Voronoi) complementary volume bounded by the edges of the dual mesh that connect the circumcenters of the triangles containing this vertex (see Figure 2a.). In the sequel, quantities associated with the dual mesh will be prefixed with 'co–', eg. co–edges, co–vertices etc. refer to edges and vertices in the dual mesh. Note that the union of the co–volumes does not exhaust Ω, there is a region near the boundary that is omitted (see Figure 2b.). Of course this construction can be completed in higher dimensions, by constructing the complementary volumes from the hyperplanes that bisect and are perpendicular to the edges of the original mesh (see eg. Nicolaides and Wu [15]). For clarity of exposition, we will use the vocabulary appropriate to plane problems below.

For an arbitrary triangulation $\mathcal{T}_h$ of Ω, the co–volumes may not be well defined in the sense that the edges for a particular co–volume may intersect at points other than the vertices. However, if each triangle $T \in \mathcal{T}_h$ has angles no larger than 90^o, each co–volume will be convex, and none will extend beyond the boundary. It will be assumed that we are dealing with such a mesh in the sequel. Placing the dual vertices at the circumcenter of each triangle guarantees that the edges of the dual mesh intersect the edges of the original mesh at right angles. It is this feature that leads to 'natural' discretizations of conservation laws.

The discrete equations may be motivated by integrating equation (1.1) over a typical co–volume A_i, and applying the divergence equation to the spatial term:

$$|A_i|\frac{de_i}{dt} - \int_{\partial A_i} \frac{\partial u}{\partial n} = |A_i| f_i,$$

where e_i, and f_i are the average values of the energy and source terms on A_i. Let $C_i \subset \mathbb{N}$ be defined by $j \in C_i$ implies there is an edge in the triangulation $\mathcal{T}_h$ connecting vertex i to vertex j. Denote the length of such an edge by h_{ij}, and the length of the perpendicular co–edge by[1] $h_{ij}^{\perp}$. If the normal derivative on this co–edge is approximated by $(u_j - u_i)/h_{ij}$, where u_i, u_j etc. are the nodal values of a piecewise linear approximation to u on $\mathcal{T}_h$, then the discrete Stefan problem becomes

$$|A_i|\frac{de_i}{dt} + \sum_{j\in C_i} \frac{h_{ij}^{\perp}}{h_{ij}}(u_i - u_j) = |A_i| f_i \qquad i = 1, 2, \ldots N_h. \tag{3}$$

[1] In three dimensions, $h_{ij}^{\perp}$ is the area of the corresponding co–face

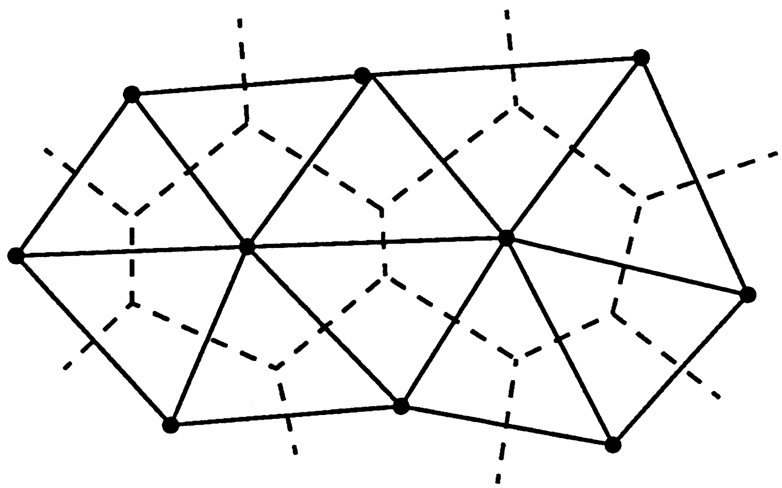


Figure 2a. Complementary Volumes.

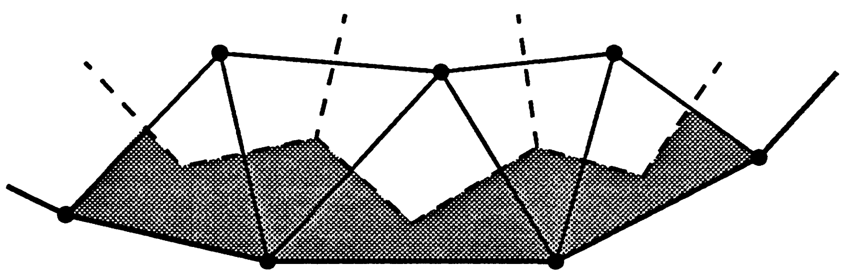


Figure 2b. Region Near Boundary.

Note that these equations are only defined for the N_h interior vertices which correspond to co–volumes. On the boundary, the nodal values of u are obtained from the Dirichlet data. We will think of $\{e_i\}_{i=1}^{N_h}$ as representing a piecewise constant approximation of e on $\cup_i A_i$. From a mathematical point of view, it suffices to approximate e by zero in the region near the boundary, $\Omega \setminus \cup_i A_i$, since this region is small, and $e \in L^\infty(\Omega)$. A more physically appealing approximation could be made by integrating equation (1.1) over a boundary co–volume, $\tilde{A}_i$, to obtain,

$$|\tilde{A}_i|\frac{de_i}{dt} + \sum_{j\in C_i} \frac{h_{ij}^\perp}{h_{ij}}(u_i - u_j) - h_i^+ \nabla u \cdot \mathbf{n}^+ - h_i^- \nabla u^- \cdot \mathbf{n}^- = |\tilde{A}_i| f_i.$$

$\nabla u \cdot \mathbf{n}^\pm$ is the normal derivative on the boundary evaluated using the piecewise linear temperature field on each side of node i (for i and j on the boundary, $h_{ij}^\perp$ is defined as the perpendicular distance from the $(i,j)^{th}$ edge to the circumcenter). This technique is used when Neumann boundary conditions are specified; in this situation the specified normal derivative is used in place of $\nabla u \cdot \mathbf{n}^\pm$. (See eg. [14]).

In order to calculate approximate solutions of the Stefan problem using equation (3), it is necessary to discretize the temporal derivative. Integrating equation (3) over a time interval $(n\tau, (n+1)\tau)$ and approximating the temperature by piecewise constant functions in time gives either the implicit discretization,

$$(4) \qquad |A_i|(e_i^{n+1} - e_i^n) + \tau \sum_{j\in C_i} \frac{h_{ij}^\perp}{h_{ij}}(u_i^{n+1} - u_j^{n+1}) = \tau |A_i| f_i^{n+\frac{1}{2}},$$

or the explicit discretization

$$(5) \qquad |A_i|(e_i^{n+1} - e_i^n) + \tau \sum_{j\in C_i} \frac{h_{ij}^\perp}{h_{ij}}(u_i^{n} - u_j^{n}) = \tau |A_i| f_i^{n+\frac{1}{2}}.$$

In the above, a supper script refers to the time level, and τ the time step, $e_i(n\tau) \simeq e_i^n$ etc., and the supper script $n + \frac{1}{2}$ indicates the time average over the interval $(n\tau, (n+1)\tau)$. Of course equations (4) and (5) could have been derived directly from equation (1) by integrating over the space–time cylinder $A_i \times (n\tau, (n+1)\tau)$.

Discrete analogues of equation (2) result if we let $\{v_i\}$ denote the nodal values of a piecewise linear function on $\mathcal{T}_h$ vanishing on the boundary, and form the discrete ℓ^2 inner product of equations (4) and (5) to obtain

$$(6) \quad \sum_i |A_i|(e_i^{n+1} - e_i^n)v_i + \tau \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}}(u_i^{n+1} - u_j^{n+1})(v_i - v_j) = \tau \sum_i |A_i| f_i^{n+\frac{1}{2}} v_i,$$

and

$$(7) \quad \sum_i |A_i|(e_i^{n+1} - e_i^n)v_i + \tau \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}}(u_i^{n} - u_j^{n})(v_i - v_j) = \tau \sum_i |A_i| f_i^{n+\frac{1}{2}} v_i.$$

In the above $\sum_{(i,j)} \dots$ indicates the sum over all edges (joining nodes i and j) in the mesh $\mathcal{T}_h$, and a discrete version of Green's theorem was utilized:

$$\sum_i \sum_{j\in C_i} \frac{h_{ij}^\perp}{h_{ij}}(u_i - u_j)v_i = \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}}(u_i - u_j)(v_i - v_j),$$

provided $v_i = 0$ at boundary vertices. The nontrivial identity

$$\int_\Omega \nabla u \cdot \nabla v = \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}}(u_i - u_j)(v_i - v_j)$$

is shown to hold in [4], for piecewise linear functions defined on $\mathcal{T}_h$.

The following function spaces will be utilized below. $V = H_0^1(\Omega)$, the Sobolev space of functions vanishing on the boundary and having square integrable derivatives, and $V_h \subset V$ will be the space of piecewise linear functions defined on $\mathcal{T}_h$ which vanish on the boundary. V_h is endowed with the inner product

$$(u, v)_{V_h} = \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}}(u_i - u_j)(v_i - v_j), \qquad u, v \in V_h.$$

Let $H = L^2(\Omega)$ and $H_h \subset H$ be the set of functions that are piecewise constant on the co–volumes A_i, and zero on $\Omega \setminus \cup_i A_i$. Note that while $V \subset H$, V_h is not a subset of H_h; however, there is a natural embedding $v \mapsto \tilde{v}$ where $v_i = \tilde{v}_i$. Dual pairings will be denoted using angled brackets, $\langle .,. \rangle$, and round brackets $(.,.)$ will denote the L^2 inner product. Usually approximate quantities calculated from the fully discrete schemes (4) and (5) will be subscripted with h, eg. e_h; however, when it is necessary to distinguish between solutions of the semi–discrete scheme (3) and the fully discrete schemes, the fully discrete solution will have a superscript of τ, eg. e_h^τ.

3. Discrete Semigroups. The Stefan problem generates a contraction semigroup in both $H^{-1}(\Omega)$, and $L^1(\Omega)$. In this section we show that the discrete operator generates semigroups in the analogous discrete spaces. The continuous semigroup also maps bounded sets in $L^2(\Omega)$ and $L^\infty(\Omega)$ to bounded sets. We show that analogous properties hold for the discrete operator. Recall that the discrete evolution operator $\mathcal{A}_h : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ is

$$\mathcal{A}_h(\mathbf{e})_i = \frac{1}{|A_i|} \sum_{j \in C_i} \frac{h_{ij}^\perp}{h_{ij}}(u_i - u_j),$$

where $u_i = K(e_i)$ if i is an interior vertex, and $u_i = 0$ otherwise, and that $K : \mathbb{R} \to \mathbb{R}$ is a monotone function vanishing a zero. Frequently we will abuse notation by writing

$$\mathcal{A}_h(\mathbf{e})_i = \frac{1}{|A_i|} \sum_{j \in C_i} \frac{h_{ij}^\perp}{h_{ij}}(K(e_i) - K(e_j)),$$

with the understanding that $K(e_i) = 0$ if i corresponds to a boundary vertex (where e_i is undefined).

LEMMA 3.1. *Let $\|.\|_{\ell^1}$ denote the discrete $L^1(\Omega)$ norm defined by $\|e\|_{\ell^1} = \sum_i |A_i||e_i|$ where the sum is over the interior nodes of $\mathcal{T}_h$. Then $\mathcal{A}_h$ is accretive in ℓ^1, ie.*

$$\|\mathbf{e}^{(2)} - \mathbf{e}^{(1)}\|_{\ell^1} \le \|\mathbf{e}^{(2)} - \mathbf{e}^{(1)} + \tau(\mathcal{A}_h(\mathbf{e}^{(2)}) - \mathcal{A}_h(\mathbf{e}^{(1)}))\|_{\ell^1},$$

for all $\tau \ge 0$.

Proof. Let $s_i = sgn(e_i^{(2)} - e_i^{(1)})$ where $sgn(.)$ denotes the signum function. Subtracting the discrete equations for $e_i^{(2)} + \tau\mathcal{A}_h(\mathbf{e}^{(2)})_i$ and $e_i^{(1)} + \tau\mathcal{A}_h(\mathbf{e}^{(1)})_i$, multiplying the difference by $|A_i|s_i$ and summing gives

$$\|\mathbf{e}^{(2)} - \mathbf{e}^{(1)} + \tau(\mathcal{A}_h(\mathbf{e}^{(2)}) - \mathcal{A}_h(\mathbf{e}^{(1)}))\|_{\ell^1} \geq$$
$$\sum_i |A_i|(e_i^{(2)} - e_i^{(1)})s_i + \tau \sum_{(i,j)} [(u_i^{(2)} - u_i^{(1)}) - (u_j^{(2)} - u_j^{(1)})](s_i - s_j).$$

The lemma now follows upon observing that $(e_i^{(2)} - e_i^{(1)})s_i = |e_i^{(2)} - e_i^{(1)}|$, and that the second term on the right is non–negative since K is monotone. □

We next show that the operator $\mathcal{A}_h$ is also accretive in the dual of V_h. A function $v \in V_h$ is uniquely identified by a vector of interior nodal values $\{v_i\}_{i=1}^{N_h} \in \mathbb{R}^{N_h}$, since $v|_{\partial\Omega} = 0$. Given $f \in V_h'$, the dual space of V_h, we define,

$$\|f\|_{V_h'} = \sup_{\|v\|_{V_h}=1} f(v).$$

The dual norm may be computed using the discrete Riesz map for V_h, $\|f\|_{V_h'} = \|v\|_{V_h}$, where $v = \mathcal{R}_h^{-1}(f) \in V_h$ is defined by

$$\sum_{(i,j)} \frac{h_{ij}^{\perp}}{h_{ij}}(v_i - v_j)(w_i - w_j) = f(w),$$

for all $w \in V_h$. We identify $e \in H_h \subset V_h'$ according to

$$\langle e, v \rangle = \sum_i |A_i| e_i v_i.$$

Since V_h is finite dimensional, we can represent any $e \in V_h'$ by a vector of nodal values $\langle e, v \rangle = \sum_i |A_i| e_i v_i$.

LEMMA 3.2. *Let $\mathcal{A}_h : V_h' \to V_h'$ be defined by*

$$\langle \mathcal{A}_h(e), v \rangle = \sum_i |A_i| \mathcal{A}_h(\mathbf{e})_i v_i,$$

where $e \in V_h'$ and $v \in V_h$ are identified with their nodal values. Then $\mathcal{A}_h$ is accretive in V_h', ie.

$$\|e^{(2)} - e^{(1)}\|_{V_h'} \leq \|e^{(2)} - e^{(1)} + \tau(\mathcal{A}_h(e^{(2)}) - \mathcal{A}_h(e^{(1)}))\|_{V_h'},$$

for all $\tau \geq 0$.

Proof. Let $v = \mathcal{R}_h^{-1}(e^{(2)} - e^{(1)})$, then

$$\begin{aligned}
&\langle e^{(2)} - e^{(1)} + \tau(\mathcal{A}_h(e^{(2)}) - \mathcal{A}_h(e^{(1)})), v \rangle \\
&= \sum_i |A_i|(e_i^{(2)} - e_i^{(1)})v_i + \tau \sum_{(i,j)} \frac{h_{ij}^{\perp}}{h_{ij}}[(u_i^{(2)} - u_i^{(1)}) - (u_j^{(2)} - u_j^{(1)})](v_i - v_j) \\
&= \|e^{(2)} - e^{(1)}\|_{V_h'}^2 + \tau \sum_i |A_i|(e_i^{(2)} - e_i^{(1)})(u_i^{(2)} - u_i^{(1)}) \\
&\geq \|e^{(2)} - e^{(1)}\|_{V_h'}^2 \\
&= \|e^{(2)} - e^{(1)}\|_{V_h'} \|v\|_{V_h}.
\end{aligned}$$

The lemma now follows by dividing both sides by $\|v\|_{V_h}$ and recognizing that the left hand side is then less than $\|e^{(2)} - e^{(1)} + \tau(\mathcal{A}_h(e^{(2)}) - \mathcal{A}_h(e^{(1)}))\|_{V_h'}$. □

We next show that discrete $L^2(\Omega)$ and $L^\infty(\Omega)$ bounds are preserved by the discrete semigroup.

LEMMA 3.3. *For* $\mathbf{e} \in \mathbb{R}^{N_h}$ *let* $\|\mathbf{e}\|_{\ell^2}^2 = \sum_i |A_i| e_i^2$, *and* $\|\mathbf{e}\|_{\ell^\infty} = \max_i |e_i|$. *If* $\mathbf{e}$ *is a solution of the resolvent equation*

$$\mathbf{e} + \tau \mathcal{A}_h(\mathbf{e}) = \mathbf{f},$$

then $\|\mathbf{e}\|_{\ell^2}^2 + 2m\tau\|u\|_{V_h}^2 \leq \|\mathbf{f}\|_{\ell^2}^2$ *and* $\|\mathbf{e}\|_{\ell^\infty} \leq \|\mathbf{f}\|_{\ell^\infty}$, *where* m *is the reciprocal of the Lipschitz constant for* K.

Proof. Taking the ℓ^2 inner product of the resolvent equation with $\mathbf{e}$ gives

$$\sum_i |A_i| e_i^2 + \tau \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}} (u_i - u_j)(e_i - e_j) = \sum_i |A_i| f_i e_i.$$

Since K is monotone, it follows that $(u_i - u_j)(e_i - e_j) \geq m(u_i - u_j)^2$. The ℓ^2 estimate now follows from an application of the Cauchy–Schwarz and geometric mean inequalities to the right hand side.

Let $k = \|\mathbf{f}\|_{\ell^\infty}$ and $s_i = sgn^+(e_i - k)$ ($sgn^+(x) = 1$ if $x \geq 0$ and zero otherwise). Subtracting k from both sides of (each component of) the resolvent equation, and forming the ℓ^2 inner product with s_i gives

$$\sum_i |A_i|(e_i - k)^+ + \tau \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}} (K(e_i) - K(e_j))(s_i - s_j) = \sum_i |A_i|(f_i - k)s_i.$$

The choice of k guarantees that the right hand side is non–positive. Since both terms on the left are non–negative, it follows that the both sides of the equation are identically zero. In particular, $(e_i - k)^+ = 0$ for each i implies $e_i \leq k = \|\mathbf{f}\|_{\ell^\infty}$. A similar argument with $s_i = sgn^-(e_i + k)$ yields $e_i \geq -k = -\|\mathbf{f}\|_{\ell^\infty}$. □

COROLLARY 3.4. *The implicit difference scheme (2.2) satisfies,*

$$\|\mathbf{e}^{n+1}\|_{\ell^\infty} \leq \|\mathbf{e}^n\|_{\ell^\infty} + \tau \sum_{m=0}^{n} \|\mathbf{f}^{m+\frac{1}{2}}\|_{\ell^\infty},$$

and

$$\|\mathbf{e}^{n+1}\|_{\ell^2}^2 + 2m\tau \sum_{m=0}^{n} \|u^{m+1}\|_{V_h}^2 \leq e^{t^n}(\|\mathbf{e}^0\|_{\ell^2}^2 + \tau \sum_{m=0}^{n} \|\mathbf{f}^{m+\frac{1}{2}}\|_{\ell^2}^2),$$

where m *is the reciprocal of the Lipschitz constant for* K.

Proof. The implicit difference scheme may be written as $(I + \tau\mathcal{A}_h)\mathbf{e}^{n+1} = \mathbf{e}^n + \tau\mathbf{f}^{n+\frac{1}{2}}$. Putting $\mathbf{f} = \mathbf{e}^n + \tau\mathbf{f}^{n+\frac{1}{2}}$ into the above gives the ℓ^∞ bound immediately. Similarly, the ℓ^2 estimate implies $\|\mathbf{e}^{n+1}\|_{\ell^2}^2 + 2m\tau\|u^{n+1}\|_{V_h}^2 \leq (1+\tau)(\|\mathbf{e}^n\|_{\ell^2}^2 + \tau\|\mathbf{f}^{n+\frac{1}{2}}\|_{\ell^2}^2)$. An elementary calculation with the difference equation $\alpha^{n+1}/(1+\tau) - \alpha^n + c\tau\beta^n/(1+\tau) \leq \tau f^n$ completes the proof. □

COROLLARY 3.5. *The resolvent equation* $(I + \tau\mathcal{A}_h) : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ *is coercive, ie.* $\langle (I + \tau\mathcal{A}_h)(\mathbf{e}), \mathbf{e}\rangle_{\ell^2}/\|\mathbf{e}\|_{\ell^2} \to \infty$ *as* $\|\mathbf{e}\|_{\ell^2} \to \infty$.

Proof. If $\mathbf{f} = (I + \tau \mathcal{A}_h)(\mathbf{e})$, the proof of the ℓ^2 estimate above revealed

$$\sum_i |A_i| e_i^2 \leq \sum_i |A_i| f_i e_i,$$

ie. $\|\mathbf{e}\|_{\ell^2}^2 \leq \langle (I + \tau \mathcal{A}_h)(\mathbf{e}), \mathbf{e} \rangle_{\ell^2}$. □

COROLLARY 3.6. *$\mathcal{A}_h$ generates a contraction semigroup in both the ℓ^1 and V_h' topologies.*

Proof. Since any coercive continuous operator on a finite dimensional space is surjective, it follows that $\mathcal{A}_h$ is m–accretive in either the ℓ^1 or V_h' norms. An application of the Crandall–Ligget theorem then shows that $\mathcal{A}_h$ generates a contraction semigroup. □

The estimates above show the the implicit scheme (4) is stable, and semigroup theory shows that the difference solutions converge to the solution of (3) as $\tau \to 0$ [7]. We next establish a stability result for the explicit scheme (5).

LEMMA 3.7. *Let*

$$\mathbf{e}^{n+1} = \mathbf{e}^n - \tau \mathcal{A}_h(\mathbf{e}^n) + \tau \mathbf{f}^{n+\frac{1}{2}},$$

then

$$\|\mathbf{e}^{n+1}\|_{\ell^2}^2 \leq (1+\tau)\left[\|\mathbf{e}^n\|_{\ell^2}^2 - 2\tau\,(m - \tau M_h)\,\|u^n\|_{V_h} + \tau \|\mathbf{f}^{n+\frac{1}{2}}\|_{\ell^2}^2 \right],$$

Where m is the reciprocal of the Lipschitz constant for K, and $M_h = \max_i (1/|A_i|) \sum_{j \in C_i} (h_{ij}^{\perp}/h_{ij})$.

Proof. We begin with the inequality

$$\begin{aligned}
\|\mathbf{e}^{n+1}\|_{\ell^2}^2 &= \|\mathbf{e}^n - \tau \mathcal{A}_h(\mathbf{e}^n) + \tau \mathbf{f}^{n+\frac{1}{2}}\|_{\ell^2}^2 \\
&\leq (\|\mathbf{e}^n - \tau \mathcal{A}_h(\mathbf{e}^n)\|_{\ell^2} + \tau \|\mathbf{f}^{n+\frac{1}{2}}\|_{\ell^2})^2 \\
&\leq (1+\tau)\|\mathbf{e}^n - \tau \mathcal{A}_h(\mathbf{e}^n)\|_{\ell^2}^2 + \tau(1+\tau)\|\mathbf{f}^{n+\frac{1}{2}}\|_{\ell^2}^2.
\end{aligned}$$

The remainder of the proof is a detailed computation of the first term on the right. We omit the superscript from $\mathbf{e}^n$ below.

$$\begin{aligned}
&\|\mathbf{e} - \tau \mathcal{A}_h(\mathbf{e})\|_{\ell^2}^2 \\
&= \|\mathbf{e}\|_{\ell^2}^2 - 2\tau \langle \mathbf{e}, \mathcal{A}_h(\mathbf{e}) \rangle_{\ell^2} + \tau^2 \|\mathcal{A}_h(\mathbf{e})\|_{\ell^2}^2 \\
&= \|\mathbf{e}\|_{\ell^2}^2 - 2\tau \sum_{(i,j)} \frac{h_{ij}^{\perp}}{h_{ij}} (u_i - u_j)(e_i - e_j) + \tau^2 \sum_i \frac{1}{|A_i|} \left(\sum_{j \in C_i} \frac{h_{ij}^{\perp}}{h_{ij}} (u_i - u_j) \right)^2 \\
&\leq \|\mathbf{e}\|_{\ell^2}^2 - 2m\tau \sum_{(i,j)} \frac{h_{ij}^{\perp}}{h_{ij}} (u_i - u_j)^2 + \tau^2 \sum_i \frac{1}{|A_i|} \left[\sum_{j \in C_i} \frac{h_{ij}^{\perp}}{h_{ij}} (u_i - u_j)^2 \right] \left[\sum_{j \in C_i} \frac{h_{ij}^{\perp}}{h_{ij}} \right] \\
&\leq \|\mathbf{e}\|_{\ell^2}^2 - 2\tau\,(m - \tau M_h) \sum_{(i,j)} \frac{h_{ij}^{\perp}}{h_{ij}} (u_i - u_j)^2
\end{aligned}$$

The final line involved the use of the inequality[2] $\sum_i \sum_{j \in C_i} a_{ij} \leq 2 \sum_{(i,j)} a_{ij}$ when $a_{ij} = a_{ji} \geq 0$. Combining the inequalities completes the proof. □

[2] This inequality would be an equality if i summed over all the nodes in the mesh. In this situation no restriction is required on the sign of a_{ij}.

COROLLARY 3.8 (STABILITY OF EXPLICIT SCHEME). *If $\tau \le m/M_h$ then*

$$\|\mathbf{e}^{n+1}\|_{\ell^2}^2 + c\tau \sum_{m=0}^{n} \|u^m\|_{V_h}^2 \le e^{t^n}(\|\mathbf{e}^0\|_{\ell^2}^2 + \tau \sum_{m=0}^{n} \|\mathbf{f}^{m+\frac{1}{2}}\|_{\ell^2}^2),$$

where $t^n = n\tau$, and $c = 2(m - \tau M_h)$.

Note that if the mesh $\mathcal{T}_h$ is constructed from equilateral triangles, then $h_{ij}^{\perp}/h_{ij} = 1/\sqrt{3}$ and $|C_i| = 6$. Also, $|A_i| = (\sqrt{3}/2)h^2$ where h is the length of the sides of the triangles, so that $M_h = 4/h^2$. The stability estimate then becomes $\tau \le mh^2/4$.

We complete this section with a summary of the bounds satisfied by solutions of both the implicit and explicit schemes. The bounds that don't follow immediately from the previous results can be proved using identical techniques.

THEOREM 3.9. *The following bounds are satisfied by both the implicit and explicit schemes. The explicit scheme requires CFL condition $m - \tau M_h \ge c > 0$ to hold uniformly (recall that m is the reciprocal of the Lipschitz constant for K, and $M_h \sim h^{-2}$ is a mesh parameter).*

- $\|e_h\|_{L^\infty[0,T;H]} \le C(\|e_0\|_{L^2(\Omega)}, \|f\|_{L^2[0,T;H]}, T)$.
- $\|u_h\|_{L^2[0,T;V]} \le C(\|e_0\|_{L^2(\Omega)}, \|f\|_{L^2[0,T;H]}, T)$.
- $\|\frac{de_h}{dt}\|_{L^2[0,T;V'_h]} \le C(\|e_0\|_{L^2(\Omega)}, \|f\|_{L^2[0,T;H]}, T)$.

Additionally, solutions of the implicit scheme satisfies the following bounds,

- $\|e_h\|_{L^\infty[0,T;L^\infty(\Omega)]} \le C(\|e_0\|_{L^\infty(\Omega)}, \|f\|_{L^1[0,T;L^\infty(\Omega)]}$.
- $\|e_h\|_{Lip[0,T;L^1(\Omega))]} \le C(\|\Delta u_0\|_{L^1(\Omega)}, \|f\|_{Lip[0,T;L^1(\Omega)]}, T)$.
- $\|u_h\|_{L^\infty[0,T;V]} \le C(\|u_0\|_V, \|f\|_{H^1[0,T;V']})$.
- $\|\frac{de_h}{dt}\|_{L^\infty[0,T;V']} \le C(\|u_0\|_V, \|f\|_{H^1[0,T;V']}, T)$.
- $\|\frac{du_h}{dt}\|_{L^2[0,T;H]} \le C(\|u_0\|_V, \|f\|_{H^1[0,T;V']}, T, m)$.

4. Convergence of the Discrete Problem. In order to prove convergence of the co-volume algorithm it is necessary to interpret the energy as an element in the dual space $V' = H_0^1(\Omega)'$; however, we shall not simply pivot through $H = L^2(\Omega)$. Let Π_0 and Π_1 be the standard projections of H onto H_h and V onto V_h respectively:

$$\Pi_0 v \in H_h \qquad (\Pi_0 v, w) = (v, w) \qquad \forall\ w \in H_h,$$

and

$$\Pi_1 v \in V_h \qquad (\nabla \Pi_1 v, \nabla w) = (\nabla v, \nabla w) \qquad \forall\ w \in V_h.$$

We next define a projection $P_h : V \to H_h$ by: $P_h v$ is the function in H_h whose value on the co-volume A_i containing the node x_i is $\Pi_1 v(x_i)$. ie. $P_h v$ is the piecewise constant function whose values correspond to the nodal values of the projection of v onto V_h (see figure 3). The role of this projection becomes clear when we write the co-volume algorithm as:

$$\left(\frac{d}{dt}e_h, P_h v\right) + (\nabla u_h, \nabla v) = (f, P_h v) \qquad \forall\ v \in\ V, \tag{8}$$

or

$$\frac{d}{dt}P_h' e_h - \Delta u_h = P_h' f \qquad in\ V',$$

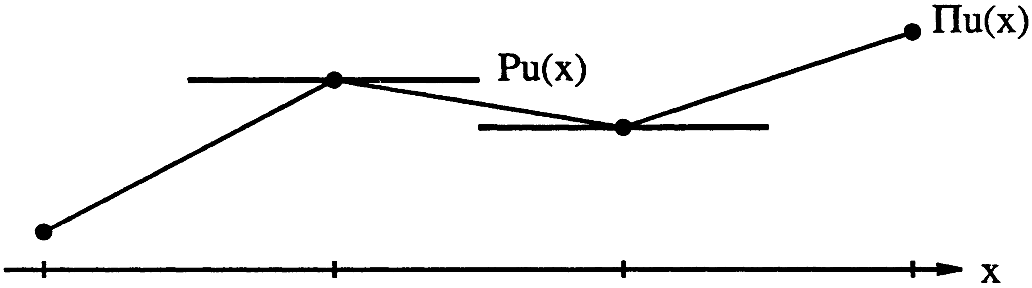


Figure 3. Projections Pu and Πu.

where $P_h' : H_h \to V'$ is the dual map, $\langle P_h' e_h, v\rangle = (e_h, P_h v)$. The following elementary lemma shows that the natural identification of e_h as an element of V' (i.e. $\langle e_h, v\rangle \equiv (e_h, v)$) is close to $P_h' e_h$.

LEMMA 4.1. *If $u \in W_0^{1,p}(\Omega)$ then $\|u - P_h u\|_{L^p(\Omega)} \le C(p)h\|u\|_{W_0^{1,p}(\Omega)}$, $1 \le p < \infty$, hence if $e_h \in H_h$,*

$$\|e_h - P_h' e_h\|_{V'} \le Ch\|e_h\|_H, \qquad \|P_h' e_h\|_{V'} \le Ch\|e_h\|_H + \|e_h\|_{V'},$$

and $\|P_h' e_h\|_{V'} \le \|e_h\|_{V_h'}$.

Proof. If $u_h = \Pi_1 u$, then $P_h u_h = P_h u$, so that

$$\|u - P_h u\|_{L^p(\Omega)} \le \|u - u_h\|_{L^p(\Omega)} + \|u_h - P_h u_h\|_{L^p(\Omega)}.$$

Standard approximation theory shows the first term is less than $Ch\|u\|_{W_0^{1,p}(\Omega)}$, and the second term can be bounded by writing $u_h(x) = u_h(x_i) + (\nabla u_h)|_T \cdot (x - x_i)$ for $x \in T$ where T is a typical triangle with vertex x_i. Since $u_h(x_i) = (P_h u_h)_i$ it follows that the leading term of $u_h - P_h u_h$ cancels. The first order remainder can then be bounded by $h\|u_h\|_{W_0^{1,p}(\Omega)} \le Ch\|u\|_{W_0^{1,p}(\Omega)}$.

The inequalities pertaining to e follow from

$$\langle e_h - P_h' e_h, u\rangle = (e_h, u - P_h u) \le Ch\|e_h\|_H\|u\|_V,$$

and

$$\langle P_h' e_h, u\rangle_V = \langle e_h, P_h u\rangle_{V_h} \le \|e_h\|_{V_h'}\|\Pi_1 u\|_{V_h} \le \|e_h\|_{V_h'}\|u\|_V.$$

□

4.1. Convergence of the Implicit Scheme. Since the discrete co–volume scheme retains the semigroup properties of the continuous problem, abstract semigroup results can be used to establish convergence of the implicit scheme. We begin by recalling the Kato–Trotter [5] and Crandall–Ligget theorems [7], and indicate how they may be used. The resolvent convergence required for the Kato–Trotter theorem is then established. This will show convergence of the discrete energy in $V' = H^{-1}(\Omega)$ for arbitrary meshes, and if the mesh is uniform, we can also establish convergence in $L^1(\Omega)$.

THEOREM 4.2 (KATO–TROTTER). *Let $X_h \subset X$, $h > 0$ be an inclusion of Banach spaces, and suppose the A_h is an m–accretive operation on X_h with associated semigroup $\mathcal{S}_h$. If $x_h \to x$ as $h \to 0$, $x_h \in X_h$, and A is an m–accretive operator in X satisfying,*

$$(9) \qquad f_h \in X_h,\ f_h \to f \in X \qquad \Rightarrow \qquad (I + A_h)^{-1} f_h \to (I + A)^{-1} f \in X,$$

then $\mathcal{S}_h(t)x_h \to \mathcal{S}(t)x$ uniformly in t on compact intervals, where $\mathcal{S}$ is the semigroup in X generated by A.

THEOREM 4.3 (CRANDALL–LIGGET). *Let A_h be an m–accretive operator in X_h, then for $f \in L^1[0, T, X_h]$, the piece-wise linear functions generated by the implicit scheme*

$$e^{n+1} - e^n + \tau A_h(e^{n+1}) \ni f^n, \qquad e^0 = e_{0h}, \qquad n = 0, 1, \cdots, N-1,$$

converge in $C[0,T;X_h]$ *to* $S_h(.)e_{0h}$*, the solution of* $de_h/dt + A_h(e_h) = f_h$*,* $e_h(0) = e_{0h}$*. (*$f^n = (1/\tau)\int_{n\tau}^{(n+1)\tau} f(s)\,ds$*,* $\tau = 1/N$*, etc).*

To apply the above results, let $X_h = \mathbb{R}^{N_h}$, with either the ℓ^1 topology so that $X_h \subset X = L^1(\Omega)$, or the V_h' topology, so that $X_h \subset X = H^{-1}(\Omega)$ (in this instance the inclusion map is P_h'). Suppose for the moment that condition (9) is satisfied for each of the above choices, then if e_h^τ denotes the discrete solution obtained with the implicit co–volume scheme, and e_h denotes the semi–discrete solution obtained by letting $\tau \to 0$, then

$$\|e - e_h^\tau\|_{C[0,T;X]} \le \|e - e_h\|_{C[0,T;X]} + \|e_h - e_h^\tau\|_{C[0,T;X_h]}.$$

By selecting h sufficiently small, the Kato–Trotter theorem implies that the first term may be made arbitrarily small. Then, for h fixed, the Crandall–Ligget theorem shows that the second term may be made arbitrarily small by selecting τ sufficiently small, establishing the convergence of the implicit scheme. Berger, Brezis and Rogers [3] used the Kato–Trotter theorem in a similar manner for a continuous in space, discrete time scheme.

Remark: Strictly speaking the Kato–Trotter theorem only pertains to the homogeneous equation, $de/dt + A(e) = 0$. However, standard arguments (see eg. [7]) using translates of the form $\tilde{A}(e) = A(e) - f$ for $f \in X$, can be used to establish convergence for a non–homogeneous right hand sides.

LEMMA 4.4. *Let* $(I + A_h)e_h = f_h$ *in* V_h'*,* $P_h'f_h \to f$ *in* $V' = H^{-1}(\Omega)$*, and suppose* K *satisfies the growth condition* $|K(s)| \ge c|s|$ *for* $|s|$ *large. Then* $P_h'e_h \to e$ *in* V'*, where* e *is the solution to* $(I - \Delta \circ K)e = f$*.*

Proof. The discrete resolvent equation may be written as

$$\langle P_h'e_h, v\rangle + (\nabla u_h, \nabla v) = \langle P_h'f_h, v\rangle, \qquad \forall\ v \in V.$$

Selecting $v = u_h$, and recalling that $\langle P_h'e_h, u_h\rangle \ge 0$, implies that $\|u_h\|_V \le \|P_h'f_h\|_{V'}$, so remains bounded. The growth condition on K (and the Poincaré inequality) then imply that $\|e_h\|_H$ is bounded. Passing to a subsequence, standard compactness results imply that

$$u_h \rightharpoonup u \ \text{in}\ V, \qquad u_h \to u \ \text{in}\ H,$$

$$e_h \rightharpoonup e \ \text{in}\ H, \qquad e_h \to e \ \text{in}\ V'.$$

Defining $\bar{u}_h = P_h u_h$, lemma 4.1 then implies that

$$\bar{u}_h \to u \ \text{in}\ H, \qquad P_h'e_h \to e \ \text{in}\ V'.$$

Passing to a further subsequence, we may assume that $\bar{u}_h(x) \to u(x)$ for almost every $x \in \Omega$. Taking the limit of the resolvent equation shows that e and u satisfy

$$\langle e, v\rangle + (\nabla u, \nabla v) = \langle f, v\rangle, \qquad \forall\ v \in V,$$

moreover, $e_h = K(\bar{u}_h)$ and the almost everywhere convergence of $\bar{u}_h(x)$ shows that $e = K(u)$ a.e, so that e is a solution of the resolvent equation in V'. Since solutions of the resolvent equation are unique, it follows that the whole sequence $\{P_h'e_h\}$ converged strongly to e in V'. □

Remark: The strong convergence of $P_h' e_h$ can be established directly without the compactness argument as in the proof of Theorem 4.11 below.

To establish convergence of the resolvent equation is $L^1(\Omega)$, we begin with a minor modification of a lemma by Stampacchia (see [11]).

LEMMA 4.5 (STAMPACCHIA). *Given $\mathbf{g} \in L^p(\Omega)^n$, $p > n$, let $v_h \in V_h$ satisfy*

$$\int_\Omega \nabla v_h \cdot \nabla w_h = \int_\Omega \mathbf{g} \cdot \nabla w_h \qquad \forall\, w_h \in V_h,$$

then $\|v_h\|_{L^\infty(\Omega)} \leq C(|\Omega|)\|\mathbf{g}\|_{L^p(\Omega)}$.

Proof. (Sketch) The only modification required of the proof in [11] is to begin by letting $\xi_h \in V_h$ be the piecewise linear function with nodal values given by $\xi_i = sgn(v_i)\max(|v_i| - k, 0)$, and observing that

$$\begin{aligned}
\|\xi_h\|_V^2 &= \int_\Omega \nabla \xi_h \cdot \nabla \xi_h = \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}} (\xi_i - \xi_j)^2 \\
&\leq \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}} (v_i - v_j)(\xi_i - \xi_j) = \int_\Omega \nabla v_h \cdot \nabla \xi_h.
\end{aligned}$$

The rest of the proof remains the same. □

COROLLARY 4.6. *Let e_h be a solution of the discrete resolvent equation $(I + \mathcal{A}_h)\mathbf{e}_h = \mathbf{f}_h$, then for $1 \leq p' < n/(n-1)$, $\|u_h\|_{W_0^{1,p'}(\Omega)} \leq C(p)\|\mathbf{f}_h\|_{\ell^1}$.*

Proof. Given $v_h \in V_h$, the resolvent equation implies that

$$\int_\Omega \nabla u_h \cdot \nabla v_h = \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}} (u_i - u_j)(v_i - v_j) = \sum_i |A_i|(f_i - e_i)v_i,$$

and by Lemma 3.1, $\|\mathbf{e}_h\|_{\ell^1} \leq \|\mathbf{f}_h\|_{\ell^1}$. For $\mathbf{g} \in L^p(\Omega)^n$, $p > n$, let $v_h \in V_h$ be the solution of the discrete equation given in Stampacchia's lemma above, then

$$\begin{aligned}
\int_\Omega \mathbf{g} \cdot \nabla u_h &= \sum_i |A_i|(f_i - e_i)v_i \\
\left|\int_\Omega \mathbf{g} \cdot \nabla u_h\right| &\leq \|\mathbf{f}_h - \mathbf{e}_h\|_{\ell^1}\|v_h\|_{\ell^\infty} \\
&\leq C\|\mathbf{f}_h\|_{\ell^1}\|\mathbf{g}\|_{L^p(\Omega)}.
\end{aligned}$$

The lemma now follows by taking a suitable supremum over $\mathbf{g}$. □

Using the techniques in [2] we establish the strong convergence of $\{e_h\}$ in $L^1(\Omega)$. Essentially we show the ideas used for the continuous problem carry over to the discrete scheme. The next two lemmas are discrete analogues of Lemmas E and F in [2]

LEMMA 4.7. *Let e_h be a solution of the resolvent equation $(I + \mathcal{A}_h)\mathbf{e}_h = \mathbf{f}_h$, then for $b > 0$,*

$$\sum_i |A_i|(|e_i| - b)^+ \leq \sum_i |A_i|(|f_i| - b)^+.$$

$(\alpha^+ \equiv \max(\alpha, 0))$.

Proof. Subtracting b from both sides of the resolvent equation and selecting $v_i = sgn^+(e_i - b)$ and then adding b to both sides and selecting $v_i = sgn^-(e_i + b)$ gives the two inequalities,

$$\begin{aligned} \sum_i |A_i|(e_i - b)^+ &\leq \sum_i |A_i|(f_i - b)^+ \\ \sum_i |A_i|(e_i + b)^- &\leq \sum_i |A_i|(f_i + b)^-. \end{aligned}$$

The lemma now follows by observing that $(|\alpha| - b)^+ = (\alpha - b)^+ + (\alpha + b)^-$, $(\alpha^- \equiv \alpha^+ - \alpha)$. □

LEMMA 4.8. *Let $\xi_h \in V_h$ be non-negative, and suppose that $u_h \in V_h$ satisfies*

$$\frac{1}{|A_i|} \sum_{j \in C_i} \frac{h_{ij}^\perp}{h_{ij}} (u_i - u_j) = f_i$$

at points where $\xi_i \neq 0$. Then

$$\sum_i |u_i| \sum_{j \in C_i} \frac{h_{ij}^\perp}{h_{ij}} (\xi_i - \xi_j) \leq \sum_i |A_i| f_i s_i,$$

for any selection $s_i \in sgn(u_i)$.

Proof. Since $|u_i|$ vanishes on the boundary nodes we use the discrete Green's formula to get

$$\begin{aligned} \sum_i |u_i| \sum_{j \in C_i} \frac{h_{ij}^\perp}{h_{ij}} (\xi_i - \xi_j) &= \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}} (|u_i| - |u_j|)(\xi_i - \xi_j) \\ &= \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}} (s_i u_i - s_j u_j)(\xi_i - \xi_j) \\ &= \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}} \{(u_i - u_j)(s_i \xi_i - s_j \xi_j) + \xi_i u_j (s_i - s_j) + \xi_j u_i (s_j - s_i)\} \\ &= \sum_i |A_i| f_i s_i \xi_i + \sum_{(i,j)} \frac{h_{ij}^\perp}{h_{ij}} \{\xi_i u_j (s_i - s_j) + \xi_j u_i (s_j - s_i)\}. \end{aligned}$$

Since each of the terms in the second sum on the right hand side is non–positive, the lemma follows. □

LEMMA 4.9. *Let $\mathbf{e}_h$ be the solutions of the resolvent equations $\mathbf{e}_h + \mathcal{A}_h(\mathbf{e}_h) = \mathbf{f}_h$ on a family of* uniform *meshes $\{\mathcal{T}_h\}_{h>0}$, and let $\mathbf{e}_h$ and $\mathbf{f}_h$ be identified with the corresponding piece-wise constant functions e_h and f_h in $L^1(\Omega)$ (extended to zero off Ω_h). If $f_h \to f$ in $L^1(\Omega)$, then $\{e_h\}_{h>0}$ converges strongly in $L^1(\Omega)$ to the solution of the continuous resolvent equation $(I - \Delta \circ K)e = f$.*

Proof. It follows immediately from Corollary 4.7 that we may pass to a sub–sequence for which $\{u_h\}$ converges weakly in $W_0^{1,p'}(\Omega)$, and hence strongly in $L^{p'}(\Omega)$. Similarly, Lemma 4.7 is precisely the criteria for weak sequential compactness in $L^1(\Omega)$, so we may assume $\{e_h\}$ is weakly convergent. To show that a subsequence of $\{e_h\}$

converges strongly in $L^1(\Omega)$, it then suffices to show that for any sub–domain $\Omega' \subset\subset \Omega$ that

$$\sup_h \int_{\Omega'} |e_h - e_h(\cdot + \delta)| \to 0 \qquad \text{as } |\delta| \to 0.$$

To accomplish this, let $\xi \in C_0^\infty(\Omega)$ satisfy $0 \le \xi \le 1$ and $\xi = 1$ on Ω', and denote $\xi(x_i)$ by ξ_i for each mesh point x_i.

We first consider δ a mesh lattice vector, ie. translation by δ carries the mesh onto itself, and assume that δ is sufficiently small to guarantee that $x \in \text{supp}(\xi) \Rightarrow x + \delta \in \Omega$. If e_i denotes the value of $\mathbf{e}_h$ at mesh point x_i, let $e_{i+\delta}$ denote the value of $\mathbf{e}_h$ at the mesh point $x_i + \delta$. Subtracting the discrete equations for e_i and $e_{i+\delta}$ yields for $x_i \in \text{supp}(\xi)$,

$$|A_i|(e_i - e_{i+\delta}) + \sum_{j\in C_i} \frac{h_{ij}^\perp}{h_{ij}}[(u_i - u_{i+\delta}) - (u_j - u_{j+\delta})] = |A_i|(f_i - f_{i+\delta}).$$

Defining $s_i = sgn(e_i - e_{i+\delta}) \in sgn(u_i - u_{i+\delta})$, Lemma 4.8 then asserts that

$$\sum_i |u_i - u_{i+\delta}| \sum_{j\in C_i} \frac{h_{ij}^\perp}{h_{ij}}(\xi_i - \xi_j) \le \sum_i |A_i|[(f_i - f_{i+\delta}) - (e_i - e_{i+\delta})]s_i\xi_i,$$

or

$$\|e_h - e_h(\cdot + \delta)\|_{L^1(\Omega')} \le \|f_h - f_h(\cdot + \delta)\|_{L^1(\Omega)} + \|P_h(u_h - u_h(\cdot + \delta))\|_{L^1(\Omega)}\|\mathcal{A}_h(\xi)\|_{\ell^\infty}.$$

An elementary calculation shows that $\|\mathcal{A}_h(\xi)\|_{\ell^\infty} \le C\|D^2\xi\|_{L^\infty(\Omega)}$, and since $\{u_h\}$ is bounded in $W_0^{1,1}(\Omega)$, Lemma 4.1 shows $\|P_h(u_h - u_h(\cdot + \delta))\|_{L^1(\Omega)} \le C\|u_h - u_h(\cdot + \delta)\|_{L^1(\Omega)}$. It follows that

$$\|e_h - e_h(\cdot + \delta)\|_{L^1(\Omega')} \le \|f_h - f_h(\cdot + \delta)\|_{L^1(\Omega)} + C\|u_h - u_h(\cdot + \delta)\|_{L^1(\Omega)}.$$

Since f_h and u_h converge strongly in $L^1(\Omega)$, it follows that the right hand side goes to zero uniformly in h as $|\delta| \to 0$.

To complete the proof for arbitrary shifts, we decompose δ into a "large" shift comprising of one of the mesh lattice vectors, and a "small" shift where the translate of each co–volume would only overlay its adjacent neighbors. Using the above estimate for the large shift with the triangle inequality, it suffices to consider these small translates. On a typical co–volume A_i we may explicitly write

$$\int_{A_i} |e_h - e_h(\cdot + \delta)| = |A_i| \sum_{j\in C_i} \rho_j |e_i - e_j|,$$

where $\rho_j = |A_i \cap A_j|/|A_i|$, so that $\sum_j \rho_j \le 1$. Letting $\{\delta_j\}_{j=1}^J$ ($J = |C_i|$), denote the set of mesh vectors that point from node i to the adjacent nodes j, we may write

$$\int_{A_i} |e_h - e_h(\cdot + \delta)| = |A_i| \sum_{j=1}^J \rho_j |e_i - e_{i+\delta_j}|.$$

Since the mesh is uniform, the quantities subscripted with j do not depend upon the particular co–volume in question, thus summing over all co–volumes in Ω' yields

$$\begin{aligned}\int_{\Omega'} |e_h - e_h(\cdot + \delta)| &= \sum_i |A_i| \sum_{j=1}^{J} \rho_j |e_i - e_{i+\delta_j}| \\ &= \sum_{j=1}^{J} \rho_j \|e_h - e_h(\cdot + \delta_j)\|_{L^1(\Omega')}.\end{aligned}$$

Since each of the δ_j are lattice vectors, we obtain

$$\|e_h - e_h(\cdot + \delta)\|_{L^1(\Omega')} \leq \sum_{j=1}^{J} \rho_j \{\|f_h - f_h(\cdot + \delta_j)\|_{L^1(\Omega)} + C\|u_h - u_h(\cdot + \delta_j)\|_{L^1(\Omega)}\}.$$

As in Lemma 4.4, it follows that the $u = K(e)$, and $(I - \Delta \circ K)e = f$ in $\mathcal{D}'(\Omega)$. Uniqueness of this solution then implies that the whole sequence $\{e_h\}$ converges to e in $L^1(\Omega)$. □

In summary we have,

THEOREM 4.10. *Let $\{e_h^\tau\}$ be the sequence of solutions given by the implicit co–volume algorithm (4) for a regular family of meshes, then*

- *If $P_h' e_{0h} \to e_0$ in V' and $P_h' f_h \to f$ in $L^1[0,T;V']$, then $P_h' e_h^\tau \to e$ in $C[0,T;V']$, provided $|K(s)| \geq c|s|$, for some $c > 0$ when $|s|$ is large.*
- *If $e_{0h} \to e_0$ in $L^1(\Omega)$ and $f_h \to f$ in $L^1[0,T;L^1(\Omega)]$ for a uniform family of meshes, then $e_h^\tau \to e$ in $C[0,T;L^1(\Omega)]$.*

4.2. Rates of Convergence. The preceding results do not give any rates of convergence for the implicit scheme. The following establishes a (sub–optimal) rate of convergence for both the implicit and explicit scheme.

THEOREM 4.11. *Let the data for the Stefan problem satisfy $e_0 \in H$ and $f \in L^2[0,T;H]$, then the discrete solution given by either the implicit co–volume algorithm or the explicit co–volume algorithm (subject to the CFL constraint $m - \tau M_h \geq c > 0$) satisfy the error estimates*

$$\|e - e_h\|_{L^\infty[0,T;V']} \leq C(h^{1/2} + \tau^{1/2}), \quad \text{and} \quad \|u - u_h\|_{L^2[0,T;H]} \leq \tilde{C}(h^{1/2} + \tau^{1/2}),$$

where C depends only upon the data e_0 and f, and $\tilde{C}$ depends additionally upon the Lipschitz constant for K.

Proof. Taking the difference of equations (2) and (8) gives

$$\langle \frac{d}{dt}(e - P_h' e_h), v\rangle + (\nabla(u - u_h), v) = (f - P_h' f, v) \qquad \forall\, v \in V.$$

It is to be understood that e_h is the temporally piece–wise linear function that interpolates the values $\{e^n\}_{n=0}^N$, and u_h is the piecewise constant function with values $\{u^n\}_{n=1}^N$ for the implicit scheme and values $\{u^n\}_{n=0}^{N-1}$ for the explicit scheme. Selecting $v \in V$ as the solution to

$$(\nabla v, \nabla w) = \langle e - P_h' e_h, w\rangle \qquad \forall\, w \in V,$$

yields

$$\frac{1}{2}\frac{d}{dt}\|e - P_h' e_h\|_{V'}^2 + \langle e - P_h' e_h, u - u_h\rangle = (f, v - Pv). \tag{10}$$

The term on the right can be bounded by $Ch\|f\|_H\|v\|_V = Ch\|f\|_H\|e - P_h'e_h\|_{V'}$. To estimate the middle term we introduce the piece wise constant functions $\bar{u}_h \in H_h$ defined by $\bar{u}_h = Pu_h$, and $\bar{e}_h \in H_h$ given as the piece–wise constant function in time having values $\{e^n\}_{n=1}^N$ for the implicit scheme, and $\{e^n\}_{n=0}^{N-1}$ for the explicit scheme. This choice gives $\bar{u}_h = K(\bar{e}_h)$ with e_h and $\bar{e}_h$ equal at the discrete times $t^n = n\tau$. Writing

$$\langle e - P_h'e_h, u - u_h\rangle = \langle e - P_h'\bar{e}_h, u - u_h\rangle + \langle P_h'(\bar{e}_h - e_h), u - u_h\rangle,$$

we bound each of the terms on the right.

$$\begin{aligned}
\langle e - P_h'\bar{e}_h, u - u_h\rangle &= (e - \bar{e}_h, u - u_h) + \langle \bar{e}_h - P_h'\bar{e}_h, u - u_h\rangle \\
&= (e - \bar{e}_h, u - \bar{u}_h) + (e - \bar{e}_h, \bar{u}_h - u_h) + \langle \bar{e}_h - P_h'\bar{e}_h, u - u_h\rangle \\
&\geq (e - \bar{e}_h, u - \bar{u}_h) - Ch(\|e - \bar{e}_h\|_H\|u_h\|_V + \|u - u_h\|_V\|\bar{e}_h\|_H).
\end{aligned}$$

For $t^n = n\tau \leq t \leq t^{n+1} = (n+1)\tau$, the second term may be bounded by

$$\begin{aligned}
\langle P_h'(\bar{e}_h - e_h), u - u_h\rangle &\leq \|e_h - \bar{e}_h\|_{V_h'}\|u - u_h\|_V \\
&\leq \tau\|\frac{de_h}{dt}|_{(t^n,t^{n+1})}\|_{V_h'}\|u - u_h\|_V.
\end{aligned}$$

Since the $\|e\|_{L^2[0,T;H]}$, $\|e_h\|_{L^2[0,T;H]}$, $\|\bar{e}_h\|_{L^2[0,T;H]}$, $\|\frac{de_h}{dt}\|_{L^2[0,T;V'_h]}$, $\|u\|_{L^2[0,T;V]}$, and $\|u_h\|_{L^2[0,T;V]}$ are all bounded, integration of (10) from time zero to t yields

$$\|e - P_h'e_h\|_{V'}^2(t) + \int_0^t (e - e_h, u - \bar{u}_h) \leq \|e_0 - e_{0h}\|_{V'}^2 + C(h + \tau).$$

The projection $e_{0h} = \Pi_0 e$ satisfies $\|e - e_{0h}\|_{V'} \leq Ch\|e\|_H$, so an L^∞ estimate on $\|e - P_h'e_h\|_{V'}$, and hence $\|e - e_h\|_{V'}$ follows immediately. A bound on the error $\|u - \bar{u}_h\|_{L^2[0,T;H]}$ follows from the inequality $(e - \bar{e}_h, u - \bar{u}_h) \geq m\|u - \bar{u}_h\|_H^2$, where m is the reciprocal of the Lipschitz constant for K. The estimate $\|u_h - \bar{u}_h\|_H = \|u_h - Pu_h\|_H \leq Ch\|u_h\|_V$ and an application of the triangle inequality completes the proof. □

5. Numerical Examples. We present some numerical examples that demonstrate the convergence of the co–volume algorithm. The examples suggest that the rate of $(h^{1/2} + \tau^{1/2})$ is pessimistic, since rates close to order one are observed. In order to try and isolate the spatial and temporal errors, we attempt to hold one of h or τ fixed, and let the other go to zero. It is relatively easy to let $\tau \to 0$ with h fixed using the explicit scheme. At worst, the program will take a long time to run. However, when attempting to let $h \to 0$, one must use the implicit scheme, and the equations at each step become difficult to solve. While it is known that they can always be solved using a Gauss Seidel algorithm [21, 22], this is not very effective when τ/h^2 is large since, this is analogous to a linear system of equations with a large difference in the modulus of eigenvalues. In this situation, each iteration makes a very small correction (of order h^2/τ) to the vector of unknowns, and this is unacceptable, since, if in one time step the phase boundary passes over a node in the mesh, the energy at that node will increase by the latent heat (set to unity in the examples), so that it will take $\sim h^2/\tau$ iterations to capture this. One other disadvantage is that when the change with each iteration is so small, it is difficult to determine when convergence has occurred. To solve the implicit equations, the following Newton scheme was used,

$$\mathbf{e}^{new} = \mathbf{e}^{old} - J^{-1}(\mathbf{e}^{old})\mathcal{F}(\mathbf{e}^{old}),$$

where

$$\mathcal{F}(\mathbf{e})_i = |A_i|(e_i - e_i^n) + \tau \sum_{j \in C_i} \frac{h_{ij}^{\perp}}{h_{ij}}(K(e_i) - K(e_j)) - \tau |A_i| f_i^{n+1/2},$$

$$J_{ij} = \frac{\partial \mathcal{F}_i}{\partial e_j} = \begin{cases} |A_i| + \tau \sum_{k \in C_i} \frac{h_{ik}^{\perp}}{h_{ik}} K'(e_i) & i = j, \\ \frac{h_{ij}^{\perp}}{h_{ij}} K'(e_j) & \text{otherwise.} \end{cases}$$

Clearly the components of J are not well defined where K' fails to exist; however, in practice this scheme works much better than simple relaxation. This Newton algorithm differs from the one proposed in [10] where a meaning to $(K^{-1})'$ is established.

When evaluating the errors in the energy, it is necessary to use numerical quadrature to integrate the discontinuous exact solution. A little thought shows that if the interfaces where the discontinuities occur are smooth, the quadrature errors will be of order h, so that it is still possible to detect order h convergence. The L^1 errors tabulated for $e - e_h^{\tau}$ are calculated over the union of the co–volumes, $\Omega_h = \cup_i A_i$, which excludes a region near the boundary. A similar problem arises when evaluating the $V' = H^{-1}(\Omega)$ norms. This norm was estimated by solving the discrete Dirichlet problem, $v_h \in V_h$,

$$\sum_{(i,j)} \frac{h_{ij}^{\perp}}{h_{ij}}(v_i - v_j)(w_i - w_j) = \sum_i \int_{A_i} (e_i - e) w_i \qquad \forall\, w_h \in V_h,$$

where e is the exact solution, and setting $\|e - P_h' e_h^{\tau}\|_{V'} \simeq \|v_h\|_V$. Again, a little thought shows that this process will be of order h, so that errors up to order h may be detected.

In each of the examples below, the energy temperature relation was taken to be $K(e) = e$ if $e < 0$, $K(e) = 0$ if $0 \le e \le 1$ and $K(e) = e - 1$ for $e > 1$ (see Figure 1).

5.1. One Dimensional Example. The one dimensional analogue of the co–volume algorithm corresponds to the staggered mesh scheme proposed in [19] and analyzed in [1] (see Figure 3). An example, taken from [17] involving the melting of a mushy region ($0 \le e \le 1$) is computed. The solution in the (x,t) plane is shown in Figure 4, and is given by

$$e(x,t) = \begin{cases} \begin{cases} 4(s_2(t) - x) & x \le s_2(t) \\ -(x - s_2(t))^2 & x > s_2(t) \end{cases} & 0 < t \le 1/4, \\ \begin{cases} s_1(t)^2 - x^2 + 1 & x \le s_1(t) \\ 4(s_2(t) - x) & s_1(t)x \le s_2(t) \\ -(x - s_2(t))^2 & x > s_2(t) \end{cases} & 1/4 < t \le 3/4, \\ \begin{cases} s_1(t)^2 - x^2 + 1 & x \le s_1(t) \\ -[x - s_1(t)]^2 \\ -2[x - s_1(t)](t - 3/4) & x > s_1(t) \end{cases} & 3/4 < t \le 1, \end{cases}$$

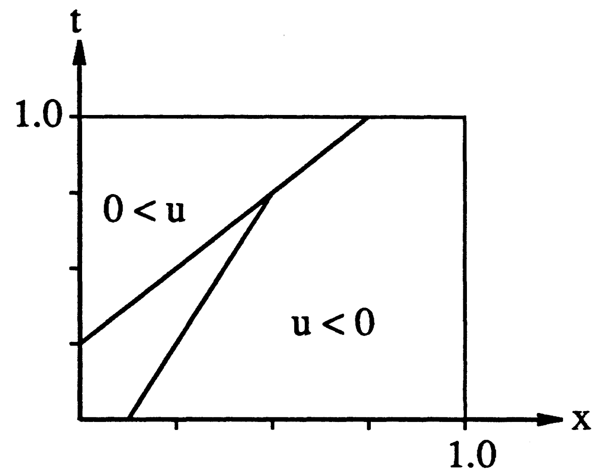
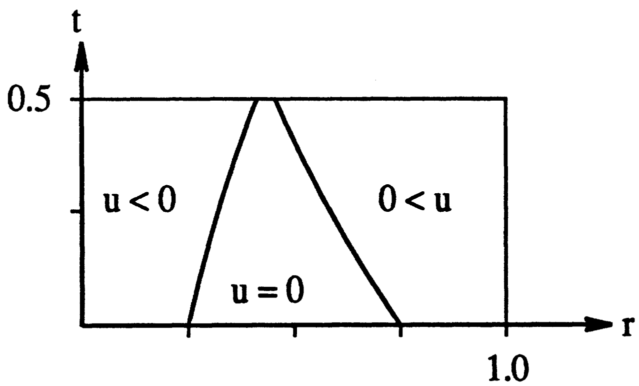
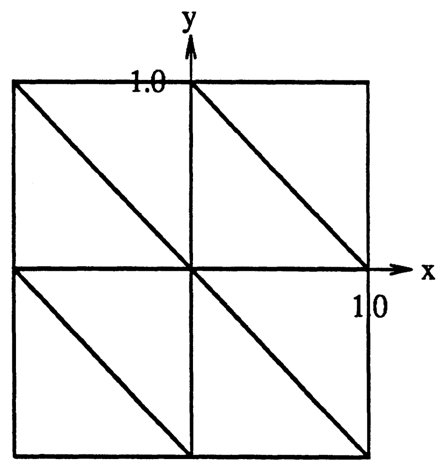


Figure 4. One Dimensional Example.



(a) Solution in (r,t) plane.



(b) A 2x2 Mesh (h = 1).

Figure 5. Two Dimensional Example.

TABLE 1
Rates of Convergence with Respect to h for the One Dimensional Problem

h	1/10		1/22		1/46
$\|\|(e-e_h)(T)\|\|_{L^1(\Omega)}$	0.019527		0.008878		0.004225
Rates		0.999674		1.006707	
$\|\|(e-P_h' e_h)(T)\|\|_{V'}$	0.001385		0.000465		0.000162
Rates		1.383272		1.430705	
$\|\|u-u_h\|\|_{L^2[0,T;H]}$	0.005693		0.002154		0.000867
Rates		1.232490		1.233936	
Final Time Step Size	1/51200		1/61952		1/67792

where $s_1(t) = t-1/4$ and $s_2(t) = (1/2)(t+1/4)$. The non-homogeneous term required to obtain this solution is

$$f(x,t) = \begin{cases} 2.0 & x \leq s_2(t) \\ x - s_2(t) + 2.0 & x > s_2(t) \end{cases} \quad 0 < t \leq 1/4,$$

$$f(x,t) = \begin{cases} 2s_1(t) + 2.0 & x \leq s_1(t) \\ 2.0 & s_1(t)x \leq s_2(t) \\ x - s_2(t) + 2.0 & x > s_2(t) \end{cases} \quad 1/4 < t \leq 3/4,$$

$$f(x,t) = \begin{cases} 2s_1(t) + 2.0 & x \leq s_1(t) \\ 2t + 0.5 & x > s_1(t) \end{cases} \quad 3/4 < t \leq 1,$$

Using the explicit algorithm, the time step τ was reduced until there was negligible change (less than 1.0^{-6}) in the error norms for each of three mesh sizes, h, on a uniform grid. The results are summarized in Table 1. The values of h of 1/10, 1/22, and 1/46 were chosen so that the phase boundary of the exact solution at $T = 1.0$ was aligned with a co–boundary, eliminating the quadrature errors discussed above. As indicated in Table 1, rates of no less than order h are observed. A similar procedure was followed to estimate the rate of convergence with respect to τ. For three time steps $\tau = 1/32$, 1/64, and 1/128, the mesh size was reduced by a factor of two until the implicit equations could no longer be solved. The error norms typically differed in their third significant figure for the last two meshes. As can be observed from Table 2, the rate of convergence with respect to τ is close to unity.

5.2. Two Dimensional Example. A radially symmetric solution in two dimensions was solved on the square $(-1,1)^2$. Since a rectangular grid was used, the radial symmetry should not artificially increase the rates of convergence. A sketch of the solution in the (r,t) plane and a 2×2 mesh ($h = 1$) is shown in Figure 5. Again a solution with a mushy region is selected to eliminate any favorable affects that sharp interfaces may have on the rates of convergence. The exact solution is given by

$$e(r,t) = \begin{cases} \phi(r,t) & \phi(r,t) \leq 0 \\ \psi(r,t) + 1 & \psi(r,t) \geq 0 \text{ and } \phi \geq 0 \\ 0.5 & \text{otherwise} \end{cases}$$

TABLE 2

Rates of Convergence with Respect to τ for the One Dimensional Problem

τ	1/32		1/64		1/128
$\|(e-e_h)(T)\|_{L^1(\Omega)}$	0.009617		0.004894		0.002486
Rates		0.974593		0.977400	
$\|(e-P_h'e_h)(T)\|_{V'}$	0.002537		0.001281		0.000648
Rates		0.986399		0.983229	
$\|u-u_h\|_{L^2[0,T;H]}$	0.019068		0.009591		0.004810
Rates		0.991450		0.995745	
Final Mesh Size	1/1534		1/1534		1/3070

TABLE 3

Rates of Convergence with Respect to h for the Two Dimensional Problem

h	1/8		1/16		1/32		1/64
$\|(e-e_h)(T)\|_{L^1(\Omega)}$	0.1875		0.1237		0.07224		0.03759
Rates		0.2772		0.7760		0.9424	
$\|(e-P_h'e_h)(T)\|_{V'}$	0.02191		0.003629		0.004891		0.002143
Rates		2.5939		-0.4306		1.1905	
$\|u-u_h\|_{L^2[0,T;H]}$	0.01334		0.004704		0.002290		0.000796
Rates		1.5038		1.0385		1.5245	

where $\phi(r,t) = (1/4)(r^2 - e^{2t}/16)$, and $\psi(r,t) = (1/4)(r^2 - (9/16)e^{-2t})$. The corresponding non–homogeneous term is

$$f(r,t) = \begin{cases} \frac{1}{32}e^{2t} - 1 & \phi(r,t) \le 0 \\ \frac{9}{32}e^{-2t} - 1 & \psi(r,t) \ge 0 \text{ and } \phi \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Table 3, analogous to Table 1, was obtained by letting $\tau \to 0$ for each of the meshes $h = 1/8, 1/16, 1/32$, and $1/64$. In all instances, $\tau = 0.5/102400$ sufficed to render the temporal errors negligible. The erratic behavior of the L^1 and V' norms at T suggests that there are significant quadrature errors associated with the numerical integration of e_h and the discontinuous right hand side f. The quadrature errors will depend upon how each of the rectangular meshes can approximate the circular fronts. While it is not clear that the L^1 norm has attained an asymptotic rate, the other two norms do appear to converge, on average, at a near unit rate.

Computational resource limitations prohibited the calculation of rates with respect to τ obtained by letting $h \to 0$. However, a recent result by Rulla [20] has established that for fixed h, the rate of convergence with respect to τ is indeed unity for the Hilbert norms, $\|(e_h - e_h^\tau)(T)\|_{V_h'}$, and $\|u_h - u_h^\tau\|_{L^2[0,T;H]}$ for the implicit scheme. Indeed, the constant of proportionality depends only upon $\|u_h(0)\|_{V_h}$, so that $\|e_h - e_h^\tau\|_{V_h'} \le C\tau$, and $\|u_h - u_h^\tau\|_{L^2[0,T;H]} \le C\tau$ where C is independent of h. Of course, this doesn't determine any rate for $\|(e-e_h)(T)\|_{L^1(\Omega)}$.

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